Asset Management Contracts and Equilibrium Prices

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Abstract

We study the joint determination of fund managers' contracts and equilibrium asset prices. Because of agency frictions, fund investors make managers' fees more sensitive to performance and benchmark performance against a market index. In equilibrium, managers underweight expensive stocks that are in high demand by other traders and have endogenously high volatility and market betas, and overweight low-demand stocks. Since benchmarking makes it risky for managers to deviate from the index, it exacerbates cross-sectional differences in alphas and the negative relationship between alpha on one hand, and volatility or beta on the other. Moreover, because underweighting expensive stocks is riskier than overweighting cheaper ones, benchmarking raises the price of the aggregate market. Socially optimal contracts provide steeper incentives and cause larger pricing distortions than privately optimal contracts.

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1 Introduction

Asset management is a large and growing industry. For example, according to French's (2008) presidential address to the American Finance Association, individual investors held directly only 21.5% of U.S. stocks in 2007. The remainder was held within financial institutions of various types, run by professional managers. The risk and return of asset managers is measured against benchmarks, and performance relative to these benchmarks affects the managers' compensation and the money that investors give them to manage. In this paper we study the implications of delegated portfolio management and benchmarking on equilibrium asset prices. Unlike most of the prior literature, we do not assume exogenous preferences or compensation contracts for asset managers, but optimize over the contract choice.

We assume a continuous-time infinite-horizon economy with a riskless asset and multiple risky assets (stocks). The main agents in our model are a representative fund manager and a representative fund investor. The investor cannot invest in the stocks directly and must employ the manager. The manager receives a fee, which is an affine function of the fund's performance and the performance of the market index. We impose the functional form of the contract exogenously, but optimize over the coefficients of the affine function. The manager can add value over the index because additional agents, buy-and-hold investors, hold a portfolio that differs from the index. Hence, stock prices adjust so that holding the stocks' residual supply, i.e., the supply net of the holdings of buy-and-hold investors, is optimal for the manager.

We first solve the model in the case where there are no agency frictions. The fee that the manager receives achieves optimal risk-sharing between him and the investor, and involves no benchmarking. Stock prices depend on effective supply. Stocks in large supply are cheap and hence offer high expected return, holding all else equal, so that the manager is induced to give them larger weight than the index weight. Moreover, the prices of these stocks are less sensitive to shocks to expected dividends. This is because a positive dividend shock raises not only expected dividends per share but also the volatility of dividends per share. The compensation for the increased volatility is larger for a stock in large effective supply, and hence the positive effect of the shock on the stock's price is smaller. The low price sensitivity to dividend shocks for stocks in large supply causes them to have low idiosyncratic volatility, as well as low total return volatility and market beta in some cases. Since these stocks have also positive alpha, our model provides a mechanism that can generate the volatility anomaly (e.g., Haugen and Baker 1996; Ang et al. 2006, 2009) and the beta anomaly (e.g., Black 1972; Black, Jensen and Scholes 1972; Frazzini and Pedersen 2013).

We next solve the model in the case where there are agency frictions. We model these frictions as an unobserved action by the manager that benefits him but reduces the fund's value. The fee that the manager receives is more sensitive to the fund's performance than under optimal risk-sharing, and involves benchmarking. Because the fee makes the manager less willing to deviate from the index, the effects of supply are amplified. Stocks in large supply become cheaper and less volatile, while stocks in small supply become more expensive and more volatile. The increase in volatility for the latter stocks can be interpreted as an amplification effect: following an increase in their expected dividends, the manager becomes even more averse to underweighting these stocks because they account for a larger fraction of market movements. As a consequence, he seeks to buy these stocks, amplifying the price increase. This effect seems consistent, for example, with the behavior of many fund managers towards technology stocks during the 1998-2000 rise in the Nasdaq. Because the agency frictions increase cross-sectional discrepancies in expected returns and volatilities, they amplify the volatility and beta anomalies relative to the case of no agency frictions.

Besides their effect on the prices of individual stocks, agency frictions also affect the aggregate market. We show that the aggregate market goes up, and its expected return goes down. Therefore, the cross-sectional effect of the frictions is asymmetric: stocks in small supply go up more than stocks in large supply go down. The intuition is that underweighting overpriced stocks is more costly for the manager than overweighting underpriced stocks because the former are more volatile and hence expose the manager to greater risk of deviating from the benchmark.

In addition to the positive results, we perform a normative analysis. We show that under the socially optimal contract, the manager has steeper incentives and is exposed to more risk than under the privately optimal contract. The inefficiency can be viewed as a free-rider problem, by interpreting our price-taking investor and manager as a continuum of identical such agents. When one investor in the continuum gives steeper performance incentives to her manager, hence exposing him to more risk, this makes the manager less willing to take risk and to exploit mispricings. Other managers, however, remain equally willing to do so, benefiting their investors. When all investors give steeper incentives to their managers, mispricings become more severe in equilibrium, and all managers remain equally willing to exploit them despite being exposed to more risk.

1.1 Literature Review (Incomplete)

Our paper is closest to a literature that investigates the effects of benchmarking on equilibrium asset prices. Brennan (1993) shows that benchmarking yields a two-factor model for expected returns, with the factors being the market and the benchmark. Cuoco and Kaniel (2011) show that benchmarking increases the prices of the assets included in the index, lowers their expected returns, and has an ambiguous effect on return volatility, depending on the presence of convexity in the performance fee structure. They also perform a normative analysis, and show that linear contracts do not achieve first-best risk-sharing, and that adding convexity to the contract can be Pareto improving. Basak and Pavlova (2013) introduce concerns for relative performance into the preferences of an institutional investor. They show that delegation raises the price level and volatility of the assets included in the index, and makes them comove more strongly. Our contribution relative to those papers is to endogenize asset management contracts. Most of our asset-pricing results also differ from those papers.

Malamud and Petrov (2014) endogenize asset management contracts in a static equilibrium setting with competitive, differentially skilled managers and convex compensation contracts. They show that the equilibrium incentives may hurt investors, who are unable to coordinate on their contract choice. In a setting in which banks need to exert effort in order to combine project cash flows with (complex) financial securities and hedge risks, Iovino (2014) shows that equilibrium contracts are inefficient, as investor do not internalize the interaction between agency frictions and security trading.

Wurgler (2010) and Baker, Bradley and Wurgler (2011) conjecture a link between benchmarking and the beta anomaly. According to their argument, benchmarking renders the risk of low- and high-beta stocks symmetric: low-beta stocks underperform when the market goes up, high-beta stocks underperform when the market goes down, and the risk is symmetric for managers who are compensated relative to a market benchmark. Therefore, managers prefer high-beta stocks if these earn higher expected returns, and this causes their alphas to be negative. As Brennan (1993) shows, however, benchmarking on its own does not generate mispricing: if the benchmark coincides with the market, the CAPM holds. Our results instead arise because of the combination of benchmarking and supply effects.

Optimal contracts for asset managers have mainly been analyzed in partial equilibrium settings. Examples include Bhattacharya and Pfleiderer (1985), Starks (1987), Kihlstrom (1988), Stoughton (1993), Heinkel and Stoughton (1994), Admati and Pfleiderer (1997), Das and Sundaram (2002), Palomino and Prat (2003), Ou-Yang (2003), Liu (2005), Dybvig, Farnsworth and Carpenter (2010), Cadenillas, Cvitanic and Zapatero (2007), Cvitanic, Wan and Zhang (2009) and Li and Tiwari (2009).

Our analysis focuses on fund managers' explicit incentives, deriving from their fees. A number of papers explore instead implicit incentives, deriving from fund flows, as well as the effects that flows have on equilibrium asset prices. Examples include Shleifer and Vishny (1997), Berk and Green (2004), Vayanos (2004), He and Krishnamurthy (2012, 2013), Kaniel and Kondor (2013), Vayanos and Woolley (2013). Dasgupta and Prat (2008), Dasgupta, Prat and Verardo (2011), Guerrieri and Kondor (2012), and Malliaris and Yan (2012) emphasize implicit incentives generated by managers' reputation concerns.

Christoffersen and Simutin (2014) test for the impact of benchmarking on institutional holdings. They find that fund managers who face greater pressure to beat their benchmarks, because they control large amounts of pension assets, substitute low-beta stocks for high-beta stocks. By tilting their portfolios, they contribute to nurture the beta and volatility anomalies. These results are consistent with our findings.

2 Model

2.1 Assets

Time t is continuous and goes from zero to infinity. There is a riskless asset with an exogenous return equal to r, and N risky assets. We refer to the risky assets as stocks, but they could also be interpreted as segments of the stock market, e.g., industry-sector portfolios or investment styles. The price S_{it} of stock $i = 1, .., N$ is determined endogenously in equilibrium. The dividend flow D_{it} of stock i is given by

$$D_{it} = b_i s_t + e_{it}, \tag{2.1}$$

where s_t is a component common to all stocks and e_{it} is a component specific to stock i. The variables $(s_t, e_{1t}, .., e_{Nt})$ are positive and mutually independent, and we specify their stochastic evolution below. The constant $b_i \geq 0$ measures the exposure of stock i to the common component s_t. We set $D_t \equiv (D_{1t}, .., D_{Nt})'$, $S_t \equiv (S_{1t}, .., S_{Nt})'$, and $b \equiv (b_1, .., b_N)'$. We denote by dR_t the vector of stocks' returns per share in excess of the riskless rate:

$$dR_t \equiv D_t dt + dS_t - rS_t dt. \tag{2.2}$$

The return per dollar invested in stock i can be derived by dividing stock i's return per share dR_{it} by the stock's price S_{it}. Stock i is in supply of $\eta_i > 0$ shares. We denote the market portfolio by $\eta \equiv (\eta_1, .., \eta_N)$, and refer to it as the index.

The variables $(s_t, e_{1t}, .., e_{Nt})$ evolve according to square-root processes:

$$ds_t = \kappa (\bar{s} - s_t) dt + \sigma_s \sqrt{s_t} dw_{st}, \tag{2.3}$$

$$de_{it} = \kappa (\bar{e}_i - e_{it}) dt + \sigma_i \sqrt{e_{it}} dw_{it}, \tag{2.4}$$

where $(\bar{s}, \bar{e}_1, .., \bar{e}_N, \sigma_s, \sigma_1, .., \sigma_N)$ are positive constants, and the Brownian motions $(w_{st}, w_{1t}, .., w_{Nt})$ are mutually independent. The square-root specification (2.3) and (2.4) allows for closed-form solutions, while also ensuring that dividends remain positive. An additional property of this specification is that the volatility of dividends per share (i.e., of D_{it}) increases with the dividend level. This property is both realistic and key for our analysis.

The constants $(\bar{s}, \bar{e}_1, .., \bar{e}_N)$ are the unconditional means of the variables $(s_t, e_{1t}, .., e_{Nt})$. The increments $(ds_t, de_{1t}, .., de_{Nt})$ of these variables have variance rates $(\sigma_s^2 s_t, \sigma_1^2 e_{1t}, .., \sigma_N^2 e_{Nt})$ conditionally and $(\sigma_s^2 \bar{s}, \sigma_1^2 \bar{e}_1, .., \sigma_N^2 \bar{e}_N)$ unconditionally. We occasionally consider the special case of "scale invariance," where the ratio of unconditional standard deviation to unconditional mean is identical across the $N+1$ processes. This occurs when the vector $(\sigma_s^2, \sigma_1^2, .., \sigma_N^2)$ is collinear with $(\bar{s}, \bar{e}_1, .., \bar{e}_N)$.

2.2 Agents

The main agents in our model are an investor and a fund manager. Both agents are competitive price-takers, and they can be interpreted as a continuum of identical investors and managers. The investor can invest in stocks directly by holding the index, or indirectly by employing the manager. Employing the manager is the only way for the investor to hold a portfolio that differs from the index, and hence to "participate" in the market for individual stocks. One interpretation of this participation friction is that the investor cannot identify stocks that offer higher returns than the index, and hence must employ the manager for non-index investing.

If the investor and the manager were the only agents in the model, then the participation friction would not matter. This is because the index is the market portfolio, so equilibrium prices would adjust to make that portfolio optimal for the investor. For the participation friction to matter, the manager must add value over the index. To ensure that this can happen, we introduce a third set of agents, buy-and-hold investors, who do not hold the index. These agents could be holding stocks for corporate-control or hedging purposes, or could be additional unmodeled fund managers. We denote their aggregate portfolio by $\eta - \theta$, and assume that $\theta \equiv (\theta_1, .., \theta_N)$ is constant over time and not proportional to η. The number of shares of stock i available to the investor and the manager is thus θ_i, and represents the residual supply of stock i to them. Stocks in large residual supply (large θ_i) must earn high expected returns in equilibrium, so that the manager is willing to give them weight larger than the index weight. By overweighting stocks that earn high expected returns, the manager adds value over the index. We assume that the residual supply of each stock is positive ($\theta_i > 0$ for all i). We refer to residual supply simply as supply from now on.[1]

The investor chooses an investment x in the index η, i.e., holds $x\eta_i$ shares of stock i. She also decides whether or not to employ the manager. Both decisions are made once and for all at $t = 0$.

[1] An alternative interpretation of our setting is that there are no buy-and-hold investors, θ is the market portfolio, and η is an index that differs from the market portfolio, e.g., does not include private equity.

If the manager is employed by the investor, then he chooses the fund's portfolio $z_t \equiv (z_{1t}, .., z_{Nt})$ at each time t, where z_{it} denotes the number of shares of stock i held by the fund. The manager can also undertake a "shirking" action $m_t \geq 0$ that delivers to him a private benefit $\left(Am_t - \frac{B}{2}m_t^2\right) dt$, where $1 \geq A \geq 0$ and $B \geq 0$, and reduces the fund's return by $m_t dt$. A literal interpretation of m_t is as cash diverted from the fund, with diversion involving a deadweight cost except when $A = 1$ and $B = 0$. Alternatively, m_t could be interpreted in reduced form as insufficient effort to lower operating costs or to identify a more efficient portfolio. When $A = 0$, the private benefit is non-positive for all values of m_t and there are no agency frictions. The investor can influence the choices of z_t and m_t through a compensation contract, offered at $t = 0$. The contract specifies a fee that the investor pays to the manager over time. It is chosen optimally within a parametrized class, described as follows. The fee is paid as a flow, and the flow fee df_t is an affine function of the fund's return $z_t dR_t - m_t dt$ and the index return ηdR_t. Moreover, the coefficients of this affine function are chosen at $t = 0$ and remain constant over time. Thus, the flow fee df_t is given by

$$df_t = \phi \left(z_t dR_t - m_t dt\right) - \chi \eta dR_t + \psi dt, \tag{2.5}$$

where (ϕ, χ, ψ) are constants. The constant ϕ is the fee's sensitivity to the fund's performance, and the constant χ is the sensitivity to the index performance. If $\chi = 0$ then the manager is paid only based on the fund's absolute performance, while if $\chi \neq 0$ then performance relative to the index also matters. We assume that the manager invests his personal wealth in the riskless asset. This is without loss of generality: since the manager is exposed to stocks through the fee, and can adjust this exposure continuously by changing the fund's portfolio, a personal investment in stocks is redundant. If the manager is not employed by the investor, then he chooses a personal stock portfolio $\bar{z}_t$, receives no fee, and has no shirking action available.[2]

2.2.1 Manager's Optimization Problem

The manager derives utility over intertemporal consumption. Utility is exponential:

$$\mathbb{E}\left[\int_0^\infty - \exp(-\bar{\rho}\bar{c}_t - \bar{\delta}t)dt\right], \tag{2.6}$$

where $\bar{\rho}$ is the coefficient of absolute risk aversion, $\bar{c}_t$ is consumption, and $\bar{\delta}$ is the discount rate. We denote the manager's wealth by $\bar{W}_t$.

The manager decides at $t = 0$ whether or not to accept the contract offered by the investor. If

[2] Ruling out the shirking action for an unemployed manager is without loss of generality: since the manager invests his personal wealth, he would not undertake the shirking action even if that action were available.

the manager accepts the contract and is hence employed by the investor, then he chooses at each time t the fund's portfolio z_t and the shirking action m_t. His budget constraint is

$$d\bar{W}_t = r\bar{W}_t dt + df_t + \left(Am_t - \frac{B}{2}m_t^2\right)dt - \bar{c}_t dt, \tag{2.7}$$

where the first term is the return from the riskless asset, the second term is the fee paid by the investor, the third term is the private benefit from shirking, and the fourth term is consumption. The manager's optimization problem is to choose controls $(\bar{c}_t, z_t, m_t)$ to maximize the expected utility (2.6) subject to the budget constraint (2.7) and the fee (2.5). We denote by $z_t(\phi, \chi, \psi)$ and $m_t(\phi, \chi, \psi)$ the manager's optimal choices of z_t and m_t, and by $\bar{V}(\bar{W}_t, s_t, e_t)$ his value function, where $e_t \equiv (e_{1t}, .., e_{Nt})'$.

If the manager is not employed by the investor, then he chooses his personal portfolio $\bar{z}_t$. His budget constraint is

$$d\bar{W}_t = r\bar{W}_t dt + \bar{z}_t dR_t - \bar{c}_t dt, \tag{2.8}$$

The manager's optimization problem is to choose controls $(\bar{c}_t, \bar{z}_t)$ to maximize (2.6) subject to (2.8). We denote by $\bar{V}_u(\bar{W}_t, s_t, e_t)$ his value function. The manager accepts the contract offered by the investor if

$$\bar{V}(\bar{W}_0, s_0, e_0) \geqslant \bar{V}_u(\bar{W}_0, s_0, e_0). \tag{2.9}$$

2.2.2 Investor's Optimization Problem

The investor derives utility over intertemporal consumption. Utility is exponential:

$$\mathbb{E}\left[\int_0^\infty -\exp(-\rho c_t - \delta t)dt\right], \tag{2.10}$$

where ρ is the coefficient of absolute risk aversion, c_t is consumption, and δ is the discount rate.

The investor chooses an investment x in the index η, and whether or not to employ the manager. Both decisions are made at $t = 0$. If the investor employs the manager, then she offers him a contract (ϕ, χ, ψ). We denote the investor's wealth by W_t. The investor's budget constraint is

$$dW_t = rW_t dt + x\eta dR_t + z_t dR_t - m_t dt - df_t - c_t dt, \tag{2.11}$$

where the first term is the return from the riskless asset, the second term is the return from the investment in the index, the third and fourth term are the return from the fund, the fifth term is the fee paid to the manager, and the sixth term is consumption. The investor's optimization problem is to choose controls $(c_t, x, \phi, \chi, \psi)$ to maximize the expected utility (2.10) subject to the budget constraint (2.11), the fee (2.5), the manager's incentive compatibility constraint

$$z_t = z_t(\phi, \chi, \psi),$$

$$m_t = m_t(\phi, \chi, \psi),$$

and the manager's individual rationality constraint (2.9). We denote by $V(W_t, s_t, e_t)$ the investor's value function.

If the investor does not employ the manager, then her budget constraint is

$$dW_t = rW_t dt + x\eta dR_t - c_t dt. \tag{2.12}$$

The investor's optimization problem is to choose controls (c_t, x) to maximize (2.10) subject to (2.12). We denote by $V_u(W_t, s_t, e_t)$ her value function. The investor employs the manager if

$$V(W_0, s_0, e_0) \geqslant V_u(W_0, s_0, e_0). \tag{2.13}$$

2.3 Equilibrium Concept

We look for equilibria in which the investor employs the manager, i.e., offer a contract that the manager accepts. These equilibria are described by a price process S_t, a compensation contract (ϕ, χ, ψ) that the investor offers to the manager, and a direct investment x in the index by the investor.

Definition 1 (Equilibrium prices and contract). *A price process S_t, a contract (ϕ, χ, ψ), and an index investment x, form an equilibrium if:*

(i) Given S_t and (ϕ, χ, ψ), $z_t = \theta - x\eta$ solves the manager's optimization problem.

(ii) Given S_t, the investor chooses to employ the manager, and (x, ϕ, χ, ψ) solve the investor's optimization problem.

The equilibrium in Definition 1 involves a two-way feedback between prices and contracts. A contract offered by the investor affects the manager's portfolio choice, and hence equilibrium prices. Equilibrium prices are determined by the market-clearing condition that the fund's portfolio z_t plus the portfolio $x\eta$ that the investor holds directly add up to the supply portfolio θ. Conversely, the contract that the investor offers to the manager depends on the equilibrium prices.

We conjecture that the equilibrium price of stock i is an affine function of s_t and e_{it}:

$$S_{it} = a_{0i} + a_{1i}s_t + a_{2i}e_{it}, \tag{2.14}$$

where (a_{0i}, a_{1i}, a_{2i}) are constants. The conjectured value functions of the manager and the investor under the equilibrium prices and when the manager is employed by the investor are

$$\bar{V}(\bar{W}_t, s_t, e_t) = -\exp\left[-\left(r\bar{\rho}\bar{W}_t + \bar{q}_0 + \bar{q}_1 s_t + \sum_{i=1}^{N} \bar{q}_{2i}e_{it}\right)\right], \tag{2.15}$$

$$V(W_t, s_t, e_t) = -\exp\left[-\left(r\rho W_t + q_0 + q_1 s_t + \sum_{i=1}^{N} q_{2i}e_{it}\right)\right], \tag{2.16}$$

respectively, where $(\bar{q}_0, \bar{q}_1, \bar{q}_{2i}, q_0, q_1, q_{2i})$ are constants. The conjectured value functions when the manager is not employed by the investor have the same form, but for different constants $(\bar{q}_{u0}, \bar{q}_{u1}, \bar{q}_{u2i}, q_{u0}, q_{u1}, q_{u2i})$.

3 Equilibrium without Agency Frictions

In this section we solve for equilibrium in the absence of agency frictions. We eliminate agency frictions by setting the parameter A in the manager's private-benefit function $Am_t - \frac{B}{2}m_t^2$ to zero. This ensures that the private benefit is non-positive for all values of m_t. When $A = 0$, the investor and the manager share risk optimally, through the contract. The equilibrium becomes one with a representative agent, whose risk tolerance is the sum of the investor's and the manager's. We compute prices in that equilibrium in closed form. We show that the combination of exponential utility and square-root dividend processes—which to our knowledge is new to the literature—yields a framework that is not only tractable but can also help address empirical puzzles about the risk-return relationship.

Theorem 3.1 (**Equilibrium Prices and Contract without Agency Frictions**). *If* $A = 0$, *then the following form an equilibrium: the price process S_t given by (2.14) with*

$$a_{0i} = \frac{\kappa}{r}\left(a_{1i}\bar{s} + a_{2i}\bar{e}_i\right), \tag{3.1}$$

$$a_{1i} = \frac{b_i}{\sqrt{(r+\kappa)^2 + 2r\frac{\rho\bar{\rho}}{\rho+\bar{\rho}}\theta b \sigma_s^2}} \equiv b_i a_1, \tag{3.2}$$

$$a_{2i} = \frac{1}{\sqrt{(r+\kappa)^2 + 2r\frac{\rho\bar{\rho}}{\rho+\bar{\rho}}\theta_i \sigma_i^2}}; \tag{3.3}$$

the contract $(\phi, \chi, \psi) = \left(\frac{\rho}{\rho+\bar{\rho}}, 0, 0\right)$*; and the index investment* $x = 0$.

Since $x = 0$, the investor does not invest directly in the index. Market-clearing hence implies that the fund holds the supply portfolio θ. Since, in addition, $\chi = 0$, the manager is compensated based only on absolute performance and not on performance relative to the index. Therefore, the manager receives a fraction $\phi = \frac{\rho}{\rho+\bar{\rho}}$ of the return of the supply portfolio, and the investor receives the complementary fraction $1-\phi = \frac{\bar{\rho}}{\rho+\bar{\rho}}$. This coincides with the standard rule for optimal risk-sharing under exponential utility.

The coefficient a_{2i} measures the sensitivity of stock i's price to changes in the stock-specific component e_{it} of dividends. A unit increase in e_{it} causes stock i's dividend flow at time t to increase by one. In the absence of risk aversion ($\rho\bar{\rho} = 0$), (3.3) implies that the price of stock i would increase by $a_{2i} = \frac{1}{r+\kappa}$. This is the present value, discounted at the riskless rate r, of the increase in stock i's expected dividends from time t onwards: the dividend flow at time t increases by one, and the effect decays over time at rate κ.

The coefficient a_{1i} measures the sensitivity of stock i's price to changes in the common component s_t of dividends. We normalize a_{1i} by b_i, the sensitivity of stock i's dividend flow to changes in s_t. This yields a coefficient a_1 that is common to all stocks, and that measures the sensitivity of any given stock's price to a unit increase in the stock's dividend flow at time t caused by an increase in s_t. In the absence of risk aversion, (3.2) implies that the price of stock i would increase by $a_1 = \frac{1}{r+\kappa}$. Hence, a_1 and a_{2i} would be equal: an increase in a stock's dividend flow would have the same effect on the stock's price regardless of whether it comes from the common or from the stock-specific component.

Risk aversion lowers a_1 and a_{2i}. This is because increases in s_t or e_{it} not only raise expected dividends but also make them riskier, and risk has a negative effect on prices when agents are risk averse. The effect of increased risk attenuates that of higher expected dividends. One would expect the attenuation to be larger when the increased risk comes from increases in s_t rather than in e_{it}.

This is because agents are more averse to risk that affects all stocks rather than a specific stock. Equations (3.2) and (3.3) imply that $a_1 < a_{2i}$ if

$$\theta b \sigma_s^2 = \left(\sum_{i=1}^{N} \theta_i b_i \right) \sigma_s^2 > \theta_i \sigma_i^2. \tag{3.4}$$

Equation (3.4) evaluates how a unit increase in stock i's dividend flow at time t affects the covariance between the dividend flow of stock i and of the supply portfolio. This covariance captures the relevant risk in our model. The left-hand side of the inequality in (3.4) is the increase in the covariance when the increase in dividend flow is caused by an increase in s_t. The right-hand side is the increase in the same covariance when the increase in dividend flow is caused by an increase in e_{it}. When (3.4) holds, the change in s_t has a larger effect on the covariance compared to the change in e_{it}. Therefore, it has a larger attenuating effect on the price.

Equation (3.4) holds when the number N of stocks exceeds a threshold, which can be zero. This is because the left-hand side increases when stocks are added, while the right-hand side remains constant. In the special case of scale invariance, (3.4) takes the intuitive form

$$\theta b \bar{s} > \theta_i \bar{e}_i. \tag{3.5}$$

The left-hand side is the dividend flow of the supply portfolio that is derived from the common component. The right-hand side is the dividend flow of the same portfolio that is derived from the component specific to stock i. Equation (3.5) obviously holds when N is large enough.

3.1 Supply Effects

We next examine how differences in supply in the cross-section of stocks are reflected into prices and return moments. We compare two stocks i and i' that differ only in supply and are otherwise identical. This amounts to determining how the price and return of stock i change when the stock's supply becomes $\theta_{i'}$ instead of θ_i, holding constant aggregate quantities such as $\theta b = \sum_{i=j}^{N} \theta_j b_j$. We perform this comparison locally, setting $\theta_{i'} = \theta_i + d\theta_i$. This amounts to computing partial derivatives with respect to θ_i.

We compute return moments both for returns per share and for returns per dollar invested. Moments of share returns can be computed in closed form. To compute closed-form solutions for moments of dollar returns, we approximate the dollar return of a stock by its share return divided by the unconditional mean of the share price. Thus, while expected dollar return, for example, is the expected ratio of share return to share price, we approximate it by the ratio of expected share return to expected share price.

Proposition 3.1 (**Price and Expected Return**). *An increase in the supply θ_i of a stock i lowers the stock's price ($\frac{\partial S_{it}}{\partial \theta_i} < 0$), and raises its expected share return ($\frac{\partial \mathbb{E}(dR_{it})}{\partial \theta_i} > 0$) and expected dollar return ($\frac{\partial \mathbb{E}\left(\frac{dR_{it}}{\mathbb{E}(S_{it})}\right)}{\partial \theta_i} > 0$).*

A stock i in small supply θ_i must offer low expected share return, so that the manager is induced to hold a small number of shares of the stock. Therefore, the stock's price must be high. The stock's expected dollar return is low because of two effects working in the same direction: low expected share return in the numerator, and high price in the denominator.

The effect of θ_i on the stock price is through the coefficient a_{2i}, which measures the price sensitivity to changes in the stock-specific component e_{it} of dividends. When θ_i is small, an increase in e_{it} is accompanied by a small increase in the covariance between the dividend flow of the stock and of the effective-supply portfolio. Therefore, the positive effect that the increase in e_{it} has on the price through higher expected dividends is attenuated by a small negative effect due to the increase in risk. As a consequence, a_{2i} is large. Since an increase in e_{it} away from its lower bound of zero has a large effect on the price, the price is high.

Note that θ_i does not have an effect through the coefficient a_{1i}, which measures the price sensitivity to changes in the common component s_t of dividends. This coefficient depends on θ_i only through the aggregate quantity θb, which is constant in cross-sectional comparisons.

Proposition 3.2 (**Return Volatility**). *An increase in the supply θ_i of a stock i lowers the stock's share return variance ($\frac{\partial \mathbb{V}\mathrm{ar}(dR_{it})}{\partial \theta_i} < 0$). It lowers its dollar return variance ($\frac{\partial \mathbb{V}\mathrm{ar}\left(\frac{dR_{it}}{\mathbb{E}(S_{it})}\right)}{\partial \theta_i} < 0$) if and only if*

$$a_1 b_i \sigma_s^2 < a_{2i} \sigma_i^2. \tag{3.6}$$

Since dividend changes have a large effect on the price of a stock that is in small supply, such a stock has high share return volatility (square root of variance). This effect is concentrated on the part of volatility that is driven by the stock-specific component, while there is no effect on the part that is driven by the common component. Whether small supply is associated with high or low dollar return volatility depends on two effects working in opposite directions: high share return volatility in the numerator, and high price in the denominator. The first effect dominates when (3.6) holds.

Since the effect of supply on volatility is concentrated on the part that is driven by the stock-specific component, (3.6) should hold if that part is large enough. This can be confirmed, for example, in the case of scale invariance. Equation (3.6) becomes

$$a_1 b_i \bar{s} < a_{2i} \bar{e}_i, \tag{3.7}$$

and has the simple interpretation that the volatility driven by the stock-specific component exceeds the volatility driven by the common component. Indeed, the conditional variance rate driven by the common component is $a_1^2 b_i^2 \sigma_s^2 s_t$ and that driven by the stock-specific component is $a_{2i}^2 \sigma_i^2 e_{it}$. Taking expectations, we find the unconditional variance rates $a_1^2 b_i^2 \sigma_s^2 \bar{s}$ and $a_{2i}^2 \sigma_i^2 \bar{e}_i$, respectively. Under scale invariance, the latter exceeds the former if (3.7) holds.

We next examine how supply affects the systematic and idiosyncratic parts of volatility. When the number N of stocks is large, these coincide, respectively, with the parts driven by the common and the stock-specific component. For small N, however, the systematic part includes volatility driven by the stock-specific component. This is especially so if a "stock" is interpreted as an industry-sector portfolio or investment style that is a large fraction of the market. To compute the systematic and idiosyncratic parts, we regress the return dR_{it} of stock i on the return $dR_{\eta t} \equiv \eta dR_t$ of the index:

$$dR_{it} = \beta_i dR_{\eta t} + d\epsilon_{it}. \tag{3.8}$$

The CAPM beta of stock i is

$$\beta_i = \frac{\mathbb{C}\mathrm{ov}(dR_{it}, dR_{\eta t})}{\mathbb{V}\mathrm{ar}(dR_{\eta t})}, \tag{3.9}$$

and measures the systematic part of volatility. The variance $\mathbb{V}\mathrm{ar}(d\epsilon_{it})$ of the regression residual measures the idiosyncratic part. These quantities are defined in per-share terms. Their per-dollar counterparts are

$$\beta_i^{\$} = \frac{\mathbb{C}\mathrm{ov}\left(\frac{dR_{it}}{\mathbb{E}(S_{it})}, \frac{dR_{\eta t}}{\mathbb{E}(S_{\eta t})}\right)}{\mathbb{V}\mathrm{ar}\left(\frac{dR_{\eta t}}{\mathbb{E}(S_{\eta t})}\right)} \tag{3.10}$$

and $\mathbb{V}\mathrm{ar}\left(\frac{d\epsilon_{it}}{\mathbb{E}(S_{it})}\right)$, where $S_{\eta t} \equiv \eta S_t$ denotes the price of the index.

Proposition 3.3 (**Beta and Idiosyncratic Volatility**). *An increase in the supply θ_i of a stock* *i lowers the stock's share beta* ($\frac{\partial \beta_i}{\partial \theta_i} < 0$) *and idiosyncratic share return variance* ($\frac{\partial \mathbb{V}\mathrm{ar}(d\epsilon_{it})}{\partial \theta_i} < 0$)*. It lowers the stock's dollar beta* ($\frac{\partial \beta_i^{\$}}{\partial \theta_i} < 0$) *if and only if*

$$a_1^2 b_i \eta b \sigma_s^2 \bar{s} - 2a_1 b_i a_{2i} \eta_i \sigma_i^2 \bar{s} - a_{2i}^2 \eta_i \sigma_i^2 \bar{e}_i < 0, \tag{3.11}$$

and lowers the stock's idiosyncratic dollar return variance ($\frac{\partial \mathbb{V}\mathrm{ar}\left(\frac{d\epsilon_{it}}{\mathbb{E}(S_{it})}\right)}{\partial \theta_i} < 0$) *if and only if*

$$\begin{aligned} &a_1^3 b_i \eta b(\eta b - 2\eta_i b_i) a_{2i} \sigma_s^2 \sigma_i^2 \bar{s}^2 \bar{e}_i - a_1 b_i (a_1 b_i \sigma_s^2 - a_{2i}\sigma_i^2) \left(\sum_{j=1}^N a_{2j}^2 \eta_j^2 \sigma_j^2 \bar{e}_j \right) \bar{s}\bar{e}_i \\ &- (2a_1 b_i \bar{s} + a_{2i}\bar{e}_i) a_{2i}^3 \eta_i^2 \sigma_i^4 \bar{e}_i^2 > 0. \end{aligned} \tag{3.12}$$

The share beta and idiosyncratic volatility are large for a stock that is in small supply because of the effect identified in Propositions 3.1 and 3.2: changes to the stock-specific component of dividends have a large effect on the price of such a stock. This yields high idiosyncratic volatility. It also yields large beta because stock-specific shocks have a large contribution to the stock's covariance with the index. Whether small supply is associated with high or low dollar beta and idiosyncratic volatility depends on two effects working in opposite directions: high share beta and idiosyncratic volatility in the numerator, and high price in the denominator. We next determine which effect dominates when the number N of stocks is large.

For large N, a stock's covariance with the index is driven mainly by the common shocks, whose effect on price does not depend on supply. Since supply affects only a small fraction of the covariance, the effect of supply on price should dominate that on share beta. Hence, dollar beta should be small for a stock that is in small supply. This can be confirmed, for example, in the case of scale invariance and symmetric stocks. We denote by $(b_c, \bar{e}_c, \eta_c, \theta_c)$ the common values of $(b_i, \bar{e}_i, \eta_i, \theta_i)$ across all stocks, by a_{2c} the common value of a_{2i}, and by $y \equiv \frac{a_{2c}\bar{e}_c}{a_1 b_c \bar{s}}$ the ratio of volatility driven by the stock-specific component to the volatility driven by the common component. We can write (3.11) as

$$N - 2y - y^2 < 0. \tag{3.13}$$

As N increases, (3.13) is satisfied for values of y that exceed an increasingly large threshold.

A stock's idiosyncratic volatility, for large N, is driven mainly by the shocks specific to that stock. Since the effect of supply is only through those shocks, while common shocks account for a potentially large fraction of the price, the effect of supply on idiosyncratic share volatility should dominate the effect of supply on price. Hence, idiosyncratic dollar return volatility should be large for a stock that is in small supply. For example, in the case of scale invariance and symmetric stocks, we can write (3.12) as

$$\begin{aligned} & N(N-2) - Ny + (N-2)y^2 - y^3 > 0 \\ & \Leftrightarrow N - 2 - y > 0. \end{aligned} \tag{3.14}$$

As N increases, (3.14) is satisfied for values of y that are below an increasingly large threshold.

To relate the effects of supply derived in Propositions 3.2 and 3.3 to cross-sectional market anomalies, we next determine how supply affects stocks' CAPM alphas, i.e., the expected returns that stocks are earning in excess of the CAPM. The CAPM alpha of stock i is

$$\alpha_i = \mathbb{E}(dR_{it}) - \beta_i \mathbb{E}(dR_{\eta t}), \tag{3.15}$$

in per-share terms. Its per-dollar counterpart is

$$\alpha_i^{\$} = \mathbb{E}\left(\frac{dR_{it}}{\mathbb{E}(S_{it})}\right) - \beta_i^{\$} \mathbb{E}\left(\frac{dR_{\eta t}}{\mathbb{E}(S_{\eta t})}\right) = \frac{\alpha_i}{\mathbb{E}(S_{it})}. \tag{3.16}$$

Proposition 3.4 (**Alpha**)**.** *An increase in the supply θ_i of a stock i raises the stock's share alpha ($\frac{\partial \alpha_i}{\partial \theta_i} > 0$). It raises its dollar alpha ($\frac{\partial \alpha_i^{\$}}{\partial \theta_i} > 0$) if and only if*

$$b_i \bar{s} + \bar{e}_i - \frac{\left[1-(r+\kappa)a_1\right]\eta b \bar{s} + \sum_{j=1}^N \left[1-(r+\kappa)a_{2j}\right]\eta_j \bar{e}_j}{a_1^2 (\eta b)^2 \sigma_s^2 \bar{s} + \sum_{j=1}^N a_{2j}^2 \eta_j^2 \sigma_j^2 \bar{e}_j} \left(a_1^2 b_i \eta b \sigma_s^2 \bar{s} - 2a_1 b_i a_{2i} \eta_i \sigma_i^2 \bar{s} - a_{2i}^2 \eta_i \sigma_i^2 \bar{e}_i\right) > 0. \tag{3.17}$$

Equation (3.17) holds when alpha is positive. It holds for all values of alpha when, for example, stocks share the same characteristics $(b_i, \bar{e}_i)$.

A stock in small supply has low share alpha because it has low expected share return (Proposition 3.1) and high share beta (Proposition 3.3). The effect of supply on dollar alpha is unambiguously positive when alpha is positive: an increase in supply raises dollar alpha because it raises share alpha and lowers the price. The effect can become ambiguous, however, when alpha is negative because an increase in share alpha corresponds to a reduction in absolute value. When stocks share the same characteristics $(b_i, \bar{e}_i)$ (but can differ in (η_i, θ_i)), an increase in supply unambiguously raises dollar alpha for both positive and negative values.

Our results have implications for the relationship between risk and return. According to the CAPM, stocks' expected returns in excess of the riskless rate should be linearly related in the cross-section to the stocks' CAPM betas. Empirically, however, the intercept of the line is positive instead of zero, and the slope is smaller than the theoretical one. As a consequence, high-beta stocks have negative alphas, while low-beta stocks have positive alphas. This is the beta anomaly, documented by Black (1972), Black, Jensen and Scholes (1972), and Frazzini and Pedersen (2013).

Related to the beta anomaly is the volatility anomaly. This is that high-volatility stocks have negative alphas, while low-volatility stocks have positive alphas. The volatility anomaly has been documented by Haugen and Baker (1996) and Ang et al. (2006). The latter paper considers idiosyncratic volatility in addition to (total) volatility, and shows that it is also negatively related to alpha.

The beta and volatility anomalies are puzzling. Yet, even more puzzling is that the negative relationship between risk and return can arise even when alpha is replaced by expected return. That is, high-risk stocks earn expected returns that are not only lower than the CAPM benchmarks, but can also be lower than the expected returns of low-risk stocks. Ang et al. (2006) document this effect both when risk is measured by volatility and when it is measured by idiosyncratic volatility. The same effect has also been shown for beta in some cases (e.g., Jylha 2014).

The results in this section suggest a mechanism that could help explain the anomalies, even in the absence of agency frictions. A negative relationship between alpha or expected return on one hand, and volatility or beta on the other, can be generated by the way that these variables depend on supply. Stocks in small supply earn low expected dollar return (Proposition 3.1) and negative alpha (Proposition 3.4). Under some conditions, they also have high dollar return volatility (Proposition 3.2), high idiosyncratic dollar return volatility (Proposition 3.3), and high dollar beta (Proposition 3.3). Under these conditions our model can explain both the negative relationship between risk and alpha, as well as the more puzzling negative relationship between risk and expected return.

The conditions for small supply to be associated with high risk are the least restrictive in the case of idiosyncratic volatility. When N is large, as is the case when our model is applied to stocks, stocks in small supply have high idiosyncratic volatility for almost all of the parameter region. Hence, our model could help explain the idiosyncratic volatility anomaly of Ang et al. (2006).

Explaining the volatility anomaly within our model requires that idiosyncratic volatility exceeds systematic volatility. This condition is plausible when our model is applied to stocks.

The beta anomaly is the hardest to explain within our model. When N is large, stocks in small supply have small beta for almost all of the parameter region. Hence, our model could explain the beta anomaly only when N is small, in which case stocks must be interpreted as market segments such as industry-sector portfolios or investment styles. Given that the anomaly holds for individual stocks, their dividends must be having a significant segment-specific component (not present in our model) and so must supply. The mechanism suggested by our model could be relevant in such cases. One could argue, for example, that during the 1998-2000 run-up of technology stocks, demand by some investors for that segment of the market pushed up prices and lowered expected returns, while also introducing a significant segment-specific component of volatility that raised beta.

4 Equilibrium with Agency Frictions

In this section we solve for equilibrium in the presence of agency frictions. We introduce agency frictions by setting the parameter A in the manager's private-benefit function $Am_t - \frac{B}{2}m_t^2$ to a positive value. For simplicity, we set the parameter B to zero. This pins down immediately the coefficient ϕ that characterizes how sensitive the manager's fee is to the fund's performance. Indeed, if $\phi < A$, then the manager will choose an arbitrarily large shirking action m_t. This forces the investor to offer $\phi \geq A$, in which case there is no shirking, i.e., $m_t = 0$.[3] When $A \leq \frac{\rho}{\rho+\bar{\rho}}$, the constraint $\phi \geq A$ is not binding, since in the equilibrium without agency frictions the investor offers $\phi = \frac{\rho}{\rho+\bar{\rho}}$. When instead $A > \frac{\rho}{\rho+\bar{\rho}}$, the constraint is binding, and the investor offers $\phi = A$.

Allowing B to be positive yields a richer theory of contract determination, both on the positive and on the normative front. The asset pricing results, however, remain essentially the same. For this reason we defer the case $B > 0$ to Section 5, where we perform a normative analysis of contracts.

Theorem 4.1 (Equilibrium Prices and Contract with Agency Frictions). *Suppose that $B = 0$. When $\frac{\rho}{\rho+\bar{\rho}} \geq A > 0$, the equilibrium in Theorem 3.1 remains an equilibrium. When $A > \frac{\rho}{\rho+\bar{\rho}}$, the following form an equilibrium: the price process S_t given by (2.14) with a_{0i} given by (3.1),*

$$a_{1i} = \frac{b_i}{\sqrt{(r+\kappa)^2 + 2r\bar{\rho}(\phi\theta - \chi\eta)b\sigma_s^2}} \equiv b_i a_1, \tag{4.1}$$

$$a_{2i} = \frac{1}{\sqrt{(r+\kappa)^2 + 2r\bar{\rho}(\phi\theta_i - \chi\eta_i)\sigma_i^2}}; \tag{4.2}$$

[3]For $\phi = A$, the manager is indifferent between all values of m_t. We assume that he chooses $m_t = 0$, as would be the case for any positive value of B, even arbitrarily small.

the contract (ϕ, χ, ψ) *with* $\phi = A$, $\psi = 0$, *and* $\chi > 0$ *being the unique solution to*

$$\hat{s}\eta b\left(a_1 - \check{a}_1\right) + \sum_{i=1}^{N} \hat{e}_i \eta_i \left(a_{2i} - \check{a}_{2i}\right) = 0, \tag{4.3}$$

where

$$\check{a}_1 \equiv \frac{1}{\sqrt{(r+\kappa)^2 + 2r\rho[(1-\phi)\theta + \chi\eta]b\sigma_s^2}}, \tag{4.4}$$

$$\check{a}_{2i} \equiv \frac{1}{\sqrt{(r+\kappa)^2 + 2r\rho[(1-\phi)\theta_i + \chi\eta_i]\sigma_i^2}}, \tag{4.5}$$

$\hat{s} \equiv s_0 + \frac{\kappa}{r}\bar{s}$, *and* $\hat{e}_i \equiv e_{i0} + \frac{\kappa}{r}\bar{e}_i$; *and the index investment* $x = 0$.

When $A > \frac{\rho}{\rho+\bar{\rho}}$, the investor renders the manager's fee more sensitive to the fund's performance compared to the equilibrium without agency frictions ($\phi = A > \frac{\rho}{\rho+\bar{\rho}}$). This exposes the manager to more risk, but eliminates his incentive to undertake the shirking action m_t. If the increase in ϕ were the only change in the contract, the manager would respond by scaling down the fund's stock holdings and investing more in the riskless asset. This would offset the increase in his personal risk exposure caused by the larger ϕ. The investor restores the manager's incentives to take risk by making the fee sensitive to the index performance ($\chi > 0$). This induces the manager to scale up the fund's stock holdings because his personal exposure to market drops becomes smaller. The increase in stock holdings, however, is according to the weights in the index η and not those in the supply portfolio θ. The fund's portfolio thus changes in response to the increases in ϕ and χ, causing equilibrium prices to change. The investor does not invest directly in the index ($x = 0$) because she can control the fund's index exposure by changing χ.

The compensation that the manager receives for performance relative to the index is analogous to relative-performance evaluation in models of optimal contracting under moral hazard (e.g., Holmstrom 1979). The mechanism is somewhat different, however. In typical moral-hazard models, relative-performance evaluation is used to insulate the agent from risk that he cannot control. In our model, instead, the agent can control his risk exposure through his choice of the fund's portfolio. Compensation based on relative performance is instead used to induce the agent to take risk.

Equations (4.1) and (4.2) show how the contract parameters (ϕ, χ) affect equilibrium prices. Prices are determined by the covariance with the portfolio $\phi\theta - \chi\eta$. This is the portfolio that describes the manager's personal risk exposure: the fee is ϕ times the fund's return, which in equilibrium is the return of the supply portfolio θ, minus χ times the return of the index portfolio

η. The covariance is multiplied by the manager's risk aversion coefficient $\bar{\rho}$. Prices are determined by the manager's risk aversion and risk exposure because the manager is marginal in pricing the stocks. We examine the properties of prices in Sections 4.1 and 4.2.

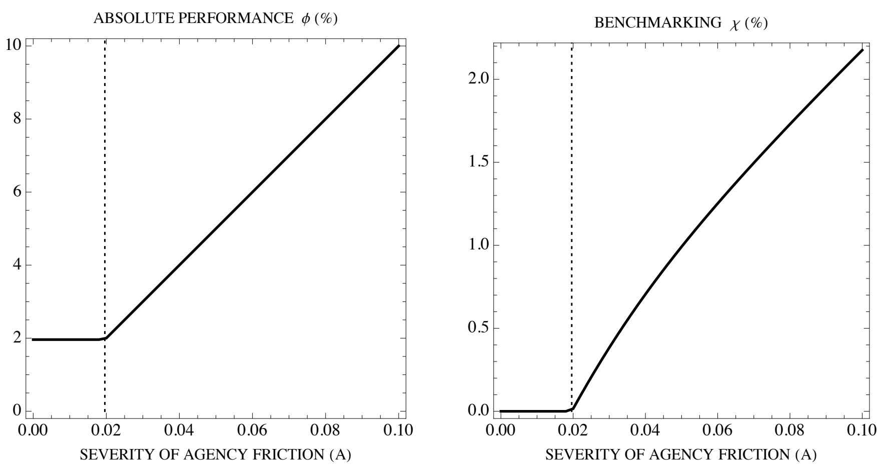
Equation (4.3), which characterizes the the contract parameter χ, can be given an intuitive interpretation. The quantity $S_{i0} = a_1 b_i \hat{s} + \sum_{i=1}^N a_{2i}\hat{e}_i$ is the price of stock i at time zero. We can also construct the counterpart $\check{S}_{i0} \equiv \check{a}_1 b_i \hat{s} + \sum_{i=1}^N \check{a}_{2i}\hat{e}_i$ of this expression for the coefficients $\check{a}_1$ and $\check{a}_{2i}$ defined in (4.4) and (4.5). This is the hypothetical price of stock i at time zero under the assumption that the stock is priced from the investor instead of the manager. The price $\check{S}_{i0}$ can be derived from S_{i0} by replacing the manager's risk exposure $\phi\theta - \chi\eta$ by the investor's exposure $(1-\phi)\theta + \chi\eta$, and the manager's risk-aversion coefficient $\bar{\rho}$ by the investor's coefficient ρ. Equation (4.3) states that the investor and the manager agree on their valuation of the index: $\eta\check{S}_0 = \eta S_0$. This is because the investor can invest directly in the index, and hence is marginal in pricing the index. The investor and the manager can disagree on their valuation of other portfolios. In particular, and as we show in the proof of Theorem 4.1, the investor values the supply portfolio more than the manager: $\theta\check{S}_0 > \theta S_0$. The investor could acquire more of the supply portfolio by lowering ϕ, but this would incentivize the manager to undertake the shirking action m_t.

Proposition 4.1. *Suppose that the private-benefit parameters (A, B) satisfy $A > \frac{\rho}{\rho+\bar{\rho}}$ and $B = 0$. Following an increase in A, the manager's fee becomes more sensitive to the fund's performance ($\frac{\partial\phi}{\partial A} > 0$) and to the index performance ($\frac{\partial\chi}{\partial A} > 0$).*

We illustrate the comparative statics of Proposition 4.1 using a numerical example. The investor's risk-aversion coefficient ρ is one. This is a normalization because we can redefine the units of the consumption good. The manager's risk-aversion coefficient $\bar{\rho}$ is 50, meaning that the manager accounts for $\frac{\rho}{\rho+\bar{\rho}} = 2\%$ of aggregate risk tolerance. The interest rate r is 4%. There are two groups of stocks, with three stocks in each group ($N = 6$). Stocks in each group have identical characteristics, and the only difference across groups is effective supply. The market index η includes one share of each stock ($\eta_i = 1$ for $i = 1, .., 6$). This is a normalization because we can redefine one share of each stock. The residual supply of stocks 1, 2, and 3, left over by the buy-and-hold investors, is 0.8 share, and that of stocks 4, 5, and 6 is 0.2 share ($\theta_1 = \theta_2 = \theta_3 = 0.8$ and $\theta_4 = \theta_5 = \theta_6 = 0.2$). The mean-reversion parameter κ is 0.1, meaning that the half-life of dividend shocks is $\frac{\log(2)}{0.1} = 6.93$ years. The stocks are symmetric in terms of the sensitivity b_i of their dividends to the common component s_t, and in terms of the long-run mean $\bar{e}_i$ and volatility parameter σ_i of the stock-specific component. The dividend sensitivities are equal to one ($b_i = 1$ for $i = 1, .., 6$). This is a normalization because we can redefine s_t. The long-run mean of the common component of dividends is 1.5 and that of the stock-specific component is one ($\bar{s} = 1.5$ and $\bar{e}_i = 1$ for $i = 1, .., 6$). The long-run means are equal to the time zero values of the corresponding variables ($s_0 = \bar{s}$ and $e_{i0} = \bar{e}_i$ for $i = 1, .., 6$). The volatility parameters satisfy scale invariance: $\frac{\sigma_s^2}{\bar{s}} = \frac{\sigma_i^2}{\bar{s}_i} \equiv R$ for $i = 1.., 6$. The common ratio R determines the volatility of asset returns, and we set it to one (hence $\sigma_s = \sqrt{1.5}$ and $\sigma_i = 1$ for $i = 1, .., 6$).

Figure 1 plots ϕ and χ, expressed as percentages, as functions of A. The figure confirms that both variables increase in A for $A > \frac{\rho}{\rho + \bar{\rho}}$.

Figure 1: Optimal Contract



The sensitivity of the manager's fee to the fund's performance (ϕ) and to the index performance (χ) as a function of the private-benefit parameter A. There are two groups of stocks, with three stocks in each group. Stocks within each group have identical characteristics, and the only difference across groups is supply. Parameter values are: $\rho = 1$, $\bar{\rho} = 50$, $r = 4\%$, $N = 6$, $\eta_i = 1$, $\theta_1 = \theta_2 = \theta_3 = 0.8$, $\theta_4 = \theta_5 = \theta_6 = 0.2$, $\kappa = 0.1$, $b_i = 1$, $\bar{s} = 1.5$, $\bar{e}_i = 1$, $\sigma_s = \sqrt{1.5}$, $\sigma_i = 1$, for $i = 1, .., 6$.

4.1 Cross-Sectional Pricing and Amplification

We perform our analysis of equilibrium prices using the numerical example presented in the previous section. Figure 2 plots several equilibrium quantities against the private-benefit parameter A. The blue line represents a stock in large supply and the red line a stock in small supply.

Asset Price Levels Recall that in the absence of agency frictions, stocks in large supply (high θ_i) are cheaper than otherwise identical stocks in small supply. This is because the manager requires a large risk premium to absorb the large supply.

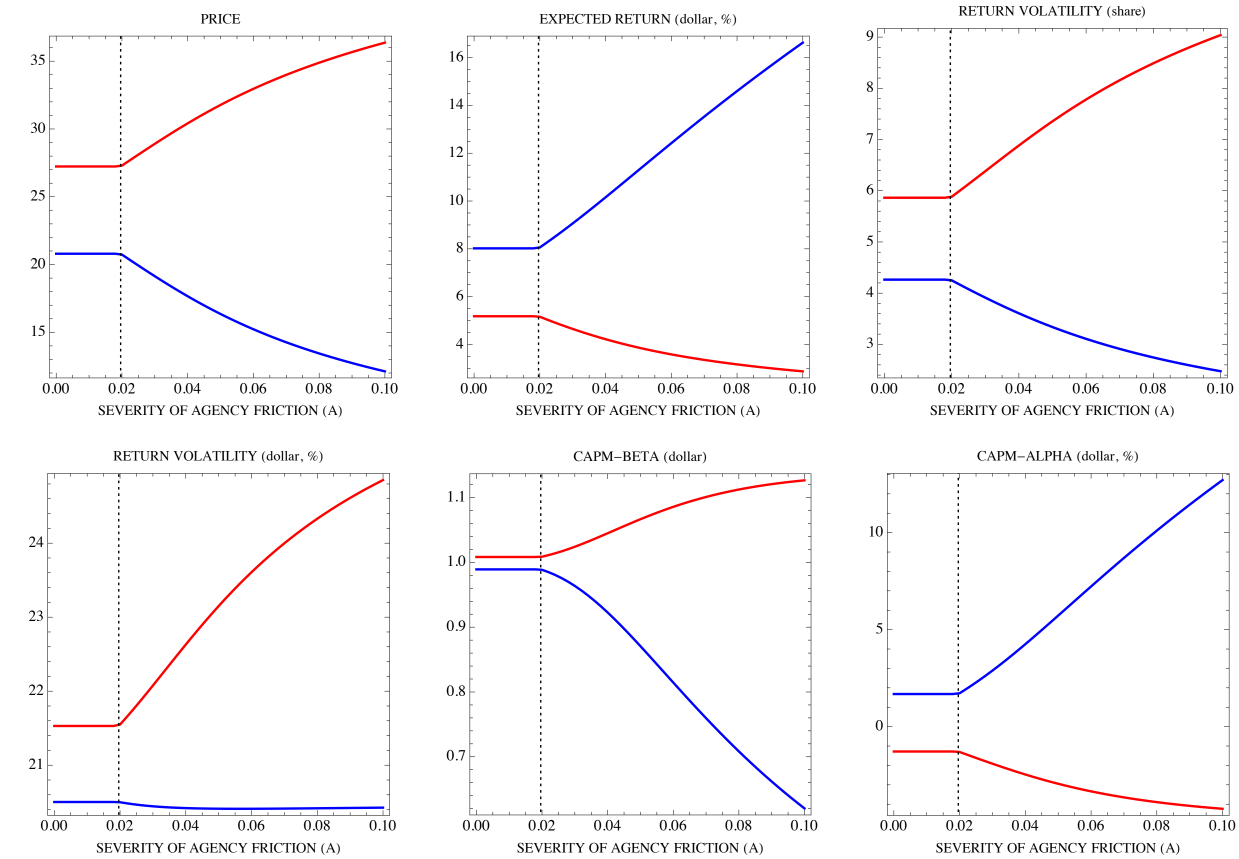
In the presence of agency frictions the dispersion in asset prices increases: stocks in small supply become even more expensive, while stocks in large supply become even cheaper. The intuition is that the manager has an incentive to reduce the deviations from the index by selecting a portfolio that is more in line with the index. Note that, without frictions, stocks in large supply are overweighed in the fund portfolio compared to the index, whereas stocks in small supply are underweighted. This is precisely how the manager can provide a superior portfolio to the investor. As a consequence, a fund portfolio closer to the benchmark is achieved by (i) reducing the demand for stocks in large supply, and (ii) increasing the demand for stocks in small supply. Since there is smaller demand for the large-supply stocks, their prices decrease. The prices of the small-supply stocks, which are higher demand, increase. The first plot in Figure 2 illustrates this result.

The effect on expected returns follows naturally from the one on asset prices. The agency friction increases the expected returns on high-θ stocks, and it decreases the expected returns on low-θ stocks. The second plot in Figure 2 illustrates this result.

Asset Volatility To illustrate the amplification effect that agency frictions have on asset volatility, consider the following example. Suppose that a stock in small supply experiences a positive shock to its dividend flow. Higher dividends imply an increase in dividend volatility, which in turn makes the manager, who is subject to benchmarking, more eager to *buy* the stock, to reduce the risk of deviating from the index. Higher demand reduces the stock's risk premium and amplifies the price increase initially induced by the positive dividend shock. Suppose now that the positive shock affects a stock in large residual supply. As in the case without agency frictions, the increase in dividend volatility induces a large increase in the risk premium that strongly attenuates the positive price effect of the shock. In addition, in the presence of benchmarking, a higher dividend volatility makes the fund manager more eager to *sell* the stock in large supply, again to be closer to the index. This raises the stock risk premium and hence reduces the price increase even further. Therefore, agency frictions amplify the supply effects on asset return volatility: small-supply stocks become even more volatile, while large supply stocks become even less volatile. The third plot in Figure 2 illustrates this result.

Anomalies Since the supply effects on the level and volatility of stock prices are exacerbated by the agency friction, the agency friction exacerbates the anomalies discussed in Section 3.1. The last three plots in Figure 2 illustrate these results. The negative cross-sectional relationship between alpha on one hand, and volatility or beta on the other, becomes stronger as the agency friction becomes more severe.

Figure 2: Agency Frictions and Cross-Sectional Asset Pricing



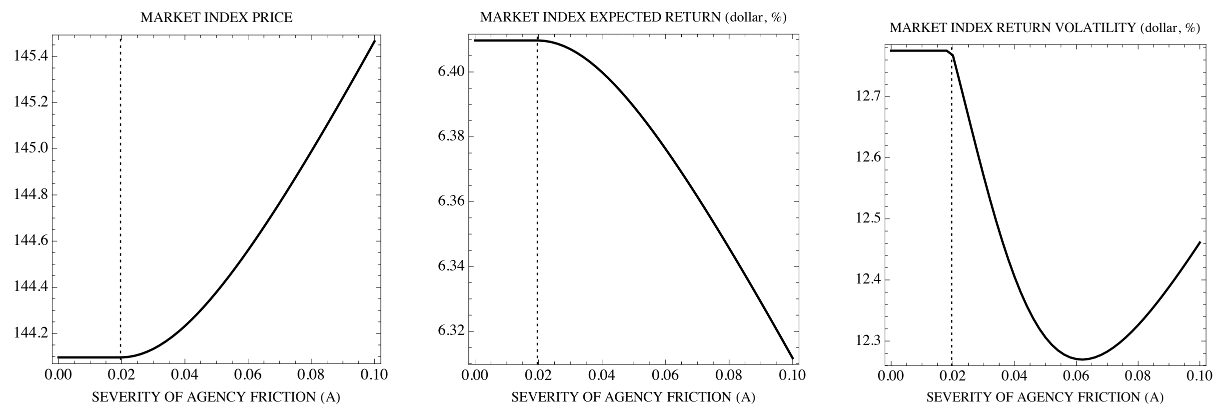
Average price $\mathbb{E}(S_{it})$, dollar expected return $\frac{1}{dt}\mathbb{E}\left(\frac{dR_{it}}{\mathbb{E}(S_{it})}\right)$, share return volatility $\sqrt{\frac{1}{dt}\mathbb{V}\text{ar}(dR_{it})}$, dollar return volatility $\sqrt{\frac{1}{dt}\mathbb{V}\text{ar}\left(\frac{dR_{it}}{\mathbb{E}(S_{it})}\right)}$, dollar beta $\beta_i^{\$}$, and dollar alpha $\frac{1}{dt}\alpha_i^{\$}$, as a function of the private-benefit parameter A. There are two groups of stocks, with three stocks in each group. Stocks within each group have identical characteristics, and the only difference across groups is supply. Parameter values are: $\rho = 1$, $\bar{\rho} = 50$, $r = 4\%$, $N = 6$, $\eta_i = 1$, $\theta_1 = \theta_2 = \theta_3 = 0.8$, $\theta_4 = \theta_5 = \theta_6 = 0.2$, $\kappa = 0.1$, $b_i = 1$, $\bar{s} = 1.5$, $\bar{e}_i = 1$, $\sigma_s = \sqrt{1.5}$, $\sigma_i = 1$, for $i = 1, .., 6$. Stocks 1 to 3, in the high-supply group are plotted in blue, and stocks 4 to 6, in the low-supply group are plotted in red.

4.2 Aggregate Market

Aggregate Market The amplification of supply effects induced by the agency friction has an impact on the aggregate market, i.e., the market index. This is because amplification affects stocks in large and small supply asymmetrically. The asymmetry can be seen from Figure 2.

The intuition for the asymmetry is as follows. Because the optimal contract involves benchmarking, the fund manager becomes less willing to deviate from the market index. Because stocks in small supply are more volatile and account for a larger fraction of the market movement than

Figure 3: Agency Frictions and the Aggregate Market



Average price $\mathbb{E}(S_{\eta t})$, dollar expected return $\frac{1}{dt}\mathbb{E}\left(\frac{dR_{\eta t}}{\mathbb{E}(S_{\eta t})}\right)$, and dollar return volatility $\sqrt{\frac{1}{dt}\mathbb{V}\text{ar}\left(\frac{dR_{\eta t}}{\mathbb{E}(S_{\eta t})}\right)}$ of the market index, as a function of the private-benefit parameter A. There are two groups of stocks, with three stocks in each group. Stocks within each group have identical characteristics, and the only difference across groups is supply. Parameter values are: $\rho = 1$, $\bar{\rho} = 50$, $r = 4\%$, $N = 6$, $\eta_i = 1$, $\theta_1 = \theta_2 = \theta_3 = 0.8$, $\theta_4 = \theta_5 = \theta_6 = 0.2$, $\kappa = 0.1$, $b_i = 1$, $\bar{s} = 1.5$, $\bar{e}_i = 1$, $\sigma_s = \sqrt{1.5}$, $\sigma_i = 1$, for $i = 1, .., 6$.

stocks in large supply, deviating from the index by underweighting the former is riskier than to deviate by overweighting the latter. Therefore, the fund manager is more willing to deviate from the index by overweighting stocks in large supply, to exploit the "under-valuation," and less willing to deviate by underweighting stocks in small supply, to exploit the "over-valuation." Therefore, the aggregate market goes up as the agency friction becomes more severe, and its expected return goes down. Figure 3 plots the effects of the agency friction on the price, expected return, and volatility of the aggregate market.

5 Social Optimality

We next examine whether the privately optimal contract, determined in Section 4, is socially optimal. We assume that a social planner chooses contract parameters (ϕ, χ, ψ) at time zero. This is the social planner's only intervention: given the contract, the manager is free to choose the fund's portfolio z_t and the shirking action m_t, and prices S_t must clear markets. We restrict the investor's direct investment in the index to be zero ($x = 0$). This is without loss of generality because the social planner can control index exposures by changing χ, and the investor would choose $x = 0$ under the socially optimal χ.

The social planner maximizes the investor's value function at time zero, subject to the manager's incentive compatibility and individual rationality constraints. This optimization problem is the same as the investor's but the social planner internalizes that a change in the contract parameters affects equilibrium prices. Formally, the value functions of the investor and the manager at time zero can be written as $V(W_0, s_0, e_0, \phi, \chi, \psi, \mathcal{S})$ and $\bar{V}(\bar{W}_0, s_0, e_0, \phi, \chi, \psi, \mathcal{S})$, respectively, where $\mathcal{S}$ consists of the parameters $(a_{01}, .., a_{0N}, a_{11}, .., a_{1N}, a_{21}, .., a_{2N})$ that describe the price process. The investor chooses (ϕ, χ, ψ) taking $\mathcal{S}$ as given. The social planner instead internalizes the dependence of $\mathcal{S}$ on (ϕ, χ).

The social planner's optimization problem involves the utility of the investor and the manager, but not of the buy-and-hold investors. These investors, however, are neutral for our normative analysis, in the sense that the contract choice does not affect their stock holdings and dividend stream. Indeed, buy-and-hold investors are endowed with the portfolio $\eta - \theta$ at time zero and do not trade. Therefore, the dividend stream that they receive from their portfolio does not depend on prices, and is hence unaffected by the contract choice.

When the parameter B in the manager's private-benefit function $Am_t - \frac{B}{2}m_t^2$ is equal to zero, as assumed in Section 4, the social planner's problem yields the same solution as the investor's. Indeed, the coefficient ϕ that characterizes the fee's sensitivity to the fund's performance must satisfy $\phi \geq A$, so that the manager does not choose an arbitrarily large shirking action m_t. Moreover, any $\phi \geq A$ yields no shirking, i.e., $m_t = 0$. When A exceeds the value $\frac{\rho}{\rho+\bar{\rho}}$ that ϕ takes in the absence of agency frictions, the constraint $\phi \geq A$ is binding. Hence, the social planner sets $\phi = A$, as does the investor.

The social planner chooses the same contract as the investor because $\phi = A$ is a corner solution. The differences in marginal trade-offs between the social planner and the investor become apparent when instead ϕ is an interior solution. Interior solutions are possible when the parameter B is positive. Theorem 5.1 generalizes the equilibrium derived in the previous section to $B > 0$. Proposition 5.1 solves the social planner's problem and shows that solutions for the investor and the social planner differ.

Theorem 5.1 (Equilibrium Prices and Contract with General Agency Frictions). *When $\frac{\rho}{\rho+\bar{\rho}} \geq A > 0$, the equilibrium in Theorem 3.1 remains an equilibrium. When $A > \frac{\rho}{\rho+\bar{\rho}}$ and $B \in [0, \underline{B}] \cup [\bar{B}, \infty)$ for two constants $\bar{B} > \underline{B}$, the following form an equilibrium: the price process S_t given by (2.14), (3.1), (4.1), and (4.2); the contract (ϕ, χ, ψ) with $A \geq \phi > \frac{\rho}{\rho+\bar{\rho}}$ and $\chi > 0$*

solving the system of equations

$$\left(\frac{\phi(1-\phi)}{B} + r\hat{s}\theta b\left(a_1 - \check{a}_1\right) + r\sum_{i=1}^{N}\hat{e}_i\theta_i\left(a_{2i} - \check{a}_{2i}\right) = 0 \quad and \quad \phi < A\right) \quad or \tag{5.1}$$

$$\left(\frac{\phi(1-\phi)}{B} + r\hat{s}\theta b\left(a_1 - \check{a}_1\right) + r\sum_{i=1}^{N}\hat{e}_i\theta_i\left(a_{2i} - \check{a}_{2i}\right) \geq 0 \quad and \quad \phi = A\right), \tag{5.2}$$

and (4.3), where $\check{a}_1$, $\check{a}_{2i}$, $\hat{s}$, and $\hat{e}_i$ are as in Theorem 4.1, and $\psi = -\frac{(A-\phi)^2}{2B}$; and the index investment $x = 0$.

The behavior of ϕ for $A > \frac{\rho}{\rho+\bar{\rho}}$ is as follows. When B is positive but close to zero, $\phi = A$ is a corner solution, as in the case $B = 0$. When B exceeds a threshold, $\phi = A$ ceases to be a corner solution, and the solution becomes interior to the interval $(\frac{\rho}{\rho+\bar{\rho}}, A)$. Intuitively, the investor's benefit from raising ϕ is that the manager has a smaller incentive to undertake the shirking action. At the same time, larger ϕ involves a cost to the investor because the manager becomes less willing to take risk and hence to exploit price differentials driven by supply, i.e., invest relatively more in high-θ stocks and less in low-θ stocks. When B increases, the manager derives a smaller benefit from shirking. Hence the investor's benefit from raising ϕ is smaller, which is why ϕ decreases below A when B exceeds a threshold. When B becomes large, and so the manager's benefit from shirking converges to zero, ϕ converges to its value $\frac{\rho}{\rho+\bar{\rho}}$ under no agency frictions.

The system of equations (4.3), (5.2), and (5.1) that determines (ϕ, χ, ψ) can have multiple solutions when $B > 0$. This means that multiple equilibria can exist. The comparison between socially and privately optimal contract shown in Proposition 5.1 applies to the privately optimal contract in any of these equilibria.

The equilibrium in Theorem 5.1 may fail to exist for intermediate values of B. This is because the investor may not be willing to employ the manager. Note that the investor is willing to employ the manager not only when the benefit of shirking is small ($B \geq \bar{B}$) but also—and more surprisingly—when it is large ($B \leq \underline{B}$). This is because of a general-equilibrium effect: when the benefit from shirking is large, equilibrium prices are more distorted, making the supply portfolio an even better investment than the index portfolio.

Proposition 5.1 (Socially Optimal Contract). *When $\frac{\rho}{\rho+\bar{\rho}} \geq A \geq 0$, the socially optimal contract (ϕ^*, χ^*, ψ^*) is as in Theorem 3.1. When $A > \frac{\rho}{\rho+\bar{\rho}}$, the socially optimal contract is as follows:*

$A \geq \phi^* > \frac{\rho}{\rho+\bar{\rho}}$ and $\chi^* > 0$ are the unique solution to the system of equations

$$\left(\frac{1-\phi}{B} + r\hat{s}\theta b\left(a_1 - \breve{a}_1\right) + r\sum_{i=1}^{N}\hat{e}_i\theta_i\left(a_{2i} - \breve{a}_{2i}\right) = 0 \quad \text{and} \quad \phi < A\right) \quad \text{or} \tag{5.3}$$

$$\left(\frac{1-\phi}{B} + r\hat{s}\theta b\left(a_1 - \breve{a}_1\right) + r\sum_{i=1}^{N}\hat{e}_i\theta_i\left(a_{2i} - \breve{a}_{2i}\right) \geq 0 \quad \text{and} \quad \phi = A\right), \tag{5.4}$$

and (4.3), where $\breve{a}_1$, $\breve{a}_{2i}$, $\hat{s}$, and $\hat{e}_i$ are as in Theorem 4.1, and $\psi^* = -\frac{(A-\phi^*)^2}{2B}$. The manager's fee under the socially optimal contract is more sensitive to the fund's performance and to the index performance than under the privately optimal contract ($\phi^* \geq \phi$, $\chi^* \geq \chi$), with the inequalities being strict when $\phi < A$.

Under the socially optimal contract, the manager has steeper incentives than under the privately optimal contract, with the inequalities being strict when the privately optimal ϕ is an interior solution. The intuition is that because the social planner internalizes price effects, he is more effective than private agents in providing incentives. Indeed, recall that the investor's benefit from raising ϕ is that the manager shirks less, and her cost is that the manager becomes less willing to take risk and exploit price differentials driven by supply. The cost is lower for the social planner. This is because when the social planner raises ϕ, equilibrium prices become more distorted, to the point where the manager remains equally willing to exploit supply-driven price differentials. Because cost and benefit are equalized at an interior solution, such a solution for the investor must be strictly smaller than for the social planner.

The inefficiency can be viewed as a free-rider problem. Interpret the investor and the manager as a continuum of identical investors and managers. When one investor in the continuum gives steeper incentives to her manager, this makes the manager less willing to exploit mispricings. Other managers, however, remain equally willing to do so, benefiting their investors. When all investors give steeper incentives to their managers, mispricings become more severe, and all managers remain equally willing to exploit them despite being exposed to more risk.

Because the social planner chooses steeper incentives than private agents, supply effects are stronger under the socially optimal contract. Thus, the volatility and beta anomalies are stronger. Agency frictions also have a larger positive effect on the price of the aggregate market.

Appendix A: Proofs

Proof of Theorem 3.1. The theorem follows from the proof of Theorem 4.1, which covers the case $A = 0$. □

Proof of Proposition 3.1. Substituting a_{0i} from (3.1), we can write the price (2.14) of stock i as

$$S_{it} = a_1 b_i \left(\frac{\kappa}{r}\bar{s} + s_t\right) + a_{2i}\left(\frac{\kappa}{r}\bar{e}_i + e_{it}\right). \tag{A.1}$$

Equation (3.2) implies that a_1 does not depend on θ_i, holding the aggregate quantity θb constant. Equation (3.3) implies that a_{2i} decreases in θ_i. Therefore, (A.70) implies that S_{it} decreases in θ_i.

Substituting a_{0i} from (3.1), we can write the share return (A.16) of stock i as

$$dR_{it} = \left\{[1 - a_1(r+\kappa)]b_i s_t + [1 - a_{2i}(r+\kappa)]e_{it}\right\}dt + a_1 b_i \sigma_s \sqrt{s_t} dw_{st} + a_{2i}\sigma_i \sqrt{e_{it}} dw_{it}. \tag{A.2}$$

The expected share return is

$$\mathbb{E}(dR_{it}) = \left\{[1 - a_1(r+\kappa)]b_i \bar{s} + [1 - a_{2i}(r+\kappa)]\bar{e}_i\right\}dt, \tag{A.3}$$

and increases in θ_i because a_{2i} decreases in θ_i.

Equations (3.2) and (3.3) imply that the terms in square brackets in (A.3) are positive and so is $\mathbb{E}(dR_{it})$. Since $\mathbb{E}(dR_{it})$ increases in θ_i, and S_{it} decreases in θ_i, the expected dollar return $\mathbb{E}\left(\frac{dR_{it}}{\mathbb{E}(S_{it})}\right)$ of stock i increases in θ_i. □

Proof of Proposition 3.2. Equation (A.2) implies that the share return variance of stock i is

$$\begin{aligned}
\mathbb{V}\text{ar}(dR_{it}) &= \mathbb{E}\left[(dR_{it})^2\right] - \left[\mathbb{E}(dR_{it})\right]^2 \\
&= \mathbb{E}\left[(dR_{it})^2\right] \\
&= \mathbb{E}\left[\left(a_1^2 b_i^2 \sigma_s^2 s_t + a_{2i}^2 \sigma_i^2 e_{it}\right)dt\right] \\
&= \left(a_1^2 b_i^2 \sigma_s^2 \bar{s} + a_{2i}^2 \sigma_i^2 \bar{e}_i\right)dt,
\end{aligned} \tag{A.4}$$

where the second step follows because the term $\mathbb{E}\left[(dR_{it})^2\right]$ is of order dt and the term $\left[\mathbb{E}(dR_{it})\right]^2$ is of order $(dt)^2$. Since a_{2i} decreases in θ_i, (A.4) implies that $\mathbb{V}\text{ar}(dR_{it})$ decreases in θ_i.

Equation (A.70) implies that the expected share price of stock i is

$$\mathbb{E}(S_{it}) = \frac{\kappa + r}{r}\left(a_1 b_i \bar{s} + a_{2i}\bar{e}_i\right). \tag{A.5}$$

Therefore, the dollar return variance of stock i is

$$\mathbb{V}\text{ar}\left(\frac{dR_{it}}{\mathbb{E}(S_{it})}\right) = \frac{a_1^2 b_i^2 \sigma_s^2 \bar{s} + a_{2i}^2 \sigma_i^2 \bar{e}_i}{\frac{(\kappa+r)^2}{r^2}\left(a_1 b_i \bar{s} + a_{2i}\bar{e}_i\right)^2} dt. \tag{A.6}$$

Differentiating with respect to a_{2i}, we find that $\mathbb{V}\text{ar}\left(\frac{dR_{it}}{\mathbb{E}(S_{it})}\right)$ increases in a_{2i} and hence decreases in θ_i if and only if (3.6) holds. □

Proof of Proposition 3.3. Equation (A.2) implies that the share return of the index is

$$dR_{\eta t} = \left\{[1 - a_1(r+\kappa)]\eta b s_t + \sum_{j=1}^N [1 - a_{2i}(r+\kappa)]\eta_j e_{jt}\right\} dt + a_1 \eta b \sigma_s \sqrt{s_t} dw_{st} + \sum_{j=1}^N a_{2j}\eta_j \sigma_j \sqrt{e_{jt}} dw_{jt}. \tag{A.7}$$

Equations (3.9), (A.2) and (A.7) imply that the share beta of stock i is

$$\beta_i = \frac{\mathbb{C}\text{ov}(dR_{it}, dR_{\eta t})}{\mathbb{V}\text{ar}(dR_{\eta t})} = \frac{\mathbb{E}\left(dR_{it} dR_{\eta t}\right)}{\mathbb{E}\left[(dR_{\eta t})^2\right]} = \frac{a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i}^2 \eta_i \sigma_i^2 \bar{e}_i}{a_1^2 (\eta b)^2 \sigma_s^2 \bar{s} + \sum_{j=1}^N a_{2j}^2 \eta_j^2 \sigma_j^2 \bar{e}_j}, \tag{A.8}$$

and decreases in θ_i holding aggregate quantities such as $\sum_{j=1}^N a_{2j}^2 \eta_j^2 \sigma_j^2 \bar{e}_j$ constant because a_{2i} decreases in θ_i.

Equation (A.5) implies that the expected share price of the index is

$$\mathbb{E}(\eta S_t) = \frac{\kappa + r}{r}\left(a_1 \eta b \bar{s} + \sum_{j=1}^N a_{2j}\eta_j \bar{e}_j\right). \tag{A.9}$$

Equations (3.10), (A.5), (A.8), and (A.9) imply that the dollar beta of stock i is

$$\beta_i^{\$} = \frac{a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i}^2 \eta_i \sigma_i^2 \bar{e}_i}{a_1^2 (\eta b)^2 \sigma_s^2 \bar{s} + \sum_{j=1}^N a_{2j}^2 \eta_j^2 \sigma_j^2 \bar{e}_j} \frac{a_1 \eta b \bar{s} + \sum_{j=1}^N a_{2j}\eta_j \bar{e}_j}{a_1 b_i \bar{s} + a_{2i}\bar{e}_i}. \tag{A.10}$$

The right-hand side of (A.10) increases in a_{2i} and hence decreases in θ_i if and only if the function

$$\frac{a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i}^2 \eta_i \sigma_i^2 \bar{e}_i}{a_1 b_i \bar{s} + a_{2i}\bar{e}_i}$$

increases in a_{2i}. This happens if and only if (3.11) holds.

Equations (A.4), (A.7), and (A.8) imply that the idiosyncratic share return variance of stock i is

$$\begin{aligned}\mathbb{V}\text{ar}(d\epsilon_{it}) &= \mathbb{V}\text{ar}(dR_{it}) - \beta_i^2 \mathbb{V}\text{ar}(dR_{\eta t}) \\ &= \left[a_1^2 b_i^2 \sigma_s^2 \bar{s} + a_{2i}^2 \sigma_i^2 \bar{e}_i - \frac{\left(a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i}^2 \eta_i \sigma_i^2 \bar{e}_i\right)^2}{a_1^2 (\eta b)^2 \sigma_s^2 \bar{s} + \sum_{j=1}^N a_{2j}^2 \eta_j^2 \sigma_j^2 \bar{e}_j}\right] dt. \qquad \text{(A.11)}\end{aligned}$$

Since a_{2i} decreases in θ_i, $\mathbb{V}\text{ar}(d\epsilon_{it})$ decreases in θ_i holding aggregate quantities such as $\sum_{j=1}^N a_{2j}^2 \eta_j^2 \sigma_j^2 \bar{e}_j$ constant if and only if

$$I(a_{2i}) \equiv (a_1^2 b_i^2 \sigma_s^2 \bar{s} + a_{2i}^2 \sigma_i^2 \bar{e}_i) \left[a_1^2 (\eta b)^2 \sigma_s^2 \bar{s} + \sum_{j=1}^N a_{2j}^2 \eta_j^2 \sigma_j^2 \bar{e}_j\right] - \left(a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i}^2 \eta_i \sigma_i^2 \bar{e}_i\right)^2$$

increases in a_{2i} holding the same quantities constant. The latter happens if and only if

$$\frac{\partial I(a_{2i})}{\partial a_{2i}^2} = \sigma_i^2 \bar{e}_i \left[a_1^2 (\eta b)^2 \sigma_s^2 \bar{s} + \sum_{j=1}^N a_{2j}^2 \eta_j^2 \sigma_j^2 \bar{e}_j - 2\eta_i \left(a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i}^2 \eta_i \sigma_i^2 \bar{e}_i\right)\right] > 0. \qquad \text{(A.12)}$$

Since asset $i' \neq i$ satisfies $(b_{i'}, \sigma_{i'}, \bar{e}_{i'}, \eta_{i'}, \theta_{i'}) = (b_i, \sigma_i, \bar{e}_i, \eta_i, \theta_i + d\theta_i)$, the following inequalities hold:

$$\begin{aligned}&(\eta b)^2 \geq (\eta_i b_i + \eta_{i'} b_{i'})(\eta b) = 2\eta_i b_i \eta b, \\ &\sum_{j=1}^N a_{2j}^2 \eta_j^2 \sigma_j^2 \bar{e}_j \geq a_{2i}^2 \eta_i^2 \sigma_i^2 \bar{e}_i + a_{2i'}^2 \eta_{i'}^2 \sigma_{i'}^2 \bar{e}_{i'} = 2\eta_i a_{2i}^2 \eta_i \sigma_i^2 \bar{e}_i,\end{aligned}$$

implying that (A.12) also holds.

Equations (A.5) and (A.11) imply that the idiosyncratic dollar return variance of stock i is

$$\mathbb{V}\text{ar}\left(\frac{d\epsilon_{it}}{\mathbb{E}(S_{it})}\right) = \frac{\left[a_1^2 b_i^2 \sigma_s^2 \bar{s} + a_{2i}^2 \sigma_i^2 \bar{e}_i - \frac{\left(a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i}^2 \eta_i \sigma_i^2 \bar{e}_i\right)^2}{a_1^2 (\eta b)^2 \sigma_s^2 \bar{s} + \sum_{j=1}^N a_{2j}^2 \eta_j^2 \sigma_j^2 \bar{e}_j}\right]}{\frac{(\kappa + r)^2}{r^2} \left(a_1 b_i \bar{s} + a_{2i} \bar{e}_i\right)^2} dt, \qquad \text{(A.13)}$$

and decreases in θ_i holding aggregate quantities constant if and only if

$$I_1(a_{2i}) \equiv \frac{(a_1^2 b_i^2 \sigma_s^2 \bar{s} + a_{2i}^2 \sigma_i^2 \bar{e}_i) \left[a_1^2 (\eta b)^2 \sigma_s^2 \bar{s} + \sum_{j=1}^N a_{2j}^2 \eta_j^2 \sigma_j^2 \bar{e}_j\right] - \left(a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i}^2 \eta_i \sigma_i^2 \bar{e}_i\right)^2}{\left(a_1 b_i \bar{s} + a_{2i} \bar{e}_i\right)^2}$$

increases in a_{2i} holding the same quantities constant. The latter happens if and only if (3.12) holds, as can be seen by computing the partial derivative of $I_1(a_{2i})$ with respect to a_{2i}. □

Proof of Proposition 3.4. Since the expected share return of stock i increases in θ_i and the stock's share beta decreases in θ_i, (3.15) implies that the stock's share alpha increases in θ_i. Equations (3.16), (A.3), and

(A.10) imply that the dollar alpha of stock i is

$$\alpha_i^\$ = \frac{b_i\bar{s}+\bar{e}_i}{a_1 b_i\bar{s}+a_{2i}\bar{e}_i} - (r+\kappa) - \frac{a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i}^2\eta_i\sigma_i^2\bar{e}_i}{a_1^2(\eta b)^2\sigma_s^2\bar{s} + \sum_{j=1}^N a_{2j}^2\eta_j^2\sigma_j^2\bar{e}_j} \frac{[1-a_1(r+\kappa)]\eta b\bar{s} + \sum_{j=1}^N [1-a_{2j}(r+\kappa)]\bar{e}_j}{a_1 b_i\bar{s}+a_{2i}\bar{e}_i}. \tag{A.14}$$

Differentiating with respect to a_{2i} holding aggregate quantities constant, we find that $\alpha_i^\$$ decreases in a_{2i} and hence increases in θ_i if and only if (3.17) holds. When $\alpha_i^\$ > 0$, the left-hand side of (3.17) is larger than

$$\begin{aligned}
&(r+\kappa)(a_1 b_i\bar{s}+a_{2i}\bar{e}_i) + \frac{[1-(r+\kappa)a_1]\,\eta b\bar{s} + \sum_{j=1}^N [1-(r+\kappa)a_{2j}]\,\eta_j\bar{e}_j}{a_1^2(\eta b)^2\sigma_s^2\bar{s} + \sum_{j=1}^N a_{2j}^2\eta_j^2\sigma_i^2\bar{e}_i} \\
&\times \left[a_1^2 b_i\eta b\sigma_s^2\bar{s} + a_{2i}^2\eta_i\sigma_i^2\bar{e}_i - \left(a_1^2 b_i\eta b\sigma_s^2\bar{s} - 2a_1 b_i a_{2i}\eta_i\sigma_i^2\bar{s} - a_{2i}^2\eta_i\sigma_i^2\bar{e}_i\right)\right] \\
&= (r+\kappa)(a_1 b_i\bar{s}+a_{2i}\bar{e}_i) + 2\frac{[1-(r+\kappa)a_1]\,\eta b\bar{s} + \sum_{j=1}^N [1-(r+\kappa)a_{2j}]\,\eta_j\bar{e}_j}{a_1^2(\eta b)^2\sigma_s^2\bar{s} + \sum_{j=1}^N a_{2j}^2\eta_j^2\sigma_i^2\bar{e}_i} \left[2a_{2i}^2\eta_i\sigma_i^2\bar{e}_i + a_1 b_i a_{2i}\eta_i\sigma_i^2\bar{s}\right] > 0,
\end{aligned}$$

and hence (3.17) holds. When $(b_i, \bar{e}_i) = (b_c, \bar{e}_c)$ for all i, (3.17) becomes

$$b_c\bar{s}+\bar{e}_c - \frac{[1-(r+\kappa)a_1]\,b_c\eta\mathbf{1}\bar{s} + \sum_{j=1}^N [1-(r+\kappa)a_{2j}]\,\eta_j\bar{e}_c}{a_1^2 b_c^2(\eta\mathbf{1})^2\sigma_s^2\bar{s} + \sum_{j=1}^N a_{2j}^2\eta_j^2\sigma_j^2\bar{e}_c} \left(a_1^2 b_c^2\eta\mathbf{1}\sigma_s^2\bar{s} - 2a_1 b_c a_{2i}\eta_i\sigma_i^2\bar{s} - a_{2i}^2\eta_i\sigma_i^2\bar{e}_c\right) > 0, \tag{A.15}$$

where $\mathbf{1} \equiv (1,..,1)'$. Equation (A.15) holds under the sufficient condition

$$\begin{aligned}
&b_c\bar{s}+\bar{e}_c - \frac{b_c\eta\mathbf{1}\bar{s} + \sum_{j=1}^N \eta_j\bar{e}_c}{a_1^2 b_c^2(\eta\mathbf{1})^2\sigma_s^2\bar{s} + \sum_{j=1}^N a_{2j}^2\eta_j^2\sigma_j^2\bar{e}_c} a_1^2 b_c^2\eta\mathbf{1}\sigma_s^2\bar{s} > 0 \\
&\Leftrightarrow b_c\bar{s}+\bar{e}_c - \frac{b_c\eta\mathbf{1}\bar{s} + \eta\mathbf{1}\bar{e}_c}{a_1^2 b_c^2(\eta\mathbf{1})^2\sigma_s^2\bar{s} + \sum_{j=1}^N a_{2j}^2\eta_j^2\sigma_j^2\bar{e}_c} a_1^2 b_c^2\eta\mathbf{1}\sigma_s^2\bar{s} > 0,
\end{aligned}$$

which holds. □

Proof of Theorem 4.1. We allow A to be zero so that the proof can also cover Theorem 3.1. The proof assumes $B=0$, but when $A=0$ the proof carries through unchanged to $B>0$. We proceed in two steps:

- *Step 1:* We fix a contract (ϕ, χ, ψ) with $\phi \geq A$, and show that for the price function (2.14) and the coefficients (a_{0i}, a_{1i}, a_{2i}) given by (3.1), (4.1), and (4.2), $z_t = \theta$ solves the optimization problem of an employed manager. Hence, markets clear provided that the manager accepts the contract (ϕ, χ, ψ) and the investor invests $x=0$ in the index.
- *Step 2:* We fix prices given by (2.14), (3.1), (4.1), and (4.2), and show that the investor decides to employ the manager, i.e., offer a contract that the manager accepts, and that the contract (ϕ, χ, ψ)

given in Theorems 3.1 and 4.1, and the index investment $x = 0$, solve the investor's optimization problem. Hence, an equilibrium exists, and is as in the theorems.

Step 1. Substituting S_{it} from (2.14) into (2.2), we can write the excess return dR_{it} of asset i as

$$dR_{it} = \mu_{it}dt + a_{1i}\sigma_s\sqrt{s_t}dw_{st} + a_{2i}\sigma_i\sqrt{e_{it}}dw_{it}, \tag{A.16}$$

where

$$\mu_{it} \equiv (a_{1i}\kappa\bar{s} + a_{2i}\kappa\bar{e}_i - a_{0i}r) + [b_i - a_{1i}(r+\kappa)]s_t + [1 - a_{2i}(r+\kappa)]e_{it}. \tag{A.17}$$

We set $\mu_t \equiv (\mu_{1t}, .., \mu_{Nt})'$.

We conjecture that the value function of an employed manager takes the form

$$\bar{V}(\bar{W}_t, s_t, e_t) = -\exp\left[-\left(r\bar{\rho}\bar{W}_t + \bar{q}_0 + \bar{q}_1 s_t + \sum_{i=1}^{N}\bar{q}_{2i}e_{it}\right)\right], \tag{A.18}$$

where $(\bar{q}_0, \bar{q}_1, \bar{q}_{21}, .., \bar{q}_{2N})$ are constants. The manager's Bellman equation is

$$\max_{\bar{c}_t, z_t, m_t}\left[-\exp(-\bar{\rho}\bar{c}_t) + \mathcal{D}\bar{V}_t - \bar{\delta}\bar{V}_t\right] = 0, \tag{A.19}$$

where $\mathcal{D}\bar{V}_t$ is the drift of $\bar{V}_t$.

Using (2.3), (2.4), (2.5), (2.7), and (A.16), we find that the dynamics of $\bar{J}_t \equiv r\bar{\rho}\bar{W}_t + \bar{q}_0 + \bar{q}_1 s_t + \sum_{i=1}^{N}\bar{q}_{2i}e_{it}$ are

$$d\bar{J}_t = \bar{G}_t dt + \bar{H}_t dw_{st} + \sum_{i=1}^{N}\bar{K}_{it}dw_{it}, \tag{A.20}$$

where

$$\begin{aligned}
\bar{G}_t &\equiv r\bar{\rho}\left[r\bar{W}_t + (\phi z_t - \chi\eta)\mu_t + \psi + (A-\phi)m_t - \bar{c}_t\right] + \kappa\left[\bar{q}_1(\bar{s} - s_t) + \sum_{i=1}^{N}\bar{q}_{2i}(\bar{e}_i - e_{it})\right], \\
\bar{H}_t &\equiv \left[r\bar{\rho}\sum_{i=1}^{N}(\phi z_{it} - \chi\eta_i)a_{1i} + \bar{q}_1\right]\sigma_s\sqrt{s_t}, \\
\bar{K}_{it} &\equiv \left[r\bar{\rho}(\phi z_{it} - \chi\eta_i)a_{2i} + \bar{q}_{2i}\right]\sigma_i\sqrt{e_{it}}.
\end{aligned}$$

Using $\bar{V}(\bar{W}_t, s_t, e_t) = -\exp(\bar{J}_t)$, (A.20), and Ito's lemma, we find that the drift $\mathcal{D}\bar{V}_t$ of $\bar{V}_t$ is

$$\mathcal{D}\bar{V}_t = -\bar{V}_t\left(\bar{G}_t - \frac{1}{2}\bar{H}_t^2 - \frac{1}{2}\sum_{i=1}^{N}\bar{K}_{it}^2\right). \tag{A.21}$$

Substituting into (A.19), we can write the manager's Bellman equation as

$$\max_{\bar{c}_t, z_t, m_t} \left[-\exp(-\bar{\rho}\bar{c}_t) - \bar{V}_t \left(\bar{G}_t - \frac{1}{2}\bar{H}_t^2 - \frac{1}{2}\sum_{i=1}^N \bar{K}_{it}^2 \right) - \bar{\delta}\bar{V}_t \right] = 0. \tag{A.22}$$

The first-order condition with respect to $\bar{c}_t$ is

$$\bar{\rho}\exp(-\bar{\rho}\bar{c}_t) + r\bar{\rho}\bar{V}_t = 0.$$

Using (A.18) to substitute for $\bar{V}_t$, and solving for $\bar{c}_t$, we find

$$\bar{c}_t = r\bar{W}_t + \frac{1}{\bar{\rho}}\left(\bar{q}_0 - \log(r) + \bar{q}_1 s_t + \sum_{i=1}^N \bar{q}_{2i} e_{it} \right). \tag{A.23}$$

The first-order condition with respect to m_t is

$$m_t = 0 \tag{A.24}$$

because $\phi \geq A$. (For $\phi > A$, the manager has a strict preference for $m_t = 0$. For $\phi = A$, the manager is indifferent between all values of m_t, and we assume that he chooses $m_t = 0$.) The first-order condition with respect to z_{it} is

$$r\bar{\rho}\phi\mu_{it} - r\bar{\rho}\phi a_{1i}\left[r\bar{\rho}\sum_{i=1}^N (\phi z_{it} - \chi\eta_i)a_{1i} + \bar{q}_1 \right]\sigma_s^2 s_t - r\bar{\rho}\phi a_{2i}\left[r\bar{\rho}(\phi z_{it} - \chi\eta_i)a_{2i} + \bar{q}_{2i} \right]\sigma_i^2 e_{it} = 0. \tag{A.25}$$

The portfolio $z_t = \theta$ solves the manager's optimization problem if (A.25) holds for $z_t = \theta$ and for all values of $(s_t, e_{1t}, .., e_{Nt})$. Substituting μ_{it} from (A.17), and dividing by $r\bar{\rho}\phi$ throughout, we can write (A.25) for $z_t = \theta$ as

$$A_{0i} + A_{1i}s_t + A_{2i}e_{it} = 0, \tag{A.26}$$

where

$$A_{0i} \equiv \kappa(a_{1i}\bar{s} + a_{2i}\bar{e}_i) - a_{0i}r,$$

$$A_{1i} \equiv b_i - a_{1i}(r+\kappa) - a_{1i}\left[r\bar{\rho}\sum_{i=1}^N (\phi\theta_i - \chi\eta_i)a_{1i} + \bar{q}_1 \right]\sigma_s^2,$$

$$A_{2i} \equiv 1 - a_{2i}(r+\kappa) - a_{2i}\left[r\bar{\rho}(\phi\theta_i - \chi\eta_i)a_{2i} + \bar{q}_{2i} \right]\sigma_i^2 e_{it}.$$

The left-hand side of (A.26) is an affine function of (s_t, e_{it}). Therefore, (A.25) holds for $z_t = \theta$ and for all values of $(s_t, e_{1t}, .., e_{Nt})$ if $A_{0i} = A_{1i} = A_{2i} = 0$. Before linking these equations to the coefficients (a_{0i}, a_{1i}, a_{2i}) given in the proposition, we determine a set of additional equations that follow from the requirement that the manager's Bellman equation (A.22) holds. Using (A.23), (A.24), and (A.25) to substitute $\bar{c}_t$, m_t, and μ_{it}, we can write (A.22) for $z_t = \theta$ as

$$\bar{Q}_0 + \bar{Q}_1 s_t + \sum_{i=1}^{N} \bar{Q}_{2i} e_{it} = 0, \tag{A.27}$$

where

$$\bar{Q}_0 \equiv \bar{q}_0 r - r\bar{\rho}\psi - \kappa \left(\bar{q}_1 \bar{s} + \sum_{i=1}^{N} \bar{q}_{2i} \bar{e}_i \right) + r - \bar{\delta} - r\log(r),$$

$$\bar{Q}_1 \equiv \bar{q}_1 (r+\kappa) + \frac{1}{2}\bar{q}_1^2 \sigma_s^2 - \frac{1}{2}\left[r\bar{\rho} \sum_{i=1}^{N} (\phi\theta_i - \chi\eta_i) a_{1i} \right]^2 \sigma_s^2,$$

$$\bar{Q}_{2i} \equiv \bar{q}_{2i}(r+\kappa) + \frac{1}{2}\bar{q}_{2i}^2 \sigma_i^2 - \frac{1}{2}\left[r\bar{\rho}(\phi\theta_i - \chi\eta_i) a_{2i} \right]^2 \sigma_i^2.$$

The left-hand side of (A.27) is an affine function of $(s_t, e_{1t}, .., e_{Nt})$. Therefore, (A.27) holds for $z_t = \theta$ and for all values of $(s_t, e_{1t}, .., e_{Nt})$ if $\bar{Q}_0 = \bar{Q}_1 = \bar{Q}_{21} = .. = \bar{Q}_{2N} = 0$.

We next show that equations $A_{0i} = A_{1i} = A_{2i} = 0$ and $\bar{Q}_0 = \bar{Q}_1 = \bar{Q}_{2i} = 0$ determine the coefficients $(a_{0i}, a_{1i}, a_{2i}, \bar{q}_0, \bar{q}_1, \bar{q}_{2i})$ uniquely, with (a_{0i}, a_{1i}, a_{2i}) being as in the proposition. This will imply that $z_t = \theta$ solves the manager's optimization problem given the prices in the proposition. Equation $A_{1i} = 0$ implies that $a_{1i} = b_i a_1$, with a_1 being independent of i. Hence, $A_{1i} = 0$ can be replaced by $A_1 = 0$ with

$$A_1 \equiv 1 - a_1(r+\kappa) - a_1 \left[r\bar{\rho}(\phi\theta - \chi\eta) b a_1 + \bar{q}_1 \right] \sigma_s^2.$$

Moreover, $\bar{Q}_1$ can be written as

$$\bar{Q}_1 = \bar{q}_1(r+\kappa) + \frac{1}{2}\bar{q}_1^2\sigma_s^2 - \frac{1}{2}\left[r\bar{\rho}(\phi\theta - \chi\eta) b a_1 \right]^2 \sigma_s^2.$$

The quadratic equation $\bar{Q}_1 = 0$ has the unique positive root[4]

$$\bar{q}_1 = \frac{\sqrt{(r+\kappa)^2 + \left[r\bar{\rho}(\phi\theta - \chi\eta) b a_1 \right]^2 \sigma_s^4} - (r+\kappa)}{\sigma_s^2}. \tag{A.28}$$

[4]Holding wealth $\bar{W}_t$ constant, the manager is better off the larger s_t is. This is because with larger s_t, dividends are more volatile, and the manager must earn higher compensation in equilibrium for investing in stocks. (In the extreme case where volatility is zero, stocks earn the same return as the riskless asset and the manager derives no utility from investing in them.) Because the manager's utility increases in s_t, the coefficient $\bar{q}_1$ must be positive. The coefficients $(\bar{q}_{21}, .., \bar{q}_{2N})$, and the counterparts of $(\bar{q}_1, \bar{q}_{21}, .., \bar{q}_{2N})$ in the investor's value function, must be positive for the same reason.

Substituting (A.28) into $A_1 = 0$, we find

$$\begin{aligned}
&1 - a_1^2 r\bar{\rho}(\phi\theta - \chi\eta)b\sigma_s^2 = a_1\sqrt{(r+\kappa)^2 + [r\bar{\rho}(\phi\theta - \chi\eta)ba_1]^2\,\sigma_s^4} \\
&\Rightarrow 1 - a_1^2\left[(r+\kappa)^2 + 2r\bar{\rho}(\phi\theta - \chi\eta)b\sigma_s^2\right] = 0 \\
&\Rightarrow a_1 = \frac{1}{\sqrt{(r+\kappa)^2 + 2r\bar{\rho}(\phi\theta - \chi\eta)b\sigma_s^2}} \qquad \text{(A.29)}
\end{aligned}$$

where the second equation follows from the first by squaring both sides and simplifying. Eqs. $a_{1i} = b_i a_1$ and (A.29) coincide with (4.1). Substituting (A.29) into (A.28) we can determine $\bar{q}_1$:

$$\bar{q}_1 = \frac{(r+\kappa)^2 + r\bar{\rho}(\phi\theta - \chi\eta)b\sigma_s^2}{\sigma_s^2\sqrt{(r+\kappa)^2 + 2r\bar{\rho}(\phi\theta - \chi\eta)b\sigma_s^2}} - \frac{r+\kappa}{\sigma_s^2}. \qquad \text{(A.30)}$$

Following the same procedure to solve the system of $A_{2i} = \bar{Q}_{2i} = 0$, we find (4.2) and

$$\bar{q}_{2i} = \frac{(r+\kappa)^2 + r\bar{\rho}(\phi\theta_i - \chi\eta_i)\sigma_i^2}{\sigma_i^2\sqrt{(r+\kappa)^2 + 2r\bar{\rho}(\phi\theta_i - \chi\eta_i)b\sigma_i^2}} - \frac{r+\kappa}{\sigma_i^2}. \qquad \text{(A.31)}$$

Finally, $A_{0i} = 0$ implies (3.1), and $\bar{Q}_0 = 0$ implies

$$\bar{q}_0 = \bar{\rho}\psi + \frac{\kappa}{r}\left(\bar{q}_1\bar{s} + \sum_{i=1}^N \bar{q}_{2i}\bar{e}_i\right) - 1 + \frac{\bar{\delta}}{r} + \log(r). \qquad \text{(A.32)}$$

Step 2. We conjecture that the value function of the investor when he employs the manager, offers contract $(\tilde{\phi}, \tilde{\chi}, \tilde{\psi})$ that satisfies $\tilde{\phi} \geq A$ and can differ from the equilibrium contract (ϕ, χ, ψ), and invests x in the index, takes the form

$$V(W_t, s_t, e_t) = -\exp\left[-\left(r\rho W_t + q_0 + q_1 s_t + \sum_{i=1}^N q_{2i}e_{it}\right)\right], \qquad \text{(A.33)}$$

where $(q_0, q_1, q_{21}, .., q_{2N})$ are constants. The investor's Bellman equation is

$$\max_{c_t}\left[-\exp(-\rho c_t) + \mathcal{D}V_t - \delta V_t\right] = 0, \qquad \text{(A.34)}$$

where $\mathcal{D}V_t$ is the drift of V_t.

When the investor offers the equilibrium contract (ϕ, χ, ψ), the manager's first-order condition (A.25) is satisfied for $z_t = \theta$, as shown in Step 1. When the investor offers contract $(\tilde{\phi}, \tilde{\chi}, \tilde{\psi})$ with $\tilde{\phi} \geq A$, (A.25) is satisfied for z_t given by

$$\tilde{\phi}z_{it} - \tilde{\chi}\eta_i = \phi\theta_i - \chi\eta_i$$
$$\Rightarrow z_{it} = \frac{\phi\theta_i + (\tilde{\chi} - \chi)\eta_i}{\tilde{\phi}}. \tag{A.35}$$

This is because (A.25) depends on $(\tilde{\phi}, \tilde{\chi}, \tilde{\psi})$ only through the quantity $\tilde{\phi}z_{it} - \tilde{\chi}\eta_i$: if $(\tilde{\phi}, \tilde{\chi}, \tilde{\psi})$ changes, then z_{it} also changes in a way that $\tilde{\phi}z_{it} - \tilde{\chi}\eta_i$ is kept constant. The economic intuition is that the manager chooses the fund's portfolio z_t to "undo" a change in contract: his personal risk exposure, through the fee, is the same under $(\tilde{\phi}, \tilde{\chi}, \tilde{\psi})$ and (ϕ, χ, ψ). The manager's personal risk exposure arises through the fee's variable component, which is $(\tilde{\phi}z_t - \tilde{\chi}\eta)dR_t$ under $(\tilde{\phi}, \tilde{\chi}, \tilde{\psi})$, and $(\phi\theta - \chi\eta)dR_t$ under (ϕ, χ, ψ). Eq. (A.35) relies on the assumption that the investor and the manager take stock prices as given and independent of the contract. Formally, the contract $(\tilde{\phi}, \tilde{\chi}, \tilde{\psi})$ in (A.25) does not affect the price coefficients (a_{i0}, a_{1i}, a_{2i}). We drop the time subscript from the portfolio z_t in (A.35) because that portfolio is constant over time.

Using (2.3), (2.4), (2.5), (2.11), (A.16), $m_t = 0$ (which holds because $\tilde{\phi} \geq A$), and (A.35), we find that the dynamics of $J_t \equiv r\rho W_t + q_0 + q_1 s_t + \sum_{i=1}^N q_{2i}e_{it}$ are

$$dJ_t = G_t dt + H_t dw_{st} + \sum_{i=1}^N K_{it} dw_{it}, \tag{A.36}$$

where

$$G_t \equiv r\rho\left[rW_t + (x\eta + z - \phi\theta + \chi\eta)\mu_t - \tilde{\psi} - c_t\right] + \kappa\left[q_1(\bar{s} - s_t) + \sum_{i=1}^N q_{2i}(\bar{e}_i - e_{it})\right],$$
$$H_t \equiv \left[r\rho(x\eta + z - \phi\theta + \chi\eta)ba_1 + q_1\right]\sigma_s\sqrt{s_t},$$
$$K_{it} \equiv \left[r\rho(x\eta_i + z - \phi\theta_i + \chi\eta_i)a_{2i} + q_{2i}\right]\sigma_i\sqrt{e_{it}}.$$

Proceeding as in Step 1, we can write the investor's Bellman equation (A.34) as

$$\max_{cb_t}\left[-\exp(-\rho c_t) - V_t\left(G_t - \frac{1}{2}H_t^2 - \frac{1}{2}\sum_{i=1}^N K_{it}^2\right) - \delta V_t\right] = 0. \tag{A.37}$$

The first-order condition with respect to c_t is

$$\rho\exp(-\rho c_t) + r\rho V_t = 0,$$

and yields

$$c_t = rW_t + \frac{1}{\rho}\left(q_0 - \log(r) + q_1 s_t + \sum_{i=1}^N q_{2i}e_{it}\right). \tag{A.38}$$

Using (A.38) to substitute c_t, we can write (A.37) as

$$Q_0 + Q_1 s_t + \sum_{i=1}^{N} Q_{2i} e_{it} = 0, \tag{A.39}$$

where

$$\begin{aligned}
Q_0 &\equiv q_0 r + r\rho\tilde{\psi} - \kappa\left(q_1\bar{s} + \sum_{i=1}^{N} q_{2i}\bar{e}_i\right) + r - \bar{\delta} - r\log(r), \\
Q_1 &\equiv q_1(r+\kappa) + \frac{1}{2}q_1^2\sigma_s^2 - r\rho(x\eta + z - \phi\theta + \chi\eta)ba_1 \\
&\quad\times\left[r\bar{\rho}(\phi\theta - \chi\eta)ba_1 - \frac{1}{2}r\rho(x\eta + z - \phi\theta + \chi\eta)ba_1 + \bar{q}_1 - q_1\right]\sigma_s^2, \\
Q_{2i} &\equiv q_{2i}(r+\kappa) + \frac{1}{2}q_{2i}^2\sigma_i^2 - r\rho(x\eta_i + z_i - \phi\theta_i + \chi\eta_i)a_{2i} \\
&\quad\times\left[r\bar{\rho}(\phi\theta_i - \chi\eta_i)a_{2i} - \frac{1}{2}r\rho(x\eta_i + z_i - \phi\theta_i + \chi\eta_i)a_{2i} + \bar{q}_{2i} - q_{2i}\right]\sigma_i^2.
\end{aligned}$$

The left-hand side of (A.27) is an affine function of $(s_t, e_{1t}, .., e_{Nt})$. Therefore, (A.27) holds for all values of $(s_t, e_{1t}, .., e_{Nt})$ if $Q_0 = Q_1 = Q_{21} = .. = Q_{2N} = 0$. Using $A_1 = 0$ we can simplify Q_1 to

$$\begin{aligned}
Q_1 &= q_1(r+\kappa) + \frac{1}{2}q_1^2\sigma_s^2 + r\rho(x\eta + z - \phi\theta + \chi\eta)ba_1 \\
&\quad\times\left[\frac{1}{2}r\rho(x\eta + z - \phi\theta + \chi\eta)ba_1\sigma_s^2 - q_1\sigma_s^2 + r + \kappa - \frac{1}{a_1}\right],
\end{aligned}$$

and using $A_{2i} = 0$ we can simplify Q_{2i} to

$$\begin{aligned}
Q_{2i} &= q_{2i}(r+\kappa) + \frac{1}{2}q_{2i}^2\sigma_i^2 + r\rho(x\eta_i + z_i - \phi\theta_i + \chi\eta_i)a_{2i} \\
&\quad\times\left[\frac{1}{2}r\rho(x\eta_i + z_i - \phi\theta_i + \chi\eta_i)a_{2i}\sigma_i^2 - q_{2i}\sigma_i^2 + r + \kappa - \frac{1}{a_{2i}}\right].
\end{aligned}$$

Using the simplified expressions, we find that the positive root of $Q_1 = 0$ is

$$q_1 = \frac{\sqrt{(r+\kappa)^2 + 2r\rho(x\eta + z - \phi\theta + \chi\eta)b\sigma_s^2} - (r+\kappa)}{\sigma_s^2} - r\rho(x\eta + z - \phi\theta + \chi\eta)ba_1, \tag{A.40}$$

and the positive root of $Q_{2i} = 0$ is

$$q_{2i} = \frac{\sqrt{(r+\kappa)^2 + 2r\rho(x\eta_i + z_i - \phi\theta_i + \chi\eta_i)\sigma_i^2} - (r+\kappa)}{\sigma_i^2} - r\rho(x\eta_i + z_i - \phi\theta_i + \chi\eta_i)a_{2i}. \tag{A.41}$$

Moreover, $Q_0 = 0$ implies

$$q_0 = -\rho\tilde{\psi} + \frac{\kappa}{r}\left(q_1\bar{s} + \sum_{i=1}^{N} q_{2i}\bar{e}_i\right) - 1 + \frac{\delta}{r} + \log(r). \tag{A.42}$$

If the investor decides to employ the manager, then she chooses a contract $(\tilde{\phi}, \tilde{\chi}, \tilde{\psi})$ and index investment x to maximize her time-zero value function $V(W_0, s_0, e_0)$. This objective is equivalent to $q_0 + q_1 s_0 + \sum_{i=1}^{N} q_{2i}e_{i0}$ because of (A.33), and the latter objective is equivalent to

$$-\rho\tilde{\psi} + q_1\hat{s} + \sum_{i=1}^{N} q_{2i}\hat{e}_i \tag{A.43}$$

because of (A.42). The maximization is subject to the manager's individual rationality (IR) constraint (2.9). To derive the time-zero value function $\bar{V}(\bar{W}_0, s_0, e_0)$ of an employed manager under a contract $(\tilde{\phi}, \tilde{\chi}, \tilde{\psi})$, we recall from (A.35) that the contract does not affect the manager's personal risk exposure ($\tilde{\phi} z_t - \tilde{\chi}\eta = \phi\theta - \chi\eta$). The contract also does not affect the manager's shirking action m_t, which is zero because $\tilde{\phi} \geq A$. Hence, the value function is as in Step 1, i.e., as under the equilibrium contract (ϕ, χ, ψ), with $(\bar{q}_1, \bar{q}_{21}, .., \bar{q}_{2N})$ given by (A.30) and (A.31), and $\bar{q}_0$ given by

$$\bar{q}_0 = \bar{\rho}\tilde{\psi} + \frac{\kappa}{r}\left(\bar{q}_1\bar{s} + \sum_{i=1}^{N} \bar{q}_{2i}\bar{e}_i\right) - 1 + \frac{\bar{\delta}}{r} + \log(r)$$

instead of (A.32). The time-zero value function $\bar{V}_u(\bar{W}_0, s_0, e_0)$ of an unemployed manager follows by the same argument. An unemployed manager can be viewed as an employed one with contract $(\tilde{\phi}, \tilde{\chi}, \tilde{\psi}) = (1, 0, 0)$ and shirking action $m_t = 0$. Hence, the value function is as in Step 1, with $(\bar{q}_1, \bar{q}_{21}, .., \bar{q}_{2N})$ given by (A.30) and (A.31), and $\bar{q}_0$ given by

$$\bar{q}_0 = \frac{\kappa}{r}\left(\bar{q}_1\bar{s} + \sum_{i=1}^{N} \bar{q}_{2i}\bar{e}_i\right) - 1 + \frac{\bar{\delta}}{r} + \log(r).$$

The manager's IR constraint (2.9) thus reduces to

$$\tilde{\psi} \geq 0.$$

The investor chooses $\tilde{\psi}$ that meets this constraint with equality: $\tilde{\psi} = 0$. Substituting into (A.43), we can write the investor's optimization problem as

$$\max_{\tilde{\phi}, \tilde{\chi}, x} \left(q_1\hat{s} + \sum_{i=1}^{N} q_{2i}\hat{e}_i\right),$$

subject to the constraint $\tilde{\phi} \geq A$. Because this problem is concave, the first-order conditions characterize an optimum. To confirm that $(\tilde{\phi}, \tilde{\chi}, x) = (\phi, \chi, 0)$ is an optimum, we thus need to check that the first-order conditions are satisfied for $(\phi, \chi, 0)$. Equation (A.35) implies that

$$\left| \frac{\partial z}{\partial \tilde{\phi}} \right|_{(\tilde{\phi}, \tilde{\chi}) = (\phi, \chi)} = -\frac{\theta}{\phi}, \tag{A.44}$$

$$\left| \frac{\partial z}{\partial \tilde{\chi}} \right|_{(\tilde{\phi}, \tilde{\chi}) = (\phi, \chi)} = \frac{\eta}{\phi}. \tag{A.45}$$

Using (A.40), (A.41), (A.44), and (A.45), we find

$$\frac{\partial}{\partial \tilde{\phi}} \left(q_1 \hat{s} + \sum_{i=1}^{N} q_{2i} \hat{e}_i \right) \Bigg|_{(\tilde{\phi}, \tilde{\chi}, x) = (\phi, \chi, 0)} = \frac{r\rho}{\phi} \left[\hat{s}\theta b \left(a_1 - \check{a}_1\right) + \sum_{i=1}^{N} \hat{e}_i \theta_i \left(a_{2i} - \check{a}_{2i}\right) \right] \equiv \frac{r\rho}{\phi} \Phi, \tag{A.46}$$

$$\frac{\partial}{\partial \tilde{\phi}} \left(q_1 \hat{s} + \sum_{i=1}^{N} q_{2i} \hat{e}_i \right) \Bigg|_{(\tilde{\phi}, \tilde{\chi}, x) = (\phi, \chi, 0)} = -\frac{r\rho}{\phi} \left[\hat{s}\eta b \left(a_1 - \check{a}_1\right) + \sum_{i=1}^{N} \hat{e}_i \eta_i \left(a_{2i} - \check{a}_{2i}\right) \right] \equiv -\frac{r\rho}{\phi} X, \tag{A.47}$$

$$\frac{\partial}{\partial x} \left(q_1 \hat{s} + \sum_{i=1}^{N} q_{2i} \hat{e}_i \right) \Bigg|_{(\tilde{\phi}, \tilde{\chi}, x) = (\phi, \chi, 0)} = -r\rho X. \tag{A.48}$$

The first-order conditions with respect to $\tilde{\chi}$ and x require that $X = 0$, which is equivalent to (4.3). The first-order condition with respect to $\tilde{\phi}$ requires that Φ is non-positive if $\phi = A$ and is equal to zero if $\phi > A$. To show that the values of (ϕ, χ) implied by these conditions are as in Theorems 3.1 and 4.1, we first characterize the solution χ of (4.3) and then determine the sign of Φ.

Given $\phi \in [0, 1]$, X is increasing in χ because a_1 is increasing in χ from (4.1), a_{2i} is increasing in χ from (4.2), $\check{a}_1$ is decreasing in χ from (4.4), and $\check{a}_{2i}$ is decreasing in χ from (4.5). It converges to ∞ when χ goes to

$$\bar{\chi} \equiv \min \left\{ \left[\frac{(r+\kappa)^2}{2r\bar{\rho}\sigma_s^2} + \phi\theta b \right] \frac{1}{\eta b}, \min_{i=1..,N} \left[\frac{(r+\kappa)^2}{2r\bar{\rho}\sigma_i^2} + \phi\theta_i \right] \frac{1}{\eta_i} \right\},$$

and to $-\infty$ when χ goes to

$$\underline{\chi} \equiv -\min \left\{ \left[\frac{(r+\kappa)^2}{2r\bar{\rho}\sigma_s^2} + (1-\phi)\theta b \right] \frac{1}{\eta b}, \min_{i=1..,N} \left[\frac{(r+\kappa)^2}{2r\bar{\rho}\sigma_i^2} + (1-\phi)\theta_i \right] \frac{1}{\eta_i} \right\}.$$

Therefore, (4.3) has a unique solution $\chi(\phi)$. Moreover, X is decreasing in ϕ because a_1 is decreasing in ϕ from (4.1), a_{2i} is decreasing in ϕ from (4.2), $\check{a}_1$ is increasing in ϕ from (4.4), and $\check{a}_{2i}$ is increasing in ϕ from (4.5). Therefore, $\chi(\phi)$ is increasing in ϕ. Since $X = 0$ for $(\phi, \chi) = (\frac{\rho}{\rho+\bar{\rho}}, 0)$, $\chi(\phi)$ has the same sign as $\phi - \frac{\rho}{\rho+\bar{\rho}}$.

We next substitute $\chi(\phi)$ into Φ, and show property $(\mathcal{P})$: Φ has the same sign as $\frac{\rho}{\rho+\bar{\rho}} - \phi$. Property $(\mathcal{P})$ will imply that the values of (ϕ, χ) are as in Theorems 3.1 and 4.1. Indeed, when $A \leq \frac{\rho}{\rho+\bar{\rho}}$, Φ cannot be negative: the first-order condition with respect to $\tilde{\phi}$ would then imply that $\phi = A \leq \frac{\rho}{\rho+\bar{\rho}}$, and property $(\mathcal{P})$ would imply that Φ has to be non-negative. Therefore, $\Phi = 0$, which implies $\phi = \frac{\rho}{\rho+\bar{\rho}}$ and $\chi(\phi) = 0$. When instead $A > \frac{\rho}{\rho+\bar{\rho}}$, ϕ cannot be strictly larger than A: the first-order condition with respect to $\tilde{\phi}$ would then imply that $\Phi = 0$, and property $(\mathcal{P})$ would imply that $\phi = \frac{\rho}{\rho+\bar{\rho}} < A$. Therefore, $\phi = A > \frac{\rho}{\rho+\bar{\rho}}$, which implies $\chi(\phi) > 0$.

Setting $\Delta \equiv a_1 - \check{a}_1$ and $\Delta_i \equiv a_{2i} - \check{a}_{2i}$, we can write Φ and (4.3) as

$$\Phi = r\hat{s}\theta b\Delta + r\sum_{i=1}^{N}\hat{e}_i\theta_i\Delta_i, \tag{A.49}$$

$$\hat{s}\eta b\Delta + \sum_{i=1}^{N}\hat{e}_i\eta_i\Delta_i = 0, \tag{A.50}$$

Eqs. (4.1) and (4.4) imply that Δ has the same sign as

$$[\rho - (\rho+\bar{\rho})\phi]\theta b + (\rho+\bar{\rho})\chi\eta b.$$

Likewise, (4.2) and (4.5) imply that Δ_i has the same sign as

$$[\rho - (\rho+\bar{\rho})\phi]\theta_i + (\rho+\bar{\rho})\chi\eta_i.$$

For $\phi = \frac{\rho}{\rho+\bar{\rho}}$, $\chi(\phi) = 0$, and hence $\Delta = \Delta_i = 0$ and $\Phi = 0$. For $\phi < \frac{\rho}{\rho+\bar{\rho}}$,

$$\begin{aligned}\Phi &= r\hat{s}\theta b\Delta + r\sum_{i=1}^{N}\hat{e}_i\theta_i\Delta_i \\ &> -\frac{(\rho+\bar{\rho})\chi(\phi)}{\rho-(\rho+\bar{\rho})\phi}\left(r\hat{s}\eta b\Delta + r\sum_{i=1}^{N}\hat{e}_i\eta_i\Delta_i\right) \\ &= 0,\end{aligned}$$

where the second step follows by distinguishing cases according to the signs of Δ and Δ_i, and the third step follows from (A.50). The inequality in the second step is strict. This is because θ is not proportional to η, and hence the components of the vector $[\rho - (\rho+\bar{\rho})\phi]\theta + (\rho+\bar{\rho})\chi\eta$ cannot all be zero. For $\phi > \frac{\rho}{\rho+\bar{\rho}}$, the same reasoning implies that $\Phi < 0$. Therefore, property $(\mathcal{P})$ holds. Note that property $(\mathcal{P})$ implies that when $A > \frac{\rho}{\rho+\bar{\rho}}$, the investor values the supply portfolio more than the manager: $\theta\check{S}_0 > \theta S_0$. This is because $\theta\check{S}_0 - \theta S_0$ has the same sign as $-\Phi$, which is positive when $\phi > \frac{\rho}{\rho+\bar{\rho}}$.

Setting $(\tilde{\phi}, \tilde{\chi}, \tilde{\psi}) = (\phi, \chi, \psi)$ in (A.40), (A.41), and (A.42), and using (A.35), we find that the coefficients

q_1, q_{2i}, and q_0 when the investor offers the equilibrium contract (ϕ, χ, ψ) are

$$q_1 = \frac{\sqrt{(r+\kappa)^2 + 2r\rho((1-\phi)\theta + \chi\eta)b\sigma_s^2} - (r+\kappa)}{\sigma_s^2} - r\rho((1-\phi)\theta + \chi\eta)ba_1, \tag{A.51}$$

$$q_{2i} = \frac{\sqrt{(r+\kappa)^2 + 2r\rho((1-\phi)\theta_i + \chi\eta_i)\sigma_i^2} - (r+\kappa)}{\sigma_i^2} - r\rho((1-\phi)\theta_i + \chi\eta_i)a_{2i}, \tag{A.52}$$

$$q_0 = -\rho\psi + \frac{\kappa}{r}\left(q_1\bar{s} + \sum_{i=1}^{N} q_{2i}\bar{e}_i\right) - 1 + \frac{\delta}{r} + \log(r). \tag{A.53}$$

The investor decides to employ the manager if (2.13) is satisfied. To derive the time-zero value function $V_u(W_0, s_0, e_0)$ of the investor when he does not employ the manager, we can follow the same steps as when she employs the manager, but with two modifications. First, we replace $x\eta + z - \phi\theta + \chi\eta$ by $x\eta$ since the investor's only exposure to stocks when he does not employ the manager is through the investment x in the index. Second, we replace $\tilde{\psi}$ by zero because the investor does not offer a contract. The value function is given by (A.33), with

$$q_{1u} \equiv \frac{\sqrt{(r+\kappa)^2 + 2r\rho x\eta b\sigma_s^2} - (r+\kappa)}{\sigma_s^2} - r\rho x\eta b a_1, \tag{A.54}$$

$$q_{2iu} = \frac{\sqrt{(r+\kappa)^2 + 2r\rho x\eta_i\sigma_i^2} - (r+\kappa)}{\sigma_i^2} - r\rho x\eta_i a_{2i}, \tag{A.55}$$

$$q_{0u} = \frac{\kappa}{r}\left(q_{1u}\bar{s} + \sum_{i=1}^{N} q_{2iu}\bar{e}_i\right) - 1 + \frac{\delta}{r} + \log(r), \tag{A.56}$$

instead of q_1, q_{2i}, and q_0, respectively. The investor's optimization problem is

$$\max_x \left[q_{1u}\hat{s} + \sum_{i=1}^{N} q_{2iu}\hat{e}_i\right].$$

The investor decides to employ the manager if

$$\max_{\tilde{\phi},\tilde{\chi},x}\left(q_1\hat{s} + \sum_{i=1}^{N} q_{2i}\hat{e}_i\right) > \max_x\left(q_{1u}\hat{s} + \sum_{i=1}^{N} q_{2iu}\hat{e}_i\right). \tag{A.57}$$

To show that (A.57) holds, we show that it holds when setting $(\tilde{\phi}, x) = (\phi, 0)$ in the left-hand side. Using (A.51), (A.52), (A.54), (A.55), and setting

$$f_1(y) \equiv \frac{\sqrt{(r+\kappa)^2 + 2r\rho y\sigma_s^2} - (r+\kappa)}{\sigma_s^2} - r\rho y a_1,$$

$$f_{2i}(y) \equiv \frac{\sqrt{(r+\kappa)^2 + 2r\rho y\sigma_i^2} - (r+\kappa)}{\sigma_i^2} - r\rho y a_{2i},$$

for a scalar y, we can write the latter condition as

$$\max_{\tilde{\chi}} \left[f_1 \left((1-\phi)\theta b + \left(\frac{\tilde{\chi}-\chi}{\phi} + \chi \right) \eta b \right) \hat{s} + \sum_{i=1}^{N} f_{2i} \left((1-\phi)\theta_i + \left(\frac{\tilde{\chi}-\chi}{\phi} + \chi \right) \eta_i \right) \hat{e}_i \right]$$
$$> \max_{x} \left[f_1(x\eta b)\hat{s} + \sum_{i=1}^{N} f_{2i}(x\eta_i)\hat{e}_i \right]. \tag{A.58}$$

The function $f_1(y)$ is concave and maximized for y given by

$$\frac{1}{\sqrt{(r+\kappa)^2 + 2r\rho y \sigma_s^2}} - a_1 = 0$$
$$\Leftrightarrow y = \frac{\bar{\rho}}{\rho + \bar{\rho}} \theta b,$$

where the second step follows from (3.2). Likewise, the function $f_2(y)$ is concave and maximized for y given by

$$\frac{1}{\sqrt{(r+\kappa)^2 + 2r\rho y \sigma_s^2}} - a_{2i} = 0$$
$$\Leftrightarrow y = \frac{\bar{\rho}}{\rho + \bar{\rho}} \theta_i,$$

where the second step follows from (3.3). For any given x, we can write

$$(1-\phi)\theta + \left(\frac{\tilde{\chi}-\chi}{\phi} + \chi \right) \eta = \lambda \frac{\bar{\rho}}{\rho + \bar{\rho}} \theta + (1-\lambda) x \eta$$

by defining $(\lambda, \tilde{\chi})$ by

$$\lambda \frac{\bar{\rho}}{\rho + \bar{\rho}} \equiv 1 - \phi,$$
$$(1-\lambda)x \equiv \frac{\tilde{\chi}-\chi}{\phi} + \chi.$$

Since $1 \geq \phi \geq \frac{\rho}{\rho+\bar{\rho}}$, $\lambda \in [0,1]$. Therefore, the arguments of $f_1(y)$ and $f_2(y)$ in the left-hand side of (A.58) are convex combinations of the corresponding arguments in the right-hand side and of the maximands of $f_1(y)$ and $f_2(y)$. Concavity of $f_1(y)$ and $f_2(y)$ then implies that the values of $f_1(y)$ and $f_2(y)$ in the left-hand side of (A.58) exceed the corresponding values in the right-hand side. Moreover, at least one of the inequalities is strict. This is because θ is not proportional to η and hence the arguments of $f_1(y)$ and $f_2(y)$ in the right-hand side of (A.58) cannot all coincide with the maximands of $f_1(y)$ and $f_2(y)$. Therefore, (A.58) holds. □

Proof of Proposition 4.1. When $A > \frac{\rho}{\rho+\bar{\rho}}$, ϕ is equal to A and hence is increasing in A. Since $\chi(\phi)$ is increasing in ϕ, χ is also increasing in A. □

Proof of Theorem 5.1. We proceed in two steps, as in the proof of Theorem 4.1.

Step 1. Same as for Theorem 4.1, except that we do not impose the restriction $\phi \geq A$, and we replace $\bar{G}_t$, (A.24), and (A.32) by

$$\bar{G}_t \equiv r\bar{\rho}\left[r\bar{W}_t + (\phi z_t - \chi\eta)\mu_t + \psi + (A-\phi)m_t - \frac{B}{2}m_t^2 - \bar{c}_t\right] + \kappa\left[\bar{q}_1(\bar{s}-s_t) + \sum_{i=1}^{N}\bar{q}_{2i}(\bar{e}_i - e_{it})\right],$$

$$m_t = \frac{A-\phi}{B}1_{\{\phi\leq A\}},$$

$$\bar{q}_0 = \bar{\rho}\left(\psi + \frac{(A-\phi)^2}{2B}1_{\{\phi\leq A\}}\right) + \frac{\kappa}{r}\left(\bar{q}_1\bar{s} + \sum_{i=1}^{N}\bar{q}_{2i}\bar{e}_i\right) - 1 + \frac{\bar{\delta}}{r} + \log(r), \tag{A.59}$$

respectively, where $1_{\mathcal{S}}$ is the indicator function of the set $\mathcal{S}$.

Step 2. Same as for Theorem 4.1, with the following changes. We replace G_t, (A.42), and (A.53) by

$$G_t \equiv r\rho\left[rW_t + (x\eta + z - \phi\theta + \chi\eta)\mu_t - \tilde{\psi} - (1-\tilde{\phi})m_t - c_t\right] + \kappa\left[q_1(\bar{s}-s_t) + \sum_{i=1}^{N}q_{2i}(\bar{e}_i - e_{it})\right],$$

$$q_0 = -\rho\left(\tilde{\psi} + \frac{(1-\tilde{\phi})(A-\tilde{\phi})}{B}1_{\{\tilde{\phi}\leq A\}}\right) + \frac{\kappa}{r}\left(q_1\bar{s} + \sum_{i=1}^{N}q_{2i}\bar{e}_i\right) - 1 + \frac{\delta}{r} + \log(r), \tag{A.60}$$

$$q_0 = -\rho\left(\psi + \frac{(1-\phi)(A-\phi)}{B}1_{\{\phi\leq A\}}\right) + \frac{\kappa}{r}\left(q_1\bar{s} + \sum_{i=1}^{N}q_{2i}\bar{e}_i\right) - 1 + \frac{\delta}{r} + \log(r), \tag{A.61}$$

respectively. The manager's individual rationality constraint becomes

$$\tilde{\psi} + \frac{(A-\tilde{\phi})^2}{2B}1_{\{\tilde{\phi}\leq A\}} \geq 0,$$

and the investor chooses $\tilde{\psi} = -\frac{(A-\tilde{\phi})^2}{2B}1_{\{\tilde{\phi}\leq A\}}$. The investor's optimization problem becomes

$$\max_{\tilde{\phi},\tilde{\chi},x}\left(-\rho\frac{(A-\tilde{\phi})(2-A-\tilde{\phi})}{2B}1_{\{\phi\leq A\}} + q_1\hat{s} + \sum_{i=1}^{N}q_{2i}\hat{e}_i\right),$$

without the constraint $\tilde{\phi} \geq A$. The first-order conditions with respect to $\tilde{\chi}$ and x are equivalent to (4.3). Using (A.46), we find that the first-order condition with respect to $\tilde{\phi}$ is

$$\frac{1-\phi}{B} + \frac{r}{\phi}\Phi = 0 \quad \text{if} \quad \phi < A, \tag{A.62}$$

$$\frac{1-\phi}{B} + \frac{r}{\phi}\Phi \geq 0 \quad \text{and} \quad \Phi \leq 0 \quad \text{if} \quad \phi = A, \tag{A.63}$$

$$\Phi = 0 \quad \text{if} \quad \phi > A. \tag{A.64}$$

Equations (A.62)-(A.64) rule out that Φ is positive. When $A \leq \frac{\rho}{\rho+\bar{\rho}}$, Φ cannot be negative: (A.64) would then imply that $\phi \leq A \leq \frac{\rho}{\rho+\bar{\rho}}$, and property $(\mathcal{P})$ would imply that Φ has to be non-negative. Therefore, $\Phi = 0$, which implies $\phi = \frac{\rho}{\rho+\bar{\rho}}$ and $\chi(\phi) = 0$. When instead $A > \frac{\rho}{\rho+\bar{\rho}}$, ϕ cannot be strictly larger than A: (A.64) would then imply that $\Phi = 0$, and property $(\mathcal{P})$ would imply that $\phi = \frac{\rho}{\rho+\bar{\rho}} < A$. Moreover, ϕ cannot be smaller than $\frac{\rho}{\rho+\bar{\rho}}$: (A.62) would then imply that $\Phi < 0$, and property $(\mathcal{P})$ would imply that $\phi > \frac{\rho}{\rho+\bar{\rho}}$. Therefore, $\phi \in (\frac{\rho}{\rho+\bar{\rho}}, A]$, which implies $\chi(\phi) > 0$. Equations (A.62) and (A.63) yield (5.1) and (5.2). The condition that the investor decides to employ the manager becomes

$$\max_{\tilde{\phi},\tilde{\chi},x} \left(-\rho \frac{(A-\tilde{\phi})(2-A-\tilde{\phi})}{2B} 1_{\{\tilde{\phi}\leq A\}} + q_1\hat{s} + \sum_{i=1}^{N} q_{2i}\hat{e}_i \right) > \max_{x} \left(q_{1u}\hat{s} + \sum_{i=1}^{N} q_{2iu}\hat{e}_i \right) \tag{A.65}$$

instead of (A.57). Equation (A.65) is satisfied for $B = 0$, as shown in Theorem 4.1. It is also satisfied for $B = \infty$ because the left-hand side of (A.65) becomes identical to that of (A.57). By continuity, it is satisfied for B close to zero and to infinity. □

Proof of Proposition 5.1. The social planner maximizes the investor's value function $V(W_0, s_0, e_0)$ at time zero, subject to the manager's incentive compatibility (IC) and individual rationality (IR) constraints. The IC constraint is that the manager's choices of (z_t, m_t) are optimal given the contract. The IR constraint is that the manager's value function $\bar{V}(\bar{W}_0, s_0, e_0)$ exceeds the value function $\bar{V}_u(\bar{W}_0, s_0, e_0)$ from being unemployed. From Steps 1 and 2 of the proof of Theorem 5.1, the social planner's problem reduces to maximizing

$$r\rho W_0 + q_0 + q_1 s_0 + \sum_{i=1}^{N} q_{2i} e_{i0}$$

subject to

$$r\bar{\rho}\bar{W}_0 + \bar{q}_0 + \bar{q}_1 s_0 + \sum_{i=1}^{N} \bar{q}_{2i} e_{i0} \geq \bar{q}_{0u} + \bar{q}_{1u} s_0 + \sum_{i=1}^{N} \bar{q}_{2iu} e_{i0},$$

where $(q_0, q_1, q_{21}, .., q_{2N})$ are given by (A.51), (A.52), and (A.61), $(\bar{q}_0, \bar{q}_1, \bar{q}_{21}, .., \bar{q}_{2N})$ are given by (A.30), (A.31), and (A.59), and $(\bar{q}_{0u}, \bar{q}_{1u}, \bar{q}_{21u}, .., \bar{q}_{2Nu})$ are the counterparts of $(\bar{q}_0, \bar{q}_1, \bar{q}_{21}, .., \bar{q}_{2N})$ for an unemployed manager. The values of $(\bar{q}_{0u}, \bar{q}_{1u}, \bar{q}_{21u}, .., \bar{q}_{2Nu})$ computed in Theorem 5.1 depend on (ϕ, χ). (In particular, $(\bar{q}_{1u}, \bar{q}_{21u}, .., \bar{q}_{2Nu}) = (\bar{q}_1, \bar{q}_{21}, .., \bar{q}_{2N})$.) This is because the manager computes his value function when unemployed under the equilibrium prices, which depend on the contract. The values of $(\bar{q}_{0u}, \bar{q}_{1u}, \bar{q}_{21u}, .., \bar{q}_{2Nu})$ computed by the social planner, however, do not depend on (ϕ, χ). This is because the social planner internalizes that when the manager is unemployed, prices change and do not depend on the contract.

Using (A.59) and (A.61), we can write the social planner's problem as

$$\max_{\phi,\chi}\left[r(W_0+\bar{W}_0)-\frac{(A-\phi)(2-A-\phi)}{2B}1_{\{\phi\le A\}}+\frac{1}{\rho}\left(q_1\hat{s}+\sum_{i=1}^{N}q_{2i}\hat{e}_i\right)+\frac{1}{\bar{\rho}}\left(\bar{q}_1\hat{s}+\sum_{i=1}^{N}\bar{q}_{2i}\hat{e}_i\right)\right]. \quad (A.66)$$

Since the investor and the manager are endowed collectively with the portfolio θ at time zero, the problem (A.66) is equivalent to

$$\max_{\phi,\chi}\left[r\theta S_0-\frac{(A-\phi)(2-A-\phi)}{2B}1_{\{\phi\le A\}}+\frac{1}{\rho}\left(q_1\hat{s}+\sum_{i=1}^{N}q_{2i}\hat{e}_i\right)+\frac{1}{\bar{\rho}}\left(\bar{q}_1\hat{s}+\sum_{i=1}^{N}\bar{q}_{2i}\hat{e}_i\right)\right]. \quad (A.67)$$

Using (A.51), (A.52),

$$\bar{q}_1=\frac{\sqrt{(r+\kappa)^2+2r\bar{\rho}(\phi\theta-\chi\eta)b\sigma_s^2}-(r+\kappa)}{\sigma_s^2}-r\bar{\rho}(\phi\theta-\chi\eta)ba_1, \quad (A.68)$$

which follows from (4.1) and (A.30),

$$\bar{q}_{2i}=\frac{\sqrt{(r+\kappa)^2+2r\bar{\rho}(\phi\theta_i-\chi\eta_i)\sigma_i^2}-(r+\kappa)}{\sigma_i^2}-r\bar{\rho}(\phi\theta_i-\chi\eta_i)a_{2i}, \quad (A.69)$$

which follows from (4.2) and (A.31), and

$$S_{i0}=b_ia_1\hat{s}+a_{2i}\hat{e}_i, \quad (A.70)$$

which follows from (2.14), (3.1), and $a_{1i}=b_ia_1$, we can write (A.67) as

$$\begin{aligned}\max_{\phi,\chi}\Bigg[&-\frac{(A-\phi)(2-A-\phi)}{2B}1_{\{\phi\le A\}}\\ &+\frac{1}{\rho}\left(\frac{\sqrt{(r+\kappa)^2+2r\rho((1-\phi)\theta+\chi\eta)b\sigma_s^2}}{\sigma_s^2}\hat{s}+\sum_{i=1}^{N}\frac{\sqrt{(r+\kappa)^2+2r\rho((1-\phi)\theta_i+\chi\eta_i)\sigma_i^2}}{\sigma_i^2}\hat{e}_i\right)\\ &+\frac{1}{\bar{\rho}}\left(\frac{\sqrt{(r+\kappa)^2+2r\bar{\rho}(\phi\theta-\chi\eta)b\sigma_s^2}}{\sigma_s^2}\hat{s}+\sum_{i=1}^{N}\frac{\sqrt{(r+\kappa)^2+2r\bar{\rho}(\phi\theta_i-\chi\eta_i)\sigma_i^2}}{\sigma_i^2}\hat{e}_i\right)\Bigg]. \quad (A.71)\end{aligned}$$

The first-order condition with respect to χ is (4.3). The first-order condition with respect to ϕ is

$$\frac{1-\phi}{B}+r\Phi=0 \quad \text{if} \quad \phi<A, \quad (A.72)$$

$$\frac{1-\phi}{B}+r\Phi\ge 0 \quad \text{and} \quad \Phi\le 0 \quad \text{if} \quad \phi=A, \quad (A.73)$$

$$\Phi=0 \quad \text{if} \quad \phi>A. \quad (A.74)$$

Since (A.71) is strictly concave, the first-order conditions characterize a unique maximum (ϕ^*, χ^*). Using the same arguments as in the proof of Theorem 5.1, we find that (ϕ^*, χ^*, ψ^*) are as in the theorem.

We finally show that $\phi^* \geq \phi$ and $\chi^* \geq \chi$ for the privately optimal (ϕ, χ), with the inequalities being strict when $\phi < A$. Since χ^* solves (4.3), it is equal to $\chi(\phi^*)$ for the function $\chi(\phi)$ defined in the proof of Theorem 4.1. Since $\chi(\phi)$ is increasing in ϕ, it suffices to show the inequalities for ϕ^*. When $\phi < A$, ϕ is determined by (A.62). Using (A.62), we can write the left-hand side of (A.72) as

$$\frac{1-\phi}{B} - \frac{\phi(1-\phi)}{B} = \frac{(1-\phi)^2}{B} > 0.$$

Since the derivative of the social planner's objective with respect to ϕ is positive at the privately optimal ϕ, $\phi^* > \phi$. When $\phi = A$, ϕ satisfies (A.63). Using (A.63), we find that the left-hand side of (A.72) is larger than $\frac{(1-\phi)^2}{B} > 0$. Since the derivative of the social planner's objective with respect to ϕ is positive at the privately optimal $\phi = A$, $\phi^* = \phi = A$. □

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