INCENTIVE CONSTRAINED RISK SHARING, SEGMENTATION, AND ASSET PRICING

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ABSTRACT

Incentive problems make assets imperfectly pledgeable. Introducing these problems in an otherwise canonical general equilibrium model yields a rich set of implications. Asset markets are endogenously segmented. There is a basis going always in the same direction, as the price of any risky asset is lower than that of the replicating portfolio of Arrow securities. Equilibrium expected returns are concave in consumption betas, in line with empirical findings. As the dispersion of consumption betas of the risky assets increases, incentive constraints are relaxed and the basis reduced. When hit by adverse shocks, relatively risk tolerant agents sell the safest assets they hold.

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1 Introduction

Financial markets allow agents to share risk by trading contingent claims, such as futures, options or Credit Default Swaps. These contingent claims are promises to make payments in the future, if some states occur. To deliver on these promises, agents must have assets generating resources they can use to pay counterparties. We refer to these promise-backing assets as collateral. This is a broad definition of collateral, encompassing securities and real assets.

Collateral, however, is imperfectly pledgeable, due to direct and indirect bankruptcy costs. Direct costs arise because when an agent defaults it is costly and difficult for creditors to seize his assets. Similarly to Kiyotaki and Moore (1997) and Kiyotaki (1998), this enables debtors to renegotiate liabilities ex post down to the collateral value minus bankruptcy costs. Ex-ante, this sets a limit to what can be credibly pledged, making collateral imperfectly pledgeable. Indirect costs arise because, when an agent is financially distressed, he no longer has the incentives to maintain the value of his assets. Incentive compatibility constraints reflecting moral hazard and cash diversion problems thus limit pledgeability, as in Bolton and Scharfstein (1990), Holmström and Tirole (1997), DeMarzo and Fishman (2007) and DeMarzo and Sannikov (2006). Significant direct and indirect costs have been documented by Campbell, Giglio, and Pathak (2011) for households' mortgage debt, and Andrade and Kaplan (1998), Bris, Welch, and Zhu (2006) and Davydenko, Strebulaev, and Zhao (2012), for non-financial firms. These costs can also be substantial for financial firms, even when their liabilities benefit from a "safe harbor" provision, as documented by Fleming and Sarkar (2014) and Jackson, Scott, Summe, and Taylor (2011) in case studies of the bankruptcy of Lehman Brothers Holdings Inc. For instance, Fleming and Sarkar (2014, p. 193) write that "it has been alleged that Lehman (...) failed to segregate collateral" and that creditors to these claims "were unable to make recovery through the close-out netting process and became unsecured creditor to the Lehman estate." In addition, "counterparties did not know when their collateral would be returned to them, nor did they know how much they would recover given the deliberateness and unpredictability of the bankruptcy process."

Imperfect collateral pledgeability limits the extent to which agents can credibly promise contingent payments, i.e., issue contingent claims, to share risk in financial markets. The goal of this paper is to study the implications of this friction for risk sharing and asset pricing. To conduct this analysis, we consider a

canonical one-period general equilibrium model with a single consumption good. At time 0, competitive risk averse agents are endowed with shares of assets (or "Lucas trees") generating consumption good output at time 1. Lucas trees are risky and heterogenous: time-1 output varies across states of nature and across trees. The agents can trade these trees but cannot "strip" their cash flows, i.e, they must buy or sell the whole vector of outputs generated by a tree in all the possible states of nature. Agents, however, can also trade a complete set of Arrow securities (one for each state). While Lucas trees are in positive net supply, Arrow securities are in zero-net supply. Any long asset positions, either in trees or in Arrow securities, can be used as collateral.

If collateral was perfectly pledgeable, the first best would be attained in equilibrium. Agents would use the output from their collateral assets to make the payments they promised when issuing Arrow securities. These payments would enable agents to exchange consumption across states and achieve efficient insurance. The consumptions of all agents would comove with aggregate output. In this complete market consumption CAPM world, only the risk associated with aggregate output would be priced. Finally, agents would be indifferent between holding a tree and a corresponding replicating portfolio of Arrow securities, since both would have the same price and the same payoff. As a result, the allocation of trees would be indeterminate.

In contrast, we analyze equilibrium when collateral is imperfectly pledgeable. Because of the above discussed direct and indirect bankruptcy costs, an agent cannot pledge all the output of his assets: Incentive compatibility implies that his pledgeable income in a given state is only a fraction of his tree portfolio's output in that state. This limits the state contingent transfers agents can promise one another and hence constrains the provision of insurance. Thus, while agents can trade a complete set of Arrow securities, risk sharing can be imperfect and the market endogenously incomplete. We show in this context that equilibrium exists and is constrained Pareto optimal (constrained optimality of equilibrium obtains because, in a one-period model, prices do not appear in incentive compatibility constraints.)

Equilibrium risk sharing is imperfect only if the first best consumption profile cannot be achieved with an allocation in which all agents only hold long positions in trees, that is, only if agents must establish short positions to share risk.[1] Without short positions, there would no liabilities and no scope for default, so that

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[1]When short positions are needed to implement risk sharing, it is optimal to issue Arrow securities, rather than shorting trees, because the latter creates unnecessary liabilities (and therefore tighten incentive constraints), which the former can avoid.

imperfect pledgeability would not be an issue.

When equilibrium short positions imply incentive constraints bind (so that risk sharing is imperfect), the prices of trees are equal to the value of their output, evaluated at Arrow securities prices, minus the shadow cost of the incentive constraints. This delivers two related implications for asset prices:

First, there is a basis between the price of a tree and that of replicating portfolios of Arrow securities, even when trees and Arrow securities are equally pledgeable. Furthermore, the basis always go in the same direction, as each tree is priced *below* its replicating portfolio of Arrow securities. The basis is a deviation from the law of one price but does not constitute an arbitrage opportunity. In order to conduct an arbitrage trade, an agent would need to sell Arrow securities and use the proceeds to buy assets. This is precluded by the incentive constraint: if the agent sold these Arrow securities, this would increase his liabilities, leading to a violation of his incentive constraint.[2]

Second, equilibrium expected excess returns reflect two premia. The first premium increases in the covariance between the tree payoff and the consumption of an unconstrained agent, whose identity varies across states. The second premium increases with the covariance between the tree payoff and the shadow price of the incentive compatibility constraint of the agents holding it.

Our model also delivers implications for tree holdings. Since incentive constraints limit agents' ability to promise future payments via the sale of Arrow securities, agents find it optimal to hold portfolios of risky trees with payoff as close as possible to their desired state-contingent consumption profile. Indeed, these portfolios allow them to limit their purchases (resp. sales) of Arrow securities, for states in which their desired consumption profile is above (resp. below) the output of their risky asset portfolio. Thus, even though the Arrow security market is complete, the tree market is endogenously segmented: agents differing in terms of risk aversion find it optimal to hold different types of trees in equilibrium.

To further illustrate equilibrium properties, we consider the simple 2-by-2 case in which there are two states, two agents' types with different relative risk aversion, one more risk tolerant and the other more-risk averse, and an arbitrary distribution of trees. Equilibrium in this simple case is as follows: As in the first

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[2]In contrast, the opposite situation, in which a tree would be worth more than the replicating portfolio of Arrow securities, could be arbitraged. An agent holding the tree (some agents must, by market clearing), could sell the tree and use the proceeds to buy the replicating portfolio. In contrast with the case in which the tree is worth less than the replicating portfolio, this arbitrage can be conducted without taking on state contingent liabilities, and therefore does not tighten incentive constraints.

best (but to a lower extent), the consumption share of the risk tolerant agent is smaller in the bad state than in the good state. That is, the more risk tolerant agent offers some insurance to the more risk averse agent. To implement this consumption allocation, the more risk tolerant agent sells (resp. buys) Arrow securities that pay in the bad (resp. good) state. Hence his incentive compatibility constraint binds in the bad state and is slack in the good state. To mitigate incentive problems, the more risk averse agent holds trees with relatively large payoff in the bad state, so that he needs to purchase less Arrow securities from the risk tolerant agent. That is, the more risk averse agent holds trees with low consumption beta, while, by market clearing, the more risk tolerant agent holds trees with high consumption beta. This is a deviation from the two-fund separation principle, which illustrates how segmentation arises in equilibrium in our model. This simple 2-by-2 case delivers a rich set of implications.

First, expected excess returns are concave in consumption betas, in line with Black (1972) and recent evidence by Frazzini and Pedersen (2014) and Hong and Sraer (2016). The intuition is the following: Within the set of high beta assets held by the risk tolerant agent, the riskier ones have higher cash flow in the good state (in which his incentive compatibility condition is slack) and lower cash flow in the bad state (in which his incentive compatibility condition binds). Consequently, the riskiest assets held by the risk tolerant agent command a lower premium than the safer assets held by this agent. Symmetrically, within the set of low beta assets held by the more risk averse agent, the safest ones command a relatively low premium, while the riskier ones command a larger premium. Thus expected excess returns are concave in beta. This contrasts with the theoretical analysis of Frazzini and Pedersen (2014), in which the security market line is flat but linear.

Second, holding aggregate risk and aggregate pledgeable income constant, the distribution of aggregate output across trees in each state matters for equilibrium outcomes. Namely, incentive constraints are less likely to impact equilibrium if the dispersion of security betas is larger. The intuition is the following. Increasing the supply of low beta assets enables the more risk averse agents to hold more assets with payoff in the bad state, reducing their demand for Arrow securities contingent on that state. This reduces the sales of these Arrow securities by the more risk tolerant agents, thus relaxing their incentive constraints. This underscores the key role of our assumption that the same production technology (tree) generating output

in different states cannot be split into several productive assets, each generating output in one state only, so that only Arrow securities can transfer payoff across states. In practice, the dispersion of security betas might be lower when many firms are conglomerates; when the productive sector relies on a relatively small number of technologies; or when many corporations are financially distressed so that safe assets are scarce. Our analysis suggests that an instrument for public policy could be to modify the distribution of securities risk profiles, which may be one of the aspects of non-conventional monetary policy.

Third, our analysis sheds light on the consequences of shocks worsening incentive problems. Suppose the more risk tolerant agent is subject to a negative shock, increasing his shadow cost of holding assets. This has joint implications for equilibrium holdings and pricing: The agent sells his least risky holdings, for which his comparative advantage is the lowest. At the same time, the basis increases for all the assets initially held by the agent, so that there is comovement among these assets.

Literature: Our model is in line with the limited commitment literature, which studies risk-sharing and asset pricing when agents cannot commit not to default on their promises, see Kehoe and Levine (1993, 2001), Holmström and Tirole (2001), Alvarez and Jermann (2000), Chien and Lustig (2009) and Gottardi and Kubler (2015). In these papers, and also in ours, while there is a complete set of Arrow securities, incentive constraints prevent full risk-sharing. This affects asset prices, as the stochastic discount factor for each state reflects the shadow cost of the incentive constraint for that state. However, these papers assume that some income (interpreted either as labour or corporate income) cannot be seized nor traded, but tradeable assets can be seized. In contrast, we assume that all income is generated by assets that can be traded and partially seized. This difference in assumption generates a difference in results: In the limited commitment literature, the law of one price holds.[3] In contrast, our model exhibits equilibrium deviations from the law of one price. Different state prices are applied to different state-contingent claims. Thus one contribution of our analysis relative to the limited commitment literature is to show that there is a basis between the prices of tradeable and imperfectly pledgeable assets and the prices of replicating portfolios of

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[3]For example, Alvarez and Jermann (2000) write (on page 776): "The price of an arbitrary asset is calculated by adding up the prices of the corresponding contingent claims." Likewise, in Holmström and Tirole (2001), there is a single vector of state prices that is used to calculate the value of all claims, either corporate or non-corporate (see equation (20) and (23) on page 1849).

Arrow securities.[4]

Deviations from the law of one price have been obtained by another important strand of literature, in particular Hindy and Huang (1995), Aiyagari and Gertler (1999), Gromb and Vayanos (2002, 2017), Coen-Pirani (2005), Fostel and Geanakoplos (2008), Gârleanu and Pedersen (2011), Geanakoplos and Zame (2014), and Brumm, Grill, Kubler, and Schmedders (2015). That literature differs from our paper and from the limited commitment literature in that it specifies plausible but exogenous financial constraints, which are not microfounded as resulting from incentive compatibility conditions. In contrast, in our paper and in the limited commitment literature, financial constraints stem from the incentive compatibility condition that the agent must prefer to hold his promises rather than deviating.[5] Thus, while market equilibrium relies on the exchange of trees and a complete set of Arrow securities, the resulting allocation can also be interpreted as the outcome of an optimal contracting process. For example, in our model in which equilibrium is constrained Pareto efficient, the equilibrium allocation is equal to that arising in the optimal contract designed by the central planner subject to incentive compatibility constraints.

Gârleanu and Pedersen (2011) offer an insightful analysis of asset pricing under margin constraints. In their model, the constraint is that the sum across assets of each asset's exogenous margin multiplied by the absolute value of the asset weight in the portfolio must be lower than or equal to one. There is one shadow price for this constraint. The basis between an asset and a replicating derivative portfolio is equal to this shadow price, multiplied by the difference in exogenous margin constraints between the asset and the replicating derivative. In our endogenous constraints model, in contrast, as in the limited commitment literature, there is one incentive compatibility constraint per state, and its shadow price is impounded in the pricing kernel. Thus, the basis between an asset and a replicating portfolio reflects the covariance (across states) between the asset's payoff and the shadow price of the constraint. This leads to our empirical prediction, absent from Gârleanu and Pedersen (2011) that expected excess returns are concave in consumption betas. Another distinct empirical prediction is that, in our model, all bases go in the same direction: assets are always priced below replicating derivative portfolio. Moreover, our predictions

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[4]Rampini and Vishwanathan (2017) also study risk-sharing under financial constraints (between a risk averse hedger and a risk neutral intermediary, via state-contingent debt). Their analysis, which does not consider the trading of assets, differs from ours, which focuses on pricing the cross section of assets.

[5]In this context the payoff of the agent in case of deviation is explicitly specified. For example in Alvarez and Jermann (2000) agents must revert to autarky, while in Chien and Lustig (2009) the agents' holdings of a Lucas tree are seized, and in our model a fraction of the output from the Lucas trees held by the agent is seized.

about bases arise without assuming exogenous differences in pledgeability between derivatives and underlying assets.

Fostel and Geanakoplos (2008), Geanakoplos and Zame (2014), Brumm, Grill, Kubler, and Schmedders (2015) and Gromb and Vayanos (2002, 2017) offer seminal analyses of general equilibrium under collateral constraints. In that literature, each financial promise must be backed by its own collateral.[6] In our framework, by contrast, the constraint applies to the portfolio of assets and Arrow securities of an agent, in line with the practice of portfolio margining.[7] As mentioned above, a key finding in our framework is that assets trade at a discount relative to replicating portfolios of derivatives. This could seem to contradict the collateral premium obtained by Fostel and Geanakoplos (2008), Geanakoplos and Zame (2014), and Brumm, Grill, Kubler, and Schmedders (2015).[8] There is no contradiction, however: it is simply that the benchmark valuation is not the same for the premium and the basis results. The *premium*, which is the difference between the price of the asset and its value evaluated at the marginal utility of the agent holding it, obtains in these papers as well as in ours. Analyzing the *discount* at which real assets trade relative to replicating portfolios of Arrow securities is a contribution of our paper relative to that literature.

Gromb and Vayanos (2002, 2017) also assume that markets are exogenously segmented. In contrast, our framework leads to endogenous market segmentation. This enables us to pin down new features and implications of segmentation. In our model, agents hold assets whose final payoffs are correlated (instead of diversifying). Thus, more risk averse agents hold low consumption beta assets, while more risk tolerant agents hold high beta assets. This implies that, in response to a shock unrelated to fundamental payoffs (e.g., a risk aversion or pledgeability shock), assets whose fundamental payoffs are correlated will comove, i.e., react similarly to the shock, because they are held by the same agents.

The next section presents our model. Section 3 defines equilibrium and states its optimality properties. Section 4 presents general results on equilibrium. Section 5 presents more specific results, obtained when there are only two types of agents.

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[6]So the same asset, generating strictly positive output in two states, cannot be used to collateralize the issuance of two Arrow securities, promising payments in these two states.

[7]For example, on http://www.cboe.com/products/portfolio-margining-rules, one can read: "The portfolio margining rules have the effect of aligning the amount of margin money ... to the risk of the portfolio as a whole, calculated through simulating market moves up and down, and accounting for offsets between and among all products held..."

[8]Similarly, new monetarist analyses point to a "liquidity premium" when assets can be used as means of payment: see for example Lagos (2010), Li, Rocheteau, and Weill (2012), Lester, Postlewaite, and Wright (2012), Venkateswaran and Wright (2013), and Jacquet (2015) who show how this premium varies with asset risk.

2 Model

2.1 Assets and Agents

There are two dates $t = 0, 1$. The state of the world ω realizes at $t = 1$ and is drawn from some finite set Ω according to the probability distribution $\{\pi(\omega)\}_{\omega \in \Omega}$, where $\pi(\omega) > 0$ for all ω. All real resources are the dividends of assets referred to as "trees." The set of tree types is taken to be a compact interval that we normalize to be $[0, 1]$, endowed with its Borel σ-algebra. The distribution of asset supplies is a positive and finite measure $\bar{N}$ over the set $[0, 1]$ of tree types. We place no restriction on $\bar{N}$: it can be discrete, continuous, or a mixture of both. The payoff of tree $j \in [0, 1]$ in state $\omega \in \Omega$ is denoted by $d_j(\omega) \geq 0$, with at least one strict inequality in some state $\omega \in \Omega$. A technical condition for our existence proof is that, for all $\omega \in \Omega$, $j \mapsto d_j(\omega)$ is continuous. Economically, continuity means that trees are finely differentiated: nearby trees in $[0, 1]$ have nearby characteristics. Continuity is a mild assumption since we do not impose any restriction on the distribution $\bar{N}$ of supplies.[9] As will become clear later, we consider a continuum of trees for two reasons. First, it will demonstrate clearly that our results do not arise because the span of tree payoffs is incomplete. Second, in Section 5, it will make it easier to explicitly characterize patterns of segmentations.

The economy is populated by finitely many types of agents, indexed by $i \in I$. The measure of type $i \in I$ agents is normalized to one. Agents of type $i \in I$ have Von Neumann Mortgenstern utility

$$U_i(c_i) \equiv \sum_{\omega \in \Omega} \pi(\omega) u_i \left[c_i(\omega) \right]$$

over time $t = 1$ state-contingent consumption. We take the utility function to be either linear, $u_i(c) = c$, or strictly increasing, strictly concave, and twice-continuously differentiable over $c \in (0, \infty)$. Without loss of generality, we apply an affine transformation to the utility function $u_i(c)$ so that it satisfies either $u_i(0) = 0$; or $u_i(0) = -\infty$ and $u_i(\infty) = +\infty$; or $u_i(0) = -\infty$ and $u_i(\infty) = 0$. In addition, if $u_i(0) = -\infty$ we assume that there exists some $\gamma_i > 1$ such that, for all c small enough, $\frac{u'_i(c)c}{|u_i(c)|} \leq (\gamma_i - 1)$. This implies the Constant Relative Risk Aversion (CRRA) bound $0 \geq u_i(c) \geq Kc^{1-\gamma_i}$ for all c small enough and some negative constant

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[9]For example, our model nests a standard specification with finitely many trees $k \in \{1, \ldots, K\}$, with state-contingent payoff $D^{(k)}(\omega)$ and positive supplies $S^{(k)} > 0$. Indeed, this discrete specification obtains by fixing a finite sequence $j_1 < j_2, \ldots < j_K$, choosing any continuous function $j \mapsto d_j(\omega)$ such that $d_{j_k}(\omega) = D^{(k)}(\omega)$ for $k \in \{1, \ldots, K\}$, and letting $\bar{N}$ be a discrete distribution with atoms at $j_1 < j_2, \ldots < j_K$, $N_{j_k} - N_{j_k-} = S^{(k)}$.

K.[10]

Finally, we assume that, at time $t = 0$, agent $i \in I$ is endowed a strictly positive share, $\bar{n}_i > 0$, in the market portfolio. Of course, agents' shares in the market portfolio must add up to one, that is $\sum_{i \in I} \bar{n}_i = 1$.

2.2 Markets, Budget Constraints, and Incentive Compatibility

Markets. At time zero, agents trade two types of assets: trees, and a complete set of Arrow securities. While trees are in positive supply, Arrow securities are in zero net supply.

We assume that agents must choose positive tree holdings. Formally, they choose a portfolio N from the set $\mathcal{M}_+$ of positive and finite measures over the set of tree types, $[0, 1]$. Positivity means that agents cannot own a negative fraction of a firm. However, we allow them to take short positions by selling a complete set of Arrow securities, subject to borrowing constraints specified below. Therefore we are explicit about the fact that short positions are liabilities, and we model these liabilities as negative positions in a portfolio of Arrow securities. The vector of agent i's positions in each of the Arrow securities is denoted by $a_i \equiv \{a_i(\omega)\}_{\omega \in \Omega}$. The position $a_i(\omega)$ can be positive (if the agent buys the Arrow security) or negative (if the agent sells the Arrow security).[11]

Budget constraints. A *price system* for trees and Arrow securities is a pair (p, q), where $p : j \mapsto p_j$ is a continuous function for the price of tree j,[12] and $q = \{q(\omega)\}_{\omega \in \Omega}$ is a vector in $\mathbb{R}^{|\Omega|}$ for the prices of Arrow securities. Given the price system (p, q), the time-zero budget constraint for agent i is:

$$\sum_{\omega \in \Omega} q(\omega) a_i(\omega) + \int p_j \, dN_{ij} \leq \bar{n}_i \int p_j \, d\bar{N}_j. \tag{1}$$

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[10]We use this technical condition in our existence proof to show that the maximum correspondence of the social planner's problem has a weakly closed graph and that, at points where some of the welfare weights are equal to zero, the maximized social welfare function is continuous in welfare weights. See the proof of Proposition A.1 in the Supplementary Appendix.

[11]In Appendix B.9 we explicitly allow for short sale of trees. We show that short-selling a tree is always weakly dominated by selling the corresponding replicating portfolio of Arrow securities. This justifies our assumption that all short positions must take the form of sales of Arrow securities.

[12]Hence, we assume that the price functional admits a dot-product representation based on a continuous function, p_j, of tree type. This is a restriction: in full generality one should allow for any continuous linear functional, which may not have such a representation. However, given our maintained assumption that $j \mapsto d_j(\omega)$ is continuous, this restriction turns out to be without loss of generality. Namely, one can show that *any* equilibrium allocation can be supported by a price functional with such a dot-product representation, based on a continuous function of tree types. See the paragraph before Proposition A.3 page 45.

At time one, agent i's consumption must satisfy:

$$c_i(\omega) = a_i(\omega) + \int d_j(\omega) \, dN_{ij}. \tag{2}$$

We denote the state-contingent consumption plan by $c_i \equiv \{c_i(\omega)\}_{\omega \in \Omega}$.

Incentive compatibility constraints. At time $t = 1$, the agent is supposed to follow the consumption plan given in (2). Instead, the agent could strategically default on his contractual obligations. As discussed in the introduction in the Lehman Brother's case, when counterparties default, collateral recovery is costly and imperfect. Debtors can take advantage of such imperfect recoverability to renegotiate liabilities. To capture this process in the simplest possible way, we assume an agent can make a take-it-or-leave-it offer to his creditors, who, if they refuse this offer can only seize fraction $1 - \delta \in (0, 1]$ of the agent's assets. Hence the consumption plan $c_i(\omega)$ must give agent i no less than a fraction δ of the positive payoff of his trees and Arrow security long positions in state $\omega \in \Omega$.[13] An alternative interpretation of δ is that it quantifies how much resources agents can divert, as in the theoretical analyses of Bolton and Scharfstein (1990), Holmström and Tirole (1997), DeMarzo and Fishman (2007) and DeMarzo and Sannikov (2006). Thus, in the terminology of Holmström and Tirole (1997), $1 - \delta$ is the pledgeable fraction of the agent's income.

Formally, consider agent i's incentives to strategically default in state $\omega \in \Omega$. The agent has two types of assets. First, the payoff $\int d_j(\omega) \, dN_{ij}$ of his long tree portfolio. Second, the payoff $a_i^+(\omega)$ of his long Arrow security position. The agent's liability is a short Arrow position, $a_i^-(\omega)$. The net position in the state ω Arrow security is $a_i(\omega) = a_i^+(\omega) - a_i^-(\omega)$. If the agent strategically defaults and renegotiates his liabilities or diverts, he obtains a fraction δ of his *long* positions in trees and Arrow securities, and he consumes:

$$\hat{c}_i(\omega) = \delta \int d_j(\omega) \, dN_{ij} + \delta \, a_i^+(\omega). \tag{3}$$

The incentive compatibility condition is such that the agent prefers repaying his promise rather than strate-

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[13]Here we assume for simplicity that δ is constant across agents and assets. In the appendix all our proofs cover the generalized case in which it is a continuous function δ_{ij} of the identity i of the agent and of the type j of the asset. The results obtained in the special case in which δ is constant generalise straightforwardly to the general case δ_{ij} and the main economic intuitions are unchanged.

gically defaulting:

$$c_i(\omega) \geq \hat{c}_i(\omega),$$

where $c_i(\omega)$ is given in (2) and $\hat{c}_i(\omega)$ in (3). Substituting (2) and (3) into the above equation, we obtain that the incentive constraint can be rewritten as a state-contingent borrowing constraint:

$$a_i^-(\omega) \leq (1-\delta)\left[\int d_j(\omega)\, dN_{ij} + a_i^+(\omega)\right]. \tag{4}$$

The left-hand side is the agent's liability in state ω. The right-hand side is the pledgeable part of the agent's assets in state ω. An immediate implication of constraint (4) is:

Lemma 1 *It is always weakly optimal to choose an Arrow position such that $a_i^+(\omega) = 0$ or $a_i^-(\omega) = 0$.*

Indeed, if $a_i^+(\omega) > 0$ and $a_i^-(\omega) > 0$, the agent could reduce both positions equally by some small amount. Because this does not change the net position, $a_i(\omega) = a_i^+(\omega) - a_i^-(\omega)$, the agent can keep his consumption the same. But this would relax (4) because the left-hand side would decrease by more than the right-hand side. Economically, this means that it is suboptimal to purchase Arrow assets, $a_i^+(\omega)$, in order to increase borrowing in the corresponding Arrow liabilities, $a_i^-(\omega)$. Indeed, increasing the long Arrow position by one unit only allows to increase the short position by $(1-\delta) < 1$. While this indeed increases the agent's gross borrowing, the net borrowing actually decreases.

Lemma 1 allows us to focus on the case in which agents choose Arrow positions such that $a_i^+(\omega) = 0$ or $a_i^-(\omega) = 0$. A key implication is that whenever an agent has a long Arrow position, he does not have any simultaneous short position and (4) is slack. In other words, long Arrow positions never create any incentive problems. As will be clear shortly, long tree positions, on the other hand, may create incentive problems.

The Lemma also leads to a simpler representation of (4) in terms of net Arrow position. Namely, if $a_i^-(\omega) > 0$, then $a_i^+(\omega) = 0$, and (4) writes as

$$-a_i(\omega) \leq (1-\delta)\int d_j(\omega)\, dN_{ij}. \tag{5}$$

If $a_i^+(\omega) > 0$, then $a_i^-(\omega) = 0$, (4) is slack, and (5) holds as well. Conversely, given $a_i^-(\omega) = 0$ or $a_i^+(\omega) = 0$,

if (5) holds, then the original constraint (4) holds too. The next step is to use (2) in order to express $a_i(\omega)$ in terms of consumption and tree payoff, so we obtain the equivalent incentive compatibility condition:

$$c_i(\omega) \geq \delta \int d_j(\omega) \, dN_{ij}, \tag{6}$$

for all $\omega \in \Omega$, where the left-hand side is the consumption plan of the agent, and the right-hand side is the part of the dividend flow the agent could get by renegotiating his liabilities, or, equivalently, diverting resources. We hereafter refer to this part of the dividend flow as the non-pledgeable dividend.

As shown by equation (4), the state-contingent payoff of trees serves as collateral for the state-contingent liability of the agent. But the amount the agent can promise is lower than the face value of the collateral, because of incentive constraints. The wedge between the output/collateral and the maximum promised payment can be interpreted as a haircut. Haircuts are increasing in δ. Haircuts are not imposed on an individual asset basis, but at the level of the aggregate position, or portfolio of the agent. This is in line with the practice of "portfolio margining."

Note that the haircut is not imposed by the regulator. It is requested by the private contracting agents to limit counterparty risk. There is however an aspect of that requirement that cannot be completely decentralized. The incentive compatibility constraint of agent i involves the Arrow securities traded by agent i with all other agents in the economy. These multiple trades must be aggregated (and cleared) to determine the total exposure of agent i to state ω, and then compared to the assets of the agent, imputing the right haircuts. This can be the role of the Central Clearing Party (CCP), which in our model can centralize and clear all trades to ensure incentive compatibility, and thus deliver a better outcome than the outcome which would arise with bilateral contracting only.

3 Definition and Elementary Properties of Equilibrium

3.1 Agent's Problem and Definition of Equilibrium

As is standard, the consumption plan, c_i, and the tree holdings, N_i, satisfy the time-zero budget constraint (1) and the time-one budget constraint (2), if and only if they satisfy the inter-temporal budget constraint:

$$\sum_{\omega \in \Omega} q(\omega) c_i(\omega) + \int p_j \, dN_{ij} \le \bar{n}_i \int p_j \, d\bar{N}_j + \sum_{\omega \in \Omega} q(\omega) \int d_j(\omega) \, dN_{ij}. \tag{7}$$

Both the budget constraint (7) and the incentive compatibility constraint (6) are only a function of (c_i, N_i), and do not depend on the Arrow security holdings a_i. Hence, we can define the consumption set of agent $i \in I$ to be $X_i \equiv \mathbb{R}_+^{|\Omega|} \times \mathcal{M}_+$, the product of the set of positive state contingent consumption plans and of the set of positive tree holdings. *The problem of agent i is, then, to maximize $U_i(c_i)$ with respect to* $(c_i, N_i) \in X_i$, subject to the intertemporal budget constraint (7) and the incentive compatibility condition (6).

Let X denote the cartesian product of all agents' consumption set. An *allocation* is a collection $(c, N) = (c_i, N_i)_{i \in I} \in X$ of consumption plans and tree holdings for every agent $i \in I$. An allocation (c, N) is *feasible* if it satisfies:

$$\sum_{i \in I} c_i(\omega) \le \sum_{i \in I} \int d_j(\omega) \, dN_{ij} \text{ for all } \omega \in \Omega \tag{8}$$

$$\sum_{i \in I} N_i = \bar{N}. \tag{9}$$

An *equilibrium* is a feasible allocation (c, N) and a price system (p, q) such that, for all $i \in I$, (c_i, N_i) solves agent's i problem given prices.

Our formulation of the agent's problem clearly shows that, since trees are tradeable and the Arrow security market is complete, an agent can always escape any incentive problems by setting $N_i = 0$. That is, the agent can always sell all of his tree holdings, and only purchase Arrow securities. But, of course, this cannot be part of a tree-market equilibrium: if all agents sell all of their trees, the market clearing condition for trees, (9), does not hold. This intuitively implies our result about bases: for the tree market to clear,

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the price of trees will have to fall below that of Arrow securities. This also highlights that, in our model, binding incentive compatibility constraints is a general equilibrium phenomenon. This is in contrast with earlier models in which non-pledgeable income is not tradeable: in these environments, binding incentive constraints would already arise in partial equilibrium contract-theoretic settings.

3.2 Incentive-Constrained Pareto Optimality

An allocation $(c, N) \in X$ is said to be *incentive-feasible* if it satisfies the incentive compatibility constraints (6) for all $(i, \omega) \in I \times \Omega$, and the feasibility constraint (8) and (9). An incentive-feasible allocation $(\hat{c}, \hat{N})$ *Pareto dominates* the incentive-feasible allocation (c, N) if $U_i(\hat{c}_i) \geq U_i(c_i)$ for all $i \in I$, with at least one strict inequality for some $i \in I$. An allocation is *incentive-constrained Pareto optimal* if it is incentive-feasible and not Pareto dominated by any other incentive-feasible allocation. In our model, we have:

Proposition 1 *Any equilibrium allocation is incentive-constrained Pareto optimal.*

The reason why optimality obtains in spite of incentive constraints is because prices do not show up in the incentive compatibility condition, so that there are no "contractual externalities". See Prescott and Townsend (1984) and Kehoe and Levine (1993) for other examples of economies in which equilibrium is constrained optimal, and Gottardi and Kubler (2015) for examples in which it is not. Because there are no contractual externalities, the proof of Proposition 1 is similar to its perfect market counterpart: if an equilibrium allocation were Pareto dominated by another incentive feasible allocation, the latter must lie outside the agents' budget set. Adding up across agents leads to a contradiction.

3.3 Existence and Uniqueness

To prove existence of equilibrium, we follow Negishi (1960). We consider the problem of a Planner who assigns Pareto weights $\alpha_i \geq 0$ to each agent $i \in I$, with $\sum_{i \in I} \alpha_i = 1$, and chooses incentive feasible allocations to maximize the social welfare function, $\sum_{i \in I} \alpha_i U_i(c_i)$. We establish the existence of Pareto weights such that, given agents' initial endowment, the social optimum can be implemented in a competitive equilibrium without making any wealth transfers between agents.

Proposition 2 *There exists an equilibrium.*

The proof follows arguments found in Negishi (1960), Magill (1981), and Mas-Colell and Zame (1991) with some differences. First, our planner is now subject to incentive compatibility constraints. Second, technical difficulties arise because the commodity space is infinite dimensional which makes it harder to apply separation theorems. We solve these difficulties by explicitly deriving first-order necessary and sufficient conditions for the Planner's problem, and using the associated Lagrange multipliers to construct equilibrium prices.

We can show uniqueness in a particular case of interest:

Proposition 3 *Suppose that there are two types of agents, $I = \{1, 2\}$, with CRRA utility, with respective RRA coefficients (γ_1, γ_2) such that $0 \leq \gamma_1 \leq \gamma_2 \leq 1$ and $\gamma_2 > 0$. Then the equilibrium consumption allocation is uniquely determined. The prices of Arrow securities and the price of trees are, $\bar{N}$-almost everywhere, uniquely determined up to a positive multiplicative constant.*

In general, the asset allocation is not uniquely determined. As will be clear below, this arises for example when none of the incentive constraints bind. In that case the allocation is not uniquely determined because it is equivalent to hold tree j or a portfolio of Arrow securities with the same cash-flows as j.

As is standard, only relative prices are pinned down, hence price levels are only determined up to a positive multiplicative constant.

Finally, asset prices are only uniquely determined $\bar{N}$-almost everywhere. In particular, the prices of assets in zero supply are not uniquely determined. This is intuitive: given the short-sale constraint, the only equilibrium requirement for an asset in zero supply is that the price is large enough so that no agent want to hold it. As a result equilibrium only imposes a lower bound on the price of trees in zero supply.[14]

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[14]To remove the indeterminacy, it would be natural to inject a small additional supply for all trees, $\bar{N} + \varepsilon \mathcal{U}_{[0,1]}$ and let $\varepsilon \to 0$.

4 Equilibrium Analysis

4.1 First Best Implementability

We first study circumstances under which the incentive compatibility constraints do not impact equilibrium outcomes.

Formally, define a $\delta = 0$ equilibrium to be a first-best allocation and price system (c^0, N^0, p^0, q^0) when $\delta = 0$, i.e., when there is no scope for incentive problems. Fix some $\delta > 0$. Then, the $\delta = 0$-equilibrium is said to be $\delta > 0$-*implementable* if there exists some $\delta > 0$-equilibrium, $(c^\delta, N^\delta, q^\delta, p^\delta)$, such that $c^0 = c^\delta$. It is clear that a $\delta = 0$-equilibrium, (c^0, N^0, p^0, q^0), is $\delta > 0$-implementable if and only if there exists a feasible asset allocation, there exists some N^δ, such that incentive constraints are satisfied for all agents and all states:

$$\sum_{i \in I} N_i^\delta = \bar{N} \tag{10}$$

$$c_i^0(\omega) \geq \delta \int d_j(\omega) \, dN_{ij}^\delta \quad \forall (i, \omega) \in I \times \Omega. \tag{11}$$

Based on this observation, we obtain:

Proposition 4 *Fix some $\delta > 0$. A $\delta = 0$ equilibrium (c^0, n^0, p^0, q^0) is $\delta > 0$-implementable if one of the following conditions is satisfied:*

- *Inada conditions are satisfied for all $i \in I$ and δ is strictly positive but small enough.*

- *There exists $\{N_i\}_{i \in I} \in \mathcal{M}_+^{|I|}$ such that $\sum_{i \in I} N_i = \bar{N}$ and $\int d_j(\omega) \, dN_{ij} = c_i^0(\omega) \; \forall (i, \omega) \in I \times \Omega$.*

- *Agents have Constant Relative Risk Aversion (CRRA) with identical coefficient.*

To understand the first bullet point, note that, with Inada conditions, consumptions are strictly positive for all agents and all states. Therefore, as long as δ is small enough, the incentive compatibility constraint (11) is satisfied for all agents and all states when each agent holds, say, an equal fraction of the market portfolio, $N_i = \bar{N}/|I|$, and simultaneously issues liabilities to attain its first-best consumption plan.

The second bullet point of the proposition states that all incentive compatibility constraints hold if two sets of conditions are satisfied. First agents can replicate their $\delta = 0$-equilibrium consumption with *positive* holdings of trees. Second, these agents holding are *feasible*, i.e., they add up to the aggregate. This means that they do not need to make any financial promise, i.e., promise to deliver consumption out of the payoff of their equilibrium holdings of trees. Clearly, if agents do not need to make any financial promise, limited pledgeability is not an issue.

The third bullet point is an example of the second: if agents have CRRA utilities with identical risk aversion, then they all consume a constant share of the aggregate endowment. Clearly, they can attain that consumption plan by holding a portfolio of trees, namely a constant share in the market portfolio.

Taken together, equations (10)-(11) and Proposition 4 help understand circumstances under which a $\delta = 0$ equilibrium cannot be implemented. Consider for example an economy composed of CRRA utility agents with heterogenous risk aversion, and assume that there is only one tree, the "market portfolio", with payoff equal to aggregate consumption. Because of heterogeneity in risk aversion, in the $\delta = 0$ equilibrium, agents' consumption shares vary across states – namely more risk averse agents tend to have higher consumption shares in states of low aggregate consumption. If $\delta \simeq 1$, agents cannot issue liabilities and their consumption must be approximately equal to the payoff of their tree portfolio. But since they can only hold the market portfolio, their consumption share must be approximately constant across states, so that the $\delta \simeq 1$ equilibrium cannot coincide with the $\delta = 0$ equilibrium.

4.2 Optimality Conditions

Since agents have concave objectives and are subject to finitely-many affine constraints, we can apply the Lagrange multiplier Theorems shown in Section 8.3 and 8.4 of Luenberger (1969) (see Proposition A.4 in the appendix for details). Let λ_i denote the Lagrange multiplier of the intertemporal budget constraint (7) and $\mu_i(\omega)$ the Lagrange multiplier of the incentive compatibility constraint (6). The first-order condition with respect to $c_i(\omega)$ is[15]

$$\pi(\omega)u_i'[c_i(\omega)] + \mu_i(\omega) = \lambda_i q(\omega). \tag{12}$$

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[15]In principle this condition only hold with an inequality if $c_i(\omega) = 0$, which may occur when utility is linear. However, we show in the Appendix (Proposition A.4) that one can always choose multipliers so that this condition holds at equality.

The first term on the left-hand side of (12) reflects that an increase in consumption increases utility and the second term reflects that it relaxes the incentive compatibility constraint (6), while the term on the right-hand side reflects that this increase in consumption tightens the budget constraint.

The first-order condition with respect to N_i is

$$\sum_{\omega \in \Omega} \mu_i(\omega)\delta d_j(\omega) \geq \lambda_i \left[\sum_{\omega \in \Omega} q(\omega) d_j(\omega) - p_j \right],\tag{13}$$

with an equality N_i–almost everywhere, that is, for almost all trees held by agent i. The left-hand side of (13) reflects that an increase in i's holdings of tree j increases the amount of non-pledgeable dividend $\delta d_j(\omega)$, which tightens the incentive constraint (6) for each ω. The right-hand side reflects that an increase in i's holdings of tree j relaxes the intertemporal budget constraint by $\sum_\omega q(\omega) d_j(\omega) - p_j$.

4.3 Asset Pricing

The pricing of risk and incentives. The first order condition with respect to consumption, (12), shows that if the incentive compatibility conditions were slack, the marginal rate of substitution between consumptions in different states would be equal across all agents, as in the standard, perfect and complete markets, model. When incentive compatibility conditions bind, in contrast, marginal rates of substitution differ across agents, reflecting the multipliers of the incentive constraints. This reflects imperfect risk-sharing in markets that are endogenously incomplete due to incentive constraints, as in Alvarez and Jermann (2000). Thus the Arrow securities pricing kernel

$$M(\omega) \equiv \frac{q(\omega)}{\pi(\omega)},$$

which in our model prices the Arrow securities, differs from its perfect and complete markets counterpart because in general, there is no agent whose marginal utility is equal to $M(\omega)$ *in all states*. Instead, $M(\omega)$ corresponds to the marginal utility of an unconstrained agent, whose type varies from state to state.

The first order condition with respect to asset holdings, (13), rewrites as

$$p_j \geq \sum_{\omega \in \Omega} q(\omega) d_j(\omega) - \delta \sum_{\omega \in \Omega} \frac{\mu_i(\omega)}{\lambda_i} d_j(\omega),\tag{14}$$

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with an equality for almost all trees held by agent i. Denote

$$A_i(\omega) \equiv \frac{\mu_i(\omega)}{\lambda_i \pi(\omega)},$$

which can be interpreted as the shadow cost of the incentive compatibility constraint of agent i in state ω. With these notations, (14) rewrites as:

$$p_j \geq \mathbb{E}\left[M(\omega)d_j(\omega)\right] - \delta \mathbb{E}\left[A_i(\omega)d_j(\omega)\right], \tag{15}$$

with an equality for almost all trees held by agent i. Equation (15) shows that the price of an asset held by i is the difference between two terms.

The first term is $\mathbb{E}\left[M(\omega)d_j(\omega)\right]$, the present value of the dividends evaluated with the pricing kernel M. It reflects the pricing of risk embedded in the prices of the Arrow securities.

The second term, $\delta \mathbb{E}\left[A_i(\omega)d_j(\omega)\right]$, is new to our setting. It reflects the pricing of incentives, as it is equal to the shadow cost incurred by agents of type i when they hold one marginal unit of asset j and their incentive constraints become tighter. It is the expected product of the shadow cost of the incentive constraint, $A_i(\omega)$, by the non-pledgeable part of the dividend, $\delta \, d_j(\omega)$.

Excess return decomposition. The pricing formula (15) leads to a natural decomposition of excess return. Define the risky return on asset j as $R_j(\omega) \equiv d_j(\omega)/p_j$ and let the risk-free return be $R_f \equiv 1/\mathbb{E}\left[M(\omega)\right]$. Then, standard manipulations of the first order condition (14) show that for almost all assets held by agents of type i:

$$\mathbb{E}\left[R_j(\omega)\right] - R_f = -R_f \operatorname{cov}\left[M(\omega), R_j(\omega)\right] + R_f \mathbb{E}\left[A_i(\omega)\delta R_j(\omega)\right] \tag{16}$$

The first term on the right-hand side of (16) can be interpreted as a risk premium. It is positive if the return on asset j, $R_j(\omega)$, is large for states in which the pricing kernel, $M(\omega)$, is low. It is similar to the standard risk premium associated with consumption betas in frictionless markets but, unlike in the frictionless CCAPM, the pricing kernel $M(\omega)$ mirrors neither aggregate nor individual consumption.

The second term on the right-hand side of (16) is a premium reflecting incentive constraints. This premium is large if non-pledgeable income, $\delta R_j(\omega)$, is large when the incentive compatibility condition is tight.

Equation (16) bears some similarities with the margin-CAPM characterized by Gârleanu and Pedersen (2011) (see in particular Proposition 2 in Gârleanu and Pedersen (2011)). In both cases, expected excess returns increase with i) a consumption beta and ii) a term reflecting financial constraints. These two terms, however, are different in Gârleanu and Pedersen (2011) and in our model. First, in Gârleanu and Pedersen (2011) the consumption beta is defined relative to the aggregate consumption, while, as discussed above, our consumption risk premium does not mirror aggregate consumption, due to the endogenous, incentive driven, market incompleteness. Second, in Gârleanu and Pedersen (2011) the premium associated with financial constraints is proportional to the product between the single multiplier associated with this constraint (which is a constant) and the exogenous margin requirement for the priced asset. Thus, in Gârleanu and Pedersen (2011) differences in this premia across assets are proportional to the exogenous differences between these margin requirements. In contrast, in our model, there are no exogenous differences in margin requirements, as δ is the same for all assets. Instead there are many incentive compatibility constraints, one for each state, and correspondingly many multipliers or shadow prices. Thus, we have that the premium associated with the incentive constraint reflects the covariance between an asset's non-pledgeable payoff and the shadow price of the incentive constraint. Consequently, assets with different distributions of payoffs across states will generically have different incentive constraint premia, even if they have the same level of pledgeability.

Deviation from the Law of One Price. Equation (15) reveals that, if the incentive compatibility constraint of the asset holder binds in at least one state, and if the dividend is strictly positive in that state, then the price of the tree is *strictly* smaller than that of the corresponding portfolio of Arrow securities, i.e., there is a basis. One could argue that this constitutes an arbitrage opportunity. However, agents cannot trade on it without tightening their incentive constraint. Thus, the basis between $\mathbb{E}\left[M(\omega)d_j(\omega)\right]$ and the price, p_j, reflects limits to arbitrage induced by incentive constraints.

A specific implication of our model is that the basis always goes in the same direction. The price of a tree

can be lower than that of the replicating portfolio of Arrow securities, but it cannot be higher.[16] If it was, an agent holding the tree could sell it and buy the replicating portfolio of Arrow securities. That arbitrage trade would be feasible because i) market clearing implies there is at least one agent holding the tree, and ii) buying Arrow securities does not tighten IC constraints. In contrast with i), when the price of the tree is lower than that of the replicating portfolio, there does not exist an agent holding the replicating portfolio of Arrow securities (since holding that portfolio is dominated). Hence arbitrage trades would imply the issuance of Arrow securities, which would tighten the IC constraint (in contrast with ii)). Thus, unlike in previous theoretical analyses, bases always go in the same direction (assets are priced below replicating derivatives) and don't reflect any exogenous differences across assets in terms of margin constraints or pledgeability. In fact, our result of the direction of the basis does not rely on our assumption that the payoffs of all assets and contracts are equally pledgeable, and would still prevail for any heterogeneity in pledgeability parameters.

In line with our model, during the 2007–2009 crisis, there was a a basis between the price of corporate bonds (corresponding in our model to trees) and that of replicating portfolios of derivative contracts (see Bai and Collin-Dufresne (2013)). To arbitrage that basis, investors would have had to purchase the cheap underlying assets (corporate bonds), and buy protection, while funding this position by levering up. Bai and Collin-Dufresne (2013) argue that, during the crisis, such arbitrage trades were difficult to carry out because of financial constraints. Our model offers a micro-funded rationale for these constraints. In particular, the borrowing constraints experienced by traders in practice are in line with the incentive constraints in our model, which limit the amount of liabilities agents can issue to fund the arbitrage trade.

Basis vs. collateral premium. Equation (15) shows that the tree is priced at a discount relative to the replicating portfolio of Arrow securities, i.e., there is a basis. This could seem to contradict the result obtained in previous literature (see, e.g., Fostel and Geanakoplos (2008)) that equilibrium prices include a collateral premium. There is no contradiction, however, as the premium obtains, both in this paper and in previous literature, relative to a different benchmark than the replicating portfolio.

To see this, consider again the trees held by some agent i. Take the first-order condition (12) with respect

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[16]This observation explains our earlier claim that the equilibrium is robust to allowing short-sales of trees. Indeed, the observation that the basis always goes in the same direction implies that short-selling tree is always less profitable than short-selling a replicating portfolio of Arrow securities. See Appendix B.9 for the complete argument.

to $c_i(\omega)$, multiply by the dividend $d_j(\omega)$ and sum across states to obtain:

$$\mathbb{E}\left[M(\omega)d_j(\omega)\right] = \mathbb{E}\left[\frac{u_i'\left[c_i(\omega)\right]}{\lambda_i}d_j(\omega)\right] + \mathbb{E}\left[A_i(\omega)d_j(\omega)\right]. \tag{17}$$

Substituting (17) into (15) the price of asset j is

$$p_j = \mathbb{E}\left[\frac{u_i'\left[c_i(\omega)\right]}{\lambda_i}d_j(\omega)\right] + \left(\mathbb{E}\left[A_i(\omega)d_j(\omega)\right] - \delta\,\mathbb{E}\left[A_i(\omega)d_j(\omega)\right]\right). \tag{18}$$

This price equation is similar to equation (5) in Fostel and Geanakoplos (2008) or that in Lemma 5.1 in Alvarez and Jermann (2000). The first term on the right-hand side of (18) is similar to what Fostel and Geanakoplos (2008) call "payoff value": it is the expected value of asset's cash flows, evaluated at the marginal utility of the agent holding the asset. The second term on the right-hand side of (18) is similar to the collateral premium in Fostel and Geanakoplos (2008) (see Lemma 1, page 1230). This premium, however, is reduced by the last term (in factor of δ) which corresponds to the basis whose analysis is one of the contributions of our paper.

4.4 Segmentation

Let

$$v_{ij} = \mathbb{E}\left[M(\omega)d_j(\omega)\right] - \delta\,\mathbb{E}\left[A_i(\omega)d_j(\omega)\right] \tag{19}$$

denote the valuation of agent i for asset j. From the first-order condition (14), one sees that $v_{ij} = p_j$ for almost all assets held by agents of type i, and otherwise $v_{ij} \leq p_j$. Therefore, the agents who hold the asset are those who value it the most, because they have the lowest shadow incentive-cost of holding it.

In the general model, it is difficult to provide a sharp characterization of the equilibrium asset allocation. But this can be done in the context of particular examples, such as the one developed in Section 5 below. In that example, different assets are held, in equilibrium, by different agents, i.e., there is endogenous segmentation. This result is in line with the assumption made in the (exogenously) segmented market literature, in particular recent work on "intermediary asset pricing" (see for example Edmond and Weill

(2012) or He and Krishnamurthy (2013)). However, the pricing formula differs from that obtained for exogenously segmented markets. In particular, in our model, assets are not priced by the marginal utility of the asset holders. Also, the extent of segmentation is determined in equilibrium and so will vary with changes in the economic environment, as illustrated in the next section.

Segmentation can also arise in incomplete market models: for example, in some models, different agents would hold different asset portfolios because of they have different attitude towards risk. A distinctive feature of our model is that segmentation persists even though the market for Arrow securities is complete. That is, it explains why we can observe segmentation when investors have access to sophisticated derivative markets.

5 Two-by-Two

To characterize equilibrium more precisely, we hereafter focus on the simple "two-by-two" case, in which there are two types of agents $i \in \{1, 2\}$, two states, $\omega \in \{\omega_1, \omega_2\}$, and an arbitrary distribution of assets. We further assume that both types of agents, $i \in \{1, 2\}$, have CRRA utility with respective coefficients of relative risk aversion $0 \leq \gamma_1 < \gamma_2 \leq 1$. That is, agent $i = 1$ is more risk tolerant, while agent $i = 2$ is more risk averse. As shown in Proposition 3, this implies that the equilibrium consumption allocation is uniquely determined, and the equilibrium prices are uniquely determined up to a multiplicative constant. As shown in Proposition 4, the restriction $\gamma_1 \neq \gamma_2$ is necessary for incentive compatibility to matter in equilibrium.

We normalize the dividend of each tree to one, i.e., $\mathbb{E}\left[d_j(\omega)\right] = 1$.[17] Given that there are only two states, any tree can be viewed as a convex combination of two extreme securities: one security that only pays off in state ω_1, and one security that only pays off in state ω_2. Therefore, one can order the trees so that, for any $j \in [0, 1]$,

$$d_j(\omega) = \frac{j}{\pi(\omega_1)} \mathbb{I}_{\{\omega=\omega_1\}} + \frac{1-j}{\pi(\omega_2)} \mathbb{I}_{\{\omega=\omega_2\}}. \tag{20}$$

Notice that (20) includes all possible payoffs in a two-state economy. That is, after the normalization $\mathbb{E}\left[d_j(\omega)\right] = 1$, any state contigent payoff can be written as (20) for some j.

We label the states so that the aggregate endowment, denoted by $y(\omega) = \int d_j(\omega) \, d\bar{N}_j$, is strictly larger

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[17]This is without loss of generality. This merely amounts to divide the dividend in all states by the expected dividend, and simultaneously scaling the asset supply up by the same constant.

in state ω_2 than in state ω_1:

$$y(\omega_2) = \frac{1}{\pi(\omega_2)} \int (1-j) \, d\bar{N}_j > y(\omega_1) = \frac{1}{\pi(\omega_1)} \int j \, d\bar{N}_j. \tag{21}$$

In other words, ω_1 is the "bad state" while ω_2 is the "good state." The tree $j = \pi(\omega_1)$ is risk free, and so its consumption beta, $\text{cov}\,[d_j(\omega), y(\omega)] \,/V\,[y(\omega)]$ is zero. Trees with $j < \pi(\omega_1)$ have lower dividend in state ω_1 than in state ω_2, and so have positive consumption beta. The smaller is j, the more positive is the beta. Vice versa, trees with $j > \pi(\omega_1)$ have negative consumption beta. The larger is j, the more negative is the beta.

5.1   Incentive Feasible Consumption Allocations

We start by studying the set of *incentive feasible* consumption allocations, that is, consumption allocations c such that (c, N) is incentive compatible for some tree allocation N. This simplifies the analysis by reducing the number of choice variables: it allows to work directly with consumption allocations, without having to explicitly describe the underlying asset allocation that makes it incentive compatible. In particular, it allows to analyze incentive-feasibility and equilibrium in an Edgeworth box. Focusing on the case in which the consumption share of agent 1 is lower in the bad state than in the good state (which, as shown below, is the case in equilibrium), our first main result is the following:

Lemma 2 *Consider a feasible consumption allocation such that $c_1(\omega_1)/y(\omega_1) < c_1(\omega_2)/y(\omega_2)$. Then, c is incentive feasible if and only if there exists $k \in [0,1]$ and $(\Delta N_1, \Delta N_2) \geq 0$, $\Delta N_1 + \Delta N_2 = \bar{N}_k - \bar{N}_{k-}$, such that:*

$$c_1(\omega_1) \;\geq\; \delta \int_{j \in [0,k)} d_j(\omega_1) d\bar{N}_j + \delta d_k(\omega_1) \Delta N_1 \tag{22}$$

$$c_2(\omega_2) \;\geq\; \delta \int_{j \in (k,1]} d_j(\omega_2) d\bar{N}_j + \delta d_k(\omega_2) \Delta N_2. \tag{23}$$

Equation (22) is the incentive compatibility condition of agent $i = 1$ in state ω_1 when he holds all trees riskier than k ($j < k$), and, if there is an atom at k in the distribution of assets, a mass ΔN_1 of that atom.

Similarly, equation (23) is the incentive compatibility condition of agent $i = 2$ when he holds all trees $j > k$, plus a mass ΔN_2 of the atom at k, if there is one. The lemma thus states that a consumption allocation is incentive feasible if and only it can be implemented by allocating risky assets to agent 1 and safer assets to agent 2 without violating their incentive compatibility constraints in states ω_1 and ω_2, respectively.

The intuition for this result can be grasped from the following two observations.

The first observation is that, since her consumption share is smaller in ω_1 than in ω_2, agent $i = 1$ tends to have incentive problems in state ω_1. To understand why, imagine that agent $i = 1$ purchases a fraction of the market portfolio equal to her average consumption share across states. In order to implement her consumption plan $c_1(\omega)$ while holding this portfolio, agent $i = 1$ has to sell Arrow securities that pay off in state ω_1, and purchase Arrow securities that pay off in state ω_2. Hence, agent $i = 1$ only has a liability in state ω_1, so that his incentive compatibility constraint can bind only in that state. Vice versa, agent $i = 2$ faces incentives problems only in state ω_2. The lemma states, then, that a consumption allocation is incentive feasible if and only if two (equations (22) and (23)) out of the four incentive compatibility constraints hold.

The second observation is that, to mitigate these incentive problems, it is best to allocate agent $i = 1$ a portfolio of trees with low payoff in state ω_1. This minimizes the amount this agent can divert in the state in which his incentive constraint binds. Symmetrically, it is best to allocate agent $i = 2$ a portfolio of trees with low payoff in state ω_2. By market clearing an equivalent way to grasp the intuition for this result is the following: Allocating to the more risk-averse agents trees with relatively high payoff in the bad state reduces the reliance of that agent on insurance sold by the more risk tolerant agent. This, in turn, relaxes incentives constraints.

The lemma is illustrated in the Edgeworth box in Figure 1. The consumption of agent $i = 1$ in state ω_1 is on the x-axis, and his consumption in state ω_2 is on the y-axis. The curves above and below the diagonal are the boundaries of the incentive feasible set. In line with the lemma, focus on the area above the diagonal, where the consumption share of agent 1 is larger in the good state, ω_2, than in the bad state ω_1. Consider a point $(c_1(\omega_1), c_1(\omega_2))$ on the boundary of the incentive set above the diagonal. It is such that (22) and (23) hold as equalities. Starting from that point, increasing the consumption of agent 1 in state ω_1 (i.e., moving to the right on the x axis) relaxes (22). On the other hand, reducing the consumption of agent 1 in state
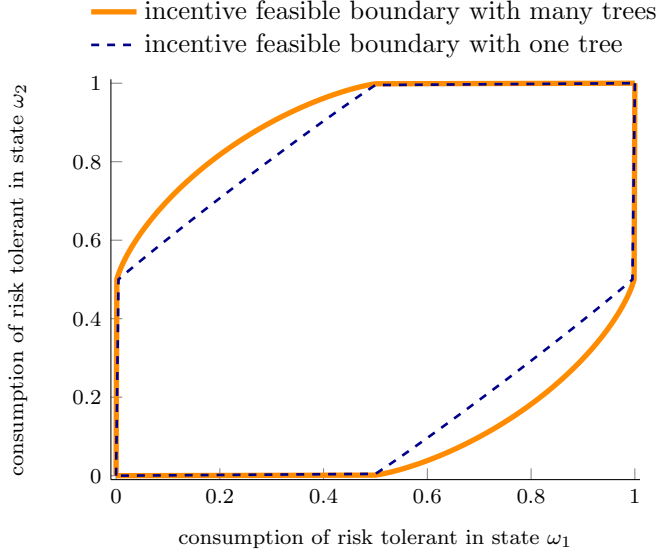
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Figure 1: The set of incentive feasible consumption allocations in an Edgeworth box, when $\pi(\omega_1) = 0.1$ and $\delta = 0.5$. In the many-trees case, tree supplies are distributed according to a beta distribution with parameters $a = b = 5$. In the one-tree case, there is just one tree equal to the market portfolio of the many-trees case.

ω_2 (i.e., moving downward on the y axis), which, by market clearing implies increasing the consumption of agent 2 in that state, relaxes (23). Therefore, the whole triangle with a right angle at $(c_1(\omega_1), c_1(\omega_2))$ and with hypothenuse on the diagonal is incentive compatible. This implies that all the area between the boundary of the incentive set and the diagonal is incentive compatible.

One sees in the figure that any allocation which gives sufficiently small consumption to one of the agents is incentive feasible. For example, if the consumption of agent $i = 1$ is sufficiently small then the consumption of agent $i = 2$ is almost equal to the aggregate endowment. As long as $\delta < 1$, this allocation can be made incentive feasible by allocating all the trees to agent $i = 2$. In equilibrium, agent $i = 1$ sells all his trees to agent $i = 2$, and agent $i = 2$ issues a liability corresponding to agent $i = 1$ consumption. This is feasible since $\delta < 1$ gives agent $i = 2$ some borrowing capacity.

Figure 1 also compares the incentive sets for one tree and for many trees (keeping aggregate output in each state constant.) The dashed line is the boundary of the incentive-feasible set when there is just one tree in strictly positive supply.[18] The solid line is the boundary when there are many trees.[19] The figure

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[18]In that case, the distribution $\bar{N}$ has just one atom. If we normalize this atom to one for simplicity, then in the Edgeworth box the boundary is the curve parameterized by $\Delta N_1 \in [0, 1]$, with cartesian coordinates $c_1(\omega_1) = \delta d(\omega_1) \Delta N_1$ and $c_1(\omega_2) = y(\omega_2) - c_2(\omega_2) = d(\omega_2) [1 - \delta + \delta \Delta N_1]$

[19]In that case we assume no atom, so the boundary is the curve parameterized by $k \in [0, 1]$, with cartesian coordinates $c_1(\omega_1) = \delta \int_0^k d_j(\omega_1) \, d\bar{N}_j$ and $c_1(\omega_2) = y(\omega_2) - c_2(\omega_2) = \int_0^1 d_j(\omega_2) \, d\bar{N}_j - \delta \int_k^1 d_j(\omega_2) d\bar{N}_j$.

illustrates that the incentive-feasible set is smaller with one tree than with many trees. Indeed, with many trees, one can replicate one-tree allocations by allocating agents shares in the market portfolio.

5.2 Equilibrium Allocations

In order to characterize equilibrium allocations, we rely on their efficiency properties. Let (c, N) denote the equilibrium allocation. As shown in Proposition 1, (c, N) is constrained Pareto efficient. That is, (c, N) solves an *incentive-constrained* planner's problem, i.e., there exists weights $(\alpha_1, \alpha_2) \in (0, 1)^2$, $\alpha_1 + \alpha_2 = 1$, such that c maximizes $\sum_{i \in I} \alpha_i U_i(c_i)$ with respect to feasible allocations satisfying the incentive compatibility conditions. Let $c^\star$ denote the solution of the corresponding *unconstrained* planner's problem. That is, $c^\star$ maximizes the same welfare function, with the same weights (α_1, α_2), with respect to feasible allocations, but without imposing the incentive compatibility conditions.

Lemma 3 *If* $(\alpha_1, \alpha_2) > 0$, *then the solutions of the unconstrained and incentive-constrained planner's problems both are such that* $c_1^\star(\omega_1)/y(\omega_1) < c_1^\star(\omega_2)/y(\omega_2)$ *and* $c_1(\omega_1)/y(\omega_1) < c_1(\omega_2)/y(\omega_2)$.

The lemma states that the risk tolerant agent, $i = 1$, receives a lower share of aggregate consumption in the low state than in the high state (as in the first best). Since consumption shares add up to one across agents, it follows that the risk averse agent, $i = 2$, enjoys a higher share of aggregate consumption in the low than in the high state. Intuitively, a consumption allocation which delivers a constant consumption share in both states to both agents is always strictly incentive feasible: it can be implemented by giving each agent a share in the market portfolio equal to that consumption share. But the risk tolerant cares relatively less about the low state, ω_1, and relatively more about the high state, ω_2. Hence, social welfare increases strictly if the risk tolerant agent, $i = 1$ insures the more risk averse agent by letting $i = 2$ have a larger share of aggregate consumption in the bad state.

Lemma 3 states that the planner always finds it optimal to pick consumption allocations above the diagonal of the Edgeworth box. Therefore, the relevant incentive constraint is the upper boundary of the incentive feasible set in Figure 1. Using Lemma 2, we then obtain our next proposition:

Proposition 5 *If* $c \neq c^\star$, *then the incentive compatibility constraint of agent* $i = 1$ *binds in state* ω_1, *while the incentive compatibility constraint of agent* $i = 2$ *binds in state* ω_2. *Moreover, there exists* $k \in [0, 1]$ *and*

$(\Delta N_1, \Delta N_2) \geq 0$, $\Delta N_1 + \Delta N_2 = \bar{N}_k - \bar{N}_{k-}$, *such that agent $i = 1$ holds all assets $j < k$, i.e.,*

$$N_1 = \bar{N}\,\mathbb{I}_{\{j<k\}} + \Delta N_1\mathbb{I}_{\{j=k\}}$$

and agent $i = 2$ holds all assets $j > k$, i.e.,

$$N_2 = \bar{N}\,\mathbb{I}_{\{j>k\}} + \Delta N_2\mathbb{I}_{\{j=k\}}.$$

Lemma 2 stated that a consumption allocation was incentive feasible if and only if it could be implemented by allocating the riskier trees to the more risk tolerant agent and the safer trees to the more risk averse agent. In line with that result, Proposition 5 states that, in equilibrium, the binding incentive constraints of the more risk tolerant agent in the bad state, and the more risk averse agent in the good state, pin down such an allocation. This equilibrium allocation can be interpreted in terms of segmentation, as different classes of investors hold different types of assets. Thus, Proposition 5 is in line with the empirical finding that households' participation to stock markets is limited (see Vissing-Jørgensen (2002)), if one interprets the more risk averse agents in our model as households and the relatively risky trees in our model as stocks.[20]

The proposition is illustrated in Figure 2. In the figure, the "incentive-constrained Pareto set" and the "unconstrained Pareto set" are, respectively, the set of consumption allocations obtained by solving the incentive-constrained and the constrained Planner's problem for all possible weights $(\alpha_1, \alpha_2) \in [0,1]^2$, $\alpha_1 + \alpha_2 = 1$. The incentive-constrained Pareto set coincides with the unconstrained Pareto set when the latter lies below the upper boundary of the incentive-feasible set. Otherwise, the incentive-constrained Pareto set coincides with the IC boundary. As α_1/α_2 increases, then the constrained Pareto efficient allocation moves monotonically to the northeast of the Edgeworth box.

The above discussion is in terms of the Pareto weigths (α_1, α_2) for the social planner's problem. The following proposition shows how it maps into a discussion of the initial endowments $(\bar{n}_1, \bar{n}_2)$ for the equilibrium implementing the second best.

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[20]In contrast with asset pricing models in which households are assumed not to hold risky assets, in our model this is an equilibrium outcome.
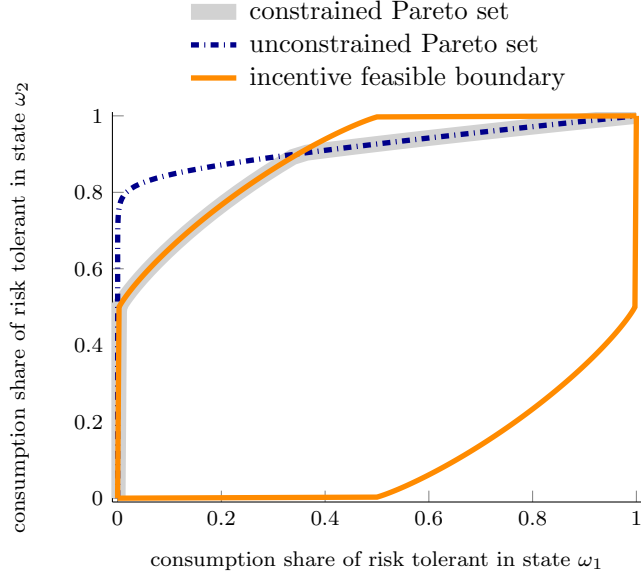
Figure 2: The set of incentive-feasible and the set of incentive-constrained Pareto allocation, when $\pi(\omega_1) = 0.9$, $\gamma_1 = 0.1$, $\gamma_2 = 1$, and $\delta = 0.5$. In the many-trees case, tree supplies are distributed according to a beta distribution with parameters $a = b = 15$. In the one-tree case, there is just one tree equal to the market portfolio of the many-trees case.

Proposition 6 *The ratio of endogenous Pareto weights, α_1/α_2, is strictly increasing in the ratio of initial endowment $\bar{n}_1/\bar{n}_2$.*

Thus, while Figure 2 reveals that incentive compatibility does not matter for extreme values of α_1/α_2, Proposition 6 enables one to restate this observation in terms of the distribution of initial endowments. When this distribution is very unequal, as agents $i = 1$ are initially endowed with a very large fraction of the market portfolio ($\bar{n}_1/\bar{n}_2$ large), or agents $i = 2$ are initially endowed with a very large fraction of the market portfolio ($\bar{n}_1/\bar{n}_2$ low), there is little scope for risk sharing between the two types of agents. Thus, even the unconstrained equilibrium involves little trading, so that incentive constrains don't bind (as is the case in Figure 2 in the north east and the south west of the Edgeworth box). In contrast, when the distribution of initial endowments is more equal ($\bar{n}_1$ close to $\bar{n}_2$) the scope for risk sharing is large. In that case the incentive constrained equilibrium allocation differs significantly from its unconstrained counterpart (as is the case in Figure 2 in the north west of the Edgeworth box). Correspondingly, the basis is zero when the distribution of endowments is very unequal, while it can be strictly positive when initial endowments are equally distributed.

5.3 Relative Supply Effects

In our model, the relative supply of trees (i.e., the distribution $\bar{N}_j$) determines equilibrium outcomes, by changing the shape of the incentive feasible set. Changing the relative supplies of trees changes equilibrium outcomes even if it does not change aggregate output and pledgeable income in each state, nor the span of the trees' payoff matrix. This is in sharp contrast with the complete markets case, in which the set of feasible allocations is not affected by the way the aggregate output is split across trees.

As an illustration, compare an economy with just one tree (the "market portfolio") to another economy in which there are two trees, one paying off only in the good state, the other paying off only in the bad state. To reason ceteris paribus, the aggregate output is the same in each state in the two economies. If incentive problems are severe, because $\delta \simeq 1$, then, when there is only one tree, incentive constraints are more likely to bind in equilibrium. In contrast, when there are two trees, each paying off only in one state, the second condition of Proposition 4 holds: all agents can attain their $\delta = 0$ equilibrium consumption just by holding trees. Hence incentive constraints do not bind, even if $\delta \simeq 1$.

When there is a single tree, its consumption β is equal to one, while with two trees one has a lower β and the other a higher one. The comparison between the single tree economy and its two tree counterpart suggests that increasing the dispersion of βs relaxes incentive constraints. To make this point, first note that in our simple two-state case, the consumption β of asset j is affine in j with coefficients that only depend on the values and probabilities of aggregate dividends. Therefore, the dispersion of βs is increasing in the dispersion of js.

To model an increase in the dispersion of the distribution of assets, from $\bar{N}$ to $\bar{N}^*$, it is natural to consider a mean preserving spread, that is a decrease in the sense of second order stochastic dominance:

$$\int_0^k \bar{N}_j^* \geq \int_0^k \bar{N}_j, \qquad \text{for all } k \in [0,1], \tag{24}$$

preserving the aggregate output in each state. From equation (21), one sees that aggregate output is preserved

in each state if and only if:

$$\int_0^1 j d\bar{N}_j^* \quad = \quad \int_0^1 j d\bar{N}_j \tag{25}$$

$$\bar{N}_1^* \quad = \quad \bar{N}_1. \tag{26}$$

In this context, we obtain the following proposition:

Proposition 7 *Consider two tree supply distributions $\bar{N}$ and $\bar{N}^*$, such that $\bar{N}^*$ is more dispersed than $\bar{N}$ in the sense that (24), (25) and (26) hold. If an allocation is incentive feasible for $\bar{N}$, then it is incentive feasible for $\bar{N}^*$.*

Proposition 7 states that the set of incentive feasible allocations increases as the tree supply distribution becomes more dispersed, i.e., the dispersion of βs increases. The proposition enables one to revisit and generalise the intuition obtained by comparing the single tree and two-tree economies: a mean-preserving spread implies that the supply distribution $\bar{N}^*$ puts more weight on the riskiest trees, and also on the safest trees, than the distribution $\bar{N}$. This implies that the set of trees held by the more risk tolerant agent is overall riskier, and correspondingly yields smaller dividends in the bad state, which relaxes the incentive compatibility condition of this agent. Symmetrically, by putting more weight on the safest trees, the supply distribution $\bar{N}^*$ leads to an overall safer set of trees held by the more risk averse agent. This reduces the dividends of these trees in the good state, which relaxes the incentive constraint of the more risk averse agent.

The tightening of incentive constraints, induced by a decrease in the dispersion of consumption betas, affects equilibrium pricing. This is illustrated in Figure 3, which plots the equilibrium expected return on the market portfolio for different dispersions of consumption betas, keeping aggregate output in each state constant. We consider symmetric beta distributions, and let the log of their shape parameter increase from 1 to 6, so that their dispersion decreases while their mean remains constant. The figure shows that, as the dispersion of consumption betas decreases, the expected return of the portfolio of Arrow securities replicating the market increases, reflecting poorer allocation of risks in the economy. Moreover, as betas get less dispersed, the basis increases, reflecting the increased shadow costs of incentive constraints. In contrast,
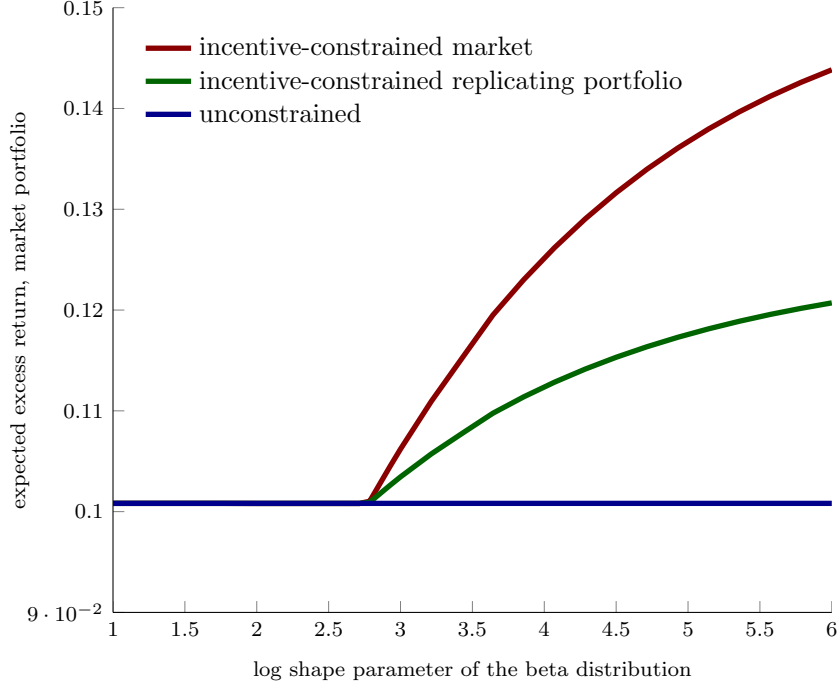
Figure 3: The expected excess return on the market portfolio as the dispersion of security beta decreases. Tree supplies are distributed according to a beta distribution with parameters $a = b$ varying from 2.7 to 403.

by construction, when there are no incentive problems the expected return on the market portfolio is not affected by changes in the dispersion of betas.

5.4 The Cross Section of Bases

Focus on the case in which incentive compatibility constraints bind. By Proposition 5, there exists a threshold k, such that the more risk-tolerant agent holds assets $j < k$, while the more risk-averse agent holds assets $j > k$. In this context, from the first-order condition (14), the basis on asset j is

$$\sum_{\omega \in \Omega} q(\omega) d_j(\omega) - p_j = \delta \frac{\mu_1(\omega_1)}{\lambda_1} d_j(\omega_1).$$

Since only relative prices are pinned down, we express this basis in relative prices, and choose as normalizing factor (or numeraire) the price of the riskless bond, $q(\omega_1) + q(\omega_2)$. For tree $j < k$ the normalised basis is thus

$$\Delta_j = \left[\frac{\mu_1(\omega_1)/\lambda_1}{q(\omega_1) + q(\omega_2)} \right] \times \left[\delta d_j(\omega_1) \right]. \tag{27}$$

The first term in the right-hand side of equation (27) is constant across all assets held by agent 1, and measures, intuitively, the tightness of the incentive constraint of agent 1. The second term is equal to the non-pledgeable cash flow of the asset in state ω_1 in which the agent holding it is constrained. This term, and correspondingly the basis, is higher for assets with a relatively large payoff in the bad state and a relatively low payoff in the high state, that is, assets with a lower consumption beta. The intuition is that the risk tolerant agent sells insurance against the bad state to the risk averse agent. However, the incentive compatibility constraint limits the amount of insurance she can sell. Since the consumption of the risk tolerant agent is low in the bad state, deviating is tempting. It implies that the shadow cost of holding a tree is higher for trees paying relatively more in the bad state, i.e., for trees with a lower consumption beta. Remember however that the risk tolerant agent holds trees with a high beta. Therefore, among trees with a high consumption beta, trees with a moderately high beta have a larger basis than trees with a very high beta.

Consider now trees $j > k$ held by agent 2. Following the same reasoning as before, the basis equals

$$\Delta_j = \left[\frac{\mu_2(\omega_2)/\lambda_2}{q(\omega_1) + q(\omega_2)} \right] \times \left[\delta d_j(\omega_2) \right].$$ \hfill (28)

Equation (28) implies that, among assets held by the risk averse agent, $i = 2$, the basis is larger for assets with a relatively large payoff in the good state and a relatively low payoff in the bad state, that is, with a higher consumption beta. The intuition is symmetric to the one above. The risk averse agent would like to sell consumption to the risk tolerant agent in the good state, but it is tempting for the risk averse agent to deviate in the good state. Thus, the shadow cost of holding a tree is higher for tree with a relatively high payoff in the good state, that is, for trees with a higher consumption beta. The risk averse agent holds trees with a low consumption beta. Therefore, among trees with a low beta, those with a moderately low beta have a larger basis than trees with a very low beta. Putting things together, we conclude that:

Proposition 8 *Suppose all trees are in positive supply. Then, the basis induced by incentive constraints is an inverse U-shape function of the consumption beta of the tree.*

The restriction that all trees are in positive supply ensures that prices are uniquely determined. Intu-
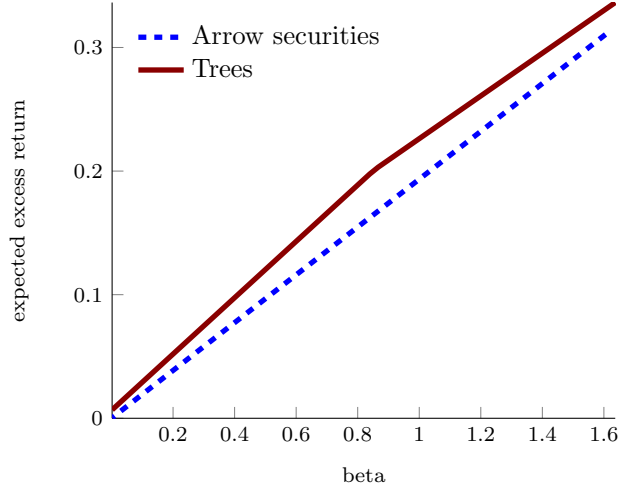
Figure 4: The Modified Security Market Line

itively, the proposition means that, after adjusting for risk, trees with either a low or a large consumption beta will tend to have a high price, and a low return. This is illustrated in Figure 4. The figure shows the security market line (SML) in our environment, which we derive explicitly in Proposition B.1 in Supplementary Appendix B.8. Since assets are held by agents who value them most, the SML is the minimum between the SML obtained from agent $i = 1$'s valuation, and that derived from agent $i = 2$'s valuation. The kink in the figure occurs at asset k, for which ownership switches from agent 1 to agent 2. The figure illustrates that, because the basis is inverse-U shaped in β, the SML is flatter at the top and steeper at the bottom, in line with Black (1972), and recent evidence in Frazzini and Pedersen (2014) and Hong and Sraer (2016).

5.5 Response to Shocks on Incentive Problems

To model shocks on incentives, fix a tree $\ell < k$ and consider a small increase in δ for tree ℓ and possibly nearby trees. Formally, assume $\delta_j = \delta + \varepsilon \phi_j$ for some continuous function ϕ_j strictly positive near ℓ, and zero everywhere else.[21] The shock worsens incentive problems for tree ℓ (and neighbouring trees). What is the effect of this shock on allocations?

Proposition 9 *Assume that the cumulative distribution of trees is continuous and strictly increasing, that*

$c \neq c^\star$, *and that $k \in (0, 1)$. The ε shock shrinks the set of trees held by agent 1: $k(\varepsilon) < k(0)$ for small $\varepsilon > 0$.*

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[21]All of our results extend to this case. In fact, our proofs in the appendix cover the case of δ which are continuously varying across agents and asset types.

When agent 1 becomes slightly worse at pledging a tree he already holds, the shadow value of his incentive-compatibility constraint increases, which makes it more costly for agent 1 to hold any tree. Thus, in equilibrium, the set of trees $[0, k)$ held by agent 1 shrinks. This means that agent $i = 1$ sells the safest assets in his portfolio, while keeping the riskiest ones. This might sound counter-intuitive if one expected that, when an agent's incentive problems become more severe, he should sell his riskiest assets. The result arises because, even after the shock, agent $i = 1$ is still in a better position to hold very risky assets than the more risk averse agent $i = 2$. Thus, as the shock reduces agent $i = 1$'s ability to hold assets overall, he divests those for which his comparative advantage is the lowest.

Now turn to the effect of the shock on prices. Equations (27) and (28) imply that when agent $i = 1$ becomes a worse pledger for tree ℓ, the basis of tree ℓ increases relative to the bases of other trees j held by $i = 1$ (that have not been directly hit by the shock), which themselves increase relative to the bases of trees j' held by $i = 2$. That is

$$\frac{\Delta_\ell(\varepsilon)}{\Delta_\ell(0)} > \frac{\Delta_j(\varepsilon)}{\Delta_j(0)} > \frac{\Delta_{j'}(\varepsilon)}{\Delta_{j'}(0)}$$

Thus, co-movement in the basis induced by incentive constraints is stronger among assets held by the same type of agents.

6 Conclusion

We introduce incentive compatibility constraints, limiting the pledgeability of collateral, in an otherwise standard general equilibrium model. In each state, agents can't pledge more than a fraction of the payoff from their holdings in that state. Hence, although a complete set of Arrow securities are available for trade, limited collateral pledgeablity reduces the ability of relatively risk tolerant agents to credibly promise insurance payments to more risk averse agents.

Thus, markets are endogenously incomplete. To cope with such incompleteness, relatively risk averse agents hold low beta assets, while relatively risk tolerant agents hold high beta assets. This reflects the risk tolerant agents' comparative advantage at holding risky assets. Correspondingly, the market is endogenously segmented. When hit by an adverse shock on incentives, agents must sell some of their holdings, which is

not unexpected. More surprisingly, relatively risk tolerant agents sell their safest assets, not their riskiest ones. This is because, while the shock reduces their ability to hold all assets, it does not eliminate their comparative advantage at holding the riskiest ones.

Incentive compatibility constraints also affect equilibrium pricing. They generate a basis between the prices of risky assets and those of replicating portfolios of derivatives (in spite of the fact that risky assets and derivatives are equally imperfectly pledgeable). The basis always goes in the same direction: the price of risky assets is below that of replicating derivative portfolios. Arbitraging the basis would imply buying the "cheap" risky assets, and selling the "expensive" derivatives, but the latter sale is ruled out by incentive compatibility. The structure of the basis is such that equilibrium expected excess returns are concave in consumption betas, in line with empirical findings. Moreover an increase in the dispersion of consumption betas relaxes incentive constraints, which reduces the basis. The latter two results are related. The concavity of the security market line reflects that assets with extreme betas are more valuable because they enable agents to share risk without creating incentive problems. An increase in the dispersion of consumption betas increases the prevalence of assets with extreme betas, which relaxes incentive constraints in equilibrium.

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A Appendix: Proofs

In this appendix we prove all of our results for the generalized model in which δ depends on the agent and (continuously) on the tree type. That is, for each, $i \in I$, the function $j \mapsto \delta_{ij}$ is continuous.

A.1 Proof of Proposition 1

Let (c, N) denote an equilibrium allocation with associated price system (q, p). Suppose it is Pareto dominated by some other incentive-feasible allocation $(\hat{c}, \hat{N})$. Then, because utility is strictly increasing, $\hat{c}_i$ must lie strictly outside the budget set of all agents for which $U_i(\hat{c}_i) > U_i(c_i)$. Otherwise, these agents would have a strict incentive to switch to $\hat{c}_i$. Likewise, $\hat{c}_i$ must lie weakly outside the budget set set of all agents for which $U_i(\hat{c}_i) = U_i(c_i)$. Otherwise, these agents would have strict incentive to increase their consumption in some state, which would respect incentive compatibility. Taken together, we obtain:

$$\sum_{\omega \in \Omega} q(\omega) \hat{c}_i(\omega) + \int p_j \, d\hat{N}_{ij} \geq \bar{n}_i \int p_j \, d\bar{N}_j + \int \sum_{\omega \in \Omega} q(\omega) d_j(\omega) \, d\hat{N}_{ij},$$

with a strict inequality for all $i \in I$ such that $U_i(\hat{c}_i) > U_i(c_i)$. Adding up across all agents we obtain that:

$$\sum_{\omega \in \Omega} q(\omega) \left\{ \sum_{i \in I} \hat{c}_i(\omega) - \sum_{i \in I} \int d_j(\omega) \, d\hat{N}_{ij} \right\} + \int p_j \left\{ \sum_{i \in I} d\hat{N}_{ij} - d\bar{N}_j \right\} > 0,$$

which contradicts the feasibility of $(\hat{c}, \hat{N})$.

A.2 Proof of Proposition 2

Our proof of existence proceeds as follows. In Section A.2.1 we define the Planner's Problem, we study some of its elementary properties, and we derive necessary and sufficient optimality conditions for a solution. In Section A.2.2, we turn to the equilibrium and derive first-order necessary and sufficient conditions for a solution to the agent's problem. Comparing the first-order conditions for the Planner and for the agent, in Section A.2.3 we show an equivalence between the set of equilibrium allocations, and the set of solutions to the Planner's problem with zero wealth transfers. We then establish the existence of a solution to the Planner's problem with zero wealth transfer. Omitted proofs are in Supplementary Appendix B.

In what follows we identify any measure with its cumulative distribution function. That is, we identify $\mathcal{M}_+$ with the set of increasing and right-continuous functions over $[0, 1]$. We denote by $\mathcal{M}$ the vector space of functions

which can be written as $F = F_1 - F_2$, where both F_1 and F_2 belong to $\mathcal{M}_+$. We endow $\mathcal{M}$ with the total variation norm. Given any sequence $N^k \in \mathcal{M}$, we said that N^k *converges strongly* towards N, and write $N^k \to N$, if $\lim_{k \to \infty} \|N^k - N\| = 0$. We say that N^k *converges weakly* towards N, and write $N^k \Rightarrow N$, if $\int f_j \, dN_j^k \to \int f_j \, dN_j$ for all continuous real-valued functions $j \mapsto f_j$ over $[0, 1]$. A set of allocations K is said to be *weakly closed* if for any weakly converging sequence $(c^k, N^k) \in K$, i.e. such that $c^k \to c$ and $N^k \Rightarrow N$, then the limit of the sequence belongs to K, i.e., $(c, N) \in K$. The set K is said to be *weakly compact* if for any sequence $(c^k, N^k) \in K$, there exist some subsequence (c^ℓ, N^ℓ) and some $(c, N) \in K$ such that $c^\ell \to c$ and $N^\ell \Rightarrow N$.

A.2.1 The Planner's Problem

Let $\mathcal{A}$ denote the simplex, i.e., the set of welfare weights $\alpha \equiv (\alpha_1, \alpha_2, \ldots, \alpha_I)$ such that $\alpha_i \geq 0$ and $\sum_{i \in I} \alpha_i = 1$. Given any $\alpha \in \mathcal{A}$, and given any $(c, N) \in X$, social welfare is defined as

$$W(\alpha, c, N) \equiv \sum_{i \in I} \alpha_i \sum_{\omega \in \Omega} \pi(\omega) u_i \left[c_i(\omega) \right].$$

In the above formula, when $u_i(0) = -\infty$, we let $\alpha_i u_i \left[c_i(\omega) \right] = 0$ if $\alpha_i = c_i(\omega) = 0$.

Given weight $\alpha \in \mathcal{A}$, the *Planner's Problem* is:

$$W^\star(\alpha) = \sup W(\alpha, c, N) \tag{29}$$

with respect to incentive-feasible allocations, i.e., with respect to $(c, N) \in X$ satisfying (6), (8) and (9). We let $\Gamma^\star(\alpha)$ denote the set of allocations solving (29). To show the existence of a solution, we rely on:

Lemma A.1 *The set of incentive feasible allocations is weakly compact.*

The proof relies on Helly's Selection Theorem (Theorem 12.9 in Stokey and Lucas (1989)) which allows to extract weakly convergence subsequences from bounded sequences in $\mathcal{M}_+$. The feasibility and incentive compatibility constraints hold in the limit by definition of weak convergence. We add to the argument in Stokey and Lucas (1989) by showing that the feasibility constraint for asset holdings is also satisfied in the limit. With this result in mind, we show in the supplementary appendix:

Proposition A.1 *The planner's value $W^\star(\alpha)$ is a continuous function of $\alpha \in \mathcal{A}$, and the maximum correspondence $\Gamma^\star(\alpha)$ is non-empty, weakly compact, convex, and has a weakly closed graph. Moreover, consider any sequence $\alpha^k \to \bar{\alpha}$ and an associated sequence of optimal allocations $(c^k, N^k) \in \Gamma^\star(\alpha^k)$. Then, if $\bar{\alpha}_i = 0$, $\lim_{k \to \infty} \alpha_i^k u' \left[c_i^k(\omega) \right] c_i^k(\omega) = 0$*

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for all $\omega \in \Omega$.

If $u_i(0) = 0$ for all $i \in I$, the result follow from the same argument as in the proof of the Theorem of the Maximum (see, for example, Theorem 3.6 in Stokey and Lucas (1989)). If $u_i(0) = -\infty$ for some i, then we need to adapt the argument because the social welfare function is not continuous at (α, c, N) such that $\alpha_i = c_i(\omega) = 0$. Likewise, the result concerning $\alpha_i^k u'\left[c_i^k(\omega)\right] c_i^k(\omega) = 0$ is obvious if $u_i(0) = 0$, but requires some additional work when $u_i(0) = -\infty$.

To compare equilibria with solutions of the Planner's Problem, we rely on first-order conditions. We first derive necessary conditions. To do so, we cannot apply the Lagrange multiplier theorems of Luenberger (1969), because they do not accommodate equality constraints. Even if we consider a "relaxed problem" where equality constraints are replaced by inequality constraints, the theorems do not apply because the relevant positive cone has an empty interior. We therefore exploit the structure of the problem to derive first-order conditions by hand. To do so we consider, for any N, the maximized objective with respect to c. We then use an Envelope Theorem of Milgrom and Segal (2002) to explicitly calculate the directional derivative of this maximized objective with respect to N. We obtain:

Proposition A.2 *Suppose $(c, N) \in X$ solves the Planner's problem given $\alpha \in \mathcal{A}$. Then there exists multipliers $\hat{q} \in \mathbb{R}_+^{|\Omega|}$ and $\hat{\mu} \in \mathbb{R}_+^{|\Omega| \times |I|}$ such that (c, N) satisfies two sets of conditions.*

- *First-order conditions:*

$$\alpha_i \pi(\omega) u_i'\left[c_i(\omega)\right] + \hat{\mu}_i(\omega) = \hat{q}(\omega), \quad \forall (i, \omega) \in I \times \Omega$$

$$\int \left[\hat{p}_j - \hat{v}_{ij}\right] dN_{ij} = 0,$$

where $\hat{v}_{ij} \equiv \sum_{\omega \in \Omega} \hat{q}(\omega) d_j(\omega) - \sum_{\omega \in \Omega} \hat{\mu}_i(\omega) \delta_{ij} d_j(\omega)$, and $\hat{p}_j \equiv \max_{i \in I} \hat{v}_{ij}$.

- *Complementary slackness conditions:*

$$\hat{q}(\omega) \left[\sum_{i \in I} \int d_j(\omega) \, dN_{ij} - \sum_{i \in I} c_i(\omega)\right] = 0 \quad \forall \omega \in \Omega$$

$$\hat{\mu}_i(\omega) \left[c_i(\omega) - \int \delta_{ij} d_j(\omega) \, dN_{ij}\right] = 0 \quad \forall (i, \omega) \in I \times \Omega.$$

Although the above conditions are also sufficient, it is convenient to state more general sufficient conditions, where $\hat{p}$ is taken to be some abstract continuous linear functional. This allows to show that any equilibrium is a solution to the Planner's Problem, even if the pricing functional cannot be represented by a continuous function. Then, using the

necessary conditions derived in Proposition A.2, one can show that the same equilibrium allocation can be supported by a pricing functional represented by a continuous function, establishing the claim in footnote 12.

Proposition A.3 *An incentive-feasible allocation* $(c, N) \in X$ *solves the Planner's problem if there exist multipliers* $\hat{q} \in \mathbb{R}_+^{|\Omega|}$, $\hat{\mu} \in \mathbb{R}_+^{|\Omega| \times |I|}$, *and a continuous linear functional* $\hat{p}$ *satisfying the following two sets of conditions.*

- *First-order conditions:*

$$
\begin{aligned}
\alpha_i \pi(\omega) u_i' \left[c_i(\omega) \right] + \hat{\mu}_i(\omega) &= \hat{q}(\omega), \quad \forall (i, \omega) \in I \times \Omega \\
\hat{p} \cdot M - \int \hat{v}_{ij} \, dM_{ij} &\geq 0 \quad \forall M_i \in \mathcal{M}_+ \text{ and } i \in I, \text{ with "} = \text{" if } M = N_i,
\end{aligned}
$$

where $\hat{v}_{ij} \equiv \sum_{\omega \in \Omega} q(\omega) d_j(\omega) - \sum_{\omega \in \Omega} \mu_i(\omega) \delta_{ij} d_j(\omega)$.

- *Complementary slackness conditions:*

$$
\begin{aligned}
\hat{q}(\omega) \left[\sum_{i \in I} \int d_j(\omega) \, dN_{ij} - \sum_{i \in I} c_i(\omega) \right] &= 0 \quad \forall \omega \in \Omega \\
\hat{\mu}_i(\omega) \left[c_i(\omega) - \int \delta_{ij} d_j(\omega) \, dN_{ij} \right] &= 0 \quad \forall (i, \omega) \in I \times \Omega.
\end{aligned}
$$

A.2.2 Optimality conditions for the Agent's Problem

Notice that the range of the constraint set in the agent's problem is finitely dimensional. In this case, the "interior point condition" for the positive cone associated with the constraint set is immediately satisfied and so one can apply the general Lagrange multiplier theorems shown in Section 8.3 and 8.4 of Luenberger (1969).

Proposition A.4 *A* $(c_i, N_i) \in X_i$ *solve the agent's problem if and only if it satisfies the intertemporal budget constraint, (7), the incentive compatibility constraint (6), and there exists multipliers* $\lambda_i \in \mathbb{R}_+$, $\mu_i \in \mathbb{R}_+^{|\Omega|}$ *satisfying the following two sets of conditions:*

- *First-order conditions:*

$$
\begin{aligned}
\pi(\omega) u_i' \left[c_i(\omega) \right] + \mu_i(\omega) &= \lambda_i q(\omega) \\
\int (p_j - v_{ij}) \, dM_{ij} &\geq 0 \quad \forall M_i \in \mathcal{M}_+, \text{ with "} = \text{" if } M_i = N_i,
\end{aligned}
$$

where $v_{ij} \equiv \sum_{\omega \in \Omega} q(\omega) d_j(\omega) - \sum_{\omega \in \Omega} \frac{\mu_i(\omega)}{\lambda_i} \delta_{ij} d_j(\omega)$.

- *Complementary slackness conditions:*

$$\lambda_i \left[\bar{n}_i \int p_j \, d\bar{N}_j + \sum_{\omega \in \Omega} q(\omega) \int d_j(\omega) \, dN_{ij} - \int p_j \, dN_{ij} - \sum_{\omega \in \Omega} q(\omega) c_i(\omega) \right] = 0$$

$$\mu_i(\omega) \left[c_i(\omega) - \int \delta_{ij} d_j(\omega) \, dN_{ij} \right] = 0 \quad \forall \omega \in \Omega.$$

There is one difference between this Proposition and the Theorems shown in Section 8.3 and 8.4 of Luenberger (1969): we are asserting that there exists multipliers such that the first-order condition with respect to $c_i(\omega)$ holds with equality. This follows from the following observation: if $c_i(\omega) = 0$, then the incentive compatibility constraint is binding, in particular $\int \delta_{ij} d_j(\omega) \, dN_{ij} = 0$. Therefore, if we raise $\mu_i(\omega)$ so that the first-order condition holds with equality, we leave the product $\mu_i(\omega) \int \delta_{ij} d_j(\omega) \, dN_{ij} = 0$ unchanged, which implies that $p \cdot N_i - \int v_{ij} \, dN_{ij} = 0$ continues to hold. Finally, since raising $\mu_i(\omega)$ decreases v_{ij}, $p \cdot M_i - \int v_{ij} \, dM_{ij}$ remains positive. Taken together, this means that we can always pick multipliers so that the first-order condition with respect to $c_i(\omega)$ holds with equality.

Finally, the following result provide a simple relationship between the value of the agent's endowment, and the marginal value of his consumption plan. This formula will be useful shortly to formulate the equilibrium fixed-point equation.

Lemma A.2 *If $(c_i, N_i) \in X_i$ solves the agent's problem, then*

$$\sum_{\omega \in \Omega} \pi(\omega) u' [c_i(\omega)] c_i(\omega) = \lambda_i \bar{n}_i \int p_j \, d\bar{N}_j.$$

The proof of the Lemma goes as follows. A solution to the agent's problem, (c_i, N_i), maximizes the Lagrangian:

$$\begin{aligned} L(\hat{c}_i, \hat{N}_i) = U_i(\hat{c}_i) \quad + \quad & \lambda_i \left[\bar{n}_i \, p \cdot \bar{N} + \sum_{\omega \in \Omega} q(\omega) \int d_j(\omega) \, d\hat{N}_{ij} - p \cdot N - \sum_{\omega \in \Omega} q(\omega) c_i(\omega) \right] \\ + \quad & \sum_{\omega \in \Omega} \mu_i(\omega) \left[\hat{c}_i(\omega) - \int \delta_{ij} d_j(\omega) \, d\hat{N}_{ij} \right], \end{aligned}$$

with respect to $(\hat{c}_i, \hat{N}_i) \in X_i$. This implies that the function $\beta \mapsto L(\beta c_i, \beta N_i)$ is maximized at $\beta = 1$. Taking first-order condition with respect to β at $\beta = 1$, and using the complementary slackness conditions, yields the desired result.

A.2.3 Existence of a Planner's Solution with Zero Wealth Transfer

By comparing the first-order conditions of the Planner and of the agent, we obtain:

Proposition A.5 *An allocation $(c, N) \in X$ is an equilibrium allocation if and only if there exists $\alpha \in \mathcal{A}$ such that:*

- *(c, N) solves the Planner's problem given α;*

- *For all $i \in I$, $\alpha_i \sum_{\omega \in \Omega} \pi(\omega) u_i' \left[c_i(\omega) \right] c_i(\omega) = \bar{n}_i \sum_{k \in I} \sum_{\omega \in \Omega} \pi(\omega) u_k' \left[c_k(\omega) \right] c_k(\omega).$*

In particular, given a solution of the Planner's problem satisfying the above two conditions, an equilibrium price system is given by the multipliers $(\hat{q}, \hat{p})$ of Proposition A.2.

Intuitively, comparing the first-order conditions of the Planner and of the agent reveals that the weight α_i must be proportional to $1/\lambda_i$, the inverse of the Lagrange multiplier on the agent's budget constraint. It then follows from Lemma A.2 that, for all agents $i \in I$:

$$\alpha_i \sum_{\omega \in \Omega} \pi(\omega) u_i' \left[c_i(\omega) \right] = \bar{n}_i \times \left[\sum_{k \in I} \frac{1}{\lambda_k} \right]^{-1} \times \int p_j \, d\bar{N}_j.$$

The second condition then follows because $\sum_{i \in I} \bar{n}_i = 1$. The final result about the price system follows from direct comparison of the first-order conditions of the agent and the planner.

We are now ready to establish the existence of an equilibrium. Let $\Delta^{\star}(\alpha)$ denote the set of transfers $\{\Delta_i^{\star}(\alpha)\}_{i \in I}$ such that:

$$\Delta_i^{\star}(\alpha) \equiv \alpha_i \sum_{\omega \in \Omega} \pi(\omega) u_i' \left[c_i(\omega) \right] c_i(\omega) - \bar{n}_i \sum_{k \in I} \alpha_k \sum_{\omega \in \Omega} \pi(\omega) u_k' \left[c_k(\omega) \right] c_k(\omega), \tag{30}$$

generated by all $(c, N) \in \Gamma^{\star}(\alpha)$, with the convention that $\alpha_i u_i'(c) c = 0$ if $\alpha_i = c = 0$. Using the Kakutani's fixed-point Theorem, as in Negishi (1960) and Magill (1981), we can show:

Proposition A.6 *There exists some $\alpha \in \mathcal{A}$, such that $0 \in \Delta^{\star}(\alpha)$.*

Based on some $\alpha \in \mathcal{A}$, using Proposition A.5, we can construct an equilibrium allocation and price system.

A.3   Proof of Proposition 3

Step 1: The equation $0 \in \Delta^{\star}(\alpha)$ has a unique solution.   Since the utility function of agent $i = 2$ is strictly concave, its allocation is uniquely determined in the Planner's problem. But since $c_1(\omega) + c_2(\omega) = \int d_j(\omega) \, d\bar{N}_j$, the consumption allocation of agent 1 is also uniquely determined. Hence $\Delta^{\star}(\alpha)$, defined in equation (30), is a function and not a correspondence. Moreover since $\Delta_1^{\star}(\alpha) + \Delta_2^{\star}(\alpha) = 0$ by construction and $\alpha_1 + \alpha_2 = 1$ by assumption, it is enough to look for a solution of $\Delta_1^{\star}(\alpha_1, 1 - \alpha_1) = 0$. That is, solving for equilibrium boils down to a one-equation in

one-unknown problem. To formulate this problem in simple terms, let

$$\mathrm{MU}_i(c_i) \equiv \sum_\omega \pi(\omega) u_i' \left[c_i(\omega) \right] c_i(\omega).$$

Notice, that with CRRA utility, $\mathrm{MU}_i(c_i) = (1 - \gamma_i) U_i(c_i)$ for $\gamma_i \neq 1$, and $\mathrm{MU}_i(c_i) = 1$ for $\gamma_i = 1$. With this notation, the one-equation-in-one-unknown problem for equilibrium is:

$$\bar{n}_2 \alpha_1 \mathrm{MU}_1(c_1) - \bar{n}_1 \alpha_2 \mathrm{MU}_2(c_2) = 0, \tag{31}$$

where (c_1, c_2) is the consumption allocation chosen by the planner given weight $\alpha \in \mathcal{A}$. We already know from Proposition A.6 that this equation has a solution. Our proof of uniqueness is based on the following observation.

Lemma A.3 *For any α' and α such that $\alpha_1' > \alpha_1$,*

$$U_1(c_1') \geq U_1(c_1) \ \text{and} \ U_2(c_2') \leq U_2(c_2)$$

$$MU_1(c_1') \geq MU_1(c_1) \ \text{and} \ MU_2(c_2') \leq MU_2(c_2)$$

for all $c \in \Gamma^\star(\alpha)$ and $c' \in \Gamma^\star(\alpha')$.

The proof can be found in the Supplementary Appendix. The inequalities on the first line are intuitive: when the weight on agent 1 increases, then his or her utility increases and that of agent 2 decreases. The inequalities on the second line follows directly because of CRRA utility with coefficient $\gamma_i \in [0, 1]$, which imply that $\mu_i(c) = (1 - \gamma_i) U_i(c)$. With this in mind we go back to the equilibrium equation (31). Let α denote some solution, and consider any $\alpha' \neq \alpha$, for example such that $\alpha_1' > \alpha_1$. Let c and c' denote the consumption allocations associated with α and α'. Then,

$$\bar{n}_2 \alpha_1' \mathrm{MU}_1(c_1') - \bar{n}_1 \alpha_2' \mathrm{MU}_2(c_2')$$

$$= \bar{n}_2 \alpha_1' \mathrm{MU}_1(c_1') - \bar{n}_1 \alpha_2' \mathrm{MU}_2(c_2') - \bar{n}_2 \alpha_1 \mathrm{MU}_1(c_1) + \bar{n}_1 \alpha_2 \mathrm{MU}_2(c_2)$$

$$= \bar{n}_2 \alpha_1' \left[\mathrm{MU}_1(c_1') - \mathrm{MU}_1(c_1) \right] - \bar{n}_1 \alpha_2' \left[\mathrm{MU}_2(c_2') - \mathrm{MU}_2(c_2) \right] + \left(\alpha_1' - \alpha_1 \right) \left[\bar{n}_2 \mathrm{MU}_1(c_1) + \bar{n}_1 \mathrm{MU}_2(c_2) \right] > 0.$$

In the above, the second line follows from subtracting $\bar{n}_2 \alpha_1 \mathrm{MU}_1(c_1) - \bar{n}_1 \alpha_2 \mathrm{MU}_2(c_2) = 0$ since α was assumed to solve (31). The third line follows from re-arranging terms and keeping in mind that $\alpha_1' - \alpha_1 = \alpha_2 - \alpha_2'$. The inequality follows from Lemma A.3, and from the fact that marginal utilities are strictly positive. Vice versa, if we consider some $\alpha' \neq \alpha$ such that $\alpha_1' < \alpha_1$, we obtain that the equilibrium equation (31) is strictly negative. Therefore, the

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equation for the weight, α, has a unique solution.

Step 2: the various uniqueness claims. Consider any equilibrium allocation, (c, N), and price system, (p, q). From Proposition A.5, we know that (c, N) solves the Planner's given the unique set of weights such that $\Delta^\star(\alpha) = 0$. But, as argued above, the consumption allocation is uniquely determined in the Planner's problem. Hence, it follows that the equilibrium allocation is uniquely determined in an equilibrium as well. Next, by direct comparison of first-order conditions, one sees that (c, N) solve the first-order conditions of the Planner's problem with weights $\alpha_i = \beta/\lambda_i$, multipliers $\hat{q}(\omega) = \beta q(\omega)$, $\hat{\mu}_i(\omega) = \alpha_i \mu_i(\omega)$, $\hat{v}_{ij} = \beta v_{ij}$ and $\hat{p}_j = \beta p_j$, where λ_i is the Lagrange multiplier on agent's i budget constraint, and $\beta \equiv \left[\sum_{k \in I} 1/\lambda_k\right]^{-1}$. But from the first-order conditions of the Planner's problem, and given that c is uniquely determined, it follows that $\hat{q}(\omega)$, $\hat{\mu}(\omega)$ and $\hat{v}(\omega)$ are uniquely determined as well. Clearly, this implies that the price of Arrow securities, q, and the private asset valuations, v, are uniquely determined up to the multiplicative constant $1/\beta$. Now turning to the price of assets, we note that the first-order condition of the agent's problem imply that $p_j = v_{ij}$ for almost all assets held by i. Since the private valuations are uniquely determined up to the multiplicative constant $1/\beta$, the same property must hold for the price assets $\bar{N}$-almost everywhere.

A.4 Proof of Proposition 4

The first bullet point of Proposition 4 follows directly from Lemma ??. Indeed, the second condition (11) of Lemma ?? holds by construction since $\delta_{ij} \in [0, 1)$.

Next, consider the second bullet point. It is well known that, in this case, in a $\delta = 0$-equilibrium, agents have constant consumption share. That is, there exists some $\{\alpha_i\}_{i \in I}$ such that $\sum_{i \in I} \alpha_i = 1$ and $c_i(\omega) = \alpha_i \sum_{j \in J} d_j(\omega)$ for all $i \in I$. One then immediately sees that $n_{ij}^\delta = \alpha_i$ satisfies the two conditions of Lemma ?? for any $\delta > 0$.

A.5 Proof of Lemma 2

As before we state proofs for our results when δ_{ij} is assumed to depend both on the type of agent holding the asset and on the type of the asset. In this case, the Proposition holds under the additional restriction that:

$$\frac{\delta_{1j} d_j(\omega_1)}{\delta_{2j} d_j(\omega_2)}, \tag{32}$$

is strictly increasing. Notice that this restriction is automatically satisfied whenever $\delta_{1j} = \delta_{2j}$ for all j. The generalization of (22)-(23) is

$$c_1(\omega_1) \geq \int_{j \in [0,k)} \delta_{1j} d_j(\omega_1) d\bar{N}_j + \delta_{1k} d_k(\omega_1) \Delta N_1 \tag{33}$$

$$c_2(\omega_2) \geq \int_{j \in (k,1]} \delta_{2j} d_j(\omega_2) d\bar{N}_j + \delta_{2k} d_k(\omega_2) \Delta N_2. \tag{34}$$

The "if" part of the Proposition. Pick the smallest possible k and the largest possible ΔN_2 such that the inequalities (33)-(34) hold. Consider the corresponding tree allocation $N_1 = \bar{N} \mathbb{I}_{\{j \in [0,k)\}} + \Delta N_1 \mathbb{I}_{\{j=k\}}$ and $N_2 = \Delta N_2 \mathbb{I}_{\{j=k\}} + \bar{N} \mathbb{I}_{\{j \in (k,1]\}}$.

By construction, the incentive constraint of agent $i = 1$ holds in state ω_1, and the incentive constraint of agent $i = 2$ holds in state ω_2.

Next, we argue that the incentive constraint of agent $i = 1$ holds in state ω_2. This is obvious if N allocates no tree to agent $i = 1$. Otherwise, if N allocates some trees to agent $i = 1$, then the incentive constraint of agent $i = 2$ must bind in state ω_2. Given $\delta_{ij} < 1$, this implies that the incentive constraint of agent $i = 1$ holds in state ω_2.

With the above observations in mind, the only incentive constraint that remains to be checked is that of agent $i = 2$ in state ω_1. If it holds with the proposed tree allocation, N, we are done. Otherwise,

$$c_2(\omega_1) < \int_{(k,1]} \delta_{2j} d_j(\omega_1) \, d\bar{N}_j + \delta_{2k} d_{2k} \Delta N_2,$$

in which case we explicitly construct another allocation of tree holdings that is incentive compatible. We proceed as follows. We start from the proportional tree allocation that delivers agents $i = 1$ and $i = 2$ their consumption in state ω_2: $\hat{N}_1 = \frac{c_1(\omega_2)}{y(\omega_2)} \bar{N}$ and $\hat{N}_2 = \frac{c_2(\omega_2)}{y(\omega_2)} \bar{N}$. By construction, with such proportional allocation, the incentive constraint of both agents hold in state ω_2. Since the consumption share of agent $i = 2$ is strictly larger in state ω_1 than in state ω_2, one sees that that agent $i = 2$ incentive compatibility constraint is slack in state ω_1. Indeed, we have:

$$c_2(\omega_1) > \frac{y(\omega_1)}{y(\omega_2)} c_2(\omega_2) = \frac{c_2(\omega_2)}{y(\omega_2)} \int d_j(\omega_1) d\bar{N}_j = \int d_j(\omega_1) d\tilde{N}_{2j} > \int \delta_{2j} d_j(\omega_1) d\tilde{N}_{2j},$$

where the first inequality states that the consumption share is larger in state ω_1 than in state ω_2, the first equality follows from rearranging and from the definition of $y(\omega_1)$, the second equality follows from the definition of N_2, and the last inequality follows because $\delta_{2j} < 1$.

Taking stock, for the original allocation N, the incentive compatibility constraints hold in state ω_2 for both

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$i = 1$ and $i = 2$, but it does not hold in state ω_1 for agent $i = 2$. For the proportional allocation $\hat{N}$, the incentive compatibility constraints also hold in state ω_2 for both $i = 1$ and $i = 2$, and it holds with strict inequality in state ω_1 for agent $i = 2$. Therefore, there is a convex combination of N and $\hat{N}$ such that the incentive compatibility constraint is binding in state ω_1 for agent $i = 2$. This implies that the incentive compatibility constraint holds in state ω_1 for agent $i = 1$. Clearly, the incentive compatibility constraint also hold in state ω_2 for both agents since they hold separately for N and $\hat{N}$.

The "only if" part of the Proposition. As before, pick the smallest possible k and the largest possible ΔN_2 such that (34) holds. Given this ΔN_2, let $\Delta N_1 \equiv \hat{N}_k - \bar{N}_{k-} - \Delta N_2$. If $k = 0$ and $\Delta N_2 = \bar{N}_0$, then (33) evidently holds. Otherwise, (34) holds with equality and we need to establish that that (33) holds as well. To that end, consider any $\hat{N}$ such that $(c, \hat{N})$ is incentive feasible. Then:

$$
\begin{aligned}
&\int_{[0,k)} \delta_{1j} d_j(\omega_1) d\bar{N}_j + \delta_{1k} d_k(\omega_1) \Delta N_1 \\
=\ &\int_{[0,k)} \delta_{1j} d_j(\omega_1) \left(d\hat{N}_{1j} + d\hat{N}_{2k} \right) + \delta_{1k} d_k(\omega_1) \Delta N_1 \\
=\ &\int_{[0,1]} \delta_{1j} d_j(\omega_1) d\hat{N}_{1j} - \int_{[k,1]} \delta_{1j} d_j(\omega_1) d\hat{N}_{1j} + \delta_{1k} d_k(\omega_1) \Delta N_1 + \int_{[0,k)} \delta_{1j} d_j(\omega_1) d\hat{N}_{2j} \\
\leq\ &c_1(\omega_1) - \int_{[k,1]} \delta_{2j} d_j(\omega_2) \frac{\delta_{1k} d_k(\omega_1)}{\delta_{2k} d_k(\omega_2)} d\hat{N}_{1j} + \delta_{1k} d_k(\omega_1) \Delta N_1 + \int_{[0,k)} \delta_{2j} d_j(\omega_2) \frac{\delta_{1k} d_k(\omega_1)}{\delta_{2k} d_k(\omega_2)} d\hat{N}_{2j} \\
=\ &c_1(\omega_1) + \frac{\delta_{1k} d_k(\omega_1)}{\delta_{2k} d_k(\omega_2)} \left[\int_{[0,1]} \delta_{2j} d_j(\omega_2) d\hat{N}_{2j} + \delta_{2k} d_k(\omega_2) \Delta N_1 - \delta_{2k} d_k(\omega_2) \left(\bar{N}_k - \bar{N}_{k-} \right) - \int_{(k,1]} \delta_{2j} d_j(\omega_2) d\bar{N}_j \right] \\
=\ &c_1(\omega_1) + \frac{\delta_{1k} d_k(\omega_1)}{\delta_{2k} d_k(\omega_2)} \left[\underbrace{\int_{[0,1]} \delta_{2j} d_j(\omega_2) d\hat{N}_{2j}}_{\leq c_2(\omega_2)} - \underbrace{\left(\delta_{2k} d_k(\omega_2) \Delta N_2 + \int_{(k,1]} \delta_{2j} d_j(\omega_2) d\bar{N}_j \right)}_{=c_2(\omega_2)} \right] \leq c_1(\omega_1),
\end{aligned}
$$

where: the second line follows by feasibility; $\bar{N} = \hat{N}_1 + \hat{N}_2$, the third line follows by rearranging and using the assumption that $(c, \hat{N})$ is incentive feasible; the fourth line follows by using the condition that (32) is strictly increasing; the fifth line by rearranging and using feasibility again; and the sixth line by our assumption that $(c, \hat{N})$ is incentive feasible and by our observation that (23) must hold with equality given our choice of k and ΔN_2.

A.6 Proof of Lemma 3

Consider first the first-best allocation, $c^\star$. The first-order condition of the Planner's problem implies

$$
\alpha_1 \left[c_1^\star(\omega) \right]^{-\gamma_1} - \alpha_2 \left[y(\omega) - c_1^\star(\omega) \right]^{-\gamma_2} = 0,
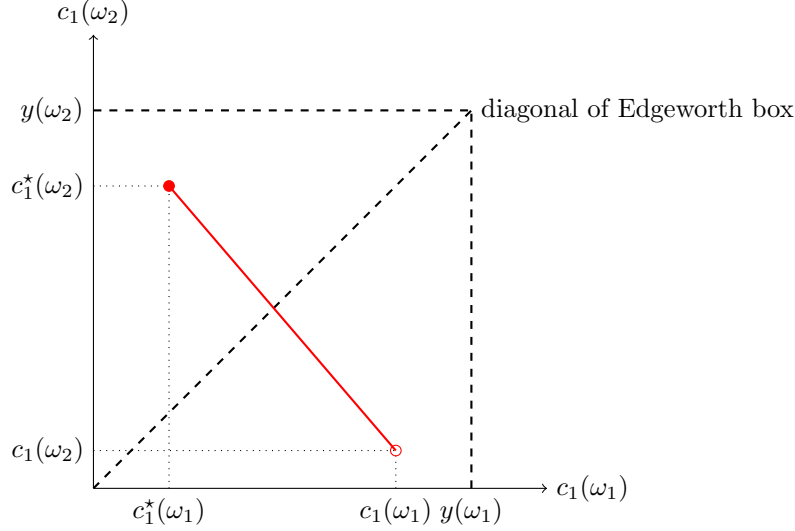$$

Figure 5: The Edgeworth box for the consumption of agent 1 in state ω_1 (x-axis) and in state ω_2 (y-axis).

for all $\omega \in \Omega$. In terms of consumption share, $c(\omega)/y(\omega)$, this equation becomes:

$$
\alpha_1 \left[\frac{c_1^{\star}(\omega)}{y(\omega)} \right]^{-\gamma_1} y(\omega)^{\gamma_2 - \gamma_1} - \alpha_2 \left[1 - \frac{c_1^{\star}(\omega)}{y(\omega)} \right]^{-\gamma_2} = 0. \tag{35}
$$

Since $\gamma_2 > \gamma_1$, this equation is strictly decreasing in the consumption share and strictly increasing in $y(\omega)$. Hence it follows that the consumption share is strictly increasing in $y(\omega)$, i.e., $c_1^{\star}(\omega_1)/y(\omega_1) < c_1^{\star}(\omega_2)/y(\omega_2)$. The inequality for $i = 2$ follows directly because consumption shares add up to one.

Now consider the equilibrium allocation, c. Assume, toward a contradiction, that $c_1(\omega_1)/y(\omega_1) \geq c_1(\omega_2)/y(\omega_2)$, i.e., the consumption shares of agent $i = 1$ lie below the diagonal of the Edgeworth box, as shown in Figure 5. Notice that since the first-best allocation, $c^{\star}$, satisfies the reverse inequality, it must lie strictly above the diagonal. This implies that $c^{\star} \neq c$. By strict concavity, social welfare evaluated at c is strictly smaller than social welfare evaluated at $c^{\star}$, and strictly smaller than social welfare at any point on the segment $(c, c^{\star}]$ linking c to $c^{\star}$, shown in red on the figure. Clearly, the segment $[c, c^{\star})$ crosses the diagonal at some point $c^{\dagger}$, which may be c. Since $c^{\dagger}$ keeps the agent's consumption share constant across states, it can be made incentive feasible by giving agents the corresponding "proportional" asset allocation, i.e., a share in the market portfolio equal to their respective consumption share, $N_i^{\dagger} = c_i^{\dagger}(\omega_i)/y(\omega_i) \, \bar{N}$. But since $\delta < 1$, it follows that all incentive constraints are slack for $(c^{\dagger}, N^{\dagger})$. Therefore, points on the segment $(c, c^{\star}]$ near $c^{\dagger}$ are incentive feasible as well. But they improve social welfare strictly relative to c, which is a contradiction.

A.7 Proof of Proposition 5

As for Lemma 2, we offer a proof in the general case when δ_{ij} is assumed to depend both on the identity of the asset holders and on the type of the asset, maintaining the restriction that

$$\frac{\delta_{1j}d_j(\omega_1)}{\delta_{2j}d_j(\omega_2)}, \tag{36}$$

is strictly increasing.

Given Lemma 2, what remains to be shown is that, for any incentive-feasible consumption allocation on the boundary, the distribution of assets is uniquely determined. We establish:

Lemma A.4 *Suppose that (22) and (23) holds with equality for some consumption allocation c, some $k \in [0,1]$ and some $(\Delta N_1, \Delta N_2) \geq 0$ such that $\Delta N_1 + \Delta N_2 = N_k - N_{k-}$. Then (c,N) is incentive feasible if and only if $N_1 = \Delta N_1 \mathbb{I}_{\{j=k\}} + \bar{N}\mathbb{I}_{\{j<k\}}$ and $N_2 = \Delta N_2 \mathbb{I}_{\{j=k\}} + \bar{N}\mathbb{I}_{\{j>k\}}$.*

The "if" part of Lemma A.4 follows because, with the proposed asset allocation, the incentive constraint of agent $i = 1$ binds in state ω_1, and that of agent $i = 2$ binds in state ω_2. It then follows that the two other incentive constraints are slack.

For the "only if" part, consider any asset allocation such that $(c, \hat{N})$ is incentive feasible. Then the incentive constraint of agent $i = 1$ in state ω_1 writes:

$$c_1(\omega_1) = \int_{[0,k)} \delta_{1j}d_j(\omega_1)\,d\bar{N}_j + \delta_{1k}d_k(\omega_1)\Delta N_1 \geq \int \delta_{1j}d_j(\omega_1)d\hat{N}_{1j}$$

Using that $d\bar{N}_j = d\hat{N}_{1j} + d\hat{N}_{2j}$ we then obtain that:

$$\int_{[0,k)} \delta_{1j}d_j(\omega_1)\,d\hat{N}_{2j} + \delta_{1k}d_k(\omega_1)\Delta N_1 \geq \int_{(k,1]} \delta_{1j}d_j(\omega_1)d\hat{N}_{1j} + \delta_{1k}d_k(\omega_1)\Delta\hat{N}_1, \tag{37}$$

where $\Delta\hat{N}_1 \equiv \hat{N}_{1k} - \hat{N}_{1k-}$. Proceeding analogously with the incentive constraint of agent $i = 2$ in state ω_2, we obtain:

$$\int_{(k,1]} \delta_{2j}d_j(\omega_2)\,d\hat{N}_{1j} + \delta_{2k}d_k(\omega_2)\Delta N_2 \geq \int_{[0,k)} \delta_{2j}d_j(\omega_2)d\hat{N}_{2j} + \delta_{2k}d_k(\omega_2)\Delta\hat{N}_2, \tag{38}$$

where $\Delta\hat{N}_2 \equiv \hat{N}_{2k} - \hat{N}_{2k-}$ Now multiply equation (37) by $\delta_{2k}d_k(\omega_2)$, and equation (38) by $\delta_{1k}d_k(\omega_1)$ and add the two inequalities. The $j = k$ terms cancel each others out because, by feasibility, $\Delta N_1 + \Delta N_2 = \Delta\hat{N}_1 + \Delta\hat{N}_2$. We

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thus obtain:

$$\int_{[0,k)} \delta_{1j} d_j(\omega_1) \delta_{2k} d_j(\omega_2) \, d\hat{N}_{2j} + \int_{(k,1]} \delta_{2j} d_j(\omega_2) \delta_{1k} d_k(\omega_1) \, d\hat{N}_{1j} \geq \int_{(k,1]} \delta_{1j} d_j(\omega_1) \delta_{2k} d_k(\omega_2) d\hat{N}_{1j} + \int_{[0,k)} \delta_{2j} d_j(\omega_2) \delta_{1k} d_k(\omega_1) d\hat{N}_{2j}.$$

After rearranging:

$$\int_{[0,k)} [\delta_{1j} d_j(\omega_1) \delta_{2k} d_j(\omega_2) - \delta_{2j} d_j(\omega_2) \delta_{1k} d_k(\omega_1)] \, d\hat{N}_{2j} \geq \int_{(k,1]} [\delta_{1j} d_j(\omega_1) \delta_{2k} d_k(\omega_2) - \delta_{2j} d_j(\omega_2) \delta_{1k} d_k(\omega_1)] \, d\hat{N}_{1j}$$

But, by (36), the integrand on the left-hand side is strictly negative over $[0, k)$, while the integrand on the right-hand side is strictly positive over $(k, 1]$. Therefore, both integrals are zero: agent $i = 2$ holds no assets in $[0, k)$ and all assets in $(k, 1]$, while agent $i = 1$ holds all assets in $[0, k)$ and no asset in $(k, 1]$. Plugging this back into the incentive compatibility constraint, we can determine each agent's holdings of asset k. Indeed, we obtain:

$$\delta_{1k} d_k(\omega_1) \Delta N_1 \geq \delta_{1k} d_k(\omega_1) \Delta \hat{N}_1 \text{ and } \delta_{2k} d_k(\omega_2) \Delta N_2 \geq \delta_{2k} d_k(\omega_1) \Delta \hat{N}_2.$$

Since $\Delta N_1 + \Delta N_2 = \Delta \hat{N}_1 + \Delta \hat{N}_2$, it follows that $\Delta N_1 = \Delta \hat{N}_1$ and $\Delta N_2 = \Delta \hat{N}_2$.

A.8 Proof of Proposition 6

With two agents, the zero-transfer equation (30) writes:

$$\bar{n}_2 \alpha_1 \mathbb{E} \left\{ u_1' [c_1(\omega)] c_1(\omega) \right\} = \bar{n}_1 \alpha_2 \mathbb{E} \left\{ u_2' [c_2(\omega)] c_2(\omega) \right\}$$

With CRRA utility, this can be simplified further:

$$\bar{n}_2 \alpha_1 \mathbb{E} \left[c_1(\omega)^{1-\gamma_1} \right] = \bar{n}_1 \alpha_2 \mathbb{E} \left[c_2(\omega)^{1-\gamma_2} \right],$$

so that:

$$\frac{\bar{n}_1}{\bar{n}_2} = \frac{\alpha_1 \mathbb{E} \left[c_1(\omega)^{1-\gamma_1} \right]}{\alpha_2 \mathbb{E} \left[c_2(\omega)^{1-\gamma_2} \right]}.$$

Now notice that, as α_1/α_2 increases, the solution of the Planner's problem moves to the northeast of the incentive-constrained Pareto set (see Lemma A.3 in the Proof of Proposition 3). Clearly, this implies a strictly increasing relationship between $\bar{n}_1/\bar{n}_2$ and α_1/α_2.

A.9   Proof of Proposition 7

Suppose c is incentive feasible under tree distribution $\bar{N}$ and is such that the consumption share of agent $i = 1$ is higher in state ω_1 than in state ω_2, as is the case in equilibrium (the opposite case is symmetric). Lemma 2 implies that there exist $k \in [0,1]$ and $(\Delta N_1, \Delta N_2) \geq 0$, $\Delta N_1 + \Delta N_2 = \bar{N}_k - \bar{N}_{k-}$ such that the incentive compatibility constraints (22) and (23) are satisfied.

To establish the claim of the proposition, that c is incentive feasible under tree distribution $\bar{N}^*$, we need to find some $k^* \in [0,1]$ and $(\Delta N_1^*, \Delta N_2^*) \geq 0$, $\Delta N_1^* + \Delta N_2^* = \bar{N}_{k^*} - \bar{N}_{k^*-}$ such that the analog of conditions (22) and (23) for the new tree allocation (described with variables with a *) are satisfied. It is thus sufficient to find some k^* and $(\Delta N_1^*, \Delta N_2^*) \geq 0$, $\Delta N_1^* + \Delta N_2^* = \bar{N}_{k^*} - \bar{N}_{k^*-}$ such that:

$$\int_{j \in [0,k^*)} j d\bar{N}_j^* + k^* \Delta N_1^* \leq \int_{j \in [0,k)} j d\bar{N}_j + k \Delta N_1 \tag{39}$$

and

$$\int_{j \in (k^*,1]} (1-j) d\bar{N}_j^* + (1-k^*) \Delta N_2^* \leq \int_{j \in (k,1]} (1-j) d\bar{N}_j + (1-k) \Delta N_2. \tag{40}$$

Let $k^* \in [0,1]$ and $(\Delta N_1^*, \Delta N_2^*) \geq 0$, $\Delta N_1^* + \Delta N_2^* = \bar{N}_{k^*} - \bar{N}_{k^*-}$, such that

$$\bar{N}_{k^*-}^* + \Delta N_1^* = \bar{N}_{k-} + \Delta N_1. \tag{41}$$

First note that (41) together with (25) and (26) imply that (39) holds if and only if (40) holds. Indeed, adding

$$\bar{N}_1^* - \left(\bar{N}_{k^*-}^* + \Delta N_1^*\right) - \int_{j \in [0,1]} j d\bar{N}_j^* = \bar{N}_1 - \left(\bar{N}_{k-} + \Delta N_1\right) - \int_{j \in [0,1]} j d\bar{N}_j$$

on each side of (39) yields (40).

Let us now show that (39) holds. Integrating the left-hand side of (39) by parts yields

$$\int_{j \in [0,k^*)} j d\bar{N}_j^* + k^* \Delta N_1^* = k^* \bar{N}_{k^*-}^* - \int_0^{k^*} \bar{N}_j^* dj + k^* \Delta N_1^*.$$

There are two cases to consider.

First case: $k^* \leq k$. Then, it follows from (41) that $\bar{N}_{k^*-}^* + \Delta N_1^* = \bar{N}_{k-} + \Delta N_1 \geq \bar{N}_{k-} \geq \bar{N}_j$ for $j \in (k^*, k)$.

Thus,

$$
\begin{aligned}
\int_{j\in[0,k^*)} j \, d\bar{N}_j^* + k^* \Delta N_1^*
&\leq k^* \bar{N}_{k^*-} - \int_0^{k^*} \bar{N}_j^* \, dj + k^* \Delta N_1^* + \int_{k^*}^k (\bar{N}_{k^*-}^* + \Delta N_1^* - \bar{N}_j) \, dj \\
&= k\left(\bar{N}_{k^*-} + \Delta N_1^*\right) - \int_0^{k^*} \bar{N}_j^* \, dj - \int_{k^*}^k \bar{N}_j \, dj \\
&= k\left(\bar{N}_{k-} + \Delta N_1\right) - \int_0^{k^*} \bar{N}_j^* \, dj - \int_{k^*}^k \bar{N}_j \, dj \\
&\leq k\left(\bar{N}_{k-} + \Delta N_1\right) - \int_0^{k^*} \bar{N}_j \, dj - \int_{k^*}^k \bar{N}_j \, dj \\
&= k\left(\bar{N}_{k-} + \Delta N_1\right) - \int_0^{k} \bar{N}_j \, dj
\end{aligned}
$$

where the second line is obtained by calculation, the third line follows from (41), the fourth line follows from the definition of second order stochastic dominance, the fifth line is obtained by calculation and is equal to the right-hand side of (39) after integration by parts.

Second case: $k < k^*$. Then

$$
\begin{aligned}
\int_{j\in[0,k^*)} j \, d\bar{N}_j^* + k^* \Delta N_1^*
&= k^*\left(\bar{N}_{k-} + \Delta N_1\right) - \int_0^{k^*} \bar{N}_j^* \, dj \\
&\leq k^*\left(\bar{N}_{k-} + \Delta N_1\right) - \int_0^{k^*} \bar{N}_j \, dj \\
&\leq k^*\left(\bar{N}_{k-} + \Delta N_1\right) - \int_0^{k} \bar{N}_j \, dj - \int_k^{k^*} \bar{N}_k \, dj \\
&\leq k^*\left(\bar{N}_{k-} + \Delta N_1\right) - \int_0^{k} \bar{N}_j \, dj - (k^* - k)\left(\bar{N}_{k-} + \Delta N_1\right) \\
&= k\left(\bar{N}_{k-} + \Delta N_1\right) - \int_0^{k} \bar{N}_j \, dj
\end{aligned}
$$

where the first line follows by integration by part of (41), the second line follows from the definition of second order stochastic dominance, the third line follows from $\bar{N}$ being increasing, the fourth line follows from $\Delta N_2 \geq 0$, the fifth line is obtained by calculation and is equal to the right-hand side of (39) after integration by parts.

A.10   Proof of Proposition 9

Notice that, since the function ϕ_ℓ is the same for both agents, we have that $\delta_{1j} d_j(\omega_1)/\delta_{2j} d_j(\omega_2) = d_j(\omega_1)/d_j(\omega_2)$ is strictly increasing, so all our results apply.

The equilibrium is uniquely pinned down by a two-equation-in-two-unknown problem, for the ratio of the two budget constraints multipliers, $r = \frac{\lambda_1}{\lambda_2}$, and the threshold k determining asset ownership. To obtain the first equation, first note that the continuity of $j \mapsto (\delta_{1j} d_j(\omega_1))/(\delta_{2j} d_j(\omega_2))$ implies that for the threshold asset, the first-order

condition with respect to asset holdings holds with an equality for both agents. Thus:

$$F(r, k) \equiv \mu_1(\omega_1)\delta_{1k}d_k(\omega_1) - r\mu_2(\omega_2)\delta_{2k}d_k(\omega_2) = 0. \tag{42}$$

where, from the first-order conditions we have that

$$
\begin{aligned}
\mu_1(\omega_1) &= r\pi(\omega_1)u_2'\left[\int_0^1 \left(1 - \delta_{1j}\mathbb{I}_{\{j<k\}}\right) d_j(\omega_1) \, d\bar{N}_j\right] - \pi(\omega_1)u_1'\left[\int_0^1 \delta_{1j}\mathbb{I}_{\{j<k\}} d_j(\omega_1) \, d\bar{N}_j\right] \\
\mu_2(\omega_2) &= \frac{1}{r}\pi(\omega_2)u_1'\left[\int_0^1 \left(1 - \delta_{2j}\mathbb{I}_{\{j\geq k\}}\right) d_j(\omega_2) \, d\bar{N}_j\right] - \pi(\omega_2)u_2'\left[\int_0^1 \delta_{2j}\mathbb{I}_{\{j\geq k\}} d_j(\omega_2) \, d\bar{N}_j\right].
\end{aligned}
$$

Notice that the continuity of the distribution of asset supplies mean that there is no atom, hence $\Delta N_1 = \Delta N_2 = 0$, i.e., the allocation of the supply of threshold assets between agents is irrelevant. The second equilibrium equation is (30) which here takes the form:

$$G(r, k) \equiv \mathbb{E}[u_1'(c_1(\omega))c_1(\omega)] - r\frac{\bar{n}_1}{\bar{n}_2}\mathbb{E}[u_2'(c_2(\omega))c_2(\omega)] = 0, \tag{43}$$

where $c_1(\omega_1) = \int_0^k \delta_{1j}d_j(\omega_1) \, d\bar{N}_j$, $c_2(\omega_1) = \int_0^1 d_j(\omega_1) \, d\bar{N}_j - c_1(\omega_1)$, $c_2(\omega_2) = \int_k^1 \delta_{2j}d_j(\omega_2) \, d\bar{N}_j$, and $c_1(\omega_2) = \int_0^1 d_j(\omega_2) \, d\bar{N}_j - c_2(\omega_2)$.

The function $F(r, k)/(\delta_{2k}d_k(\omega_2))$ is strictly increasing and continuous in both r and k. Moreover, one can explicitly solve for r as a function of k, $\rho(k)$. This function is strictly decreasing and, because of the Inada condition $u_i'(0) = +\infty$, goes to infinity as k goes to zero, $\lim_{k\to 0} \rho(k) = \infty$, and goes to zero as k goes to one, $\lim_{k\to 1} \rho(k) = 0$.

Since $\bar{N}_j$ is strictly increasing, it follows that both $c_1(\omega_1)$ and $c_1(\omega_2)$ are strictly increasing in k while both $c_2(\omega_1)$ and $c_2(\omega_2)$ are strictly decreasing in k. Recall that the coefficient of relative risk aversion are both less than one, $0 \leq \gamma_1 < \gamma_2 \leq 1$. Therefore, the function $G(r, k)$ is strictly decreasing in r and strictly increasing in k. Plugging in the function $\rho(k)$ defined above, we obtain a strictly increasing function $k \mapsto G(\rho(k), k)$. Given our earlier observation that $\lim_{k\to 0} \rho(k) = \infty$ and $\lim_{k\to 1} \rho(k) = 0$, it follows that $k \mapsto G(\rho(k), k)$ is strictly negative when $k \simeq 0$, and strictly positive when $k \simeq 1$. Thus, the equilibrium threshold is the unique solution of $G(\rho(k), k) = 0$. Clearly $c_1(\omega_1)$ increases with ε, while $c_2(\omega_2)$ stays the same. This implies that $\rho(k)$ shifts down, and that $G(\rho(k), k)$ shifts down as well. Hence $k(\varepsilon') < k(\varepsilon)$ if $\varepsilon' > \varepsilon$.

$$\frac{dk}{d\varepsilon} < 0.$$

B Supplementary appendix

B.1 Proof of Lemma A.1

For this proof, in order to apply some of the results in Chapter 12 of Stokey and Lucas (1989), we extend measures $M \in \mathcal{M}_+$ to the entire real line, $\mathbb{R}$, by setting $M_j = 0$ for all $j < 0$, and $M_j = M_1$ for all $j \geq 1$. Now consider a sequence (c^k, N^k) of incentive feasible allocations. Given that c^k belongs to a finite dimensional space and is bounded, it has a converging subsequence. Given that $\sum_{i \in I} N_i^k = \bar{N}$, N_{ij} is bounded above by $\bar{N}_j$ for all $(i, j) \in I \times \mathbb{R}$, an an application of Helly's selection Theorem (Theorem 12.9 in Stokey and Lucas (1989) easily extended to finite measures instead of distributions) shows that for each $i \in I$, N_i^k has a subsequence such that N_i^ℓ converging weakly in $\mathcal{M}_+$. Taken together, this means that there exists a subsequence (c^ℓ, N^ℓ) of (c^k, N^k) and some $(c, N) \in X$ such that $c^\ell \to c$ and $N_i^\ell \Rightarrow N_i$ for each $i \in I$.

What is left to show is that (c, N) is incentive feasible. Given that $j \mapsto d_j(\omega)$ and $j \mapsto \delta_{ij}$ are continuous, the definition of weak convergence allows us to assert that, since the feasibility constraint for consumption, (8), and the incentive compatibility constraints, (6), hold for each (c^ℓ, N^ℓ), then it must also hold in the limit for (c, N). The only difficulty is to show that the feasibility constraint for holdings is also satisfied. For this we rely on the characterization of weak convergence provided by Theorem 12.8 in Stokey and Lucas (1989), easily extended to finite measures. It asserts that N_i^ℓ converges pointwise at each continuity point of their limit, N_i. Therefore, for any $j \in \mathbb{R}$ such that all $(N_i)_{i \in I}$ are continuous, we have:

$$\sum_{i \in I} N_{ij}^\ell \to \sum_{j \in I} N_{ij}.$$

But recall that the feasibility constraint for holdings is satisfied for each j: $\sum_{i \in I} N_{ij}^\ell = \bar{N}_j$. Together with the above, this implies that

$$\sum_{i \in I} N_{ij} = \bar{N}_j,$$

for all $j \in \mathbb{R}$ such as all $(N_i)_{i \in I}$ are continuous. Now recall that the N_i's are increasing functions, and so have countably many discontinuity points. This implies that for any $j \in \mathbb{R}$, there is a sequence of $j_n \downarrow j$ such that j_n is a continuity point for all $(N_i)_{i \in I}$. Hence, for all j_n, we have

$$\sum_{i \in I} N_{ij_n} = \bar{N}_{j_n}.$$

Since $j \mapsto N_{ij}$ and $\bar{N}_j$ are all right continuous functions we can take the limit $j_n \downarrow j$ and obtain that $\sum_{i \in I} N_{ij} = \bar{N}_j$

for all $j \in \mathbb{R}$, as required.

B.2   Proof of Proposition A.1

In all what follow we let:

$$y(\omega) \equiv \sum_{j \in J} d_j(\omega), \; \underline{y} \equiv \min_{\omega \in \Omega} y(\omega), \; \text{and} \; \bar{y} \equiv \max_{\omega \in \Omega} y(\omega).$$

Proof that $\Gamma^\star(\alpha)$ is not empty.   We first show that the supremum is achieved. The only difficulty with this proof arises when $\alpha_i > 0$ and $u_i(0) = -\infty$ for some $i \in I$, because in this case the objective is unbounded as $c_i(\omega) \to 0$. However, in the planner's problem, one can restrict attention to $c_i(\omega)$ that are bounded away from zero. To see this, we first note that $c_i(\omega) = y(\omega)/I$ is incentive-feasible, implying that:

$$W^\star(\alpha) \geq \sum_{i \in I} \alpha_i \sum_{\omega \in \Omega} \pi(\omega) u_i \left[y(\omega)/I \right] \geq \sum_{i \in I} \min\{ u_i \left[\underline{y}/I \right], 0 \} \equiv \underline{W}.$$

Also, for each i such that $\alpha_i > 0$ and $u_i(0) = -\infty$, we have that

$$W(\alpha, c, n) \leq \alpha_i \pi(\omega) u_i \left[c_i(\omega) \right] + \sum_{k \neq i} \alpha_k \max\{ u_i(\bar{y}/I), 0 \}.$$

Now consider the equation

$$\alpha_i \pi(\omega) u_i \left[c_i(\omega) \right] + \sum_{k \neq i} \alpha_k \max\{ u_i(\bar{y}/I), 0 \} = \underline{W}.$$

Since $u_i(0) = -\infty$, the left-hand side is smaller than the right-hand side when $c \to 0$. Since $\underline{W} \leq 0$ by construction, the left-hand side is larger than the right-hand side when $c \to \infty$. Given the strict monotonicity of $u_i(c)$, it follows that the equation has a unique solution, which is decreasing and continuous in α_i. Let $\underline{c}_i(\alpha_i)$ be half of the minimum of these solutions across all $\omega \in \Omega$. By construction, for all allocation (c, n) such that $c_i(\omega) < \underline{c}_i$ for some $\omega \in \Omega$, $W(\alpha, c, n) < \underline{W}$. If we let $\underline{c}_i(\alpha_i) = 0$ for other i, that is for $i \in I$ such that $\alpha_i = 0$ or $u_i(0) = 0$, then, in the Planner's problem, one can restrict attention to allocation such that $c_i(\omega) \geq \underline{c}_i(\alpha_i)$, which we write as $c \geq \underline{c}(\alpha)$. Notice that, by construction, the objective of the planner is continuous over $c \geq \underline{c}(\alpha)$.

Now to show that there is a solution consider any sequence (c^k, N^k) of incentive-feasible allocation such that $W(\alpha, c^k, N^k) \to W^\star(\alpha)$. From the above remark we can focus on a sequence such that $c^k \geq \underline{c}(\alpha)$. Now, by Lemma A.1, there exists some incentive feasible allocation (c, N) and a subsequence (c^ℓ, N^ℓ) such that $c^\ell \to c$ and $N^\ell \to N$. Going to the limit in the Planner's objective, we obtain that $W(\alpha, c, N) = W^\star(\alpha)$.

Proof that $\Gamma^\star(\alpha)$ is weakly compact. The argument is the same as in the last paragraph, except that we now consider a sequence $(c^k, N^k) \in \Gamma^\star(\alpha)$.

Proof that $\Gamma^\star(\alpha)$ convex-valued. This follows because the objective is concave and the constraints linear.

Proof that $W^\star(\alpha)$ is continuous and $\Gamma^\star(\alpha)$ has a weakly closed graph. Consider any $\bar{\alpha} \geq 0$ such that $\sum_{i \in I} \bar{\alpha}_i = 1$ and any sequence $\alpha^k \to \bar{\alpha}$ and an associated sequence $(c^k, N^k) \in \Gamma^\star(\alpha^k)$. Without loss of generality for this proof, assume that $W^\star(\alpha^k)$ converges to some limit,[22] and that (c^k, N^k) converges weakly towards some incentive feasible allocation (c, N).[23] We want to show that $W^\star(\alpha^k) \to W^\star(\alpha)$ and that $(c, N) \in \Gamma^\star(\alpha)$. Let $I_0 = \{i \in I : \alpha_i = 0 \text{ and } u_i(0) = -\infty\}$. We have:

$$W^\star(\alpha^k) = \sum_{i \notin I_0} \alpha_i^k \sum_{\omega \in \Omega} \pi(\omega) u_i \left[c_i^k(\omega) \right] + \sum_{i \in I_0} \alpha_i^k \sum_{\omega \in \Omega} \pi(\omega) u_i \left[c_i^k(\omega) \right]. \tag{44}$$

By our maintained assumptions, both the left-hand side and the first term on the right-hand side have a limit as $k \to \infty$. Hence, the second term on the right-hand side has a limit as well. We argue that this limit must be negative. Indeed, for $i \in I_0$, if $\lim c_i^k(\omega) > 0$, then $\lim \alpha_i^k u_i \left[c_i^k(\omega) \right] = 0$. If $\lim c_i^k(\omega) = 0$, then $\alpha_i^k u_i \left[c_i^k(\omega) \right] \leq 0$ for k large enough. Hence,

$$\lim \sum_{i \in I_0} \alpha_i^k \sum_{\omega \in \Omega} \pi(\omega) u_i \left[c_i^k(\omega) \right] \leq 0.$$

Therefore:

$$\lim W^\star(\alpha^k) \leq \sum_{i \notin I_0} \bar{\alpha}_i \sum_{\omega \in \Omega} \pi(\omega) u_i \left[\lim c_i^k(\omega) \right] \leq W^\star(\bar{\alpha}), \tag{45}$$

since $(\lim c^k, \lim N^k)$ is incentive feasible.

To show the reverse inequality, for all $i \in I_0$, choose some $\phi_i > 0$ such that $1 + \phi_i(1 - \gamma_i) < 1$, where $\gamma_i > 1$ is the assumed CRRA bound for $u_i(c)$. Let $\beta(\alpha) \equiv \sum_{i \in I_0} (\alpha_i)^{\phi_i}$. Since $\lim \alpha_i^k = 0$ for all $i \in I_0$, we have that $\lim \beta(\alpha^k) = 0$, hence $\beta(\alpha^k) < 1$ for all k large enough. Now take any $(c, N) \in \Gamma^\star(\bar{\alpha})$. For all $i \in I_0$ we have that $\bar{\alpha}_i = 0$, which clearly implies that $c_i(\omega) = N_i = 0$, i.e., $i \notin I_0$ consume the aggregate endowment and holds the aggregate tree supplies. Therefore, if we scale down the consumption and asset holding of $i \notin I_0$ by $1 - \beta(\alpha^k)$, we keep the allocation of $i \notin I_0$ incentive compatible and we free up $\beta(\alpha^k)y(\omega)$ consumption, and $\beta(\alpha^k)\bar{N}$ assets. We can then re-distribute this consumption and these assets by giving to each agent $i \in I_0$ a consumption equal to $y(\omega) \left(\alpha_i^k \right)^{\phi_i}$ and an asset allocation equal to a fraction $\left(\alpha_i^k \right)^{\phi_i}$ of the market portfolio, $\bar{N}$. Because the consumption of $i \in I_0$ is proportional

---

[22]Indeed, since $W^\star(\alpha)$ is bounded below by $\underline{W}$ and is clearly bounded above, to show convergence towards $W^\star(\alpha)$ it is sufficient to show that every convergent subsequence of $W^\star(\alpha^k)$ converges towards $W^\star(\alpha)$.

[23]From Lemma A.1, we can always find a convergence subsequence with this property.

to its portfolio payoff, the allocation of $i \in I_0$ is incentive compatible. Therefore, this process of scaling down the consumption of $i \notin I_0$ and redistributing to $i \in I_0$, leads to an incentive feasible allocation. Hence, we have that:

$$W^\star(\alpha^k) \geq \sum_{i \notin I_0} \alpha_i^k \sum_{\omega \in \Omega} \pi(\omega) u_i \left[c_i(\omega)(1 - \beta(\alpha))\right] + \sum_{i \in I_0} \alpha_i^k \sum_{\omega \in \Omega} \pi(\omega) u_i \left[y(\omega) \left(\alpha_i^k\right)^{\phi_i}\right].$$

The first term converges to $W^\star(\bar{\alpha})$. Using the assumed CRRA bound, $0 < |u(c)| < |K|c^{1-\gamma_i}$ for c close to zero, one sees that the second term goes to zero: indeed $\alpha_i^k \left|u_i \left[y(\omega) \left(\alpha_i^k\right)^{\phi_i}\right]\right|$ is bounded above by $|K|y(\omega)^{1-\gamma_i} \left(\alpha_i^k\right)^{1+(1-\gamma_i)\phi_i}$, which goes to zero since $\lim \alpha_i^k = 0$ and ϕ_i was chosen so that $1 + \phi_i(1-\gamma_i) > 0$. Hence, we obtain that $\lim W^\star(\alpha^k) \geq W^\star(\bar{\alpha})$. Taken together we have that

$$\lim W^\star(\alpha^k) \geq \sum_{i \notin I_0} \bar{\alpha}_i \sum_{\omega \in \Omega} \pi(\omega) u_i \left[\lim c_i^k(\omega)\right] = W^\star(\bar{\alpha}). \tag{46}$$

Taken together, (45) and (46) imply that

$$\lim W^\star(\alpha^k) = \sum_{i \notin I_0} \alpha_i \sum_{\omega \in \Omega} u_i \left[\lim c_i^k(\omega)\right] = W(\bar{\alpha}) \text{ and } \lim \sum_{i \in I_0} \alpha_i^k \sum_{\omega \in \Omega} \pi(\omega) u_i \left[c_i^k(\omega)\right] = 0.$$

This establishes that $W^\star(\alpha)$ is continuous and that $\Gamma^\star(\alpha)$ has a closed graph.

Proof that $\lim \alpha_i^k u' \left[c_i^k(\omega)\right] c_i^k(\omega) = 0$ **if** $\lim \alpha_i^k = 0$. Consider any sequence $\alpha^k \to \bar{\alpha}$ and any associated sequence (not necessarily converging) (c^k, N^k) in $\Gamma^\star(\alpha^k)$. Since we have shown that $\Gamma^\star(\alpha)$ has a weakly closed graph, it follows that any converging subsequence of (c^k, N^k) has a limit belonging to $\Gamma^\star(\bar{\alpha})$. Since the Planner finds optimal to give zero consumption to agents with zero weight, it follows that $\lim c_i^k(\omega) = 0$ for all i such that $\bar{\alpha}_i = 0$.

If $u_i(0) = 0$, then the result that $\lim \alpha_i^k u' \left[c_i^k(\omega)\right] c_i^k(\omega) = 0$ follows from the inequality $0 \leq u_i'(c)c \leq u_i(c)$. If $u_i(0) = -\infty$, we need a different argument. Write $W^\star(\alpha^k) = W_1^k + W_2^k$, where

$$W_1^k \equiv \sum_{i \notin I_0} \alpha_i \sum_{\omega \in \Omega} \pi(\omega) u_i \left[c_i^k(\omega)\right] \text{ and } W_2^k \equiv \sum_{i \in I_0} \alpha_i \sum_{\omega \in \Omega} \pi(\omega) u_i \left[c_i^k(\omega)\right].$$

By assumption, we have that $\lim \left(W_1^k + W_2^k\right) = W^\star(\bar{\alpha})$. Notice that W_1^k is bounded. Indeed, it is clearly bounded above because the constraint set is bounded. It is bounded below because, for any $i \notin I_0$ such that $u_i(0) = -\infty$, $\bar{\alpha}_i > 0$ and so α_i^k and hence $\underline{c}_i(\alpha_i^k)$ is bounded away fro zero for k large enough. Given boundedness, we can extract some convergent subsequence W_1^ℓ of W_1^k. Since consumption and asset holdings are incentive feasible, it follows from Lemma A.1 that there exists a weakly convergent subsequence (c^p, N^p) of (c^ℓ, N^ℓ). Clearly, $\lim W_1^p = \lim W_1^\ell$. But,

using the results of the previous paragraph, we have that $\lim W_1^p = W^\star(\bar{\alpha})$. Hence all convergent subsequences of W_1^k have the same limit $W^\star(\bar{\alpha})$, implying that $\lim W_1^k = W^\star(\bar{\alpha})$ and that $\lim W_2^k = 0$. It follows that, asymptotically as $k \to \infty$, the aggregate consumption of agents $i \notin I_0$ is arbitrarily close to $y(\omega)$, and the consumption of each agent $i \in I_0$ is arbitrarily close to zero. Therefore, for all k large enough, all terms in W_2^k are negative. Hence, for k large enough, we that for all $i \in I_0$, $W_2^k \leq \alpha_i^k \pi(\omega) u_i \left[c_i^k(\omega) \right] \leq 0$. Since $\lim W_2^k = 0$, it follows that $\lim \alpha_i^k \pi(\omega) u_i \left[c_i^k(\omega) \right] = 0$ as well. The result then follows from the CRRA bound $0 \leq u_i'(c)c \leq \gamma_i |u_i(c)|$.

B.3   Proof of Proposition A.2

Fix any feasible N and let:

$$W(\alpha \mid N) = \max \sum_{i \in I} \alpha_i U_i(c_i)$$

with respect to $c \in \mathbb{R}_+^{|\Omega| \times |I|}$, and subject to

$$\sum_{i \in I} c_i(\omega) \leq \sum_{i \in I} \int d_j(\omega) dN_{ij} \quad \forall \omega \in \Omega$$

$$c_i(\omega) \geq \int \delta_{ij} d_j(\omega) dN_{ij} \quad \forall (i, \omega) \in I \times \Omega.$$

From Corollary 28.3 in Rockafellar (1970), $c \in \mathbb{R}_+^{|\Omega| \times |I|}$ is an optimal solution only if there exists multipliers $\hat{q} \in \mathbb{R}_+^{|\Omega|}$ and $\hat{\mu} \in \mathbb{R}_+^{|\Omega| \times |I|}$ such that:

$$\alpha_i \frac{\partial U_i}{\partial c_i(\omega)} + \hat{\mu}_i(\omega) \quad \leq \quad \hat{q}(\omega)$$

$$\hat{q}(\omega) \left[\sum_{i \in I} \int d_j(\omega) dN_{ij} - \sum_{i \in I} c_i(\omega) \right] \quad = \quad 0, \quad \forall \omega \in \Omega$$

$$\hat{\mu}_i(\omega) \left[c_i(\omega) - \int \delta_{ij} d_j(\omega) dN_{ij} \right] \quad = \quad 0, \quad \forall (i, \omega) \in I \times \Omega.$$

Notice that we can always choose multipliers such that the top first-order condition, with respect to $c_i(\omega)$, holds with equality. Indeed, if it holds with a strict inequality for some $\hat{\mu}_i(\omega)$ and some (i, ω), then $c_i(\omega) = 0$ and so the incentive constraint holds with equality. So increasing $\hat{\mu}_i(\omega)$ leaves the complementary slackness conditions unchanged.

Now consider any other feasible $\hat{N} \in \mathcal{M}_+$. Clearly, for any $h \in [0, 1]$, $(1 - h)N + h\hat{N} = N + h(\hat{N} - N)$ is also feasible. In the optimization problem associated with $W\left(\alpha \mid N + h \left[\hat{N} - N \right] \right)$, we take the derivative of the Lagrangian with respect to h. We evaluate this derivative at $h = 0$, given some optimal consumption for $W(\alpha \mid N)$

and Lagrange multipliers satisfying the first order-conditions. We obtain:

$$
\begin{aligned}
L_h &= \sum_{i \in I} \int \left[\sum_{\omega \in \Omega} \hat{q}(\omega) d_j(\omega) - \sum_{\omega \in \Omega} \hat{\mu}_i(\omega) \delta_{ij} d_j(\omega) \right] \left[d\hat{N}_{ij} - dN_{ij} \right] \\
&= \sum_{i \in I} \int \hat{v}_{ij} \left[d\hat{N}_{ij} - dN_{ij} \right],
\end{aligned}
$$

where, for any set of Lagrange multipliers, $\hat{v}_{ij} \equiv \sum_{\omega \in \Omega} \hat{q}(\omega) d_j(\omega) - \sum_{\omega \in \Omega} \hat{\mu}_i(\omega) \delta_{ij} d_j(\omega)$. Notice that $\hat{q}(\omega)$ is uniquely determined[24] but $\hat{\mu}_i(\omega)$ may not, when $c_i(\omega) = 0$. One easily sees in particular that any

$$
0 \le \hat{\mu}_i(\omega) \le \hat{q}(\omega) - \alpha_i \frac{\partial U_i}{\partial c_i(\omega)}
$$

solves the first-order conditions. Let $\hat{V}_{ij}$ denote the corresponding interval of $\hat{v}_{ij}$. It follows from Corollary 5 in Milgrom and Segal (2002) that the right-derivative of $W\left(\alpha \mid N + h \left[\hat{N} - N \right] \right)$ at $h = 0$ is

$$
\frac{d}{dh} W\left(\alpha \mid N + h \left[\hat{N} - N \right] \right) \Big|_{h=0^+} = \min_{\hat{v}_{ij} \in \hat{V}_{ij}} \sum_{i \in I} \int \hat{v}_{ij} \left[d\hat{N}_{ij} - dN_{ij} \right].
$$

Now notice that $\int \hat{v}_{ij} dN_{ij}$ does not depend on the particular choice of $\hat{v}_{ij}$. Indeed, whenever $\hat{v}_{ij}$ is not uniquely determined, it is because $c_i(\omega) = 0$ for some $\omega \in \Omega$. But from the incentive compatibility constraint, it then follows that $\int \delta_{ij} d_j(\omega) \, dN_{ij} = 0$, and so $\hat{\mu}_i(\omega) \int \delta_{ij} d_j(\omega) \, dN_{ij} = 0$ as well. Since $\hat{N}_{ij}$ is a positive measure, $\int \hat{v}_{ij} d\hat{N}_{ij}$ is minimized when $\hat{v}_{ij}$ is smallest, which occurs when $\hat{\mu}_i(\omega)$ is largest, that is, when it is chosen so that the first-order condition with respect to $c_i(\omega)$ holds with equality.

Taken together, we obtain that a necessary condition for a feasible N to be optimal is that:

$$
\sum_{i \in I} \int \hat{v}_{ij} \left[d\hat{N}_{ij} - dN_{ij} \right] \le 0, \tag{47}
$$

for all feasible $\hat{N}$, where $\hat{v}_{ij} = \sum_{\omega \in \Omega} \hat{q}(\omega) d_j(\omega) - \sum_{\omega \in \Omega} \hat{\mu}_i(\omega) \delta_{ij} d_j(\omega)$ and $\hat{\mu}_i(\omega)$ is chosen so that the first-order condition with respect to $c_i(\omega)$ holds with equality. The proof is concluded by the following Lemma:

Lemma B.1 *Condition (47) holds if and only if $\int \left[\max_{k \in I} \hat{v}_{kj} - \hat{v}_{ij} \right] dN_{ij} = 0$ for all $i \in I$.*

For necessity, consider the correspondence $\Gamma(j) \equiv \arg\max_{k \in I} \hat{v}_{kj}$. By the Measurable Selection Theorem (Theorem

---

[24]Indeed for any $\omega \in \Omega$, there exists some $i \in I$ such that the incentive compatibility constraint does not bind. For this $i \in I$, $c_i(\omega) > 0$ and so the first-order condition holds with equality. If $u_i(c)$ is linear, then $\alpha_i \partial U_i / \partial c_i(\omega) = \alpha_i$ is uniquely determined. If $u_i(c)$ is strictly concave, then $c_i(\omega)$ is uniquely determined and so is $\alpha_i \partial U_i / \partial c_i(\omega)$. Using the first-order condition, it then follows that $\hat{q}(\omega)$ is uniquely determined.

7.6 in Stokey and Lucas (1989)), there exists a measurable selection $\gamma(j)$. Consider then the asset allocation:

$$\hat{N}_{ij} = \int_0^j \mathbb{I}_{\{\gamma(k)=i\}} d\bar{N}_k,$$

which gives the supply of asset k to one agent with the highest valuation, $v_{\gamma(k)k}$. Condition (47) implies that:

$$
\begin{aligned}
0 \geq \sum_{i \in I} \int \hat{v}_{ij} \left[d\hat{N}_{ij} - dN_{ij} \right] &= \sum_{i \in I} \int \hat{v}_{ij} \mathbb{I}_{\{\gamma(j)=i\}} d\bar{N}_j - \sum_{i \in I} \hat{v}_{ij} \, dN_{ij} \\
&= \int \max_{k \in I} \hat{v}_{kj} d\bar{N}_j - \sum_{i \in I} \int \hat{v}_{ij} dN_{ij} \\
&= \sum_{i \in I} \int \left(\max_{k \in I} \hat{v}_{kj} - \hat{v}_{ij} \right) dN_{ij},
\end{aligned}
$$

where the second equality follows because $\sum_{i \in I} \hat{v}_{ij} \mathbb{I}_{\{\gamma(j)=i\}} = \hat{v}_{kj}$, and the third equality follows because $\bar{N} = \sum_{i \in I} N_i$. But each term in the sum is positive since $\max \hat{v}_{kj} - \hat{v}_{ij} \geq 0$. It thus follows that each term in the sum is zero, and we are done.

For sufficiency, write

$$
\begin{aligned}
\sum_{i \in I} \hat{v}_{ij} \left[d\hat{N}_{ij} - dN_{ij} \right] &= \sum_{i \in I} \hat{v}_{ij} \, d\hat{N}_{ij} - \sum_{i \in I} \int \max_{k \in I} v_{kj} dN_{ij} \\
&= \sum_{i \in I} \hat{v}_{ij} \, d\hat{N}_{ij} - \int \max_{k \in I} v_{kj} d\bar{N}_j \\
&= \sum_{i \in I} \left[\hat{v}_{ij} - \max_{k \in I} v_{kj} \right] d\hat{N}_{ij} \leq 0.
\end{aligned}
$$

where the last equality follows because $\hat{N}$ is feasible, so $\bar{N}_j = \sum_{i \in I} \hat{N}_{ij}$.

B.4 Proof of Proposition A.3

Consider any (c, N) and multipliers $\hat{q}$, $\hat{\mu}$ and $\hat{p}$ satisfying the first-order conditions in the Proposition. Now let $(\hat{c}, \hat{N})$ denote any other feasible allocation. We have:

$$
\begin{aligned}
& \sum_{i \in I} \alpha_i U_i(c_i) - \sum_{i \in I} \alpha_i U_i(\hat{c}_i) \\
\geq \quad & \sum_{i \in I} \sum_{\omega \in \Omega} \frac{\partial U_i}{\partial c_i(\omega)} \left[c_i(\omega) - \hat{c}_i(\omega) \right] = \sum_{i \in I} \sum_{\omega \in \Omega} \left[\hat{q}(\omega) - \hat{\mu}_i(\omega) \right] \left[c_i(\omega) - \hat{c}_i(\omega) \right] \\
= \quad & \sum_{\omega \in \Omega} \hat{q}(\omega) \left[\sum_{i \in I} c_i(\omega) - \sum_{i \in I} \int d_j(\omega)\, dN_{ij} \right] - \sum_{\omega \in \Omega} \hat{q}(\omega) \left[\sum_{i \in I} \hat{c}_i(\omega) - \sum_{i \in I} \int d_j(\omega)\, d\hat{N}_{ij} \right] \\
& - \sum_{i \in I} \sum_{\omega \in \Omega} \hat{\mu}_i(\omega) \left[c_i(\omega) - \int \delta_{ij} d_j(\omega) dN_{ij} \right] + \sum_{i \in I} \sum_{\omega \in \Omega} \hat{\mu}_i(\omega) \left[\hat{c}_i(\omega) - \int \delta_{ij} d_j(\omega) d\hat{N}_{ij} \right] \\
& + \sum_{i \in I} \int \hat{v}_{ij} \left[dN_{ij} - d\hat{N}_{ij} \right] \geq \sum_{i \in I} \int \hat{v}_{ij} \left[dN_{ij} - d\hat{N}_{ij} \right],
\end{aligned}
$$

where the last inequality follows from the complementarity slackness for (c, N), and from the feasibility of $(\hat{c}, \hat{N})$. Now since both N and $\hat{N}$ are feasible, we have that:

$$
\hat{p} \cdot \bar{N} = \hat{p} \cdot \sum_{i \in I} N_{ij} = \hat{p} \cdot \sum_{i \in I} \hat{N}_{ij}.
$$

Hence, adding and subtracting $\hat{p} \cdot \bar{N}$, we obtain:

$$
\sum_{i \in I} \int \hat{v}_{ij} \left[dN_{ij} - d\hat{N}_{ij} \right] = \sum_{i \in I} \left[\hat{p} \cdot \hat{N}_{ij} - \int \hat{v}_{ij}\, d\hat{N}_{ij} \right] - \sum_{i \in I} \left[\hat{p} \cdot N_{ij} - \int \hat{v}_{ij}\, dN_{ij} \right] \geq 0
$$

where the last inequality follows from the first-order condition with respect to N.

B.5 Proof of Proposition A.5

Necessity. let (c, N, p, q) be an equilibrium. Since $\bar{n}_i > 0$, it follows from the firs-order conditions to the agent's problem that $\lambda_i > 0$. By direct comparison of first-order conditions, one can then verify that the equilibrium allocation solves the Planner's Problem with weights

$$
\alpha_i = \frac{1/\lambda_i}{\sum_{k \in I} 1/\lambda_k}.
$$

The associated Lagrange multipliers are $\hat{\mu}_i(\omega) = \alpha_i \mu_i(\omega)$, $\hat{q}(\omega) = \beta q(\omega)$ and $\hat{v}_{ij} = \beta v_{ij}$ and $\hat{p} = \beta p$, where $\beta \equiv \left[\sum_{i \in I} 1/\lambda_i \right]^{-1}$. Finally, we have from Lemma A.2 that:

$$\alpha_i \sum_{\omega \in \Omega} \frac{\partial U_i}{\partial c_i(\omega)} c_i(\omega) + \bar{n}_i \hat{p} \cdot \bar{N}.$$

Adding up across all $i \in I$ and using $\sum_{i \in I} \bar{n}_i = 1$ yields the desired condition.

Sufficiency. Consider any solution of the Planner's problem satisfying the conditions stated in the Proposition. Notice that the second condition implies that $\alpha_i > 0$. Using Proposition A.2 we obtain associated multipliers $\hat{q}$, $\hat{\mu}$ and $\hat{p}$. Consider then the candidate equilibrium prices $q(\omega) = \hat{q}(\omega)$ and $p = \hat{p}$. Then, by direct comparison of first-order conditions, one sees that the component (c_i, N_i) of the Planner's allocation solves agent $i \in I$'s problem, except perhaps for the budget feasibility condition. The associated multipliers are $\lambda_i = 1/\alpha_i$, $\mu_i(\omega) = \hat{\mu}_i(\omega)/\alpha_i$ and $v_{ij} = \hat{v}_{ij}$. To complete the proof, we thus need to verify that (c_i, N_i) satisfies budget feasibility. For this we calculate the transfer that agent i requires to make the planner's solution budget feasible:

$$
\begin{aligned}
T_i &\equiv \sum_{\omega \in \Omega} q(\omega) c_i(\omega) + p \cdot N_i - \bar{n}_i p \cdot \bar{N} - \sum_{\omega \in \Omega} q(\omega) \int d_j(\omega) \, dN_{ij} \\
&= \sum_{\omega \in \Omega} \left[\alpha_i \frac{\partial U_i}{\partial c_i(\omega)} + \hat{\mu}_i(\omega) \right] c_i(\omega) + \hat{p} \cdot N_i - \bar{n}_i \hat{p} \cdot \bar{N} - \sum_{\omega \in \Omega} \hat{q}(\omega) \int d_j(\omega) \, dN_{ij} \\
&= \sum_{\omega \in \Omega} \alpha_i \frac{\partial U_i}{\partial c_i(\omega)} c_i(\omega) - \bar{n}_i \hat{p} \cdot N + \int [\hat{p}_j - \hat{v}_{ij}] \, dN_{ij} \\
&= \sum_{\omega \in \Omega} \alpha_i \frac{\partial U_i}{\partial c_i(\omega)} c_i(\omega) - \bar{n}_i \hat{p} \cdot N.
\end{aligned}
$$

where we substituted in the Planner's first order conditions. Since aggregate ressource feasibility, implies aggregate budget feasibility, it follows that $\sum_{i \in I} T_i = 0$. Since, in addition, $\sum_{i \in I} \bar{n}_i = 1$, we obtain that:

$$\hat{p} \cdot N = \sum_{k \in I} \sum_{\omega \in \Omega} \alpha_k \frac{\partial U_k}{\partial c_k(\omega)} c_k(\omega),$$

hence $T_i = 0$ and budget balance holds since (c, N) satisfied the second condition stated in the Proposition.

B.6 Proof of Proposition A.6

Proof that $\Delta^\star(\alpha)$ is convex-valued. To show that $\Delta^\star(\alpha)$ is convex valued, we note that when $u_i(c)$ is strictly concave, $c_i(\omega)$ is uniquely determined, and so the term

$$\pi(\omega)u_i'\left[c_i(\omega)\right]c_i(\omega)$$

is the same for all $(c, N) \in \Gamma^\star(\alpha)$. When $u_i(c)$ is linear, then $u'(c)c = c$ is linear. Taken together, this means that the function defining $\Delta^\star(\alpha)$ preserves the convexity of $\Gamma^\star(\alpha)$.

Proof that $\Delta^\star(\alpha)$ has a closed graph. Consider any converging sequence of α^k and $\Delta^k \in \Delta^\star(\alpha^k)$, generated by a sequence $(c^k, N^k) \in \Gamma^\star(\alpha^k)$. Since $\Gamma^\star(\alpha^k)$ is included in the set if incentive feasible allocations, which by Lemma A.1 we know is weakly compact, we can extract a weakly convergent subsequence (c^ℓ, N^ℓ) of (c^k, N^k). Since we know from Proposition A.1 that $\Gamma^\star(\alpha)$ has a weakly closed graph, it follows that $\lim(c^\ell, N^\ell) \in \Gamma^\star\left(\lim \alpha^\ell\right)$. If $u_i'(c)$ is continuously differentiable at $\lim c_i^\ell(\omega)$, then by continuity we have:

$$\lim \left(\alpha_i^\ell u_i'\left[c_i^\ell(\omega)\right]c_i^\ell(\omega)\right) = \left(\lim \alpha_i^\ell\right) \times u_i'\left[\lim c_i^\ell(\omega)\right] \times \left(\lim c_i^\ell(\omega)\right).$$

If $u_i(c)$ is not continuously differentiable at $\lim c_i^\ell(\omega)$ then given our maintained assumption that $u_i(c)$ is continuously differentiable over $(0, \infty)$, it must be that $\lim c_i^\ell(\omega) = 0$ and $u_i'(0) = -\infty$. Since $\lim c_i^\ell(\omega) = 0$ is part of a social optimum, it must be that $\lim \alpha_i^\ell = 0$. But we know in this case from Proposition A.1 that

$$\lim \alpha_i^\ell u_i'\left[c_i^\ell(\omega)\right]c_i^\ell(\omega) = 0 = \lim \alpha_i^\ell u_i'\left[\lim c_i^\ell(\omega)\right]\lim c_i^\ell(\omega).$$

Taken together, we obtain that $\lim \Delta^\ell = \lim \Delta^k \in \Delta^\star(\lim \alpha^\ell) = \Delta^\star(\lim \alpha^k)$.

Proof that $\Delta^\star(\alpha)$ is bounded. Otherwise, there would exists some sequence α^k and $\Delta^k \in \Delta^\star(\alpha^k)$ such that $\max |\Delta_i^k| \to \infty$. Since α^k belongs to a compact set we can extract a converging subsequence α^ℓ. Since $\Delta^\star(\alpha)$ has a closed graph $\lim \Delta^\ell \in \Gamma^\star(\lim \alpha^\ell)$ and so must be finite, which is a contradiction.

An auxiliary fixed-point problem. Let M be such that $\max |\Delta_i| \leq M$ for all $\Delta \in \Delta^\star(\alpha)$ and $\alpha \in \mathcal{A}$. Let $\mathcal{D}$ be the set of transfers $\Delta = (\Delta_1, \ldots, \Delta_I)$ such that $\sum_{i \in I} \Delta_i = 0$ and $\max |\Delta_i| \leq M$. Finally, let $K(\alpha, \Delta)$ be the

function $\mathcal{A} \times \mathcal{D} \to \mathcal{A}$ such that

$$K_i(\alpha, \Delta) = \frac{(\alpha_i - \Delta_i)^+}{\sum_{k \in I}(\alpha_k - \Delta_k)^+},$$

where x^+ denotes the positive part of x. For each $(\alpha, \Delta) \in \mathcal{A} \times \mathcal{D}$, let the set $\Phi(\alpha, \Delta)$ be the product of the singleton $\{K(\alpha, \Delta)\}$ and the set $\Delta^\star(\alpha)$. By construction, $\Phi(\alpha, \Delta) \subseteq \mathcal{A} \times \mathcal{D}$. Since $\sum_{k \in I}(\alpha_k - \Delta_k)^+ \geq \sum_{k \in I}(\alpha_k - \Delta_k) = 1 > 0$ it follows that $K_i(\alpha, \Delta)$ is a continuous function over $\mathcal{A} \times \mathcal{D}$. Given our earlier result that $\Delta^\star(\alpha)$ has a closed graph, this implies that the correspondence $\Phi(\alpha, \Delta)$ has a closed graph as well. This allows to apply Kakutani's fixed point Theorem (see Corollary 17.55 in Aliprantis and Border (1999)) and assert that Φ has a fixed point, i.e., there exists some $(\alpha, \Delta) \in \mathcal{A} \times \mathcal{D}$ such that

$$
\begin{aligned}
\alpha_i &= \frac{(\alpha_i - \Delta_i)^+}{\sum_{k \in I}(\alpha_k - \Delta_k)^+} \text{ for all } i \in I \\
\Delta &\in \Delta^\star(\alpha).
\end{aligned}
$$

Proof that all fixed-points are such that $\Delta_i = 0$ for all $i \in I$. Next, we show that a fixed point of Φ has the property that $\Delta_i = 0$ for all $i \in I$. Indeed if $\alpha_i = 0$, then from the definition of $\Delta^\star(\alpha)$ we have that $\Delta_i \leq 0$, and from the fixed-point equation that $(-\Delta_i)^+ = 0 \Leftrightarrow \Delta_i \geq 0$. Hence, if $\alpha_i = 0$, then $\Delta_i = 0$. If $\alpha_i > 0$, then from the fixed point equation

$$\alpha_i \times \sum_{k \in I}(\alpha_k - \Delta_k)^+ = \alpha_i - \Delta_i \Rightarrow \Delta_i = \alpha_i \times \left[1 - \sum_{k \in I}(\alpha_k - \Delta_k)^+\right].$$

Hence, all Δ_i such that $\alpha_i > 0$ have the same sign. Since $\Delta_i = 0$ when $\alpha_i = 0$, it follows that all Δ_i have the same sign. But since $\sum_{i \in I} \Delta_i = 0$, this implies that $\Delta_i = 0$ for all $i \in I$.

B.7 Proof of Lemma A.3

Consider two sets of weights α and α' with corresponding optimal allocations $(c, N) \in \Gamma^\star(\alpha)$ and $(c', N') \in \Gamma^\star(\alpha')$. Since the constraint set of the planner does not depend on α, (c, N) and (c', N') are both incentive feasible given α and α'. Hence, optimality implies that:

$$\alpha_1 U_1(c_1) + \alpha_2 U_2(c_2) \geq \alpha_1 U_1(c_1') + \alpha_2 U_2(c_2') \Leftrightarrow \alpha_1\left[U_1(c_1) - U_1(c_1')\right] + \alpha_2\left[U_2(c_2) - U_2(c_2')\right] \geq 0.$$

Vice versa:

$$\alpha_1' \left[U_1(c_1') - U_1(c_1) \right] + \alpha_2' \left[U_2(c_2') - U_2(c_2) \right] \geq 0.$$

Adding up these two inequality and using that, since the weight add up to one, $\alpha_1' - \alpha_1 = \alpha_2 - \alpha_2'$, we obtain:

$$\left[\alpha_1' - \alpha_1 \right] \left\{ \left[U_1(c_1') - U_1(c_1) \right] - \left[U_2(c_2') - U_2(c_2) \right] \right\},$$

which implies that:

$$U_1(c_1') - U_1(c_1) \geq U_2(c_2') - U_2(c_2).$$

But then we must have that

$$U_1(c_1') - U_1(c_1) \geq 0 \geq U_2(c_2') - U_2(c_2).$$

because otherwise either (c, N) or (c', N') would not be constrained Pareto optima.

B.8   Modified Security Market Line

Proposition B.1 *Suppose the distribution of tree supplies is strictly increasing. Let $R_j(\omega) = \frac{d_j(\omega)}{p_j}$ be the return of asset j, $R_m(\omega) = \int_0^1 \frac{p_j}{\int_0^1 p_\ell d\bar{N}_\ell} R_j(\omega) d\bar{N}_j$ the market return, and $\beta_j = \frac{Cov(R_m, R_j)}{V(R_m)}$ the market beta of asset j. Then, β_j is a continuous and strictly decreasing function of j. Moreover, the expected return of tree j is a piecewise linear function of β_j:*

$$\mathbb{E}[R_j - R_f] = \beta_j \left(\mathbb{E}[R_m - R_f] - \theta_m \right) + \theta_j, \tag{48}$$

where

$$\theta_j = \theta_k - \phi \max(\beta_j - \beta_k, 0) - \psi \max(\beta_k - \beta_j, 0), \tag{49}$$

and $R_f = \left(\sum_{\omega \in \Omega} q(\omega) \right)^{-1}$ is the risk-free rate, $\theta_j = \Delta_j / p_j$, is the (per dollar invested) discount induced by incentive constraints for asset j, k is the marginal tree, $\phi > 0$, $\psi > 0$, and $\theta_m = \int_0^1 \frac{p_j}{\int_0^1 p_\ell d\bar{N}_\ell} \theta_j d\bar{N}_j$ is the average discount induced by incentive constraints. Equation (48) also holds for financial assets by setting $\theta_j = 0$.

Proof that $j \mapsto \beta_j$ is strictly decreasing.   Since there are only two states of nature, correlations are either equal to one, zero, or minus one. It follows from $R_m(\omega_1) < R_m(\omega_2)$ that $\beta_j = \frac{\sigma(R_j)}{\sigma(R_m)} \text{Sign}[d_j(\omega_2) - d_j(\omega_1)]$, where:

$$\left(\sigma(R_j) \right)^2 = \sum_{\omega \in \Omega} \pi(\omega) \left(\frac{d_j(\omega) - \bar{d}_j}{p_j} \right)^2 = \sum_{\omega \in \Omega} \pi(\omega)(1 - \pi(\omega))^2 \left(\frac{d_j(\omega_2) - d_j(\omega_1)}{p_j} \right)^2$$

Equation (13) implies that $p_j = a_i(\omega_1)d_j(\omega_1) + a_i(\omega_2)d_j(\omega_2)$, where i denotes the agent holding asset j and $a_i(\omega) > 0$.

Thus:

$$\beta_j = \frac{1}{\sigma(R_m)}\left(\sum_{\omega\in\Omega}\pi(\omega)(1-\pi(\omega))^2\right)^{\frac{1}{2}}\frac{\frac{d_j(\omega_2)}{d_j(\omega_1)}-1}{a_i(\omega_1)+a_i(\omega_2)\frac{d_j(\omega_2)}{d_j(\omega_1)}}. \tag{50}$$

$\frac{d_j(\omega_2)}{d_j(\omega_1)} \mapsto \beta_j$ is clearly continuous away from the marginal asset k. And it is also continuous at the marginal asset since p_j is continuous at $j = k$. For $j \neq k$, we can take the derivative:

$$\frac{d\beta_j}{d\frac{d_j(\omega_2)}{d_j(\omega_1)}} = \frac{1}{\sigma(R_m)}\left(\sum_{\omega\in\Omega}\pi(\omega)(1-\pi(\omega))^2\right)^{\frac{1}{2}}\frac{a_i(\omega_1)+a_i(\omega_2)}{\left(a_i(\omega_1)+a_i(\omega_2)\frac{d_j(\omega_2)}{d_j(\omega_1)}\right)^2} > 0.$$

Proof of equation (48). There is a different pricing kernel for each agent. For assets j held by agent i, the pricing kernel is:

$$1 = \mathbb{E}\left[\frac{q(\omega)}{\pi(\omega)}R_j(\omega)\right] - \delta\frac{\mu_i(\omega_i)}{\lambda_i}R_j(\omega_i).$$

Denoting the risk-free rate as $R_f = \left(E\left[\frac{q(\omega)}{\pi(\omega)}\right]\right)^{-1}$, the usual manipulations lead to:

$$\mathbb{E}[R_j(\omega) - R_f] = -R_f Cov\left(\frac{q(\omega)}{\pi(\omega)}, R_j(\omega)\right) + \theta_j,$$

where $\Delta_j = R_f\delta\frac{\mu_i(\omega_i)}{\lambda_i}R_j(\omega_i)$. Since there are two states of nature, $\frac{q(\omega)}{\pi(\omega)}$ can be written as an affine function of the market return with slope κ. Thus:

$$\mathbb{E}[R_j(\omega) - R_f] = -\kappa R_f Cov\left(R_m(\omega), R_j(\omega)\right) + \theta_j, \tag{51}$$

where $\theta_j = R_f\delta\frac{\mu_i(\omega_i)}{\lambda_i}R_j(\omega_i) = \frac{\Delta_j}{p_j}$. Multiplying by $\frac{p_j}{\int_0^1 p_\ell d\bar{N}_\ell}$ and integrating over j, we obtain the pricing kernel for the market portfolio:

$$\mathbb{E}[R_m(\omega) - R_f] = -\kappa R_f Var\left(R_m(\omega)\right) + \theta_m, \tag{52}$$

where $\Delta_m = \int_0^1 \frac{p_j}{\int_0^1 p_\ell d\bar{N}_\ell}\theta_j d\bar{N}_j$. Combining (51) and (51) yields the modified CAPM formula (48).

Next, we show that θ_j can be written as a piecewise linear function of β_j with a kink at the marginal asset β_k. $R_j(\omega_1) = \frac{d_j(\omega_1)}{p_j} = \frac{1}{a_i(\omega_1)+a_i(\omega_2)b_j}$, where i denotes the agent holding asset j and $b_j \equiv \frac{d_j(\omega_2)}{d_j(\omega_1)}$. Equation (50) implies that β_j can be written as a function of b_j: $\beta_j = \rho_0\frac{b_j-1}{a_i(\omega_1)+a_i(\omega_2)b_j}$, where $\rho_0 = \frac{1}{\sigma(R_m)}\left(\sum_{\omega\in\Omega}\pi(\omega)(1-\pi(\omega))^2\right)^{\frac{1}{2}}$. Inverting this function, we can write b_j as a function of β_j: $b_j = \frac{\rho_0+\beta_j a_i(\omega_1)}{\rho_0-\beta_j a_i(\omega_2)}$. Thus: $R_j(\omega_1) = \frac{\rho_0-\beta_j a_i(\omega_2)}{(a_i(\omega_1)+a_i(\omega_2))\rho_0}$. Similarly: $R_j(\omega_2) = \frac{\rho_0+\beta_j a_i(\omega_1)}{(a_i(\omega_1)+a_i(\omega_2))\rho_0}$. It implies that Δ_j is linear and decreasing in β_j for assets j held by agent 1 and linear and increasing for asset held by agent 2. It follows from the continuity of θ_j at the marginal asset k that

θ_j can be written as (49).

B.9   Long and Short Positions in All Assets

In the model studied in the text, it is assumed that agents can only take short position in Arrow securities. In this appendix we generalize our model and assume that agents can take short positions in all assets: Arrow securities and trees. We show that the equilibrium is robust: the same collection of prices and allocation remains an equilibrium when agents can take short positions in all assets.

B.9.1   Distinguishing Long and Short Positions

To make this argument we introduce new notations that explicitly distinguish long positions from short positions in an agent's tree portfolio. Namely, we denote the net tree position by $N_i = N_i^+ - N_i^-$, where N_i^+ is a positive measure for the long position, and N_i^- is a positive measure for the short position. Likewise, the net Arrow security position is $a_i(\omega) = a_i^+(\omega) - a_i^-(\omega)$. We take all short positions to be liabilities: that is a short position on a particular tree is a state contingent promise to pay $d_j(\omega)$ in state $\omega \in \Omega$. The incentive compatibility constraint in state ω now states that the agent prefers paying off the liability induced by the total short position, rather than diverting a fraction δ of the long position.

$$\delta \left[a_i^+(\omega) + \int d_j(\omega) \, dN_{ij}^+(\omega) \right] \leq a_i^+(\omega) - a_i^-(\omega) + \int d_j(\omega) \left[dN_{ij}^+(\omega) - dN_{ij}^-(\omega) \right]$$

$$\Longleftrightarrow \quad a_i^-(\omega) + \int d_j(\omega) \, dN_{ij}^-(\omega) \leq (1 - \delta) \left[a_i^+(\omega) + \int d_j(\omega) \right] dN_i^+(\omega). \tag{53}$$

The other constraints for agent i are the budget constraint at $t = 0$

$$\sum_{\omega \in \Omega} q(\omega) \left[a_i^+(\omega) - a_i^-(\omega) \right] + \int p_j \left[dN_{ij}^+(\omega) - dN_{ij}^-(\omega) \right] \leq \bar{n}_j \int p_j \, d\bar{N}_j, \tag{54}$$

and the budget constraint at $t = 1$

$$c_i(\omega) = a_i^+(\omega) - a_i^-(\omega) + \int d_j(\omega) \left[dN_{ij}^+(\omega) - dN_{ij}^-(\omega) \right]. \tag{55}$$

This formulation allows net positions to differ from gross positions. That is, we could have $a_i^+(\omega) > 0$ and $a_i^-(\omega) > 0$ for some $\omega \in \Omega$, or we could have $N_i^+(J) > 0$ and $N_i^-(J) > 0$ for some measurable set J.

The *agent's problem with long and short positions in all assets* is to choose state-contingent consumption plan, c_i,

and a portfolio $(a_i^+, a_i^-, N_i^+, N_i^-)$ in order to maximize the expected utility $U_i(c_i)$, subject to the constraint (53)-(55)

B.9.2 Equilibrium Robustness

Next, we show that our equilibrium is robust to allowing agents to take long and short positions in all assets. To see this, consider an equilibrium of the paper. For trees in strictly positive supply, it is always the case that $p_j \leq \sum_{\omega \in \Omega} q(\omega) d_j(\omega)$, as shown in Lemma 4. For trees in zero supply there is some indeterminacy but we can always pick a price such that this inequality holds as well. Now let the solution of the agent's problem be (c_i, a_i, N_i) and let $(\hat{c}_i, \hat{a}_i^+, \hat{a}_i^-, \hat{N}_i^+, \hat{N}_i^-)$ be such that:

$$\hat{c}_i(\omega) \equiv c_i(\omega)$$

$$\hat{a}_i^+(\omega) \equiv \max\{a_i(\omega), 0\} \text{ and } \hat{a}_i^-(\omega) \equiv -\min\{a_i(\omega), 0\}$$

$$\hat{N}_i^+ \equiv N_i \text{ and } \hat{N}_i^- \equiv 0.$$

Then, we have:

Proposition B.2 *Given the price system* (p, q), $(\hat{c}_i, \hat{a}_i^+, \hat{a}_i^-, \hat{N}_i^+, \hat{N}_i^-)$ *is a solution of the agent's problem with long and short positions in all assets.*

The proof is straightforward and follows because short positions in Arrow securities are more profitable than short positions in trees. This is because $p_j \leq \sum_{\omega \in \Omega} q(\omega) d_j(\omega)$. Hence, by replacing the short position in tree N_i^- by a payoff-identical short position in Arrow securities, $\int d_j(\omega) dN_{ij}^-$, the agent can generate (weakly) higher profits at time zero, and satisfy all the other constraints. Therefore, to find a solution to the agent's problem, we can restrict attention to tree portfolios such that $N_i^- = 0$. In that case the collateral constraint becomes

$$a_i^-(\omega) \leq (1 - \delta) \left[a_i^+(\omega) + \int d_j(\omega) \, dN_{ij}^+ \right]. \tag{56}$$

This is the same constraint as in Section 2.2, and so the result follows.