The delegated Lucas-tree*

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Very preliminary. Please do not circulate.
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Abstract

We introduce delegation into a standard Lucas exchange economy, where in equilibrium some investors trade on their own account, but others decide to delegate trading in financial assets to funds. Flow-performance incentive functions describe how much capital fund clients provide to funds at each date as a function of past performance. Our model implies that with increasing elasticity flow-performance incentives the average fund outperforms the market in recessions and underperforms in expansions. When the share of capital that is delegated is low, all funds follow the same strategy. However, when the equilibrium share of delegated capital is high funds with identical incentives utilize heterogeneous trading strategies, trade among themselves, and fund returns are dispersed in the cross-section. Delegation affects the Sharpe ratio through two channels: discount rate and capital flow. The two work in opposite directions leading in general to an inverse U-shape relation between the share of capital that is delegated and the Sharpe ratio. Impact of delegation and its equilibrium share on other asset price properties such as price-dividend ratios is considered as well. In contrast to increasing elasticity incentives, with constant elasticity flow-performance relations there is no trade in equilibrium even if some funds have a convex relationship and others concave.

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1 Introduction

While it is undisputed that financial intermediaries play a central role in financial markets, our understanding of the impact of these intermediaries on asset prices is fairly limited, as was highlighted by the recent financial crisis.[1] Yet, during the last few decades, there has been a gradual but profound change in the way money is invested in financial markets. While in 1980 the percentage of equity in the US held by open end funds was less than 5% by 2007 it went up to more than 30%. At the same time, while almost 50% of US equities were held directly in 1980, by 2007 this proportion decreased to around 20% (see French (2008)). What are the equilibrium implications of this shift? In particular, how do trading strategies, and prices change?

To analyze the link between the incentives of financial institutions and asset prices, we introduce delegation into a Lucas exchange economy, by introducing financial intermediaries (funds) that make dynamic investment decisions on behalf of their clients. Our main assumption is that clients invest a larger share of their wealth with the fund, if the manager performs better relative to the market. Instead of deriving this flow-performance relationship from first principles, we allow for a rich set of exogenous specifications focusing on the empirically plausible case when the flow-performance relationship has increasing elasticity and, consequently, it is globally non-concave.

Our equilibrium model implies that with increasing elasticity flow-performance incentives the average fund outperforms the market in recessions and underperforms in expansions. When the equilibrium share of capital that is delegated to funds is low, all funds follow the same strategy. However, when the equilibrium share of delegated capital is high funds with identical incentives utilize heterogeneous trading strategies, trade among themselves, and fund returns are dispersed in the cross-section. In contrast to increasing elasticity incentives, with constant elasticity flow-performance relations there is no trade in equilibrium even if some funds have a convex relationship and others concave. In general there is an inverse U-shape relation between the share of capital that is delegated and the Sharpe ratio of the market portfolio. We also identify conditions under which price-dividend ratios are procyclical, monotone increasing, or monotone decreasing as a function of the realized dividend growth.

We study an exchange economy where the endowment process is represented by a Lucas tree paying a stochastic dividend each period. There are two financial assets: a stock with is a claim on the endowment process, and a riskless bond which is in zero net supply. The

[1] See, for example, the presidential address of Allen (2001) for an elaborate discussion on the importance of the role of financial intermediaries.

economy consists of two type of agents, both with log utility: investors and fund managers. Investors are the owners of the capital. They can decide whether to trade directly in financial markets or instead delegate those decisions to fund managers. Trading directly imposes on investors a utility cost. This utility cost represents the cost of acquiring the knowledge to understand how capital markets work, as well as the utility cost imposed by making regular time consuming investment decisions. Investors can avoid this cost by becoming a client and delegating the determination of their portfolio to a fund. However, when they delegate they need to pay fund managers a fee each period that is determined by the fund and depends on how much assets under management a client has with the fund.

Investors, arrive and die according to independent Poission processes with constant intensity, while managers live forever. Newborn investors decide for life whether to be clients of managers or to trade directly in financial markets. Clients' allocate capital to funds to manage each period depending on funds' past relative performance, where the relationship between last period's return compared to the market and new capital flow is described by each manager's *incentive function*. We interpret the incentive function as a short-cut for an un-modeled learning process by clients on managers'. Its empirical counterpart is the flow-performance relationship. We are agnostic as to whether the learning process is rational or not. Furthermore, instead of deriving the incentive function from first principles, we allow for a rich set of possible exogenous specifications including examples with both convex and concave intervals. However, consistent with the finding of Chevallier and Ellison (1997), we focus on those implying that flows respond to performance with increasing elasticity. These functions are globally non-concave.

To obtain an analytically tractable model we combine a few simplifying assumptions. First, investors and managers have log utility. Second, the arrival and death rate of investors is constant and i.i.d across period. Third, managers are forced to consume their fees, and can not trade on their own account. Fourth, incentive functions are assumed to be piece-wise constant elasticity functions. That is, within each segment the incentive function has constant elasticity. However, the elasticity can freely vary across segments allowing for flat, linear, concave or convex segments within the same incentive function.

The combination of log utility and piece-wise constant elasticity functions is the key methodological contribution that allows us to derive analytical formulas for the trading pattern and asset prices under various incentive functions, even when managers with different incentive functions co-exist. This combination results in a locally concave, but globally non-concave portfolio problem for managers. The first property keeps the framework tractable, while the second property ensures that we do not lose the general insight connected to convex incentives. While in most of the paper we focus on incentive functions with at most two segments and on an ex ante identical group of fund managers, in the last section we show that our framework is suitable for analyzing the interaction of two groups of fund managers with differing and more general incentive functions.

We show that increasing elasticity fund incentive functions lead the average fund to choose a smaller than one market-beta: implying that consistent with evidence in Moskowitz (2000), Kosowski (2006), Lynch and Wachter (2007), Kacperczyk, Van Nieuweburgh and Veldkamp (2010), and Glode (2010) the average fund overperforms the market in recessions and underperforms in expansions. To understand the intuition behind this result, consider the case when financial markets are only populated by direct traders and the first fund manager enters. She can decide whether to take a sufficiently contrarian position to overperform and get high capital flows in the recession, or to take a sufficiently leveraged position to get high capital flows in the expansion. Because the probability of an expansion is larger than the probability of a recession, her optimal leveraged position implies less volatility in terms of her relative return across states than her optimal contrarian position. Because of her globally non-concave utility, she values this volatility and picks the contrarian position.

While the average fund always overperforms in recessions, the cross sectional distribution of fund returns depends on the equilibrium share of delegation in the economy. For small share of delegation, all funds choose the same portfolio. However, as the share of delegation increases, there is a threshold above which fund managers follow heterogeneous strategies, even though funds are identical-ex ante and all have the same incentive function. In particular, above the threshold as the share of delegation increases a group of decreasing size follows the same "contrarian strategy" of smaller than one market beta, while a group with increasing size follows a leveraged strategy by borrowing up and investing more than 100% of their assets under management in the stock. The size of the second group remains below 50%. Interestingly, the increase in the share of delegation decreases the extent of both underperformance and overperformance.

Accounting for the fact that over the last three decades the share of delegation has increased considerable (Allen (2001) and French (2008)), this result is broadly consistent with evidence in Wermers (2000) that shows that over time the mutual fund industry has moved toward becoming more fully invested in common stocks, as opposed to bonds and cash. It is also consistent with the increased use of leveraged strategies in the money management industry prior to the 2008 financial crisis. Our model suggests that this could be a consequence of the interaction of the shape of the flow-performance relationship and the larger share of total delegation. The idea is that when most capital is delegated, the only way managers can achieve large relative returns compared to the market is if they follow heterogeneous strategies. In equilibrium, a group of managers leverage up, so gets the large capital inflow in the high state, while some follow strategies which generate extra capital flow in the low state. Thus, there is gains from trade. The size of these two groups are determined in equilibrium in a way that prices make each manager indifferent between the two strategies.

We show that typically the Sharpe-ratio follows an inverted U-pattern as the share of delegation increases. This is the outcome of the change in relative strength of two effects. First, an extra unit of return is appreciated more when it increases capital flows through the incentive function for the marginal agent. We call this the capital-flow effect. Second, an extra unit of return is appreciated more when the wealth of the marginal agent is lower. This is the standard wealth effect. The first effect is increasing in the share of delegation in the region when each fund follows the same contrarian strategy and it is constant in the region with heterogeneous strategies. The second effect changes little in the first region and typically decreases in the second region. Thus, the capital flow effect dominates for low levels of delegation and the wealth effect dominates for large levels of delegation implying the result.

We note that despite the i.i.d. growth rate, the price-dividend ratio varies both with the state of the economy and with the share of delegation. The reason is that as direct traders and managers follow different strategies, the aggregate share of consumption varies along both dimensions. We characterize parameters which imply a procyclical price-dividend ratio and monotonically increasing or decreasing price-dividend ratios.

Our paper is related to several branches of the literature. First, papers that study the effects of delegated portfolio management on asset prices (e.g. Shleifer-Vishny 1997, Vayanos, 2003, Cuoco-Kaniel, 2007, Dasgupta-Prat, 2006, 2008, Guerrieri-Kondor, 2010, Vayanos-Woolley, 2008). Most of these papers use models that are fairly different than the Lucas-tree exchange economy, making it hard to compare the results to standard consumption based asset pricing models. Apart from our work, to our knowledge the only other exception is He and Krishnamurthy (2008) who also studies the effect of delegation in a standard Lucas-economy. The main difference is that in He and Krishnamurthy (2008) managers are not directly interested in managing larger funds. In their model, fund flows effect the decision of managers only indirectly, through the effect of flows on equilibrium prices. Therefore, they cannot asses the effect of convexities in the flow-performance relationship on asset prices through managers' incentives. In contrast, this is our main focus.

Second, the literature on consumption based asset pricing with heterogeneous risk aversion (e.g. Dumas (1989), Wang (1996), Chan-Kogan (2002), Bhamra-Uppal, (2007), Longstaff-Wang (2008)). Models in these papers have similar structure to ours but very different implications. The closest to our work is Chan and Kogan (2002) who assumes that agents value their consumption relative to others. In flavor, this is similar to our assumption that fund flows are a function of relative performance. However, the two structures have very different implications. For example, unlike in our structure, in Chen and Kogan trade among identical traders is never required in equilibrium. This demonstrates well that heterogeneous incentives and heterogeneous risk aversion have different asset pricing implications. Also, just like in any other paper with heterogeneous risk-aversion, in Chan and Kogan less risk averse agents always lend to more risk averse agents. In our paper, the average mutual fund lends to direct traders even if mutual fund's incentives are globally non-concave.

additional related literature to be added here....

The structure of the paper is as follows. In the next section we present the general model. We discuss the general set up, our equilibrium concept and the main properties of the equilibrium. In Section 3, we present and discuss the derived empirical implications. Finally, we conclude.

2 The general model

In this section we introduce our framework, define our equilibrium concept and present sufficient conditions for the existence of such an equilibrium and its basic properties.

2.1 The Economy

We consider a discrete-time, infinite-horizon exchange economy with complete financial markets and a single perishable consumption good. There is only one source of uncertainty and participants trade in financial securities to share risk.

The economy is populated by investors and fund managers. The endowment process is owned by investors. However, trading directly in financial markets imposes a utility cost on investors. Instead of trading directly, investors also have the option to become a client of a fund. When delegating, clients don't suffer the utility cost they would bear if they traded directly, but give up the ability to determine their stock to bond mix.

In equilibrium the investor population includes two subsets: direct traders who trade in financial markets on their own behalf, and clients who use the services of fund managers. Clients invest a proportion of the endowment through each manager depending on the past relative performance of the manager and described by their *incentive function*. The empirical counterpart of the incentive function is the flow-performance relationship.[2] Instead of deriv-

[2]There is a large empirical literature exploring the relationship between past performance and future fund flows. This literature shows that there is a positive relationship between performance and flows for most type of financial intermediaries but the shape of this relationship is effected by the type of the fund. See Chevalier and Ellison (1997), Sirri and Tufano (1998), and Chen et al. (2003)) for evidence on mutual

ing the incentive function from first principles, we take it exogenously in the spirit of Shleifer and Vishny (1997). However, we allow for a rich set of possible exogenous specifications.[3]

Securities. The aggregate endowment process is described by the binomial tree

$$\delta_{t+1} = y_t \delta_t$$

where the growth process y_t has two i.i.d. states: $s_t = H, L$. The dividend growth is either high y_H or low y_L, with $y_H > y_L$. The probability of the high and the low states are $p > \frac{1}{2}$ and $(1-p) < \frac{1}{2}$ respectively.

Investment opportunities are represented by a one period riskless bond and a risky stock. The riskless bond is in zero net supply. The stock is a claim to the dividend stream δ_t and is in unit supply. The price of the stock and the interest rate on the bond are q_t and R_t respectively.

The return on a portfolio with portfolio weights of α in the stock and $1-\alpha$ in the risk free bond is denoted by

$$\rho_{t+1}(\alpha) \equiv \alpha \left(\frac{q_{t+1} + \delta_{t+1}}{q_t} - R_t \right) + R_t. \tag{1}$$

Fund Managers. We assume a continuum of managers with a total mass of one. Managers (M) derive utility from inter-temporal consumption, and have *log* utility. In period, t, each manager determines the fraction ψ_t^M of beginning of period assets under management w_t^M she will receive as a fee. We assume the manager must consume her fee $\psi_t^M w_t^M$.[4] She then invests the remaining $(1-\psi_t^M)w_t^M$ in a portfolio with α_t^M share in the stock and $\left(1-\alpha_t^M\right)$ share in the bond.

As we will show below, the amount of capital investors delegate to a given manager at a beginning of a period are proportional to assets under management at the end of the previous period. This proportion has two components. The first Γ_t is a state dependent scaling factor that is endogenously determined in equilibrium and that the manager takes as given. It impacts all funds similarly, and depends on the overall allocation to funds. The second $g(\cdot)$ is a relative performance incentive function that depends on a fund's return relative to the

funds, Bares, Gibson, and Gyger (2002), Brown, Goetzman, and Ibbotson (1999), Edwards and Cagalyan (2001), and Kat and Menexe (2002) for evidence on hedge funds and Kaplan and Schoar (2004) for evidence on private equity partnerships.

[3] In extensions of the model we allow for two groups of fund managers with differing incentive functions to coexist.

[4] The assumption that managers cannot invest their fees is a major simplification allowing us not to keep track the private wealth of fund managers. Note also that on one hand, we are allowing ψ_t to be conditional on any variable in the managers' information set in t. That is, we do not constrain our attention to proportional fees ex ante. On the other hand, our assumptions imply that fees are proportional in equilibrium, managers effectively maximize capital under management and fees do not play any role in the portfolio decision.

market portfolio

$$\upsilon_{t+1}^{M} \equiv \frac{\rho_{t+1}\left(\alpha_t^M\right)}{\left(\frac{q_{t+1}+\delta_{t+1}}{q_t}\right)} \tag{2}$$

and can be a nonlinear and non-concave function of this relative return. In addition, we allow the incentive function $g(\cdot)$ to depend non linearly on the fees ψ_t^M charged by the fund.

Managers solve

$$\max_{\{\psi_t^M,\alpha_t^M\}} E\left[\sum_t \beta^t \ln \psi_t^M w_t^M\right] \tag{3}$$

$$\begin{aligned} s.t.\;\; w_{t+1}^M &= g\left(\upsilon_{t+1}^M, \psi_t^M\right) w_{t+1,-}^M \\ w_{t+1,-}^M &\equiv \rho_{t+1}\left(\alpha_t^M\right)\left(1-\psi_t^M\right) w_t^M \end{aligned}$$

where $w_{t+1,-}^M$ are the the assets under management at the end of the previous period, after the time $t+1$ return has been realized, but before investors decide how much to allocate to the fund to manage between $t+1$ and $t+2$.

Investors. The endowment process is owned by investors. The mass of investors is normalized to 1, where at the beginning of each period $(1-\lambda)$ fraction of investors die and the same fraction is born. We assume that the aggregate capital of newborn investors is the same as the aggregate capital of those who died. Similar to managers, investors derive utility from inter-temporal consumption, and have *log* utility.

When an investor is born he or she needs to decide whether they will be a client (C) delegating their portfolio decisions throughout their life to a fund manager (M) or whether they will be a direct trader (D), trading on their own behalf. We assume that being a direct trader imposes a utility cost f, as one needs acquire the knowledge to understand how capital markets work and one needs to make regular time consuming investment decisions. The decision of whether to be a client or a direct trader is made once at birth and is irreversible.

When investors are born they are neither clients nor direct traders. First they optimally decide how to divide their wealth at the initial date between consumption and investment. Then they consider whether to be a client or a direct trader. If they decide to become a client they are randomly matched with a particular manager.

After the initial date, the amount of assets a client delegates to her manager each period depends on the managers performance relative to the market. Specifically, each period the amount of capital a client delegates is based on the incentive function $g(\cdot)$. At each date t

a client delegates a fraction $g(\cdot)$ of their wealth to their manager to delegate and consumes the rest.

Let $\beta^I = \lambda\beta$ be the effective discount factor that accounts for the fact that at each period each investor dies with probability $1-\lambda$.

Denoting by ψ_t^I the optimal fraction of initial wealth a time t newborn investor consumes, if the investor decides to become a direct trader her life-time expected utility is given by

$$\ln w_t\psi_t^I + V^D = \ln w_t\psi_t^I + \max_{\{\psi_u^D,\alpha_u^D\}} E\left[\sum_{u>t}(\beta^I)^{u-t}\ln\left(\psi_u^D w_u\right)\right] - f \tag{4}$$

$$s.t.\ \ w_{u+1} \quad = \quad \rho_{u+1}\left(\alpha_u^D\right)\left(1-\psi_u^D\right)w_u$$

A clients life time utility is given by

$$\ln w_t\psi_t^I + V^C = \ln w_t\psi_t^I + E\left[\sum_{u>t}(\beta^I)^{u-t}\ln\left(\left(1-g\left(\upsilon_u,\psi_u^M\right)\right)w_u\right)\right] \tag{5}$$

$$s.t.\ \ w_{u+1} \quad = \quad \rho_{t+1}\left(\alpha_t^M\right)\left(1-\psi_t^M\right)\left(g\left(\upsilon_t,\psi_u^M\right)w_u\right)$$

A time t newborn investor expected lifetime utility is given by

$$V^I = \max_{\chi\in\{0,1\},\psi_t^I} \ln w_t\psi_t^I + \chi E[V^C] + (1-\chi)E[V^D] \tag{6}$$

Relative Performance Incentive Functions. The incentive function $g(\cdot)$ describes how existing clients respond to the performance of a given manager. We assume it belongs to the following class:

$$g\left(\upsilon,\psi^M\right) = \begin{cases} \left(1-\psi^M\right)^{j-1} Z_B\,\upsilon^{n_B-1} & if \quad \upsilon<\kappa \\ \left(1-\psi^M\right)^{j-1} Z_A\,\upsilon^{n_A-1} & if \quad \upsilon\geq\kappa \end{cases} \tag{7}$$

where the "kink" $\kappa \geq 1$, $n_A \geq n_B > 1$, $Z_A > 0$ and in addition we assume that the g is continuous by imposing the restriction

$$Z_A = Z_B k^{n_B-n_A}.$$

This specification allows for varying shapes. For example, $n_B = 1$ implies that relative performance does not matter in the region where relative performance is below the "kink". Similarly, $n_A = 2$, $1 < n_A < 2$, $n_A > 2$ imply a linear, concave, and convex segment respectively. We can think of the relative size of the n coefficients across the two segments of the incentive function as the "log-log convexity" of the incentive function in the following sense. In each segment m

$$\frac{\partial \ln g\left(\upsilon, \psi^M\right)}{\partial \ln \upsilon} = n,$$

therefore, the restriction $n_A \geq n_B$ implies that $\ln g\left(\cdot\right)$ is convex in the log of the relative return. [5] In this sense, a specification with increasing elasticity can be interpreted as convexity in incentives.

2.2 A recursive formulation

We can reformulate the different optimization problems in recursive form. We will show that in solving the portfolio choice problems of different agents the two primitive state variables are the relative investment by managers compared to all investment in the stock market in the previous period,

$$\Omega_{t-1},$$

and the subsequent dividend shock s_t.

Direct trader:

$$\begin{aligned} V^D\left(w_t, s_t, \Omega_{t-1}\right) &= \max_{\psi_t^D, \alpha} \ln \psi_t^D w_t^D + \beta^I E V^D\left(w_{t+1}, s_{t+1}, \Omega_t\right) \\ w_{t+1}^D &= \rho\left(\alpha_t^D\right)\left(1 - \psi_t^D\right) w_t^D \end{aligned} \tag{8}$$

Client:

[5] Our structure can be read as a piece-wise linear approximation of the empirical finding of Chevalier-Ellison (1997) on the flow-performance relationship of mutual funds. With our notation, they find that a non-parametric estimation of

$$\frac{w_{t+1} - w_t}{w_t} - \rho_{t+1}\left(\alpha_t^M\right)\left(1 - \psi_t^M\right) = \tilde{g}\left(\rho_t\left(\alpha^M\right)\left(1 - \psi_t^M\right) - \frac{q_t + \delta_t}{q_{t-1}}\right)$$

results in a convex $\tilde{g}\left(\cdot\right)$.Taking logs of both sides of our incentive function (7) and approximating arithmetic returns with log returns and using that

$$\frac{w_{t+1} - w_t}{w_t} - \rho_{t+1}\left(\alpha_t^M\right)\left(1 - \psi_t^M\right) \approx \ln \frac{w_{t+1}}{w_t \rho_{t+1}\left(\alpha_t^M\right)\left(1 - \psi_t^M\right)}$$

shows that for our assumption to be consistent with Chevalier-Ellison (1997), we need $\ln g\left(\right)$ to be convex. This is exactly what we achieve with our formalization.

Each client stays with the same manager, after the initial random matching. To distinguish individual managers we use lower case m to denote a manager when we need to emphasize that a client is linked to a specific manager and M otherwise.

Clients observe the past relative return of that particular manager, υ_t^m.Even though each client sticks with the same manager, the manager's strategy might change. Thus, the next υ_{t+1}^m and the next strategy of the manager α_t^m is a random variable from the point of view of the client.

$$\begin{aligned} V^C\left(w_t, \upsilon_t^m, s_t, \Omega_{t-1}\right) &= \ln w_t\left(1 - g\left(\upsilon_t^m, \psi_t^M\right)\right) + \qquad (9) \\ &\quad + \lambda\beta E V^C\left(\rho_{t+1}\left(\alpha_t^M\right)\left(1-\psi_t^M\right) w_t, \upsilon_{t+1}^m, s_{t+1}, \Omega_t\right) \qquad (10) \end{aligned}$$

Newborn investor

For an investor that is born at the beginning of period t entering with wealth w_t

$$\begin{aligned} V^I\left(w_t, s_t, \Omega_{t-1}\right) = \max_{\chi\in\{0,1\},\psi_t^I} \ln w_t \psi_t^I &+ \chi\beta^I E V^C\left(\rho_{t+1}\left(\alpha_t^M\right)\left(1-\psi_t^M\right) w_t\left(1-\psi_t^I\right), s_{t+1}, \Omega_t\right) \\ &+ \left(1-\chi\right)\beta^I\left(E V^D\left(\rho_{t+1}\left(\alpha_t^D\right) w_t\left(1-\psi_t^I\right), s_{t+1}, \Omega_t\right) - f\right). \end{aligned}$$

Manager

$$\begin{aligned} V^M\left(w_t, s_t, \Omega_{t-1}\right) &= \max_{\{\psi, \alpha_t^M\}} \ln \psi_t^M w_t^M + \beta E V^M\left(w_{t+1}, s_{t+1}, \Omega_t\right) \qquad (11) \\ s.t.\;\; w_{t+1}^M &= \Gamma_t g\left(\upsilon_{t+1}^M, \psi_t^M\right) w_{t+1,-}^M \end{aligned}$$

where $w_{t+1,-}^M$, υ_{t+1}^M, and $g\left(\upsilon_{t+1}^M, \psi_t^M\right)$ were defined above.

Since we allow mixed actions for managers, that is, they might decide to submit a demand α' with probability μ and α'' with probability $(1-\mu)$. In this case we can rewrite their problem as

$$V^M\left(w_t, s_t, \Omega_{t-1}\right) = \max_{\{\psi, \alpha', \alpha'', \mu\}} \ln \psi_t^M w_t^M + \mu\beta E V^M\left(w_{t+1}', s_{t+1}, \Omega_t\right) + (1-\mu)\beta E V^M\left(w_{t+1}'', s_{t+1}, \Omega_t\right)$$

where, for example,

$$w_{t+1}' = \Gamma_t g\left(\frac{\rho_{t+1}\left(\alpha'\right)}{\frac{q_{t+1}+\delta_{t+1}}{q_t}}, \psi_t^M\right)\rho_{t+1}\left(\alpha'\right)\left(1-\psi_t^M\right) w_t^M$$

We conjecture and later verify that the value functions of different types take the form

$$
\begin{aligned}
V^C\left(w_t, \upsilon_t^m, s_t, \Omega_{t-1}\right) &= \frac{1}{1-\beta^I}\ln w_t + \Lambda^C\left(\upsilon_t^m, s_t, \Omega_{t-1}\right) && (12)\\
V^D\left(w_t, s_t, \Omega_{t-1}\right) &= \frac{1}{1-\beta^I}\ln w_t + \Lambda^D\left(s_t, \Omega_{t-1}\right) && (13)\\
V^M\left(w_t, s_t, \Omega_{t-1}\right) &= \frac{1}{1-\beta}\ln w_t + \Lambda^M\left(s_t, \Omega_{t-1}\right). && (14)
\end{aligned}
$$

2.3 Interior Equilibrium

In this section, we define the equilibrium, derive sufficient conditions for its existence and provide some general characterizations. In the following sections we use these characterization to analyze the equilibrium in more detail.

For the rest of the paper, we use the time script only when it is necessary to avoid confusion. Otherwise, variables with no subscript refer to period t and we denote variables referring to $t+1$ period by prime.

In our proposed equilibrium, we have potentially two state variables: the relative investment by fund managers compared to all investment in the stock market Ω_t and the dividend shock s_t. However, we restrict attention to stationary equilibria in which the aggregate relative investment by fund managers compared to all other investment is constant.

Definition 1 *An Ω^* **equilibrium** is a price process q_s for the stock and R_s for the bond, a relative investment by fund managers compared to all investment Ω^*, consumption and strategy profiles for newborn investors, direct investors, and managers such that*

1. *given the equilibrium prices*
   - *the initial consumption choice of newborn investors ψ_t^I and the decision on whether to become a direct trader or a client are optimal for each newborn investor,*
   - *consumption choices ψ_t^m and trading strategies α_t^m are optimal for each manager,*
   - *consumption choices ψ_t^D and trading strategies α_t^D are optimal for direct traders,*
   - *each clients consumes a positive amount in each period :*
     $\delta_t + q_t - g\left(\upsilon_t, \psi_1^m\right) w_{t,-} > 0$
2. *prices q_s, and R_s clear both good and asset markets,*
3. *the law of motion $\sigma_{s'}$ for the relative investment by fund managers compared to all investment satisfies*
   $$\Omega^* = \sigma_{s'}\left(\Omega^*\right).$$

2.3.1 Characterization and Existence

Before the formal characterization of the equilibrium it is useful to understand the intuition behind a manager's portfolio problem. The interaction of log utility and piecewise constant-elasticity incentive functions imply that the problem of the manager is locally-concave but globally non-concave in the portfolio choice, α. The dashed line on Figure 1 demonstrates this by depicting the expected utility of a manager for various α in a particular case when all other traders hold the market. The possible choices are divided to three groups. *Contrarian portfolios* has a smaller than unity exposure to the market risk, thus they overperform the market in the low state. Furthermore, this overperformance in the low state is sufficient high to generate the extra capital flows implied by the high elasticity segment of the incentive function. *Moderate portfolios* are sufficiently close to the market portfolio thus they do not overperform the market sufficiently to generate the extra capital flows in any of the states. *Aggressive portfolios* have a larger than unity exposure to the market, thus, these overperform in the high state sufficiently to generate extra capital flows. Within each of these segments there is a single optimal portfolio. Consequently, managers effectively compare three possible strategies. We will say that a manager who chooses the optimal contrarian/moderate/aggressive portfolio follows the *Contrarian*, *Moderate*, *Aggressive strategy*, respectively. The relative ranking of these three strategies depend on equilibrium prices. Finding an equilibrium amounts to finding the combination of equilibrium prices and optimal strategies which clear the market. The solid line depicts the expected utility for equilibrium prices for one particular set of parameters. As we see in this particular case, equilibrium prices imply that managers are indifferent between the optimal contrarian and the optimal aggressive strategies. In fact, this is an asymmetric equilibrium where a certain fraction of managers follow one of the strategies and the other fraction follows the other one. We will see below that for some parameters the equilibrium has this property.

Now, we are ready to formally match parameter combinations to the corresponding Ω^* equilibrium. For this, it is useful to define the *individual shape-adjusted probability of a high state*

$$\xi_{lh} \equiv \frac{pn_h}{pn_h + (1-p)\, n_l}, \tag{15}$$

ξ_{lh} is simply the probability of a high state adjusted to the relative elasticity of the incentive function in the high state, for a strategy in which the performance relative to the market is above (below) the kink in the high state if $h = A$ ($h = B$) and above (below) the kink in the low state if $l = A$ ($l = B$). This implies that a Contrarian, Moderate, Aggressive strategy correspond to $\xi_{lh} = \xi_{BA}, p, \xi_{AB}$, respectively.

Note that a direct trader is isomorphic to a fund manager with an incentive function in which $Z_A = Z_B = n_A = n_B = 1$, implying $\xi^D_{lh} \equiv p$.

In an asymmetric equilibrium managers follow heterogeneous strategies.[6] Correspondingly we denote by

$$\mu_{lh}$$

the fraction of managers that follow a strategy in which the performance relative to the market is above (below) the kink in the high state if $h = A$ ($h = B$) and above (below) the kink in the low state if $l = A$ ($l = B$).

The *aggregate shape-adjusted probability of a high state* is

$$\tilde{\xi}(\Omega) \equiv \Omega \sum_{lh} \mu_{lh}(\Omega)\, \xi_{lh} + (1-\Omega)\, p. \tag{16}$$

$\tilde{\xi}(\Omega)$ is a weighted average of the individual shape adjusted probabilities. When the incentive function elasticity is constant $n_h = n_l$ and $\tilde{\xi}(\Omega) = \xi_{lh} = p$.

As we will show later, at each point in time fund managers will follow at most two different strategies. We can therefore write the aggregate shape-adjusted probability of a high state as

$$\tilde{\xi}(\Omega) \equiv \Omega\left(\mu_1(\Omega)\, \xi_1 + (1-\mu_1(\Omega))\xi_2\right) + (1-\Omega)\, p. \tag{17}$$

Proposition 1 *For any set of other parameters there is a $\hat{Z}$ a $\hat{\lambda}$ and an interval $\left[\underline{f}, \bar{f}\right]$ that if $Z_B < \hat{Z}, \lambda \leq \hat{\lambda}$ and $f \in \left[\underline{f}, \bar{f}\right]$ then an interior equilibrium exists. Furthermore, for any $\Omega \in [0,1]$ there is a corresponding $f(\Omega) \in \left[\underline{f}, \bar{f}\right]$ that with that choice in the interior equilibrium $\Omega^* = \Omega$.*

We identify different equilibria types depending on equilibrium fund managers strategy types. We show below that the following are the only viable combinations in equilibrium:

Cont-Agg: some managers follow a Contrarian strategy and some an Aggressive strategy,

Cont-Mod: some managers follow a Contrarian strategy and some a Moderate strategy,

Cont: all managers follow a Contrarian strategy.

Mod: all managers follow a moderate strategy.

Proposition 2 *Suppose that $Z_B < \hat{Z}, \lambda \leq \hat{\lambda}$. There are critical values $\hat{p}(k, Z, n_B, n_A) \in \left(\frac{1}{2}, 1\right)$ and $\hat{k}_{high}(n_A, n_B), \hat{k}_{low}(n_A, n_B)$ that*

[6] Searching for asymmetric equilibria is necessary as as we will show there are very simple and intuitive cases where there is no symmetric equilibrium.

1. *if $k > \hat{k}_{high}$, there is a unique interior equilibrium and it is a Moderate (Mod) equilibrium ,*

2. *if $\hat{k}_{low} < k < \hat{k}_{high}$, there is a unique interior equilibrium and its type depends on p as follows:*

$p \in \left(\frac{1}{2}, \hat{p}\right)$	$p \in (\hat{p}, 1)$
Mod	$Cont - Mod$

3. *if $k < \hat{k}_{low}$,there is a unique interior equilibrium and its type depends on p as follows:*

$p \in \left(\frac{1}{2}, \hat{p}\right)$	$p \in (\hat{p}, 1)$
$Cont - Agg$	$Cont - Mod$

In the sequel instead of tracking the stock price q, and the stock price next period $q_{s'}$, it is more convenient to track the price-dividend ratio

$$\pi = \frac{q}{\delta}.$$

and the price dividend ratio next period

$$\pi_{s'} = \frac{q_{s'}}{\delta'}.$$

The following proposition characterizes equilibrium consumption choices and trading strategies of individual investors, direct traders and managers.

Proposition 3 *In an Ω^* equilibrium,*

1. *the optimal consumption rules of investors, direct traders and managers are given by*

$$\psi_t^I = \psi_t^D = \left(1 - \beta^I\right), \quad \psi_t^M = \left(1 - \beta^M\right), \tag{18}$$

where $\beta^M = \frac{\beta j}{1-\beta(1-j)}$.

2. *direct traders optimal trading strategy is*

$$\alpha_t^D = \frac{1-p}{1 - \frac{y_H(1+\pi_H)}{R\pi}} + \frac{p}{1 - \frac{y_L(1+\pi_L)}{R\pi}} \tag{19}$$

3. *fund managers optimal trading strategies*

- *Contrarian:*
$$\alpha_t^m = \frac{1-\xi_{AB}}{1-\frac{y_H(1+\pi_H)}{R\pi}} + \frac{\xi_{AB}}{1-\frac{y_L(1+\pi_L)}{R\pi}} \tag{20}$$

- *Aggresive:*
$$\alpha_t^m = \frac{1-\xi_{BA}}{1-\frac{y_H(1+\pi_H)}{R\pi}} + \frac{\xi_{BA}}{1-\frac{y_L(1+\pi_L)}{R\pi}} \tag{21}$$

- *Moderate* :
$$\alpha_t^m = \frac{1-\xi_{BB}}{1-\frac{y_H(1+\pi_H)}{R\pi}} + \frac{\xi_{BB}}{1-\frac{y_L(1+\pi_L)}{R\pi}} = \frac{1-p}{1-\frac{y_H(1+\pi_H)}{R\pi}} + \frac{p}{1-\frac{y_L(1+\pi_L)}{R\pi}} \tag{22}$$

Note that the portfolio of a direct trader is the same as the portfolio of a manager following a moderate strategy. This implies that in a moderate equilibrium all managers and direct traders follow the same strategy of holding the market, $\alpha_t^M = \alpha^D = 1$. Because of this, sometimes we refer to the moderate equilibrium as the indexed equilibrium.

In contrast, in $Cont - Agg$ and $Cont - Mod$ equilibria managers potentially follow two heterogeneous strategies. For both of these equilibria types we refer to the contrarian strategy as the primary strategy and will denote it typically by 1 and the other strategy, either aggressive or index, as the secondary strategy and denote it typically by 2.

The set of funds following a particular strategy varies with the aggregate wealth share managed by funds. I particular, there is a threshold $\hat{\Omega} \in (0,1)$ that

$$\mu_1(\Omega^*) = \mu_{Cont}(\Omega^*) = \left\{ \begin{array}{ccc} 1 & if & \Omega^* \leq \hat{\Omega} \\ \hat{\mu}_1(\Omega^*) & if & \Omega^* > \hat{\Omega} \end{array} \right\}. \tag{32}$$

where $\hat{\mu}(\Omega^*)$ is a monotonically decreasing function. That is, all funds follow a contrarian strategy as long as the wealth share managed by funds, Ω^*, is relatively small, but an increasing measure of them follows the secondary strategy as the share increases.

Given that managers has to be indifferent between the two equilibrium strategies for any $\Omega^* > \hat{\Omega}$, for any $\Omega^* > \hat{\Omega}$, $\tilde{\xi}(\Omega^*)$ is the constant $\bar{\xi}$ determined by a corresponding indifference condition specified in the next proposition. Thus, $\hat{\mu}(\Omega^*)$ can be backed out from (16) as:

$$\hat{\mu}(\Omega^*) = \frac{\bar{\xi} - \Omega^*\xi_1 - (1-\Omega^*)p}{\Omega^*(\xi_1 - \xi_2)} = \frac{\bar{\xi} - \xi_2}{\xi_1 - \xi_2} + \left(\frac{1}{\Omega^*} - 1\right)\frac{\bar{\xi} - p}{\xi_1 - \xi_2}.$$

For any, $\Omega^* < \hat{\Omega}$, by (16),

$$\tilde{\xi}(\Omega^*) = \Omega^*\xi_1 + (1-\Omega^*)p.$$

Finally, $\tilde{\xi}\left(\hat{\Omega}\right)=\bar{\xi}$ gives

$$\hat{\Omega}\equiv\frac{\bar{\xi}-p}{\xi_1-p}.$$

The following proposition summarizes these results,

Proposition 4 *In either a $Cont-Agg$ or a $Cont-Mod$ equilibrium denote the contrarian strategy shape-adjusted probability by ξ_1, and the other strategy's shape-adjusted probability by ξ_2, and let $\hat{\Omega}\equiv\frac{\bar{\xi}-p}{\xi_1-p}$, then the mass of fund managers who follow the contrarian strategy is given by*

$$\mu_1\left(\Omega^*\right)=\mu_{Cont}\left(\Omega^*\right)=\left\{\begin{array}{cc} 1 & if \quad \Omega^*\leq\hat{\Omega} \\ \frac{\bar{\xi}-\xi_2}{\xi_1-\xi_2}+\left(\frac{1}{\Omega^*}-1\right)\frac{\bar{\xi}-p}{\xi_1-\xi_2} & if \quad \Omega^*>\hat{\Omega}\end{array}\right\}. \tag{33}$$

where $\bar{\xi}$ is the solution of

$$p\ln k^{n_B-n_A}\frac{\left(\frac{\xi_{BA}}{\xi}\right)^{n_A}}{\left(\frac{\xi_{AB}}{\xi}\right)^{n_B}}=(1-p)\ln k^{n_B-n_A}\frac{\left(\frac{1-\xi_{AB}}{1-\xi}\right)^{n_A}}{\left(\frac{1-\xi_{BA}}{1-\xi}\right)^{n_B}} \tag{34}$$

and

$$p\ln\frac{\left(\frac{p}{\xi}\right)^{n_A}}{\left(\frac{\xi_{AB}}{\xi}\right)^{n_B}}=(1-p)\ln k^{n_B-n_A}\frac{\left(\frac{1-\xi_{AB}}{1-\xi}\right)^{n_A}}{\left(\frac{1-p}{1-\xi}\right)^{n_B}} \tag{35}$$

in a $Cont-Agg$ and in a $Cont-Mod$ equilibrium respectively.

The aggregate equilibrium quantities are summarized in the following proposition

Proposition 5 *In an Ω^* equilibrium*

1. *the price dividend ratio is*

$$\begin{aligned}\pi_H &= \frac{\beta^I\left(1-\lambda\Upsilon_H\right)+\lambda\bar{g}_H-\left(1-\beta^M\right)\Gamma_H\bar{g}_H}{1-\beta^I\left(1-\lambda\Upsilon_H\right)-\lambda\bar{g}_H+\left(1-\beta^M\right)\Gamma_H\bar{g}_H}\\ \pi_L &= \frac{\beta^I\left(1-\lambda\Upsilon_L\right)+\lambda\bar{g}_L-\left(1-\beta^M\right)\Gamma_L\bar{g}_L}{1-\beta^I\left(1-\lambda\Upsilon_L\right)-\lambda\bar{g}_L+\left(1-\beta^M\right)\Gamma_L\bar{g}_L}\end{aligned}$$

where

$$\begin{aligned}\Upsilon_H &= 1-\frac{\left(1-\Omega^*\right)p}{\tilde{\xi}\left(\Omega^*\right)},\\ \Upsilon_L &= 1-\frac{\left(1-\Omega^*\right)\left(1-p\right)}{1-\tilde{\xi}\left(\Omega^*\right)},\end{aligned}$$

and

$$\bar{g}_s = \Omega^* \left(\mu_1 \left(\Omega^*\right) g_{1s} + \left(1 - \mu_1 \left(\Omega^*\right)\right) g_{2s}\right),$$

and g_{1s} *and* g_{2s} *are fund managers' respective relative performance incentive functions in state* s.

2. *The aggregate flow scaling factors* Γ_H, Γ_L *are given by*

$$\begin{aligned} \Gamma_H &= \Omega^* \frac{\beta^I \left(1 - \lambda \Upsilon_H\right) + \bar{g}_H \lambda}{\bar{g}_H \left(\beta^M \left(1 - \Omega^*\right) + \Omega^*\right)} \\ \Gamma_L &= \Omega^* \frac{\beta^I \left(1 - \lambda \Upsilon_L\right) + \bar{g}_L \lambda}{\bar{g}_L \left(\beta^M \left(1 - \Omega^*\right) + \Omega^*\right)} \end{aligned}$$

3. *the interest rate is*

$$R = \frac{\theta}{\pi}$$

where θ *solves*

$$\tilde{\xi}\left(\Omega^*\right) \frac{1}{y_H \left(1 + \pi_H \left(\Omega^*\right)\right)} + \left(1 - \tilde{\xi}\left(\Omega^*\right)\right) \frac{1}{y_L \left(1 + \pi_L \left(\Omega^*\right)\right)} = \frac{1}{\theta\left(\Omega^*\right)}, \tag{36}$$

3 Implications

In this section, we discuss the equilibrium and analyze its implications. We focus on the interaction of non-concave incentives and the increased level of delegation in financial markets. We contrast our findings with existing empirical work and present additional testable implications. We start the discussion with briefly presenting two benchmarks. Then we discuss our results connected to the distribution of relative returns, then proceed to the Sharpe-ratio and the price dividend ratio. Finally, we discuss implications to the size of the credit market and the dispersion of portfolios.

3.1 Benchmark cases: no delegation and constant-elasticity incentive functions

Our natural benchmark is when the share of delegation is zero, $\Omega^* = 0$, so the financial market is populated by direct traders. For example, this is the case when the utility cost of direct trading is zero. It is simple to check that our model reduces to the standard Lucas-model where all traders hold the market and the price-dividend ratio and riskfree rate are constant:

$$\pi_H = \pi_L = \frac{\beta^I}{1-\beta^I},$$

$$R = \frac{1}{\beta^I}\frac{y_H y_L}{p y_L + (1-p) y_H},$$

and the Sharpe ratio is constant as well and given by

$$S = \frac{p^{\frac{1}{2}}(1-p)^{\frac{1}{2}}\|y_H - y_L\|}{p y_L + (1-p)\, y_H}.$$

Let us consider also a second benchmark. Consider a case when the utility cost is in the range which implies a positive share of delegation, but managers' incentive function has a constant elasticity. That is, $n_A = n_B$. In the following proposition we characterize the main properties of the equilibrium in this case.

Proposition 6 *If the incentive function has constant elasticity, $n_A = n_B$, and $Z_B \leq 1$ and if either $\beta^M Z_B > \beta$ or $\lambda < \beta^M \frac{\beta + Z_B}{\beta(1+\beta^M)}$ holds then for any Ω^* there is an f implying that an Ω^*-equilibrium exists. Furthermore, in any implied Ω^* equilibrium*

1. *both direct traders and managers hold the market,*
2. *the Sharpe ratio is equal to the Sharpe-ratio with no delegation*
3. *the price dividend ratio is constant across states.*

The proposition illustrates that delegation has little effect on the equilibrium if the incentive functions have constant elasticity. For any Ω^*, each agent holds the market, the price-dividend ratio is constant across states and the Sharpe ratio is unaffected by delegation[7] To see this, note that with only a single segment both the individual and the aggregate shape adjusted probability is p, $\xi^i_{lh} = \tilde{\xi}(\Omega^*) = p$. Thus, (22) implies the same strategy for all managers. By market clearing, this strategy must be that each manager holds the market. Therefore, relative returns are always 1.This implies that we can create examples, where a group of traders with convex incentives (fund managers) trade with a group with standard incentives (direct traders) and still they do not take positions against each other. This illustrates well that with log utility, not the convexity of the incentive functions matter but its log-log convexity. That is, whether the elasticity of flows with respect to returns is increasing

[7]Note, however, that the level of delegation, Ω^*, might still effect the level of the price dividend ratio. This is so, because managers and direct traders consume a different constant fraction of their wealth.

or not. [8] This is because of the interaction of log utility and constant elasticity incentive functions. The marginal utility from a dollar linearly increases in elasticity parameter n. Given that n is the same across states, the marginal rate of substitution is not affected by n. Thus, the marginal rate of substitution is the same for both agents. Hence, there are no gains from trade.

3.2 Relative returns and exposure to market risk

Propositions (2)-(4) describe the trading strategies in equilibrium. We can see immediately that when reaching the increasing elasticity segment of the incentive function would be too painful, because the kink, κ, is too large, then in equilibrium both direct traders and fund managers hold the market. The resulting indexed equilibrium has the same properties as our second benchmark: when fund managers have a constant elasticity incentive function. Because in this equilibrium delegation has little effect, in the rest of the paper, we restrict our attention to the segment of parameter space when the equilibrium is not of this type.

Observe that when share of delegation is low, $\Omega^* < \hat{\Omega}$, all fund managers follow a contrarian strategy in all other equilibria. We show in the appendix the the relative return of a manager following the contrarian strategy is simply

$$\upsilon_H = \frac{\xi_{AB}}{\tilde{\xi}(\Omega^*)} = \frac{\frac{n_B}{pn_B+(1-p)n_A}}{\frac{n_B}{pn_B+(1-p)n_A}\Omega^* + (1-\Omega^*)} < 1 \tag{37}$$

$$\upsilon_L = \frac{1-\xi_{AB}}{1-\tilde{\xi}(\Omega^*)} = \frac{\frac{n_A}{pn_B+(1-p)n_A}}{\frac{n_A}{pn_B+(1-p)n_A}\Omega^* + (1-\Omega^*)} > \kappa > 1 \tag{38}$$

in the high state and low state, respectively. That is, they buy bonds, and their exposure of the market is smaller than one, so in recessions they overperform the market and their clients reward them with extra capital flows, while in booms they underperform the market and clients withdraw capital. For latter reference, if they were to choose the aggressive strategy or the moderate strategy, their relative return would be

$$\upsilon_H = \frac{\xi_{BA}}{\tilde{\xi}(\Omega^*)} = \frac{\frac{pn_A}{pn_A+(1-p)n_B}}{\tilde{\xi}(\Omega^*)},\ \upsilon_L = \frac{1-\xi_{BA}}{1-\tilde{\xi}(\Omega^*)} = \frac{\frac{(1-p)n_B}{pn_A+(1-p)n_B}}{1-\tilde{\xi}(\Omega^*)} \tag{39}$$

in the former case and

$$\upsilon_H = \frac{p}{\tilde{\xi}(\Omega^*)} = \frac{p}{\tilde{\xi}(\Omega^*)},\ \upsilon_L = \frac{1-p}{1-\tilde{\xi}(\Omega^*)} = \frac{1-p}{1-\tilde{\xi}(\Omega^*)} \tag{40}$$

[8] The importance of considering the interaction of utility function and the incentives was also pointed out by Ross (2006).

in the latter case.

To see the intuition behind the equilibrium choice of managers, consider the case of the first fund manager who enters a market which is populated only by direct traders, $\Omega^* \approx 0$. The manager has three choices. She can hold a moderate portfolio, but then she will never outperform the market sufficiently to get the extra capital flows in any of the states. Or she can either follow an aggressive strategy which leads to overperformance in the high state or a contrarian strategy leading to overperformance in the low state. How do these two compare? Managers choose the contrarian strategy, because the optimal contrarian strategy implies relative returns with larger variance but the same mean of 1. Because of their globally non-concave incentives they value this larger variance. To see this, observe that the variance is proportional to the absolute difference of relative returns across states and

$$\left|\frac{\xi_{AB}}{\tilde{\xi}(0)} - \frac{1-\xi_{AB}}{1-\tilde{\xi}(0)}\right| = \frac{n_A - n_B}{pn_B + (1-p)\,n_A} > \frac{n_A - n_B}{pn_A + (1-p)\,n_B} = \left|\frac{\xi_{BA}}{\tilde{\xi}(0)} - \frac{1-\xi_{BA}}{1-\tilde{\xi}(0)}\right|$$

for any $p > \frac{1}{2}$. This case is also demonstrated by the dashed curve on Figure 1. As it is apparent, the optimal contrarian strategy implies a more extreme portfolio and a larger expected utility than the optimal aggressive portfolio. Technically, the increasing elasticity flow-performance relationship implies a more volatile contrarian strategy because it changes the way managers weight the two states in a particular way. While for the direct trader the relative weight of the high state to the low state corresponds to the relative probabilities, $\frac{p}{1-p}$, for a manager it will be determined by the elasticity weighted probabilities,

$$\frac{\xi_{AB}}{1-\xi_{AB}} = \frac{p}{1-p}\frac{n_B}{n_A},\ \frac{\xi_{BA}}{1-\xi_{BA}} = \frac{p}{1-p}\frac{n_A}{n_B}.$$

if the manager follows a contrarian or an aggressive strategy, respectively. Because $p > \frac{1}{2}$, the first distortion is larger in the sense that $|p - \xi_{AB}| > |p - \xi_{BA}|$. Thus, the resulting optimal aggressive portfolio is less extreme then the optimal contrarian portfolio.

Note that our argument is the classic idea of risk-shifting, but with a twist. Risk shifting implies that agents with globally non-concave incentives might prefer to take on larger variance, that is, they gamble. However, in our case this not necessarily implies a leveraged position. Because managers have non-concave incentives in relative instead of absolute return, in this particular case, the contrarian strategy is the larger gamble.[9]

As the share of delegation, Ω^*, increases, prices increasingly work against fund managers and they find the contrarian strategy less attractive. Because $\tilde{\xi}(\Omega^*)$ decreases with Ω^*,

[9] A similar point regarding funds increasing tracking error volatility in the presence of benchmarks has been made in Cuoco and Kaniel (2010), and Basak, Pavlova and Shapiro (2007).

the relative return of managers increases in the high state and decreases in the low state. That is, both the overperformance in the low state and the underperformance in the high state is less severe. At some threshold $\hat{\Omega}$, managers become indifferent between the optimal contrarian strategy and, depending on the parameter values, either the optimal moderate strategy or the optimal aggressive strategy. For market clearing as the market share of fund managers grows above this threshold, a decreasing set of managers has to choose the contrarian strategy. Thus, ex ante identical managers speculate against each other to an increasing extent. The idea is simple. As they start to dominate the market, the only way they can overperform the market at least in some states, if they speculate against each other.

As $\tilde{\xi}(\Omega^*) = \bar{\xi}$ is constant when the share of delegation is larger than $\hat{\Omega}$, relative returns of individual fund managers are constant. However, as the relative return of both the moderate and the aggressive strategy is higher in the high state and lower in the low state than that of the contrarian strategy, the average fund manager is increasingly exposed to the market. Since, by definition, at $\Omega^* = 1$ the average manager holds the market, the average manager must have a portfolio which overperforms in the low state and underperforms in the high state for any $\Omega^* < 1$.

Before we contrast our findings with empirical facts, we summarize them in the following two corollaries.

Corollary 1 *For any $\Omega^* < 1$, the representative fund's exposure to the market is always smaller than 1, so it overperforms the market in recessions and underperforms in booms. Furthermore, the exposure increases with the share of delegated capital in the sense that the extent of both the overperformance and the underperformance gets smaller. (When $\Omega^* = 1$, the representative fund holds the market.)*

Corollary 2 *When $\Omega^* < \hat{\Omega}$ funds follow homogeneous strategies. Thus, each fund's exposure to the market is always smaller than 1, so it overperforms the market in recessions and underperforms in booms. For $\Omega^* > \hat{\Omega}$, funds follow heterogeneous strategies. One group has smaller than 1 exposure to the market and one has larger than 1 exposure to the market. The relative size of the second group is increasing with Ω^*. However, the exposure of each group remains constant.*

Consistently with Corollary 1, evidence shows that mutual funds perform better in recessions than in booms (e.g., Moskowitz (2000), and Glode (2008), Kacperczyk, Van Nieuweburgh and Veldkamp (2010), Kosowski (2006), Lynch and Wachter (2007)). For example, Moskowitz (2000) notes that the absolute performance of the average fund manager is 6%

higher in recessions in booms.[10] Also, Wermers (2000) and French (2008) show that although funds keep less then 100 percent of their capital under management in equities, this proportion has increased over time.

Regarding Corollary 2, there is some anecdotal evidence that the heterogeneity in strategies in the money management industry are indeed increasing over time. In general there has been increased use of leveraged strategies For example, both the capital of leveraged mutual funds and leveraged ETFs increased rapidly in the boom years before the 2008-2009 (see Lo and Patel (2007)). Note also that our framework cannot distinguish between two possible interpretations of aggressive strategies. First, we can interpret it as a strategy which borrows bonds and invests more than 100% into equities. Second, we can interpret it as a fund which picks equities which has higher than 1 market-beta. Because of the lack of systematic evidence on this, we think of this result as a testable prediction for the future: As the market share of delegation goes up, we should expect larger cross-sectional dispersion in funds' returns.

3.3 Sharpe-ratio and the price dividend ratio

We find that typically, as the share of delegation increases, the Sharpe-ratio follows an inverse U pattern. It increase as long as $\Omega^* < \hat{\Omega}$, and decreases for $\Omega^* > \hat{\Omega}$. As $\Omega^* = 0$ corresponds to the standard Lucas model, this also implies that the presence of delegation increases the Sharpe-ratio at least as long as $\Omega^* < \hat{\Omega}$. Figure .... illustrates this for a wide range of parameters. Although we find this pattern robust to all the parameter variations we experimented with, analytically, we prove only a weaker statement.

Proposition 7 *If Z_B is sufficiently small then the Sharpe ratio*

1. *is increasing when the share of delegated investment is close to zero,*
2. *and decreasing when the share of delegated investment is close to 1.*

To help to understand this result, the following lemma decomposes the Sharpe-ratio in an intuitive way.

Lemma 1 *The state price of the low state relative to the high state is*

$$\frac{y_H}{y_L} X(\Omega^*),$$

[10]Note however, that the last three of these papers finds overperformance in terms of Jensen-alpha as opposed to in terms of larger exposure to the market risk. Given that in our model funds cannot generate alpha, these results are not translate to our proposition one-to-one.

and the Sharpe-ratio is

$$S(\Omega^*) = \frac{p^{\frac{1}{2}}(1-p)^{\frac{1}{2}} \|y_H X(\Omega^*) - y_L\|}{p y_L + (1-p) y_H X(\Omega^*)}. \tag{41}$$

where $X(\Omega^*)$ is the product of the capital-flow effect and the wealth effect defined as follows.

$$X(\Omega^*) \equiv \underbrace{\frac{\frac{\left(1-\tilde{\xi}(\Omega^*)\right)}{1-p}}{\frac{\tilde{\xi}(\Omega^*)}{p}}}_{\textit{capital-flow effect}} \underbrace{\frac{1+\pi_H}{1+\pi_L}}_{\textit{wealth effect}}. \tag{42}$$

The capital flow effect is 1 at $\Omega^* = 0$ and larger than 1 for any $\Omega^* > 0$. Furthermore, it is monotonically increasing in Ω^* for any $\Omega^* < \hat{\Omega}$ and constant for any $\Omega^* > \hat{\Omega}$.

The Lemma shows that both the deviation of relative state prices and the Sharpe ratio from the standard model is driven by the term $X(\Omega^*)$. In the standard model, the relative state prices is $\frac{y_H}{y_L}$, that is, $X \equiv 1$. The term $X(\Omega^*)$ is determined by the relative size of two effects: the capital flow effect and the wealth effect. The capital-flow effect is similar to the classic cash-flow effect in asset pricing. Depending on the shape of the incentive function, a dollar return in a given state might attract more or less future capital flows. The first term in (38) shows the relative capital-flow generating ability of a dollar in the low state versus the high state for the average manager. As we discussed above, for any non-trivial equilibrium the representative manager has a market exposure small than 1. Thus, her incentive function is relatively more sensitive in the low state. This implies that she finds an additional unit of return more valuable in that state which pushes the relative state price and the Sharpe ratio up. The comparatives statics of the capital-flow effect in Ω^* are direct consequence of our observations on the aggregate shape-adjusted probability $\tilde{\xi}(\Omega^*)$. We can conclude that the capital-flow effect always pushes the Sharpe-ratio up compared to the standard model, and it is non-decreasing in the share of delegation.

The second effect is the wealth effect. When the price dividend ratio is higher in the low state, this increases the relative wealth of the marginal agent in the low state, which pushes both the relative state price and the Sharpe ratio down. As we discuss below, the comparative statics on the price dividend ratio is non-trivial. Still, our numerical simulations show that some qualitative properties of the wealth effect are robust across most parameterization. Namely, the wealth effect is decreasing in the share of delegation Ω^* when $\Omega^* > \hat{\Omega}$ and its change is small when when $\Omega^* < \hat{\Omega}$. Thus, relative state price and the Sharpe-ratio increases in Ω^* when $\Omega^* < \hat{\Omega}$ because of the capital-flow effect and decreases in Ω^* when $\Omega^* > \hat{\Omega}$, because of the wealth effect.

Turning to the price-dividend ratio, note that by market clearing, it is determined by the consumption share of the marginal agent as

$$1+\pi_s=\frac{1}{\frac{\int_{i\in I,M}\psi_t^i w_t^i di}{q_t+\delta_t}\Big|_{s_t=s}}. \tag{43}$$

Given that the consumption share varies with both the relative wealth of agents and the state, the price dividend ratio also varies. In general, the price dividend ratio can be both procyclical or countercyclical and it can both increase or decrease with Ω^* The following Proposition states that whenever the average share delegated by clients is small because Z_B is small, then the price-dividend ratio is procyclical and decreases in the share of delegation.

Proposition 8 *If Z_B is sufficiently small then the price dividend ratio*

1. *decreases in both states as the share of delegated investment increases*
2. *is procyclical.*

3.4 Credit market and speculation

As opposed to standard representative agent models, in our model traders typically do not hold the market portfolio. Thus, they have to lend to and borrow from each other. This gives a significant role to credit markets. In this part, we focus on the extent of this role as the share of delegation changes.

Note that in any equilibrium, the only group of traders who lend to other groups is the group of managers who follow a contrarian strategy. Thus, we can define the size of the credit market as the total capital which this group invests in bonds. Using (20) and (36) simple algebra shows that this is

$$\frac{1-\frac{\frac{pn_B}{pn_B+(1-p)n_A}}{\tilde{\xi}(\Omega^*)}}{1-\frac{\theta(\Omega^*)}{y_H(1+\pi_H(\Omega^*))}}\mu_{Cont}(\Omega^*)\,\Omega^* \tag{44}$$

Another way the characterize the role of credit markets is to measure the cross-sectional dispersion between the portfolios of managers. We use the expression

$$\max_{m\in M}\alpha^m-\min_{m\in M}\alpha^m \tag{45}$$

for this purpose. The following Lemma describes the relationship between the portfolio of managers, the size of the credit market and the Sharpe-ratio whenever $\Omega^*>\hat{\Omega}$.

Lemma 2 *When the share of delegation is larger than $\hat{\Omega}$, whenever the Sharpe-ratio is decreasing in Ω^*, both the size of the credit market, (40), and the dispersion in managers' portfolios,(41) increases as the share of delegation increases.*

4 Extensions

In this part we illustrate the general properties of our framework by presenting a more general variant of the model. In this variant, instead of the interaction of direct traders and fund managers, we consider the interaction of two groups of fund managers whose incentive function differs. We also allow for considerably more general incentive functions.

After presenting the differences in the set-up, we establish sufficient conditions for an equilibrium to exist and emphasize the analogies and the differences across the two variants.

4.1 Modifying the set-up

Fund Managers. The economy is populated by two groups of fund managers $i = 1, 2$. We assume each group is comprised of a continuum of managers with a total mass of one. Analogously to the baseline model, managers in group i solve

$$\max_{\{\psi_t^i, \alpha_t^i\}} E\left[\sum_t \beta^t \ln \psi_t^i w_t^i\right] \qquad s.t.\ \ w_{t+1}^i = g^i\left(\upsilon_{t+1}^i\right) w_{t+1,-}^i \tag{46}$$

where $g^i(\cdot)$ is the incentive function and

$$\begin{aligned} w_{t+1,-}^i &\equiv \rho_{t+1}\left(\alpha_t^i, s_{t+1}\right)\left(1 - \psi_t^i\right) w_t^i \\ \upsilon_{t+1}^i(\alpha_t^i, s_{t+1}) &\equiv \frac{\rho_{t+1}\left(\alpha_t^i, s_{t+1}\right)}{\left(\frac{q_{t+1}(s_{t+1}) + \delta_{t+1}(s_{t+1})}{q_t}\right)} \end{aligned}$$

are the the assets under management at the end of the previous period (i.e., after the time $t+1$ return has been realized, but before investors decide how much to allocate to the fund to manage between $t+1$ and $t+2$) and the return on her portfolio relative to the market.

Clients. Suppose that all investors are client at time zero and $\lambda = 1$. That is, clients live forever and there are no direct traders. Clients delegate their portfolio decisions two both groups of managers described by the incentive function of each, $g^i(\cdot)$. Thus, they consume

$$\delta_t + q_t - \left(g^1\left(\upsilon_{t+1}^1\right) w_{t,-}^1 + g^2\left(\upsilon_{t+1}^2\right) w_{t,-}^2\right).$$

Incentive Functions. We allow more general incentive functions. In particular, incentive functions belong to the following class:

$$w_{t+1}^i = \; Z_k^i \left(\upsilon_{t+1}^i\right)^{n_m^i-1} w_{t+1,-}^i \quad if \quad \upsilon_{t+1}^i \in \kappa_m^i \tag{47}$$

where $n_k^i \geq 1$, $Z_k^i > 0$ and we pick $K^i - 1$ positive and increasing "kinks", $\kappa_k^i, k = 1, ...K^i - 1$. The kinks divide the positive segment of the real line to K intervals. We refer to the the segment $[\kappa_{k-1}, \kappa_k]$ as segment k.

4.2 The Asymmetric Interior Equilibrium

In this section, we define the equilibrium, derive sufficient conditions for its existence and provide some general characterizations. Given that in this version, there are no direct traders, let us redefine Ω as the relative share of group 1:

$$\Omega_t \equiv \frac{w_t^1}{w_t^2 + w_t^1}.$$

In our proposed equilibrium, instead of being constant, Ω serves as a new state variable. We use the following definitions in our equilibrium concept.

Definition 2 *An* ***lh−portfolio*** *is a portfolio, α, for which a fund's return relative the the market υ_{t+1} is in the $l-th$ segment of the incentive function following a low shock $s_{t+1} = L$ and in h-th segment following a high shock $s_{t+1} = H$.*

If α is an ***lh*-portfolio**, then

$$\begin{aligned} l &= \sum_{m=1}^{M^i} 1_{\{\upsilon_{t+1}(\alpha,L)\in\kappa_m^i\}}\, k \\ h &= \sum_{m=1}^{M^i} 1_{\{\upsilon_{t+1}(\alpha,H)\in\kappa_m^i\}} k. \end{aligned}$$

In an asymmetric equilibrium managers in a given group i follow heterogeneous strategies. However, we conjecture that in a given state Ω there is a unique $lh-$portfolio which is optimal for managers in group i. We refer to this locally optimal portfolio as $\alpha_{lh}^i(\Omega)$. Therefore, to describe asymmetric strategy profiles it is sufficient to specify measures $\mu_{lh}^i(\Omega)$ of managers in group i which choose corresponding $\alpha_{lh}^i(\Omega)$ portfolios.

Definition 3 *A* ***strategy profile*** $\mathcal{P}^i(\Omega)$ *is a triplet* $(\mathcal{H}^i(\Omega), \mathcal{M}^i(\Omega), \mathcal{A}^i(\Omega))$ *where for any* Ω*:*

- $\mathcal{H}^i(\Omega)$ *is a set of* lh *index-pairs,*

- $\mathcal{A}^i(\Omega)$ *is a set of corresponding* $\alpha^i_{lh}(\Omega)$*-portfolios,*

- $\mathcal{M}^i(\Omega)$ *is a set of* $\mu^i_{lh}(\Omega):[0,1]\times\mathcal{H}^i(\Omega)\to(0,1]$ *functions such that* $\sum_{lh\in\mathcal{H}^i(\Omega)}\mu^i_{lh}(\Omega)=1$.

The above definition nests also the case where all managers of group i follow the same strategy. In that case $\mathcal{H}^i(\Omega)$ and $\mathcal{A}^i(\Omega)$ are singletons, and $\mu^i_{lh}(\Omega)=1$ for $lh\in\mathcal{H}^i(\Omega)$.

The next definition defines the asymmetric interior equilibrium.

Definition 4 *An* ***asymmetric interior equilibrium*** *is a price process* $q_{s'}(\Omega)$ *for the stock and* $R(\Omega)$ *for the bond, a law of motion* $\Omega'=\Omega_{s'}(\Omega)$ *for the relative wealth share of group 1, and strategy profiles* $\mathcal{P}^i(\Omega)$ *for* $i=1,2$ *such that*

1. *consumption choices* $\psi^i(\Omega)$ *and trading strategies* $\alpha^i_{lh}(\Omega)$ *are optimal for managers given the equilibrium prices and the law of motion for the relative wealth share of group 1,*

2. *prices* $q_{s'}(\Omega)$*, and* $R(\Omega)$ *clear both good and asset markets,*

3. *the law of motion for the relative share of group 1 is consistent with managers:*

$$\Omega'=\sigma_{s'}(\Omega).$$

4.2.1 Characterization and Existence

Just as in the baseline model, in this variant we use the concept of *individual shape-adjusted probability of a high state*

$$\xi^i_{lh}\equiv\frac{pn^i_h}{pn^i_h+(1-p)\,n^i_l},\tag{48}$$

and the *aggregate shape-adjusted probability of a high state*

$$\tilde{\xi}(\Omega)\equiv\Omega\sum_{lh\in\mathcal{H}^1(\Omega)}\mu^1_{lh}(\Omega)\,\xi^1_{lh}+(1-\Omega)\sum_{lh\in\mathcal{H}^2(\Omega)}\mu^2_{lh}(\Omega)\,\xi^2_{lh}.\tag{49}$$

We also define the end of period share of wealth of a given group i after a given shock s' relative to the total, cum-dividend value as

$$W^i_{s'}(\Omega)=\frac{\sum_{lh\in\mathcal{H}^i(\Omega)}\mu^i_{lh}(\Omega)\,Z_{m_{lh}(s')}\left(\upsilon^i_{t+1}(\alpha^i_{lh}(\Omega),s')\right)^{n_{m_{lh}(s')}-1}w^i_{t+1,-}(\alpha^i_{lh}(\Omega),s')}{q_{s'}+\delta'}$$

and the aggregate version is

$$\tilde{W}_{s'}(\Omega) = W^1_{s'}(\Omega) + W^2_{s'}(\Omega).$$

The representative manager in group i solves the problem

$$\begin{aligned} V\left(w^i_t, \Omega_{t-1}, s_{t-1}\right) &= \max_{\psi^i_t, \alpha^i_t} \ln \psi^i_t w^i_t + \beta E\left(V\left(w^i_{t+1}, \Omega_t, s_t\right)\right) \qquad (50) \\ &\quad s.t. \\ w^i_{t+1} &= g^i\left(\upsilon^i_{t+1}\right) w^i_{t+1,-} \end{aligned}$$

Proposition 9 *In an asymmetric interior equilibrium,*

1. *the optimal consumption rule of agent i is*

$$\psi^i(\Omega) = (1-\beta), \qquad (51)$$

2. *her optimal trading strategy is*

$$\alpha^i_{lh}(\Omega) = \frac{1-\xi^i_{lh}}{1-\frac{y_H(1+\pi_H(\Omega))}{R(\Omega)\pi(\Omega)}} + \frac{\xi^i_{lh}}{1-\frac{y_L(1+\pi_L(\Omega))}{R(\Omega)\pi(\Omega)}} \qquad (52)$$

for some $lh \in \mathcal{H}^i(\Omega)$

3. *the total share of capital of group i is*

$$\begin{aligned} W^1_H(\Omega) &= \Omega \sum_{lh\in\mathcal{H}^1(\Omega)} \mu^1_{lh}(\Omega) Z^1_h \left(\frac{\xi^1_{lh}}{\tilde{\xi}(\Omega)}\right)^{n^1_h}. \qquad (53) \\ W^1_L(\Omega) &= \Omega \sum_{lh\in\mathcal{H}^1(\Omega)} \mu^1_{lh}(\Omega) Z^1_l \left(\frac{1-\xi^1_{lh}}{1-\tilde{\xi}(\Omega)}\right)^{n^1_l} \qquad (54) \\ W^2_H(\Omega) &= (1-\Omega) \sum_{lh\in\mathcal{H}^2(\Omega)} \mu^2_{lh}(\Omega) Z^2_h \left(\frac{\xi^2_{lh}}{\tilde{\xi}(\Omega)}\right)^{n^2_h} \qquad (55) \\ W^2_L(\Omega) &= (1-\Omega) \sum_{lh\in\mathcal{H}^2(\Omega)} \mu^2_{lh}(\Omega) Z^2_l \left(\frac{1-\xi^2_{lh}}{1-\tilde{\xi}(\Omega)}\right)^{n^2_l}, \qquad (56) \end{aligned}$$

4. *the law of motion is*

$$\Omega_{s'}(\Omega) = \frac{W^1_{s'}(\Omega)}{\tilde{W}_{s'}(\Omega)}. \qquad (57)$$

5. *the price dividend ratio is*

$$\pi_{s'}(\Omega) = \frac{\beta \tilde{W}_{s'}(\Omega)}{1 - \beta \tilde{W}_{s'}(\Omega)} \tag{58}$$

where $\tilde{W}_{s'}(\Omega) = W^1_{s'}(\Omega) + W^2_{s'}(\Omega)$ *is the total share of endowment of all managers,*

6. *the interest rate is*

$$R(\Omega) = \frac{\theta(\Omega)}{\pi(\Omega)}$$

where $\theta(\Omega)$ *solves*

$$\tilde{\xi}(\Omega) \frac{1}{y_H (1 + \pi_H(\Omega))} + \left(1 - \tilde{\xi}(\Omega)\right) \frac{1}{y_L (1 + \pi_L(\Omega))} = \frac{1}{\theta(\Omega)}, \tag{59}$$

7. *the state price of the low state relative to the high state is*

$$\frac{y_H}{y_L} X(\Omega)$$

where

$$X(\Omega) \equiv \frac{\frac{\left(1-\tilde{\xi}(\Omega)\right)}{1-p}}{\frac{\tilde{\xi}(\Omega)}{p}} \frac{1+\pi_H}{1+\pi_L}.$$

8. *and the Sharpe-ratio is*

$$S(\Omega) = \frac{p^{\frac{1}{2}} (1-p)^{\frac{1}{2}} \| y_H X(\Omega) - y_L \|}{p y_L + (1-p) y_H X(\Omega)}. \tag{60}$$

The following proposition describes sufficient conditions for existence of an asymmetric interior equilibrium.

Proposition 10 *The strategy profiles* $\mathcal{P}^i(\Omega)$, *prices* $\pi_{s'}(\Omega), \theta(\Omega)$ *and law of motion* $\Omega' = \Omega_{s'}(\Omega)$ *characterized by (68)-(53) form an asymmetric interior equilibrium, if for any* $\Omega \in [0,1]$, *and* $i = \{1, 2\}$

1. *for any* $l'h' \in \mathcal{H}^i(\Omega)$ *and* $l''h''$

$$p \ln \frac{Z^i_{h''} \left(\frac{\xi^i_{l''h''}}{\tilde{\xi}(\Omega)}\right)^{n^i_{h''}}}{Z^i_{h'} \left(\frac{\xi^i_{l'h'}}{\tilde{\xi}(\Omega)}\right)^{n^i_{h}}} + (1-p) \ln \frac{Z^i_{l''} \left(\frac{1-\xi^i_{l''h''}}{1-\tilde{\xi}(\Omega)}\right)^{n^i_{l''}}}{Z^i_{l'} \left(\frac{1-\xi^i_{l'h'}}{1-\tilde{\xi}(\Omega)}\right)^{n^i_{l'}}} \leq 0 \tag{61}$$

holds with equality whenever $l''h'' \in \mathcal{H}^i(\Omega)$, *and holds with strict inequality whenever* $l''h'' \notin \mathcal{H}^i(\Omega)$.

2. *for any* $lh \in \mathcal{H}^i(\Omega)$, $\alpha^i_{lh}(\Omega)$, *defined in (48), is an* $lh-$*portfolio, that is,*

$$h = \sum_{m=1}^{M^i} 1_{\left\{\frac{\xi^i_{lh}}{\tilde{\xi}(\Omega)} \in \kappa^i_m\right\}} m, \text{ and } l = \sum_{m=1}^{M^i} 1_{\left\{\frac{1-\xi^i_{lh}}{1-\tilde{\xi}(\Omega)} \in \kappa^i_m\right\}} m \tag{62}$$

3. *owners of capital consume a positive amount,*

$$\Omega \sum_{lh \in \mathcal{H}^1(\Omega)} \mu^1_{lh}(\Omega) Z^1_l \left(\frac{1-\xi^1_{lh}}{1-\tilde{\xi}(\Omega)}\right)^{n^1_l} + (1-\Omega) \sum_{lh \in \mathcal{H}^2(\Omega)} \mu^2_{lh}(\Omega) Z^2_l \left(\frac{\xi^2_{lh}}{\tilde{\xi}(\Omega)}\right)^{n^2_l} \leq 1 \tag{63}$$

$$\Omega \sum_{lh \in \mathcal{H}^1(\Omega)} \mu^1_{lh}(\Omega) Z^1_h \left(\frac{\xi^1_{lh}}{\tilde{\xi}(\Omega)}\right)^{n^1_h} + (1-\Omega) \sum_{lh \in \mathcal{H}^2(\Omega)} \mu^2_{lh}(\Omega) Z^2_h \left(\frac{\xi^2_{lh}}{\tilde{\xi}(\Omega)}\right)^{n^2_h} \leq 1 \tag{64}$$

We prove these propositions in the Appendix.

The logic of the proof is a general version of our method in the baseline case. Just as before, the difficulty of finding the equilibrium comes from the possible convexities in the incentive function. In particular, as the problem might not be concave in the portfolio choice, α, the first order condition might not be sufficient to find the equilibrium choice. However, because of the interaction of log-utility and our piece-wise constant elasticity specification of the incentive function, there would not be such convexity issues, if the manager should not have to consider the various segments of her incentive function. For example, suppose we modify the incentive function of a manager in a way that regardless of her portfolio she is compensated according to the parameters Z^i_l, n^i_l (Z^i_h, n^i_h) after a low (high) shock, where l (h) is a given segment of the original incentive function. Then the manager's problem has a well behaving first order condition. In fact, for given prices $\pi_{s'}(\Omega), R(\Omega)$, her optimal portfolio is $\alpha^i_{lh}(\Omega)$ defined in (48).

Now consider the following modified economy. Suppose that we modify the incentive function of each manager in a similar way by choosing an lh index pair. This index pair might be different across managers even within the same group. Think of the sets $\mathcal{M}^i(\Omega)$ as the distribution of lh index pairs across managers. Then all managers choose the portfolio (48) for the given lh and prices $\pi_{s'}(\Omega), R(\Omega)$ and consume according to (47). Aggregating across their first order conditions and imposing that their total holding of the stock has to sum up to 1, implies that prices satisfy (55). The choice (48) and expression (55) imply that the relative return υ^i_{t+1} of a manager with guess lh is

$$\frac{\xi^i_{lh}}{\tilde{\xi}(\Omega)}, \ \frac{1-\xi^i_{lh}}{1-\tilde{\xi}(\Omega)} \tag{65}$$

after a high shock and low shock, respectively. This in turn, gives expressions (49)-(52) and the law of motion (53). Market clearing in the goods market implies the price-dividend ratio (54). Finally, conditions (59)-(60) ensure that the consumption of the owners of capital is non-negative. Thus, we have just established that expressions (47)-(60) is an equilibrium of the modified economy.

Now consider the original economy. The equilibrium of the modified economy might not be an equilibrium of the original economy because of two potential problems. First, some of the $\alpha^i_{lh}(\Omega)-$ portfolios with a measure $\mu^i_{lh}(\Omega) > 0$ in $\mathcal{M}^i(\Omega)$ might result in relative returns which are not in the given lh segments of the original incentive function. That is, $\alpha^i_{lh}(\Omega)$ might not be an $lh-$portfolio under the equilibrium prices. Condition (58) makes sure that this is not the case. Second, given that in the original economy the choice of the portfolio influences at which segment of her incentive function the manager would be compensated, the manager might prefer to deviate to another lh segment. Condition (57) ensures that such deviation is not profitable.

Given Proposition 10, finding the equilibrium amounts to guessing and verifying on which segments managers are likely to be evaluated for a given Ω. That is guessing and verifying a particular choice of $\mathcal{H}^i(\Omega), \mathcal{M}^i(\Omega)$.

4.2.2 Comparison

Let us compare the asymmetric interior equilibrium characterized by Proposition 9 and the Ω^* equilibrium of the baseline model characterized by Propositions 3-5. It is apparent that optimal portfolios are determined the same way. Differences in portfolios are determined by differences in the ratio $\frac{\xi^i_{lh}}{\bar{\xi}(\Omega)}$, the proportion of individual shape-adjusted probabilities to their aggregate counterpart. This ratio also gives the relative returns of managers in equilibrium. Relative prices of bonds and stocks are also determined in the same way specified in (36) and (55). Furthermore, in both cases the Sharpe-ratio is determined by the relative size of the capital-flow effect and the wealth effect.

Despite the formal similarities note that in the baseline model the restrictions on the parameters of the incentive functions determine which strategy is played. This is described in Proposition 2. In contrast, we do not know in general which lh strategies will be played in the variant with two managers. This depends on the exact shape of the incentive functions of the two groups. In fact, this is the major challenge in analyzing a particular asymmetric interior equilibrium in the general case.

The modification that in the general variant there are no newborn investors picking whether to be direct traders or fund managers has two major consequences. We already noted that Ω is not constant anymore. It fluctuates with the relative performance and the corresponding share of delegated capital of the two groups described by (49)-(53). Also, the market clearing condition in the good market changes. Although, the price dividend ratio is still determined analogously to (39), because clients now consume $\tilde{W}_s(\Omega)$ share while managers consume $(1-\beta)\tilde{W}_s(\Omega)$ share of total wealth $(\delta_t + q_t)$, expression (39) implies (54).

5 Conclusion

In this paper we have introduced delegation into a standard Lucas exchange economy, where in equilibrium some investors trade on their own account, but others (clients) decide to delegate trading in financial assets to funds. Flow-performance incentive functions describe how much capital fund clients provide to funds at each date as a function of past performance.

Given the significantly increased fraction of capital that is managed by delegated portfolio management intermediaries over the past 30 year, our analysis has focused on the equilibrium link between the endogenously determined equilibrium share of delegated capital and asset price dynamics, as well as fund return dynamics. In addition, we have also considered how realized dividend growth impacts these different quantities. The basic setup of our economy is intentionally close to the original Lucas model, allowing us a clear comparison of how delegation changes equilibrium dynamics in the Lucas economy. Specifically, we assume dividend growth is i.i.d., so that in the Lucas economy price dividend ratios, and the risk free rate are constant.

Our model implies that with increasing elasticity flow-performance incentives the average fund outperforms the market in recessions and underperforms in expansions; consistent with empirical evidence. When the share of capital that is delegated is low, all funds follow the same strategy. However, when the equilibrium share of delegated capital is high funds with identical incentives utilize heterogeneous trading strategies, trade among themselves, and fund returns are dispersed in the cross-section. Delegation affects the Sharpe ratio through two channels: discount rate and capital flow. The two work in opposite directions leading in general to an inverse U-shape relation between the share of capital that is delegated and the Sharpe ratio. We also show that delegation can lead to pro-cyclical price-dividend ratios.

Our methodological contribution is to simplify the flow-performance relationship into a piece-wise constant elasticity function. The combination of log utility and piece-wise constant elasticity enables us to derive explicit expression for different model quantities. Using this methodology the model can be extended to analyze economies where different type of financial intermediaries co-exist. This extension is left for future work.

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6 Appendix

(preliminary)

A Existence and characterization of equilibria

In this part we prove propositions 1-5 and 9-10. However, we proceed in reverse order. We establish first that if the conditions in Proposition 10 hold then an asymmetric interior equilibrium exists with the properties described in Proposition 9. Then we prove that in our baseline models these conditions indeed hold and the characterization is similar with certain differences due to the differences in the set up.

A.1 Proof of Propositions 9-10

The logic of the proof is at is described in the main text. First we show that Proposition 9 described an equilibrium of a modified problem where managers are evaluated at a given l segment of their incentive function after a low shock and after a given h segment after a high shock. Then we show that under conditions described in Proposition 10, the equilibrium of the modified problem is the equilibrium in the original problem.

A.1.1 Equilibrium in the modified problem

Fix the sets $\mathcal{M}^i(\Omega), \mathcal{H}^i(\Omega)$. Modify the incentive functions of each manager in a way that for any Ω, for each $lh \in \mathcal{H}^i(\Omega)$, $\mu^i_{lh}(\Omega)$ measure of managers face the incentive function

$$w^i_{t+1} = \left\{ \begin{array}{lll} Z^i_h \left(\upsilon^i_{t+1}\right)^{n^i_h - 1} w^i_{t+1,-} & if & s_{t+1} = H \\ Z^i_l \left(\upsilon^i_{t+1}\right)^{n^i_l - 1} w^i_{t+1,-} & if & s_{t+1} = L \end{array} \right\}.$$

We show by a series of lemmas, that in this modified economy, expressions in Proposition 9characterize an equilibrium. We will use the observation that the market clearing on the bond market implies

$$\begin{aligned} \alpha^1\left(\bar{w}^1 - \bar{c}^1\right) + \alpha^2\left(\bar{w}^2 - \bar{c}^2\right) &= q \\ \Omega\alpha^1 + (1-\Omega)\,\alpha^2 &= 1. \end{aligned} \tag{66}$$

Lemma 3 *Strategies $(\mathcal{M}^i(\Omega), \mathcal{H}^i(\Omega), \mathcal{A}^i(\Omega))$ and Euler-equation (55) imply that the total share of capital of each group after each shock is given by (49)-(52), law of motion of Ω is* *given by (53) and the price dividend ratio is given by (54). Furthermore, relative returns are given by (61).*

Proof. By (48) and the market clearing conditions

$$\begin{aligned} w^1 - c^1 + w^2 - c^2 &= q \\ w^1 - c^1 &= \frac{\left(w^1 - c^1\right) q}{w^1 - c^1 + w^2 - c^2} = \Omega q \\ w^2 - c^2 &= \beta \left(1 - \Omega\right) q. \end{aligned} \tag{67}$$

imply that the return of a manager in group 1 after a low shock at the end of the period is

$$\left(\alpha^1 \left(\frac{\delta_{t+1} + q_{t+1}}{q_t} - R_t\right) + R_t\right) =$$

$$= R_t \left(\alpha^1 \left(\frac{\delta_{t+1} + q_{t+1}}{q_t R_t} - 1\right) + 1\right)$$

If $s_{t+1} = L$, then this equal to

$$\begin{aligned} R_t \left(\alpha^1 \left(\Omega\right) \left(\frac{y_L \left(1 + \pi_L \left(\Omega\right)\right)}{\theta \left(\Omega\right)} - 1\right) + 1\right) &= R_t \left(\left(1 - \xi^i\right) \frac{\left(\frac{y_L (1+\pi_L(\Omega))}{\theta(\Omega)} - 1\right)}{1 - \frac{y_H (1+\pi_H(\Omega))}{\theta(\Omega)}} + \left(1 - \xi^i\right)\right) = \\ &= R_t \left(\left(1 - \xi^i\right) \left(\frac{\frac{y_L (1+\pi_L(\Omega))}{\theta(\Omega)} - \frac{y_H (1+\pi_H(\Omega))}{\theta(\Omega)}}{1 - \frac{y_H (1+\pi_H(\Omega))}{\theta(\Omega)}}\right)\right) = \\ &= R_t \left(\left(1 - \xi^i\right) \left(\frac{y_H \left(1 + \pi_H \left(\Omega\right)\right) - y_L \left(1 + \pi_L \left(\Omega\right)\right)}{y_H \left(1 + \pi_H \left(\Omega\right)\right) - \theta \left(\Omega\right)}\right)\right) \end{aligned}$$

Using that from (55), we can rewrite this as

$$\frac{\left(1 - \xi^1_{lh}\right)}{1 - \tilde{\xi}\left(\Omega\right)} \frac{\delta_t}{q_t} y_L \left(1 + \pi_L \left(\Omega\right)\right). \tag{68}$$

similarly, after a high shock it is

$$\frac{\xi^1_{lh}}{\tilde{\xi}\left(\Omega\right)} \frac{\delta_t}{q_t} y_H \left(1 + \pi_H \left(\Omega\right)\right) \tag{69}$$

and for the manager in the second group it is

$$\frac{\left(1-\xi_{lh}^{2}\right)}{1-\tilde{\xi}\left(\Omega\right)}\frac{\delta_t}{q_t}y_L\left(1+\pi_L\left(\Omega\right)\right) \tag{70}$$

$$\frac{\xi_{lh}^{2}}{\tilde{\xi}\left(\Omega\right)}\frac{\delta_t}{q_t}y_H\left(1+\pi_H\left(\Omega\right)\right) \tag{71}$$

after a low shock and a high shock respectively. Thus, after a high shock

$$\left(\beta^M\right)^{b-1}Z\left(\frac{\rho_{t+1}\left(\alpha_t^1\right)}{\frac{q_{t+1}+\delta_{t+1}}{q_t}}\right)^{n-1}\rho_{t+1}\left(\alpha_t^1\right)\left(w_t^i-c_t^i\right)=$$

$$=\left(\beta^M\right)^{b-1}Z\left(\frac{\xi_{lh}^1}{\tilde{\xi}\left(\Omega\right)}\right)^{n-1}\frac{\xi_{lh}^1}{\tilde{\xi}\left(\Omega\right)}\frac{\delta_t}{q_t}y_H\left(1+\pi_H\left(\Omega\right)\right)\left(w_t^i-c_t^i\right)$$

integrating over all managers in group i it implies (49)-(52) and (53). For example,

$$\Omega_H\left(\Omega\right)=$$

$$=\frac{\sum\limits_{lh\in\mathcal{H}^1(\Omega)}\mu_{lh}^1\left(\Omega\right)Z_h^1\left(\frac{\xi_{lh}^1}{\tilde{\xi}\left(\Omega\right)}\right)^{n_h^1}\delta_t y_H\left(1+\pi_H\left(\Omega\right)\right)\Omega}{\sum\limits_{lh\in\mathcal{H}^1(\Omega)}\mu_{lh}^1\left(\Omega\right)Z_h^1\left(\frac{\xi_{lh}^1}{\tilde{\xi}\left(\Omega\right)}\right)^{n_h^1}\delta_t y_H\left(1+\pi_H\left(\Omega\right)\right)\Omega+\sum\limits_{lh\in\mathcal{H}^2(\Omega)}\mu_{lh}^2\left(\Omega\right)Z_h^2\left(\frac{\xi_{lh}^2}{\tilde{\xi}\left(\Omega\right)}\right)^{n_h^2}\left(1-\Omega\right)\left(\delta_t y_H\left(1+\pi_H\left(\Omega\right)\right)\right)\beta}=$$

$$=\frac{W_H^1\left(\Omega\right)}{\tilde{W}_H\left(\Omega\right)}.$$

Note, that the market clearing condition for the good market is

$$\delta_{t+1}=\left(\delta_{t+1}+q_{t+1}\right)\left(\left[1-\tilde{W}_{s'}\left(\Omega\right)\right]+\left(1-\beta\right)\tilde{W}_{s'}\left(\Omega\right)\right)=\left(\delta_{t+1}+q_{t+1}\right)\left(1-\beta\tilde{W}_{s'}\left(\Omega\right)\right)$$

which implies

$$\pi_{s'}\left(\Omega\right)=\frac{q_{t+1}}{\delta_{t+1}}=\frac{\beta\tilde{W}_{s'}\left(\Omega\right)}{1-\beta\tilde{W}_{s'}\left(\Omega\right)}$$

Also, (64)-(67) imply the formula for the relative returns. For example, after a high shock we get

$$\frac{\alpha^i\left(\frac{\delta_{t+1}+q_{t+1}}{q_t}-R_t\right)+R_t}{\frac{\delta_{t+1}+q_{t+1}}{q_t}}=\frac{\frac{\xi_{lh}^i}{\tilde{\xi}(\Omega)}\frac{\delta_t}{q_t}y_H\left(1+\pi_H\left(\Omega\right)\right)}{\frac{\delta_{t+1}+q_{t+1}}{q_t}}=\frac{\xi_{lh}^i}{\tilde{\xi}\left(\Omega\right)}.$$

■

Lemma 4 *Strategies* $\left(\mathcal{M}^i\left(\Omega\right),\mathcal{H}^i\left(\Omega\right),\mathcal{A}^i\left(\Omega\right)\right)$ *and the market clearing condition (62) implies (55).*

Proof. By simple substitution

$$\frac{\Omega \sum\limits_{lh\in\mathcal{H}^1(\Omega)} \mu^1_{lh}(\Omega)\,\xi^1_{lh} + (1-\Omega)\sum\limits_{lh\in\mathcal{H}^2(\Omega)} \mu^2_{lh}(\Omega)\,\xi^2_{lh}}{1-\frac{y_L\left(1+\pi_L(\Omega)\right)}{\theta(\Omega)}} +$$

$$+\frac{\Omega \sum\limits_{lh\in\mathcal{H}^1(\Omega)} \mu^1_{lh}(\Omega)\left(1-\xi^1_{lh}\right) + (1-\Omega)\sum\limits_{lh\in\mathcal{H}^2(\Omega)} \mu^2_{lh}(\Omega)\left(1-\xi^2_{lh}\right)}{1-\frac{y_H\left(1+\pi_H(\Omega)\right)}{\theta(\Omega)}} =$$

$$= \tilde{\xi}(\Omega)\frac{1}{1-\frac{y_L(1+\pi_L(\Omega))}{\theta(\Omega)}} + \left(1-\tilde{\xi}(\Omega)\right)\frac{1}{1-\frac{y_H(1+\pi_H(\Omega))}{\theta(\Omega)}} = 1$$

which gives (55). ■

Lemma 5 *In the modified economy, prices given by (54) and (55) imply that any manager has a value function which is logarithmic in wealth and her consumption and portfolio choices are described by (47) and (48).*

Proof. For any $t \geq 1$, conjecture that the value function has the form of

$$V^i\left(w^i, \Omega_{t-1}, s_{t-1}\right) = \frac{1}{1-\beta}\ln w^i + \Lambda^i\left(\Omega_{t-1}, s_{t-1}\right). \tag{72}$$

Under our conjecture we can write problem as

$$\begin{aligned} V\left(w^i, \Omega_{t-1}, s_{t-1}\right) &= \max_{\alpha^i_t,\psi^i} \ln \psi^i w^i + \frac{\beta}{1-\beta} E\left(\ln\left(\beta^M\right)^{b-1} Z_{m^i_{lh}(s')}\left(\upsilon^i_{t+1}\right)^{n_{m^i_{lh}(s')}-1} w^i_{t+1,-}\right) \\ &+ \beta E\left(\Lambda\left(\Omega_t, s_{t+1}\right)\right) \\ \Omega_t &= \Omega_{s_t}\left(\Omega_{t-1}, s_{t-1}\right) \end{aligned}$$

for the given lh. Let us fix an arbitrary α^i_t. The first order condition in c^i has the form of

$$\frac{1}{\psi^i} = \frac{\beta}{1-\beta}\frac{1}{1-\psi^i}$$

which gives

$$1-\psi^i = \beta$$

We rewrite the problem as

$$\begin{aligned}
V\left(w^i,\Omega,s_t\right) = {} & \max_{\alpha^i} \ln\left(1-\beta\right) w^i + \\
& + \frac{\beta}{1-\beta} p \ln\left(\beta^M\right)^{b-1} Z_h \left(\frac{\rho_{t+1}\left(\alpha_t^1,H\right)}{\frac{q_{t+1}(H)+\delta_{t+1}(H)}{q_t}}\right)^{n_h-1} \rho_{t+1}\left(\alpha_t^1,H\right)\beta w^i + \\
& + \frac{\beta}{1-\beta}\left(1-p\right)\ln\left(\beta^M\right)^{b-1} Z_l \left(\frac{\rho_{t+1}\left(\alpha_t^1,L\right)}{\frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t}}\right)^{n_l-1} \rho_{t+1}\left(\alpha_t^1,L\right)\beta w^i \\
& + \beta\left(p\Lambda\left(\Omega_t,H\right)+\left(1-p\right)\Lambda\left(\Omega_t,L\right)\right)
\end{aligned}$$

Note that this problem is strictly concave in α. The first order condition is

$$pn_h \frac{\frac{q_{t+1}(H)+\delta_{t+1}(H)}{q_t}-R_t}{\alpha^i\left(\frac{q_{t+1}(H)+\delta_{t+1}(H)}{q_t}-R_t\right)+R_t} + \left(1-p\right)n_l \frac{\left(\frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t}-R_t\right)}{\alpha^i\left(\frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t}-R_t\right)+R_t} = 0$$

which is equivalent to

$$\begin{aligned}
& \xi_{lh}^i \frac{\frac{q_{t+1}(H)+\delta_{t+1}(H)}{q_t}-R_t}{\alpha^i\left(\frac{q_{t+1}(H)+\delta_{t+1}(H)}{q_t}-R_t\right)+R_t} + \\
& + \left(1-\xi_{lh}^i\right)\frac{\left(\frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t}-R_t\right)}{\alpha^i\left(\frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t}-R_t\right)+R_t} = 0.
\end{aligned} \tag{73}$$

Solving for α^i gives $\alpha_{lh}^i\left(\Omega\right)$.

Define $\widehat{\mathcal{H}^i}\left(\Omega\right)$ which, for every Ω, contains a single index pair lh, the one which is assigned to the given manager in the modified economy.Substituting back α^i and ψ^i into the value function implies that our conjecture is correct with the choice of function $\Lambda\left(\Omega_{t-1},s_{t-1}\right)$

solving

$$
\begin{aligned}
\Lambda\left(\Omega_{t-1}, s_{t-1}\right) = & \ln\left(1-\beta\right)+ \qquad (74)\\
& +\beta\frac{1}{1-\beta}p\ln\sum_{lh\in\mathcal{H}^i(\Omega_t)} 1_{lh\in\widehat{\mathcal{H}^i}(\Omega_t)} Z_h\left(\frac{\xi^1_{lh}}{\tilde{\xi}\left(\Omega_t\right)}\right)^{n_h}\frac{1}{\pi_{s_t}\left(\Omega_{t-1}\right)}y_H\left(1+\pi_H\left(\Omega_t\right)\right)\beta+\\
& +\beta\frac{1}{1-\beta}\left(1-p\right)\ln\sum_{lh\in\mathcal{H}^i(\Omega_t)} 1_{lh\in\widehat{\mathcal{H}^i}(\Omega_t)} Z_l\left(\frac{1-\xi^1_{lh}}{1-\tilde{\xi}\left(\Omega_t\right)}\right)^{n_l}\frac{1}{\pi_{s_t}\left(\Omega_{t-1}\right)}y_L\left(1+\pi_L\left(\Omega_t\right)\right)\beta\\
& +\beta\left(p\Lambda\left(\Omega_t, H\right)+\left(1-p\right)\Lambda\left(\Omega_t, L\right)\right)\\
\Omega_t = & \ \Omega_{s_t}\left(\Omega_{t-1}, s_{t-1}\right)
\end{aligned}
$$

which has the conjectured form.

For the value function in period 0, suppose that we start the system at relative wealth Ω_0 and and total wealth share $\tilde{W}_0$. Thus, we can write the price dividend ratio in period 0 as

$$\pi_0\left(\Omega_{-1}\right) = \frac{\tilde{W}_0}{1-\beta\tilde{W}_0}.$$

Then we can run the same argument as in the case of $t \geq t$, with the only change that we from (70), we rewrite $\Lambda^i\left(\Omega_{t-1}, s_{t-1}\right)$ for period 0 as

$$
\begin{aligned}
\Lambda^i_0\left(\tilde{W}_0, \Omega_0\right) = & \ln\left(1-\beta\right)+ \qquad (75)\\
& +\beta\frac{p}{1-\beta}\ln\sum_{lh\in\mathcal{H}^i(\Omega_0)} 1_{lh\in\widehat{\mathcal{H}^i}(\Omega_0)}\left(\beta^M\right)^{b-1} Z^i_h\left(\frac{\xi^i_{lh}}{\tilde{\xi}\left(\Omega_0\right)}\right)^{n^i_h}\frac{\tilde{W}_0}{1-\beta\tilde{W}_0}y_H\frac{1}{1-\beta\tilde{W}_1\left(H\right)}\beta+\\
& +\beta\frac{\left(1-p\right)}{1-\beta}\ln\sum_{lh\in\mathcal{H}^i(\Omega_0)} 1_{lh\in\widehat{\mathcal{H}^i}(\Omega_0)}\left(\beta^M\right)^{b-1} Z^i_l\left(\frac{1-\xi^i_{lh}}{1-\tilde{\xi}\left(\Omega_0\right)}\right)^{n^i_l}\frac{\tilde{W}_0 y_L}{1-\beta\tilde{W}_0}\frac{1}{1-\beta\tilde{W}_1\left(L\right)}\beta\\
& +\beta\left(p\Lambda^i_0\left(\Omega_H\left(\Omega_0\right), \tilde{W}_H\left(\Omega_0\right)\right)+\left(1-p\right)\Lambda^i_0\left(\Omega_L\left(\Omega_0\right), \tilde{W}_L\left(\Omega_0\right)\right)\right)
\end{aligned}
$$

which gives the result. ■

A.1.2 Original problem

Here we show that the equilibrium of the modified problem is an equilibrium of the original problem if conditions in Proposition 10 are satisfied.

For consistency, we need that the portfolio described by (48) is indeed an lh-portfolio. That is, (58) has to be satisfied. Also, the consumption of clients has to be positive, which implies the conditions (59) and (60). Finally, from (68) and (70), the payoff-difference from a deviation from the assigned $lh-$portfolio, $\alpha^i_{l'h'}\left(\Omega\right)$ to another locally optimal $\alpha^i_{l''h''}\left(\Omega\right)$ is given by the left hand side of (57). Thus, condition (57) ensures that a deviation from the assigned strategy is suboptimal.

A.2 Proof of Propositions 1-5

We proceed by two major steps. First we take a given Ω^* and show that conditions 10 are satisfied if direct traders and managers follow strategies given in Proposition 2-4. Applying the corresponding argument in the proof of Proposition 9, this also implies that for given price-dividend ratios, this strategies are optimal and $\theta(\Omega^*)$ is determined by 36, relative returns in the high and low states are given by

$$\frac{\xi_{lh}}{\tilde{\xi}(\Omega^*)}, \frac{1-\xi_{lh}}{1-\tilde{\xi}(\Omega^*)}$$

where $\xi_{lh} = \xi_{AB}, p, \xi_{BA}$ corresponds to contrarian, moderate and aggressive strategies validating (**??**)-(**??**), and

$$\begin{aligned} g_{jH} &= \left(\beta^M\right)^{b-1} Z_h \left(\frac{\xi_{lh}}{\tilde{\xi}(\Omega^*)}\right)^h \\ g_{jL} &= \left(\beta^M\right)^{b-1} Z_l \left(\frac{1-\xi_{lh}}{1-\tilde{\xi}(\Omega^*)}\right)^l \end{aligned}$$

for $j = 1, 2$ where $lh = AB, BB, BA$ corresponding to contrarian, moderate and aggressive strategies. Also, the relative returns in equilibrium gives the aggregate share of total capital $(\delta_t + q_t)$ owned by all managers after the shock realized,

$$\begin{aligned} \tilde{\Upsilon}_H &\equiv \Upsilon_t \left(\frac{\xi_1}{\tilde{\xi}(\Omega^*)}\mu_1(\Omega^*) + \frac{\xi_2}{\tilde{\xi}(\Omega^*)}\mu_2(\Omega^*)\right) \\ \tilde{\Upsilon}_L &\equiv \Upsilon_t \left(\frac{1-\xi_1}{1-\tilde{\xi}(\Omega^*)}\mu_1(\Omega^*) + \frac{1-\xi_2}{1-\tilde{\xi}(\Omega^*)}\mu_2(\Omega^*)\right) \end{aligned}$$

where the subscripts 1 and 2 stand for the two possible strategies followed in equilibrium. The definition of $\tilde{\xi}(\Omega^*)$ gives the expressions in Proposition 5.

Second, we show that for any Ω^* there is a $\hat{Z}, \hat{\lambda}$ and f implying that investors are indifferent whether to be clients or direct traders and Γ_s, π_s given in Proposition 5 are consistent with a Ω^* equilibrium.

A.2.1 Equilibrium portfolio strategies under given Ω^*

We show that for a given set of parameters, the sufficient conditions in Proposition 10 hold in the given equilibrium described by Proposition 2-5. In the first part, we introduce the analytical formulas for deviations from the prescribed equilibrium strategies. In the second part, we show that condition (57) holds for $\Omega^* = 0$. In the third part, we show that condition (57) holds for $\Omega^* > 0$. In the last part, we show that condition (58) holds for any Ω^*. We will show that

$$
\begin{aligned}
\hat{k}_{high} &\equiv \exp\left(\frac{\ln \frac{n_A}{n_B}}{\left(1-\frac{n_B}{n_A}\right)}+1\right)\\
\hat{k}_{low} &= \exp\left(\frac{n_B \ln n_B + n_A \ln n_A - (n_A+n_B)\ln \frac{n_B+n_A}{2}}{n_A - n_B}\right)
\end{aligned}
$$

and $\hat{p}_{BA-BB}$ and $\hat{p}_{BA-AB}$ are given by the unique solution in $\left[0,\frac{1}{2}\right]$ of

$$
\begin{aligned}
\Delta^{BA-BB}\left(\hat{p}_{BA-BB}\right) &\equiv 0.\\
\hat{p}_{BA-AB}\exp\left(\frac{\Delta^{BA-BB}\left(\hat{p}_{BA-AB}\right)}{\hat{p}_{BA-AB}\left(n_A-n_B\right)}\right)+\left(1-\hat{p}_{BA-AB}\right)\exp\left(\frac{\Delta^{AB-BB}\left(\hat{p}_{BA-AB}\right)}{\left(n_A-n_B\right)\left(1-\hat{p}_{BA-AB}\right)}\right) &\equiv 1,
\end{aligned}
$$

respectively where

$$
\Delta^{l_1h_1-l_2h_2}(p) \equiv p\ln\frac{A_{h_1}}{A_{h_2}}\frac{\left(\frac{\xi^1_{l_1h_1}}{p}\right)^{n_{h_1}}}{\left(\frac{\xi^1_{l_2h_2}}{p}\right)^{n_{h_2}}}+(1-p)\ln\frac{A_{l_1}}{A_{l_2}}\frac{\left(\frac{1-\xi^1_{l_1h_1}}{1-p}\right)^{n_{l_1}}}{\left(\frac{1-\xi^1_{l_2h_2}}{1-p}\right)^{n_{l_2}}}.
$$

Differences in the value functions Note that

$$
\begin{aligned}
&\frac{1-\beta}{\beta}\left(V^{l_1h_1}_{l'h'-l''h''}\left(\Omega^*\right)-V^{l_2h_2}_{l'h'-l''h''}\left(\Omega^*\right)\right)=\\
&\qquad p\ln\frac{Z_{h_1}}{Z_{h_2}}\frac{\left(\frac{\xi^1_{l_1h_1}}{\tilde{\xi}_{l'h'-l''h''}}\right)^{n_{h_1}}}{\left(\frac{\xi^1_{l_2h_2}}{\tilde{\xi}_{l'h'-l''h''}}\right)^{n_{h_2}}}+(1-p)\ln\frac{Z_{l_1}}{Z_{l_2}}\frac{\left(\frac{1-\xi^1_{l_1h_1}}{1-\tilde{\xi}_{l'h'-l''h''}}\right)^{n_{l_1}}}{\left(\frac{1-\xi^1_{l_2h_2}}{1-\tilde{\xi}_{l'h'-l''h''}}\right)^{n_{l_2}}}=\\
&\qquad\qquad=\Delta^{l_1h_1-l_2h_2}(p)+p\ln\frac{\left(\frac{p}{\tilde{\xi}_{l'h'-l''h''}}\right)^{n_{h_1}}}{\left(\frac{p}{\tilde{\xi}_{l'h'-l''h''}}\right)^{n_{h_2}}}+(1-p)\ln\frac{\left(\frac{1-p}{1-\tilde{\xi}_{l'h'-l''h''}}\right)^{n_{l_1}}}{\left(\frac{1-p}{1-\tilde{\xi}_{l'h'-l''h''}}\right)^{n_{l_2}}}.
\end{aligned}
$$

which implies (for example)

$$V_{BA-BB}^{BA}(\Omega^*) - V_{BA-BB}^{BB}(\Omega^*) =$$
$$= \left\{ \begin{array}{cc} \Delta^{BA-BB}(p) - p(n_A - n_B)\ln\left(\Omega^* \frac{n_A}{pn_A + (1-p)n_B} + (1-\Omega^*)\right) & for \quad \Omega^* < \hat{\Omega}_{BA-BB} \\ \Delta^{BA-BB}(p) - p(n_A - n_B)\ln\left(\hat{\Omega}_{BA-BB} \frac{n_A}{pn_A + (1-p)n_B} + \left(1 - \hat{\Omega}_{BA-BB}\right)\right) & otherwise \end{array} \right\}$$

where $\hat{\Omega}_{BA-BB}$ is defined as

$$\Delta^{BA-BB}(p) \equiv p(n_A - n_B)\ln\left(\hat{\Omega}\frac{n_A}{pn_A + (1-p)n_B} + \left(1 - \hat{\Omega}\right)\right)$$

Similarly,

$$V_{BA-AB}^{BA}(\Omega^*) - V_{BA-AB}^{AB}(\Omega^*) =$$
$$= \left\{ \begin{array}{cc} \begin{array}{c} \Delta^{BA-AB}(p) - (n_A - n_B)p\ln\left(\frac{\Omega^* n_A}{pn_A + (1-p)n_B} + (1-\Omega^*)\right) - \\ (n_A - n_B)(1-p)\ln\left(\frac{\Omega^* n_B}{pn_A + (1-p)n_B} + (1-\Omega^*)\right) \end{array} & for \quad \Omega^* < \hat{\Omega}_{BA-AB} \\ \begin{array}{c} \Delta^{BA-AB}(p) - (n_A - n_B)p\ln\left(\frac{\hat{\Omega}_{BA-AB} n_A}{pn_A + (1-p)n_B} + \left(1 - \hat{\Omega}_{BA-AB}\right)\right) - \\ -(n_A - n_B)(1-p)\ln\left(\frac{\hat{\Omega}_{BA-AB} n_B}{pn_A + (1-p)n_B} + \left(1 - \hat{\Omega}_{BA-AB}\right)\right) \end{array} & otherwise \end{array} \right\}$$

where $\hat{\Omega}_{BA-AB}$ is determined by

$$\begin{aligned} \Delta^{BA-AB}(p) \quad \equiv \quad & (n_A - n_B)p\ln\left(\Omega^*\frac{n_A}{pn_A + (1-p)n_B} + (1-\Omega^*)\right) - \\ & -(n_A - n_B)(1-p)\ln\left(\Omega^*\frac{n_B}{pn_A + (1-p)n_B} + (1-\Omega^*)\right). \end{aligned}$$

A.3 $\Delta(p)$ functions

In the following Lemmas we show that under the classification in Proposition 2, at least when $\Omega^* = 0$, deviations to other locally optimal lh strategies are globally suboptimal.

Lemma 6 *If*

$$-\left(1 - \frac{n_B}{n_A}\right)(\ln k + 1) - \ln\frac{n_B}{n_A} < 0$$

then $V_x^{BA}(0) - V_x^{BB}(0) < 0$ *for all* p. *If*

$$-\left(1 - \frac{n_B}{n_A}\right)(\ln k + 1) - \ln\frac{n_B}{n_A} > 0$$

then there is a $\hat{p}_{BA-BB}$ *that* $V_x^{BA}(0) - V_x^{BB}(0) > 0$ *for all* $p < \hat{p}_{BA-BB}$ *and* $V_x^{BA}(0) - V_x^{BB}(0) < 0$ *for all* $p > \hat{p}_{BA-BB}$. *Furthermore,* $\hat{p}_{BA-BB} >)\frac{1}{2}$ *iff*

$$\left(\frac{1}{2}(n_B \ln n_B + n_A \ln n_A) - \frac{(n_A + n_B)}{2} \ln \frac{n_B + n_A}{2}\right) < (n_A - n_B) \frac{1}{2} \ln k$$

Proof. Note that

$$\Delta^{BA-BB}(p) = -(n_A - n_B) p \ln k - n_A p \ln \frac{((1-p) n_B + p n_A)}{n_A} - (1-p) n_B \ln \frac{((1-p) n_B + p n_A)}{n_B}.$$

Observe that

$$\begin{aligned} \Delta^{BA-BB}(0) &= 0 \\ \Delta^{BA-BB}(1) &= -(n_A - n_B) \ln k \\ \frac{\partial^2 \Delta^{BA-BB}(p)}{\partial^2 p} &= \frac{-(n_A - n_B)^2}{((1-p) n_B + p n_A)} < 0 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \Delta^{BA-BB}(p)}{\partial p} = &-(n_A - n_B) \ln k - (n_A - n_B) \ln((1-p) n_B + p n_A) + \\ &+ n_A \ln n_A - n_B \ln n_B - (n_A - n_B) \end{aligned}$$

Given that the second derivative is negative, if

$$\frac{\partial \Delta^{BA-BB}(p)}{\partial p}|_{p=0} < 0$$

then $\Delta^{BA-BB}(p)$ is decreasing everywhere, while if

$$\frac{\partial \Delta^{BA-BB}(p)}{\partial p}|_{p=0} > 0$$

then $\Delta^{BA-BB}(p)$ is either increasing everywhere, or increasing until a given point and then decreasing. As $n_B < n_A$ implies $\Delta^{BA-BB}(1) > 0$, this implies that $\Delta^{BA-BB}(p) > 0$ for all p. As $n_B > n_A$ implies $\Delta^{BA-BB}(1) < 0$,

$$\frac{\partial \Delta^{BA-BB}(p)}{\partial p}|_{p=0} < 0$$

implies $\Delta^{BA-BB}(p) < 0$ for all p and

$$\frac{\partial \Delta^{BA-BB}(p)}{\partial p}|_{p=0} > 0$$

implies the existence of $\hat{p}$ of the Lemma. The last part of the Lemma comes from the observation that

$$\Delta^{BA-BB}\left(\frac{1}{2}\right) = -(n_A - n_B)\frac{1}{2}\ln k + \left[\frac{1}{2}(n_B \ln n_B + n_A \ln n_A) - \frac{(n_A + n_B)}{2}\ln\frac{n_B + n_A}{2}\right]$$

where the term in the bracket is positive if $n_A > n_B$, as the function $x \ln x$ is convex . ■

Lemma 7 *If $n_B > n_A$ then $\Delta^{AB-BB}(p) > 0$ for all p. If $n_A > n_B$ and*

$$\left(1 - \frac{n_B}{n_A}\right)(\ln k + 1) + \ln\frac{n_B}{n_A} > 0$$

then $V_x^{AB}(0) - V_x^{BB}(0) < 0$ for all p. If $n_A > n_B$ and

$$\left(1 - \frac{n_B}{n_A}\right)(\ln k + 1) + \ln\frac{n_B}{n_A} < 0$$

then there is a $\hat{p}_{AB-BB}$ that $V_x^{AB}(0) - V_x^{BB}(0) < 0$ for $p < \hat{p}_{AB-BB}$. Furthermore $\hat{p}_{AB-BB} < \frac{1}{2}$, iff

$$\frac{1}{2}(n_B \ln n_B + n_A \ln n_A) - \frac{(n_B + n_A)}{2}\ln\frac{n_B + n_A}{2} > \frac{1}{2}(n_A - n_B)\ln k$$

Proof. Note that

$$\Delta^{AB-BB}(p) \equiv -(1-p)(n_A - n_B)\ln k - pn_B \ln\frac{pn_B + (1-p)n_A}{n_B} - (1-p)n_A \ln\frac{(pn_B + (1-p)n_A)}{n_A}.$$

The statement comes from simple analysis of the cases $n_A > n_B$ and $n_A < n_B$ observing that

$$\begin{aligned}
\Delta^{AB-BB}(0) &= -(n_A - n_B)\ln k \\
\Delta^{AB-BB}(1) &= 0 \\
\frac{\partial \Delta^{AB-BB}(p)}{\partial p} &= (n_A - n_B)(\ln k + 1) + (n_A - n_B)\ln(pn_B + (1-p)n_A) + n_B \ln n_B - n_A \ln n_A \\
\frac{\partial \Delta^{AB-BB}(p)}{\partial p}|_{p=1} &= (n_A - n_B)(\ln k + 1) + n_A \ln\frac{n_B}{n_A} \\
\frac{\partial^2 \Delta^{AB-BB}(p)}{\partial^2 p} &= -\frac{(n_B - n_A)^2}{(pn_B + (1-p)n_A)} < 0
\end{aligned}$$

■

and that

$$\Delta^{AB-BB}\left(\frac{1}{2}\right)=-\frac{1}{2}\left(n_A-n_B\right)\ln k+\left[\frac{1}{2}\left(n_B\ln n_B+n_A\ln n_A\right)-\frac{\left(n_B+n_A\right)}{2}\ln\frac{n_B+n_A}{2}\right]$$

where the term in the bracket is positive if $n_A>n_B$, as $x\ln x$ is a convex function.

Lemma 8 $\Delta^{BA-AB}(p)>0$.

Proof. Consider that

$$\Delta^{BA-AB}(p)\equiv\left(n_A-n_B\right)\left((1-p)-p\right)\ln k+\ln\frac{\left(\frac{n_B}{((1-p)n_B+pn_A)}\right)^{(1-p)n_B}\left(\frac{n_A}{((1-p)n_B+pn_A)}\right)^{pn_A}}{\left(\frac{n_A}{((1-p)n_A+pn_B)}\right)^{(1-p)n_A}\left(\frac{n_B}{((1-p)n_A+pn_B)}\right)^{pn_B}}.$$

We need $\Delta(p)>0$ for a BA equilibrium and $\Delta(p)<0$ for an AB equilibrium. Observe that

$$\begin{aligned}\Delta^{BA-AB}(1)&=-\left(n_A-n_B\right)\ln k\\ \Delta^{BA-AB}(0)&=\left(n_A-n_B\right)\ln k\\ \Delta^{BA-AB}\left(\frac{1}{2}\right)&=0.\end{aligned}$$

Also

$$\begin{aligned}\frac{\partial\Delta(p)}{\partial p}&=\left[-2\left(n_A-n_B\right)\ln k-2n_B\ln n_B+2n_A\ln n_A-2\left(n_A-n_B\right)\right]\\&\quad-\left(n_A-n_B\right)\left[\ln\left((1-p)n_B+pn_A\right)\left((1-p)n_A+pn_B\right)\right].\end{aligned}$$

As $n_A>n_B$ then $\Delta(p)$ is positive at $p=0$ and negative at $p=1$. Given that it is a continuous function, it's derivative cannot be positive for all $p\in[0,1]$. As the term in the second bracket is maximal for $p=\frac{1}{2}$, monotonically increasing for $p<\frac{1}{2}$ and monotonically decreasing for $p>\frac{1}{2}$ and the term in the first bracket is constant in p, there cannot be a minimum in $p\in\left(0,\frac{1}{2}\right)$ or a maximum at $p\in\left(\frac{1}{2},0\right)$. Thus, $p>\frac{1}{2}$ implies $\Delta(p)<0$ and $p<\frac{1}{2}$ implies $\Delta(p)>0$. ■

comparing thresholds $\hat{\Omega}$ Now we proceed for $\Omega^*>0$. In this part, we show that in a given $l'h'-l''h''$ equilibrium, there is no $lh\neq l''h''$ that

$$V^{lh}_{l'h'-l''h''}\left(\Omega^*\right)-V^{l'h'}_{l'h'-l''h''}\left(\Omega^*\right)>0$$

for any $\Omega^* > 0$. We already know that this is true for $\Omega^* = 0$. Given the monotonicity in Ω^* of any

$$V^{lh}_{l'h'-l''h''}(\Omega^*) - V^{l'h'}_{l'h'-l''h''}(\Omega^*)$$

functions, and given that for $\Omega^* > \hat{\Omega}, \tilde{\xi}(\Omega^*)$ is constant, thus

$$V^{lh}_{l'h'-l''h''}(\Omega^*)$$

is also constant for any lh, we only have to show that in a $l'h' - l''h''$ equilibrium,

$$\hat{\Omega}_{l'h'-l''h''} < \hat{\Omega}_{l'h'-lh}$$

for any $lh \neq l''h'', l'h'$ where $\hat{\Omega}_{l'h'-lh}$ is the $\hat{\Omega}$ in a given $l'h' - lh$ equilibrium defined as

$$V^{lh}_{l'h'-l''h''}\left(\hat{\Omega}_{l'h'-lh}\right) = V^{l'h'}_{l'h'-l''h''}\left(\hat{\Omega}_{l'h'-lh}\right).$$

This amounts to a comparison between $\hat{\Omega}_{BA-BB}(p)$ and $\hat{\Omega}_{BA-AB}(p)$ defined implicitly by the functions

$$\Delta^{BA-BB}(p) \equiv p(n_A - n_B)\ln\left(\hat{\Omega}_{BA-BB}\frac{n_A}{pn_A + (1-p)n_B} + \left(1 - \hat{\Omega}_{BA-BB}\right)\right) \tag{76}$$

$$\begin{aligned}\Delta^{BA-AB}(p) \equiv\ & (n_A - n_B)p\ln\left(\hat{\Omega}_{BA-AB}\frac{n_A}{pn_A + (1-p)n_B} + (1-\Omega^*)\right) - \\ & -(n_A - n_B)(1-p)\ln\left(\hat{\Omega}_{BA-AB}\frac{n_B}{pn_A + (1-p)n_B} + \left(1 - \hat{\Omega}_{BA-AB}\right)\right)\end{aligned} \tag{77}$$

Lemma 9 *Suppose $n_A > n_B$.If*

$$\frac{n_B \ln n_B + n_A \ln n_A - (n_A + n_B)\ln\frac{n_B+n_A}{2}}{n_A - n_B} < \ln k \tag{78}$$

then $\hat{\Omega}_{BA-BB}(p) < \hat{\Omega}_{BA-AB}(p)$ whenever the functions exist. If

$$\frac{n_B \ln n_B + n_A \ln n_A - (n_A + n_B)\ln\frac{n_B+n_A}{2}}{n_A - n_B} > \ln k \tag{79}$$

then there exist a $\hat{p} < \frac{1}{2}$ that $\hat{\Omega}_{BA-BB}(p) < \hat{\Omega}_{BA-AB}(p)$ for all $p < \hat{p}$ and $\hat{\Omega}_{BA-BB}(p) > \hat{\Omega}_{BA-AB}(p)$ for all $p > \hat{p}$

Proof. First, I show that the system

$$\begin{aligned} \Delta^{BA-BB}(p) &\equiv p(n_A - n_B)\ln\left(\Omega^* \frac{n_A}{pn_A + (1-p)n_B} + (1-\Omega^*)\right) \quad (80)\\ \Delta^{BA-BB}(p) - \Delta^{BA-AB}(p) &= \Delta^{AB-BB}(p) = (n_A - n_B)(1-p)\ln\left(\Omega^* \frac{n_B}{pn_A + (1-p)n_B} + (1-\Omega^*)\right) \end{aligned}$$

has no solution if (74) holds and a single solution $\left(\hat{p}, \hat{\Omega}\right)$ where $\hat{\Omega} = \hat{\Omega}_{BA-BB}(\hat{p}) = \hat{\Omega}_{BA-AB}(\hat{p})$ if (75) holds. For this, note that the system is equivalent to

$$\begin{aligned} \exp\left(\frac{\Delta^{BA-BB}(p)}{p(n_A - n_B)}\right) &\equiv \left(\Omega^* \frac{n_A}{pn_A + (1-p)n_B} + (1-\Omega^*)\right)\\ \exp\left(\frac{\Delta^{AB-BB}(p)}{(n_A - n_B)(1-p)}\right) &= \left(\Omega^* \frac{n_B}{pn_A + (1-p)n_B} + (1-\Omega^*)\right), \end{aligned}$$

hence, any solution of the system has to satisfy

$$p\exp\left(\frac{\Delta^{BA-BB}(p)}{p(n_A - n_B)}\right) + (1-p)\exp\left(\frac{\Delta^{AB-BB}(p)}{(n_A - n_B)(1-p)}\right) \equiv 1.$$

From

$$\begin{aligned} \frac{\Delta_{BA-BB}(p)}{p(n_A - n_B)} &= -\ln k - \frac{n_A}{n_A - n_B}\ln\frac{((1-p)n_B + pn_A)}{n_A}\\ &\quad -\frac{(1-p)}{p}\frac{n_B}{n_A - n_B}\ln\frac{((1-p)n_B + pn_A)}{n_B}\\ \frac{\Delta_{AB-BB}(p)}{(1-p)(n_A - n_B)} &= -\ln k - \frac{n_A}{n_A - n_B}\ln\frac{(pn_B + (1-p)n_A)}{n_A}\\ &\quad -\frac{p}{1-p}\frac{n_B}{n_A - n_B}\ln\frac{pn_B + (1-p)n_A}{n_B} \end{aligned}$$

observe that this function is symmetric in the sense that if

$$\Pi(p) \equiv p\exp\left(\frac{\Delta^{BA-BB}(p)}{p(n_A - n_B)}\right)$$

then

$$\tilde{\Pi}(p) \equiv \Pi(p) + \Pi(1-p) = p\exp\left(\frac{\Delta^{BA-BB}(p)}{p(n_A - n_B)}\right) + (1-p)\exp\left(\frac{\Delta^{AB-BB}(p)}{(n_A - n_B)(1-p)}\right).$$

Also

$$\frac{\partial \Pi(p)}{\partial p} = e^{\frac{\Delta_{BA-BB}(p)}{p(n_A-n_B)}}\left(1+p\frac{\partial\left(\frac{\Delta_{BA-BB}(p)}{p(n_A-n_B)}\right)}{\partial p}\right) =$$
$$= e^{\frac{\Delta_{BA-BB}(p)}{p(n_A-n_B)}}\frac{1}{p}\frac{n_B}{n_A-n_B}\ln\frac{((1-p)n_B+pn_A)}{n_B} > 0.$$

and

$$\Pi(0) = \tilde{\Pi}(0) = \tilde{\Pi}(1) = \frac{1}{k} < 1.$$

Thus, $\tilde{\Pi}(p)$ is increasing for $p < \frac{1}{2}$ and decreasing for $p > \frac{1}{2}$ and its maximum is at $p = \frac{1}{2}$. If (74) holds, then

$$\tilde{\Pi}\left(\frac{1}{2}\right) = \Pi\left(\frac{1}{2}\right) < 1,$$

which implies that $\tilde{\Pi}(p) = 1$ does not have a solution. However, if (75) holds, then $\tilde{\Pi}(p) = 1$ has two solutions. If we denote the first by $\hat{p} < \frac{1}{2}$ then the second one is $(1-\hat{p})$. However, $\hat{\Omega}_{BA-BB}(p)$ exists for a given p, only if $\Delta_{BA-BB}(p) > 0$. It is easy to $\Delta_{BA-BB}(\hat{p}) > 0$, but $\Delta_{BA-BB}(1-\hat{p}) < 0$. This concludes the proof. ■

Condition (58) In this part we show that if in a given equilibrium, a given strategy characterized by (48) for a given lh, is preferred to the BB strategy, then this implies that (48) is an lh portfolio.

1. $V_{BA}^{BA}(\Omega^*) - V_{BA}^{BB}(\Omega^*) > 0$ implies $\frac{\frac{pn_A}{pn_A+(1-p)n_B}}{\xi_{BA}(\Omega^*)} = \frac{\frac{n_A}{pn_A+(1-p)n_B}}{\Omega^*\frac{n_A}{pn_A+(1-p)n_B}+(1-\Omega^*)} > k$

$$0 < V_{BA}^{BA}(\Omega^*) - V_{BA}^{BB}(\Omega^*) =$$
$$= V_{BA}^{BA}(0) - V_{BA}^{BB}(0) - p(n_A-n_B)\ln\left(\Omega^*\frac{pn_A}{pn_A+(1-p)n_B} + p(1-\Omega^*)\right) =$$
$$= (n_A-n_B)p\ln\frac{\frac{n_A}{((1-p)n_B+pn_A)}}{k} + n_B\ln\frac{n_A^p n_B^{(1-p)}}{((1-p)n_B+pn_A)}$$
$$-p(n_A-n_B)\ln\left(\Omega^*\frac{n_A}{pn_A+(1-p)n_B}+(1-\Omega^*)\right) =$$
$$= (n_A-n_B)p\ln\frac{\frac{n_A}{((1-p)n_B+pn_A)}}{k\left(\Omega^*\frac{n_A}{pn_A+(1-p)n_B}+(1-\Omega^*)\right)} +$$
$$+ n_B\ln\frac{n_A^p n_B^{(1-p)}}{((1-p)n_B+pn_A)}$$

As $\frac{n_A n_B}{((1-p)n_B+pn_A)} < 1$ because of the inequality of arithmetic and geometric means,

$$\frac{\frac{n_A}{((1-p)n_B+pn_A)}}{\left(\Omega^* \frac{n_A}{((1-p)n_B+pn_A)} + (1-\Omega^*)\right)} > k$$

must hold.

2. $V_{AB}^{AB}(\Omega^*) - V_{AB}^{BB}(\Omega^*) > 0$ implies $\frac{\frac{(1-p)n_A}{(1-p)n_A+pn_B}}{1-\xi_{AB}(\Omega^*)} = \frac{\frac{n_A}{(1-p)n_A+pn_B}}{\Omega^* \frac{n_A}{(1-p)n_A+pn_B}+(1-\Omega^*)} > k$

$$\begin{aligned}
0 < V_{AB}^{AB}(\Omega^*) - V_{AB}^{BB}(\Omega^*) &= \\
&= V_{AB}^{AB}(0) - V_{AB}^{BB}(0) - (1-p)(n_A - n_B)\ln\left(1 - \tilde{\xi}_{AB}\left(\hat{\Omega}\right)\right) = \\
= -(1-p)(n_A-n_B)\ln k - pn_B \ln \frac{pn_B + (1-p)n_A}{n_B} &- (1-p)n_A \ln \frac{(pn_B+(1-p)n_A)}{n_A} - \\
&- (1-p)(n_A-n_B)\ln\left(\Omega^* \frac{n_A}{((1-p)n_B+pn_A)} + (1-\Omega^*)\right) = \\
= (1-p)(n_A-n_B)\ln \frac{\frac{n_A}{(pn_B+(1-p)n_A)}}{k\left(\Omega^* \frac{n_A}{((1-p)n_B+pn_A)} + (1-\Omega^*)\right)} &+ n_B \ln \frac{n_B^p n_A^{(1-p)}}{(pn_B+(1-p)n_A)}
\end{aligned}$$

the second part is negative, so $\frac{\frac{n_A}{(pn_B+(1-p)n_A)}}{\left(\Omega^* \frac{n_A}{pn_A+(1-p)n_B}+(1-\Omega^*)\right)} > k$ must hold.

3.$V_{BA-\mu}^{AB}(\Omega^*) - V_{BA-\mu}^{BB}(\Omega^*) > 0$ implies $\frac{\frac{(1-p)n_A}{(1-p)n_A+pn_B}}{1-\xi_{BA}\left(\hat{\Omega}\right)} = \frac{\frac{n_A}{(1-p)n_A+pn_B}}{\hat{\Omega}\frac{n_B}{(1-p)n_A+pn_B}+\left(1-\hat{\Omega}\right)} > k$ for all $\Omega^* > \hat{\Omega}$

$$\begin{aligned}
0 < V_{AB}^{AB}(\Omega^*) - V_{AB}^{BB}(\Omega^*) &= \\
&= V_{AB}^{AB}(0) - V_{AB}^{BB}(0) - (1-p)(n_A - n_B)\ln\left(1 - \tilde{\xi}_{BA}\left(\hat{\Omega}\right)\right) = \\
= -(1-p)(n_A-n_B)\ln k - pn_B \ln \frac{pn_B + (1-p)n_A}{n_B} &- (1-p)n_A \ln \frac{(pn_B+(1-p)n_A)}{n_A} - \\
&- (1-p)(n_A-n_B)\ln\left(\hat{\Omega} \frac{n_B}{((1-p)n_B+pn_A)} + \left(1-\hat{\Omega}\right)\right) = \\
= (1-p)(n_A-n_B)\ln \frac{\frac{n_A}{(pn_B+(1-p)n_A)}}{k\left(\hat{\Omega} \frac{n_B}{((1-p)n_B+pn_A)} + \left(1-\hat{\Omega}\right)\right)} &+ n_B \ln \frac{n_B^p n_A^{(1-p)}}{(pn_B+(1-p)n_A)}
\end{aligned}$$

the second part is negative so $\frac{\frac{n_A}{(pn_B+(1-p)n_A)}}{k\left(\hat{\Omega}\frac{n_B}{((1-p)n_B+pn_A)}+\left(1-\hat{\Omega}\right)\right)} > 1$.

A.3.1 Finding $\hat{Z}, \hat{\lambda}, f, \Gamma_s, \pi_s$ implying a Ω^* equilibrium

Let us conjecture that there is an f which makes investors indifferent whether to be clients or direct traders for a given Ω^*. We will verify this in the following part.

Here, we calculate the total investment of different groups to derive Γ_s, π_s which implies Ω^*.

Note that the total capital of clients whom invested with managers following the first type of equilibrium strategy resulting in υ_{1s} relative return in state s is

$$\begin{aligned} \int_{M'} \rho_{t+1}\left(\alpha_t^M, s_{t+1}\right)\left(1-\psi_t^M\right) w_t^M &= \\ &= \upsilon_{1s}\left(\delta_{t+1}+q_{t+1}\right)\mu_1\Omega_t \end{aligned}$$

thus, the total investment by the surviving part of these managers is

$$\lambda g\left(\upsilon_{1s}, \beta^M\right)\upsilon_{1s}\left(\delta_{t+1}+q_{t+1}\right)\mu_1\Omega_t.$$

using the same argument for clients with managers from the second group gives that the total investment by all surviving managers is

$$\lambda\left(\delta_{t+1}+q_{t+1}\right)\Omega_t\left(g\left(\upsilon_{1s}, \beta^M\right)\upsilon_{1s}\mu_1+g\left(\upsilon_{2s}, \beta^M\right)\upsilon_{2s}\mu_2\right)=\lambda\left(\delta_{t+1}+q_{t+1}\right)\bar{g}_s$$

Then, to be consistent with the individual incentive functions of managers, and conjecturing that the parameter Γ_t depends only on the state $s=H,L$, the following equality has to hold

$$(1-\lambda)\left(\delta_{t+1}+q_{t+1}\right)\beta^I\int_{j\in I}\chi^j di+\lambda\left(\delta_{t+1}+q_{t+1}\right)\bar{g}_s=\left(\delta_{t+1}+q_{t+1}\right)\Gamma_s\bar{g}_s$$

where the first term is the aggregate investment by newborns who choose to be clients, the left hand side is the aggregate investment by all clients and the right hand side is all the received capital by managers. For this, in total a mass of

$$\int_{j\in I}\chi^j dj=\bar{g}_s\frac{\Gamma_s-\lambda}{(1-\lambda)\beta^I}$$

newborns have to choose to be clients. We make sure below that this expression is indeed between 0 and 1.

Note that the above analysis implies the following aggregate dynamics on consumption and investment of newborns, clients and direct traders.

	newborns	clients	direct traders
wealth	$(1-\lambda)(\delta_t+q_t)$	$\tilde{\Upsilon}_s\lambda(\delta_t+q_t)$	$\lambda\left(1-\tilde{\Upsilon}_s\right)(\delta_t+q_t)$
consumption	$(1-\lambda)\left(1-\beta^I\right)(\delta_t+q_t)$	$(\delta_t+q_t)\lambda\left(\tilde{\Upsilon}_s-\bar{g}_t\right)$	$\lambda\left(1-\tilde{\Upsilon}_s\right)\left(1-\beta^I\right)(\delta_t+q_t)$
investment		$\Gamma_s\bar{g}_s(\delta_t+q_t)$	$(\delta_t+q_t)\beta^I\left(\left(1-\lambda\tilde{\Upsilon}_s\right)+(1-\lambda)\left(1-\int_{j\in I}\chi^j dj\right)nj\right)$

Managers receive $\Gamma_t\bar{g}_t(\delta_t+q_t)$ capital from clients, consume $\left(1-\beta^M\right)$ part of it and invest β^M part of it. Thus, by definition the share of delegated capital is

$$\Omega^* = \frac{\beta^M\Gamma_s\bar{g}_s}{\beta^M\Gamma_s\bar{g}_s+\beta^I\left(1-\lambda\tilde{\Upsilon}_s\right)+\bar{g}_s(\lambda-\Gamma_s)}$$

in both states, which gives the expressions for Γ_H, Γ_L of Proposition 5. We get the expression for the price-dividend ratio by the market clearing condition for the good market

$$\delta_t = (\delta_t+q_t)\left((1-\lambda)\left(1-\beta^I\right)+\lambda\left(\tilde{\Upsilon}_s-\bar{g}_t\right)+\lambda\left(1-\tilde{\Upsilon}_s\right)\left(1-\beta^I\right)+\left(1-\beta^M\right)\Gamma_s\bar{g}_s\right).$$

The threshold $\hat{Z}$ comes from the requirement that g_{1s}, g_{2s} always have to be smaller than 1, while the threshold $\hat{\lambda}$ comes from the requirement that $\int_{j\in I}\chi^j dj$ is between zero and 1.

To conclude the proof, in next part we verify that for any Ω^*, there is an f which makes investors indifferent between being direct traders or clients and that this relationship is continuous.

A.3.2 Equilibrium value functions

Suppose the equilibrium strategies of managers where a measure μ follows a strategy leading to relative return

$$\upsilon''_H \equiv \frac{\xi_{l''h''}}{\tilde{\xi}(\Upsilon^*)},\ \upsilon''_L \equiv \frac{1-\xi_{l''h''}}{1-\tilde{\xi}(\Upsilon^*)}$$

while a measure $(1-\mu)$ follow strategies leading to

$$\upsilon'_H \equiv \frac{\xi_{l'h'}}{\tilde{\xi}(\Upsilon^*)},\ \upsilon'_L \equiv \frac{1-\xi_{l'h'}}{1-\tilde{\xi}(\Upsilon^*)}.$$

Let also denote the corresponding absolute returns in period t as $\rho''_{t,H}, \rho''_{t,L}, \rho'_{t,H}, \rho'_{t,L}$ respectively. Write

$$\begin{aligned}\Lambda'^C_{s_t} &\equiv \Lambda^C\left(\upsilon'_{s_t}, \Upsilon^*, s_t\right)\\ \Lambda''^C_{s_t} &\equiv \Lambda^C\left(\upsilon''_{s_t}, \Upsilon^*, s_t\right)\end{aligned}$$

for $s_t = H, L$ and let

$$E\Lambda^C \equiv \mu(\Upsilon^*)\left(p\Lambda_H''^C + (1-p)\Lambda_L''^C\right) + (1-\mu(\Upsilon^*))\left(p\Lambda_H'^C + (1-p)\Lambda_L'^C\right)$$

Conjecture that in equilibrium we can write the lifetime utility of a client with initial wealth w as

$$\begin{aligned} V^C(w, \upsilon', s) &= \frac{1}{1-\beta^I}\ln w + \Lambda_s'^C \\ V^C(w, \upsilon'', s) &= \frac{1}{1-\beta^I}\ln w + \Lambda_s''^C \end{aligned}$$

then for $\upsilon_s = \upsilon_s', \upsilon_s''$ and $g_s = g_s', g_s''$ and $\rho_{t,s} = \rho_{t,s}', \rho_{t,s}''$

$$\begin{aligned} V^C(w, \upsilon, s_{t-1}) &= \ln w(1-g_s) + \beta^I EV^C(w_{t+1}, \upsilon_t, \Upsilon_t, s_t) = \\ &= \ln w(1-g_s) + \beta^I \frac{1}{1-\beta^I} p\left(\ln \rho_{t,H} w_t \beta^M g_s + (1-p)\left(\ln \rho_{t,L} w_t \beta^M g_s\right)\right) + \beta^I E\Lambda^C \\ &= \frac{1}{1-\beta^I}\ln w_t + \ln(1-g_s) + \beta^I \frac{1}{1-\beta^I}\ln \beta^M g_s + \beta^I \frac{1}{1-\beta^I}\ln\frac{1}{\pi_t} \\ &\quad + \beta^I \frac{1}{1-\beta^I}\left(\left(p\ln \pi_t \rho_{t,H}\right) + (1-p)\left(\ln \pi_t \rho_{t,L}\right)\right) + \\ &\quad + \beta^I E\Lambda^C \end{aligned}$$

Note that in our equilibrium

$$\pi_t \rho_{t,s} = \upsilon_s y_s (1+\pi_s).$$

Thus, the conjecture is correct if

$$\begin{aligned} \Lambda_H'^C &= \ln(1-g_H') + \beta^I \frac{1}{1-\beta^I}\ln \beta^M g_H' + \beta^I \frac{1}{1-\beta^I}\ln\frac{1}{\pi_H} + \\ &\quad \beta^I \frac{1}{1-\beta^I} p\left(\ln \upsilon_H' y_H(1+\pi_H) + (1-p)\ln \upsilon_H' y_L(1+\pi_L)\right) + \beta^I E\Lambda^C \end{aligned}$$

$$\begin{aligned} \Lambda_H''^C &= \ln(1-g_H'') + \beta^I \frac{1}{1-\beta^I}\ln \beta^M g_H'' + \beta^I \frac{1}{1-\beta^I}\ln\frac{1}{\pi_H} + \\ &\quad \beta^I \frac{1}{1-\beta^I} p\left(\ln \upsilon_H'' y_H(1+\pi_H) + (1-p)\ln \upsilon_H'' y_L(1+\pi_L)\right) + \beta^I E\Lambda^C \end{aligned}$$

$$\Lambda_L'^C = \ln\left(1-g_L'\right) + \beta^I \frac{1}{1-\beta^I} \ln \beta^M g_L' + \beta^I \frac{1}{1-\beta^I} \ln \frac{1}{\pi_L} + \beta^I \frac{1}{1-\beta^I} p\left(\ln \upsilon_L' y_H\left(1+\pi_H\right) + (1-p)\ln \upsilon_L' y_L\left(1+\pi_L\right)\right) + \beta^I E\Lambda^C$$

$$\Lambda_L''^C = \ln\left(1-g_L''\right) + \beta^I \frac{1}{1-\beta^I} \ln \beta^M g_L'' + \beta^I \frac{1}{1-\beta^I} \ln \frac{1}{\pi_L} + \beta^I \frac{1}{1-\beta^I} p\left(\ln \upsilon_L'' y_H\left(1+\pi_H\right) + (1-p)\ln \upsilon_L'' y_L\left(1+\pi_L\right)\right) + \beta^I E\Lambda^C$$

which implies

$$E\Lambda^C\left(1-\beta^I\right)^2 = \mu \begin{pmatrix} p\left(\left(1-\beta^I\right)\ln\left(1-g_H'\right) + \beta^I \ln \beta^M g_H' + \beta^I \ln \upsilon_H''\right) + \\ (1-p)\left(\left(1-\beta^I\right)\ln\left(1-g_L''\right) + \beta^I \ln \beta^M g_H'' + \beta^I \ln \upsilon_L''\right) \end{pmatrix} + \mu \begin{pmatrix} p\left(\left(1-\beta^I\right)\ln\left(1-g_H''\right) + +\beta^I \ln \beta^M g_H'' + \beta^I \ln \upsilon_H'\right) \\ +(1-p)\left(\left(1-\beta^I\right)\ln\left(1-g_L'\right) + \beta^I \ln \beta^M g_H'' + \beta^I \ln \upsilon_L'\right) \end{pmatrix} + \beta^I\left(p \ln \frac{y_H\left(1+\pi_H\right)}{\pi_H} + (1-p)\ln \frac{y_L\left(1+\pi_L\right)}{\pi_L}\right)$$

Similarly, writing the value function of direct traders as

$$V^D\left(w, s_{t-1}\right) = \frac{1}{1-\beta^I} \ln w + \Lambda^D_{s_{t-1}}$$

then

$$\begin{aligned} V^D\left(w, s_{t-1}\right) &= \ln w_t\left(1-\beta^I\right) + \beta^I E V^C\left(w_{t+1}, \Upsilon_t\right) = \\ &= \ln w_t\left(1-\beta^I\right) + \beta^I \frac{1}{1-\beta^I} p\left(\ln \rho_{t,H}\left(\alpha^{D*}\right) w_t \beta^I + \Lambda^D_H\right) \\ &+ \beta^I \frac{1}{1-\beta^I}(1-p)\left(\ln \rho_{t,L}\left(\alpha^{D*}\right) w_t \beta\lambda + \Lambda^D_L\right) \\ &= \frac{1}{1-\beta^I}\ln w_t + \ln\left(1-\beta^I\right) + \beta^I \frac{1}{1-\beta^I} \ln \beta^I \\ &+ \beta^I \frac{1}{1-\beta^I}\left(p \ln \rho_{t,H}\left(\alpha^{D*}\right) + (1-p)\ln \rho_{t,L}\left(\alpha^{D*}\right)\right) + \\ &+ \beta^I\left(p\Lambda^D_H + (1-p)\Lambda^D_L\right) \end{aligned}$$

$$\Lambda_H^D = \ln\left(1-\beta^I\right) + \beta^I \frac{1}{1-\beta^I} \ln \beta^I + \beta^I \frac{1}{1-\beta^I} \ln \frac{1}{\pi_H} +$$
$$\beta^I \frac{1}{1-\beta^I} p \left(\ln \pi_t \rho_{t,H}\left(\alpha^D\right) + (1-p) \ln \pi_H \rho_{t,L}\left(\alpha^D\right)\right) + \beta^I \left(p\Lambda_H^D + (1-p)\Lambda^D\right)$$
$$\Lambda_L^C = \ln\left(1-\beta^I\right) + \beta^I \frac{1}{1-\beta^I} \ln \beta^I + \beta^I \frac{1}{1-\beta^I} \ln \frac{1}{\pi_L} +$$
$$\beta^I \frac{1}{1-\beta^I} p \left(\ln \pi_L \rho_{t,H}\left(\alpha^D\right) + (1-p) \ln \pi_L \rho_{t,L}\left(\alpha^D\right)\right) + \beta^I \left(p\Lambda_H^D + (1-p)\Lambda_L^D\right)$$

implying

$$\left(p\Lambda_H^D + (1-p)\Lambda_L^D\right)\left(1-\beta^I\right)^2 = \left(1-\beta^I\right)\ln\left(1-\beta^I\right) + \beta^I \ln \beta^I +$$
$$+\beta^I \left(p \ln \frac{1}{\pi_H} + (1-p)\ln \frac{1}{\pi_L}\right) +$$
$$\beta^I \left(p \ln \frac{p}{\tilde{\xi}(\Upsilon)} y_H (1+\pi_H) + (1-p)\ln \frac{1-p}{1-\tilde{\xi}(\Upsilon)} y_L (1+\pi_L)\right).$$

Thus,

$$\left(E\Lambda^C - E\Lambda^D\right)\left(1-\beta^I\right)^2 = \mu \left(\begin{array}{c} p\left(\left(1-\beta^I\right)\ln \frac{\left(1-g''_H\right)}{1-\beta^I} + \beta^I \ln \frac{\beta^M}{\beta^I} + \beta^I \ln \frac{\upsilon''_H}{p}\right) + \\ +(1-p)\left(\left(1-\beta^I\right)\ln \frac{\left(1-g''_L\right)}{1-\beta^I} + \beta^I \ln \frac{\beta^M}{\beta^I} + \beta^I \ln \frac{\upsilon''_L}{1-p}\right) \end{array}\right)$$
$$+(1-\mu)\left(\begin{array}{c} p\left(\left(1-\beta^I\right)\ln \frac{\left(1-g'_H\right)}{1-\beta^I} + \beta^I \ln \frac{\beta^M}{\beta^I} + \beta^I \ln \frac{\upsilon'_H}{p}\right) \\ +(1-p)\left(\left(1-\beta^I\right)\ln \frac{\left(1-g'_L\right)}{1-\beta^I} + \beta^I \ln \frac{\beta^M}{\beta^I} + \beta^I \ln \frac{\upsilon'_L}{1-p}\right) \end{array}\right)$$

Thus, the expected value of a new born if he becomes a client is

$$EV^C\left(\rho\left(\alpha_t^M\right)\beta^M \beta^I w_t, \Upsilon_t, s\right) =$$
$$= \frac{1}{1-\beta^I} \ln \beta^M \beta^I w_t + \frac{1}{1-\beta^I}\left(\begin{array}{c} \mu\left(p \ln \upsilon''_H + (1-p)\ln \upsilon''_L\right) + \\ +(1-\mu)\left(p\ln \upsilon'_H + (1-p)\ln \upsilon'_L\right) \end{array}\right) +$$
$$+\frac{1}{1-\beta^I}\left(p \ln \frac{y_H(1+\pi_H)}{\pi_H} + (1-p)\ln \frac{y_L(1+\pi_L)}{\pi_L}\right) + E\Lambda^C$$

if he becomes a direct trader it is

$$
\begin{aligned}
EV^{D}\left(\rho\left(\alpha_{t}^{D}\right)w_{t}\beta^{I},\Upsilon_{t},s\right) &= \frac{1}{1-\beta^{I}}\ln\beta^{I}w_{t}+\\
&+\frac{1}{1-\beta^{I}}\left(p\ln\frac{1}{\pi_{H}}+(1-p)\ln\frac{1}{\pi_{L}}\right)+\\
+\frac{1}{1-\beta^{I}}p\ln\frac{p}{\tilde{\xi}\left(\Upsilon\right)}y_{H}\left(1+\pi_{H}\right)&+\frac{1}{1-\beta^{I}}\left(1-p\right)\ln\frac{1-p}{1-\tilde{\xi}\left(\Upsilon\right)}y_{L}\left(1+\pi_{L}\right)+E\Lambda^{D}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\left(EV^{C}-EV^{D}\right)\left(1-\beta^{I}\right)^{2} &= \left(1-\beta^{I}\right)\ln\beta^{M}+\\
&+\mu\left(\begin{array}{c}
p\left(\left(1-\beta^{I}\right)\ln\frac{\left(1-g_{H}'\right)}{1-\beta^{I}}+\beta^{I}\ln\frac{\beta^{M}g_{H}'}{\beta^{I}}+\ln\frac{\upsilon_{H}''}{p}\right)+\\
(1-p)\left(\left(1-\beta^{I}\right)\ln\frac{\left(1-g_{L}'\right)}{1-\beta^{I}}+\beta^{I}\ln\frac{\beta^{M}g_{L}'}{\beta^{I}}+\ln\frac{\upsilon_{L}''}{1-p}\right)
\end{array}\right)+\\
&+(1-\mu)\left(\begin{array}{c}
p\left(\left(1-\beta^{I}\right)\ln\frac{\left(1-g_{H}''\right)}{1-\beta^{I}}+\beta^{I}\ln\frac{\beta^{M}g_{H}''}{\beta^{I}}+\ln\frac{\upsilon_{H}'}{p}\right)+\\
(1-p)\left(\left(1-\beta^{I}\right)\ln\frac{\left(1-g_{L}''\right)}{1-\beta^{I}}+\beta^{I}\ln\frac{\beta^{M}g_{L}''}{\beta^{I}}+\ln\frac{\upsilon_{L}'}{1-p}\right)
\end{array}\right)
\end{aligned}
$$

Picking $f=\left(EV^{C}-EV^{D}\right)$ satisfies our conditions.

B Other proofs

B.1 Proof of Lemma 1

The first part of the proposition is a trivial consequence of (61) and the structure of equilibrium strategies. For the Sharpe ratio and relative state prices, observe that reading (55) as $E\left(\phi_{s}\right)=\frac{1}{R}$ where ϕ_{s} is the state price, one can see that

$$
\begin{aligned}
\phi_{H} &= \frac{\tilde{\xi}\left(\Omega^{*}\right)}{p}\frac{1}{\frac{1}{\pi_{s_{t}}\left(\Omega_{t-1}^{*}\right)}y_{H}\left(1+\pi_{H}\left(\Omega_{t}^{*}\right)\right)}\\
\phi_{L} &= \frac{\left(1-\tilde{\xi}\left(\Omega^{*}\right)\right)}{1-p}\frac{1}{\frac{1}{\pi_{s_{t}}\left(\Omega_{t-1}^{*}\right)}y_{L}\left(1+\pi_{L}\left(\Omega_{t}^{*}\right)\right)}=.
\end{aligned}
$$

Writing $X(\Omega^*) = \frac{\phi_L}{\phi_H}$.gives the Sharpe-ratio by

$$\begin{aligned} S(\Omega^*) &= \frac{\sqrt{Var(\phi_s)}}{E(\phi_s)} = \frac{p^{\frac{1}{2}}(1-p)^{\frac{1}{2}}\left\| \frac{(1-\tilde{\xi}\Omega^*)}{1-p} y_H (1+\pi_H \Omega^*) - \frac{\tilde{\xi}(\Omega^*)}{p} y_L (1+\pi_L(\Omega^*)) \right\|}{\tilde{\xi}(\Omega^*) y_L (1+\pi_L(\Omega^*)) + \left(1-\tilde{\xi}(\Omega^*)\right) y_H (1+\pi_H(\Omega^*))} \\ &= \frac{p^{\frac{1}{2}}(1-p)^{\frac{1}{2}} \| y_H X(\Omega^*) - y_L \|}{p y_L + (1-p) y_H X(\Omega^*)}. \end{aligned}$$

B.2 Proof of Propositions 7 and 8

(draft)

The propositions come from simple algebra after substituting in expressions for Γ_s into the expressions for $S(\Omega^*)$ and π_s and noting that $\lim_{Z_B \to 0} \bar{g}_s = 0$.

B.3 Proof of Lemma 2

The result is implied by the following observations. From (36) and from that $\tilde{\xi}(\Omega^*) = \bar{\xi}$ for $\Omega^* > \hat{\Omega}$,

$$\bar{\xi} = 1 - \frac{\frac{y_H(1+\pi_H(\Omega^*))}{\theta(\Omega^*)} - 1}{\frac{y_H(1+\pi_H(\Omega^*))}{y_L(1+\pi_L(\Omega^*))} - 1} \tag{81}$$

for any $\Omega^* > \hat{\Omega}$. Thus,

$$\begin{aligned} 0 &= \frac{\partial \frac{\frac{y_H(1+\pi_H(\Omega^*))}{\theta(\Omega^*)} - \frac{y_H(1+\pi_H(\Omega^*))}{y_L(1+\pi_L(\Omega^*))}}{1-\frac{y_H(1+\pi_H(\Omega^*))}{y_L(1+\pi_L(\Omega^*))}}}{\partial \Omega^*} = \\ &= \frac{\left(\frac{\partial\left(\frac{y_H(1+\pi_H(\Omega^*))}{\theta(\Omega^*)}\right)}{\partial \Omega^*} - \frac{\partial\left(\frac{y_H(1+\pi_H(\Omega^*))}{y_L(1+\pi_L(\Omega^*))}\right)}{\partial \Omega^*}\right)\left(1 - \frac{y_H(1+\pi_H(\Omega^*))}{y_L(1+\pi_L(\Omega^*))}\right)}{\left(1 - \frac{y_H(1+\pi_H(\Omega^*))}{y_L(1+\pi_L(\Omega^*))}\right)^2} + \\ &\quad \frac{\left(\frac{\partial\left(\frac{y_H(1+\pi_H(\Omega^*))}{y_L(1+\pi_L(\Omega^*))}\right)}{\partial \Omega^*}\right)\left(\frac{y_H(1+\pi_H(\Omega^*))}{\theta(\Omega^*)} - \frac{y_H(1+\pi_H(\Omega^*))}{y_L(1+\pi_L(\Omega^*))}\right)}{\left(1 - \frac{y_H(1+\pi_H(\Omega^*))}{y_L(1+\pi_L(\Omega^*))}\right)^2} \end{aligned}$$

which implies

$$\frac{\frac{y_H(1+\pi_H(\Omega^*))}{y_L(1+\pi_L(\Omega^*))} - 1}{\frac{y_H(1+\pi_H(\Omega^*))}{\theta(\Omega^*)} - 1} = \frac{\frac{\partial\left(\frac{y_H(1+\pi_H(\Omega^*))}{y_L(1+\pi_L(\Omega^*))}\right)}{\partial \Omega^*}}{\frac{\partial\left(\frac{y_H(1+\pi_H(\Omega^*))}{\theta(\Omega^*)}\right)}{\partial \Omega^*}}.$$

Given that $\bar{\xi}$ is always smaller than one, (77) implies that the left hand side is positive. Thus, the sign of $\frac{\partial\left(\frac{y_H(1+\pi_H(\Omega^*))}{y_L(1+\pi_L(\Omega^*))}\right)}{\partial\Omega^*}$ and $\frac{\partial\left(\frac{y_H(1+\pi_H(\Omega^*))}{\theta(\Omega^*)}\right)}{\partial\Omega^*}$ are the same. Observe that the sign of the effect of Ω^* on $\frac{\partial\left(\frac{y_H(1+\pi_H(\Omega^*))}{y_L(1+\pi_L(\Omega^*))}\right)}{\partial\Omega^*}$ is the same that its effect on the wealth-effect component of the Sharpe-ratio. Furthermore, as the capital flow effect is constant in Ω^* for $\Omega^* > \hat{\Omega}$, we can conclude that the sign of $\frac{\partial\left(\frac{y_H(1+\pi_H(\Omega^*))}{\theta(\Omega^*)}\right)}{\partial\Omega^*}$ is the same as the sign of $\frac{\partial S(\Omega^*)}{\Omega^*}\Omega^*$ for $\Omega^* > \hat{\Omega}$.

Second, note that using (32) and (33) in a Cont-Mod equilibrium

$$\frac{\partial\mu_{Cont}(\Omega^*)\,\Omega^*}{\partial\Omega^*} = 0$$

while in a Cont-Agg equilibrium

$$\frac{\partial\mu_{Cont}(\Omega^*)\,\Omega^*}{\partial\Omega^*} = \frac{\xi_{BA} - p}{\xi_{BA} - \xi_{AB}} > 0.$$

Putting together these two points gives the result.

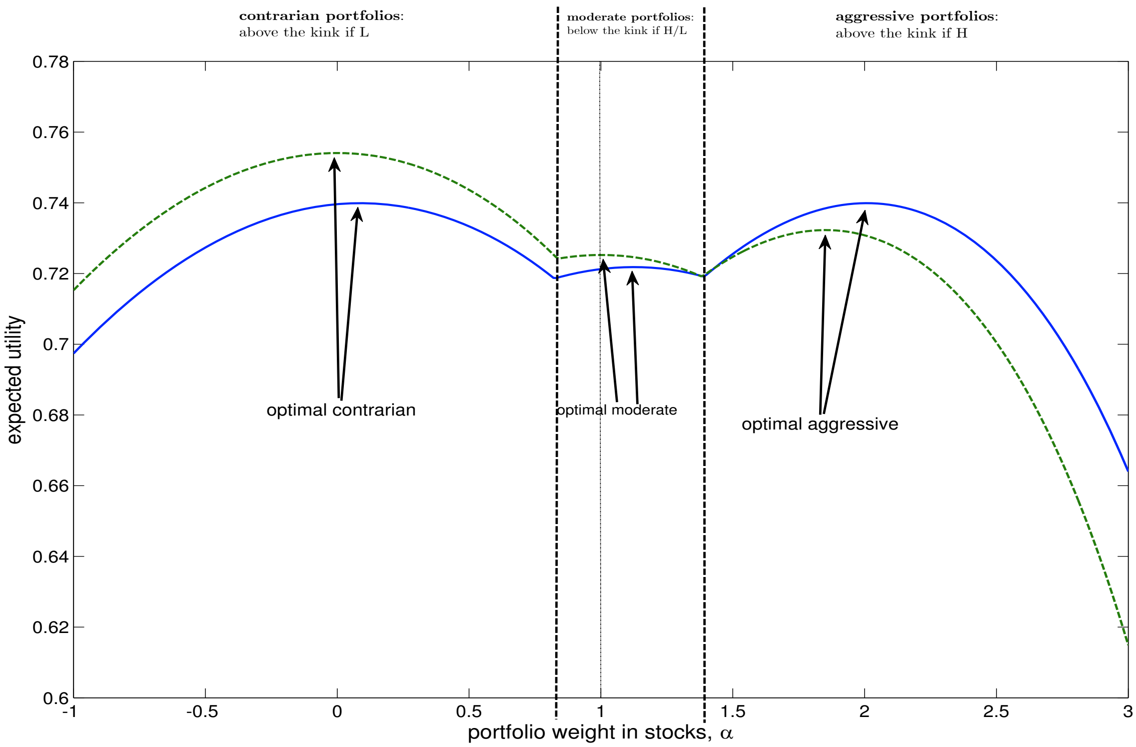

Figure 1: The graph plots the expected utility of a representative manager as function of her portfolio choice, α, for two different set of prices. The dashed line corresponds to the case when the invested capital share of managers, Ω^*, is zero. In this case all other traders hold the market. The solid line corresponds to the case when Ω^*=1. The parameters are set to $\lambda = 0.5, \beta = 0.95, p = 0.7, y_H = 1.2, y_L = 0.8, Z_B = 0.3, n_A = 3,$ and $n_B = 2$.

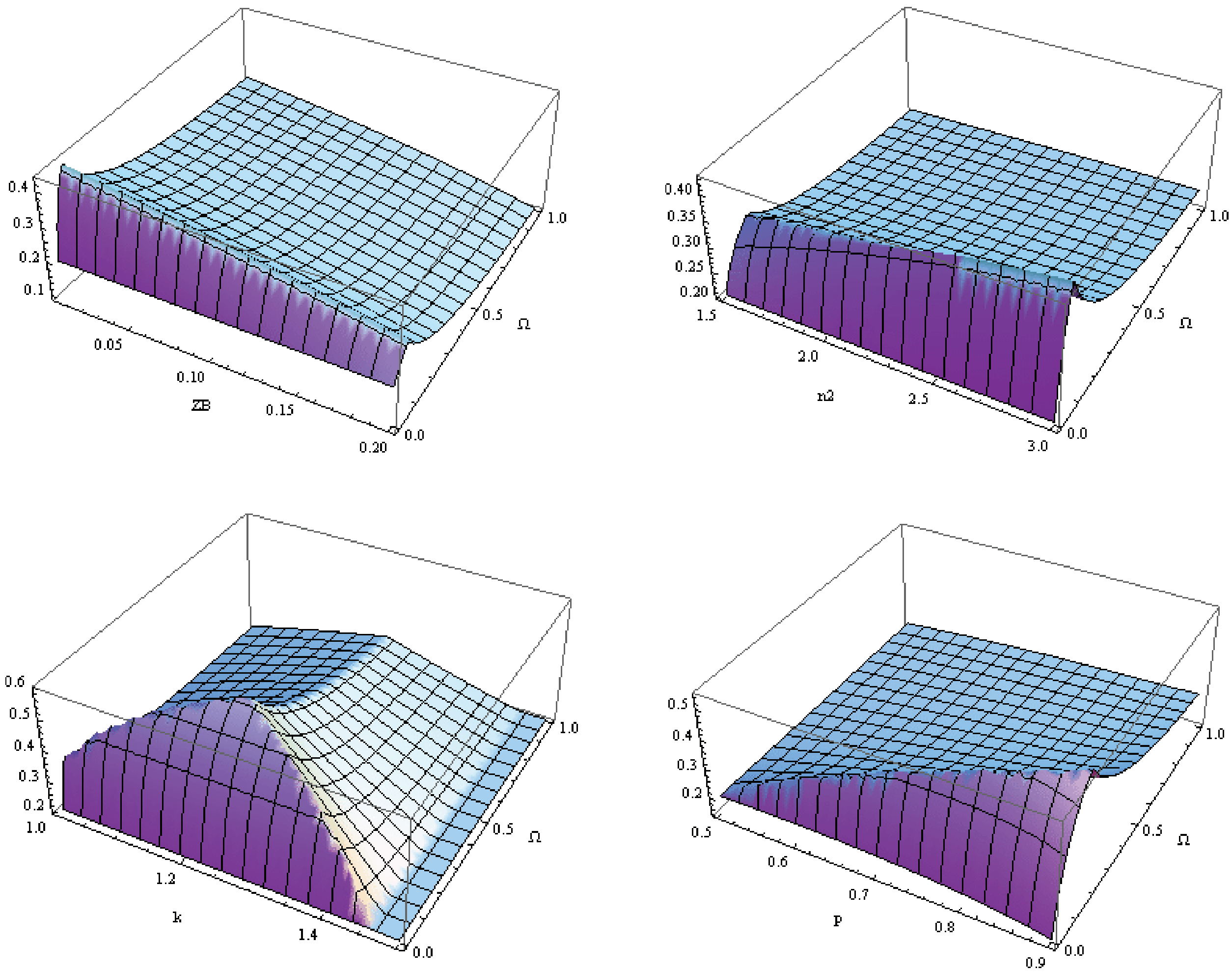

Figure 2: The graphs plot the Sharpe Ratio of the market portfolio. In each of the graphs we vary Ω^* and one additional parameter. The parameters are set to $\lambda = 0.95, \beta = 0.95, p = 0.7, y_H = 1.2, y_L = 0.8, Z_B = 0.01, n_A = 3,$ and $n_B = 1.05$.