

An Institutional Theory of Momentum and Reversal

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Abstract

We propose a rational theory of momentum and reversal based on delegated portfolio management. A competitive investor can invest through an index fund or an active fund run by a manager with unknown ability. Following a negative cashflow shock to assets held by the active fund, the investor updates negatively about the manager's ability and migrates to the index fund. While prices of assets held by the active fund drop in anticipation of the investor's outflows, the drop is expected to continue, leading to momentum. Because outflows push prices below fundamental values, expected returns eventually rise, leading to reversal. Fund flows generate comovement and lead-lag effects, with predictability being stronger for assets with high idiosyncratic risk. We derive explicit solutions for asset prices, within a continuous-time normal-linear equilibrium.

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1 Introduction

Two of the most prominent financial-market anomalies are momentum and reversal. Momentum is the tendency of assets with good (bad) recent performance to continue overperforming (underperforming) in the near future. Reversal concerns predictability based on a longer performance history: assets that performed well (poorly) over a long period tend to subsequently underperform (overperform). Closely related to reversal is the value effect, whereby the ratio of an asset's price relative to book value is negatively related to subsequent performance. Momentum and reversal have been documented extensively and for a wide variety of assets, ranging from individual stocks to industry- and country-level stock portfolios, to bonds, commodities and currencies.¹

Momentum and reversal are viewed as anomalies because they are hard to explain within the standard asset-pricing paradigm with rational agents and frictionless markets. The prevalent explanations of these phenomena are behavioral, and assume that agents react incorrectly to information signals.² In this paper we show that momentum and reversal can arise in markets where all agents are rational. We depart from the standard paradigm by assuming that investors delegate the management of their portfolios to financial institutions, such as mutual funds and hedge funds.

Our explanation emphasizes the role of fund flows, and the flows' relationship to fund performance. It can be summarized as follows. Suppose that a negative shock hits the fundamental value of some assets. Investment funds holding these assets realize low returns, triggering outflows by investors who update negatively about the ability of the managers running these funds. As a consequence of the outflows, funds sell assets they own, and this depresses further the prices of the assets hit by the original shock. If, in addition, outflows are gradual because of institutional constraints (e.g., lock-up periods, institutional decision lags), the selling pressure causes prices to decrease gradually, leading to momentum.³ At the same time, because outflows push prices below fundamental value, expected returns eventually rise, leading to reversal.

¹Jegadeesh and Titman (1993) document momentum for individual US stocks, predicting returns over horizons of 3-12 months by returns over the past 3-12 months. DeBondt and Thaler (1985) document reversal, predicting returns over horizons of up to 5 years by returns over the past 3-5 years. Fama and French (1992) document the value effect. This evidence has been extended to stocks in other countries (Fama and French 1998, Rouwenhorst 1998), industry-level portfolios (Grinblatt and Moskowitz 1999), country indices (Asness, Liew, and Stevens 1997, Bhojraj and Swaminathan 2006), bonds (Asness, Moskowitz and Pedersen 2008), currencies (Bhojraj and Swaminathan 2006) and commodities (Gorton, Hayashi and Rouwenhorst 2008). Asness, Moskowitz and Pedersen (2008) extend and unify much of this evidence and contain additional references.

²See, for example, Barberis, Shleifer and Vishny (1998), Daniel, Hirshleifer and Subrahmanyam (1998), Hong and Stein (1999), and Barberis and Shleifer (2003).

³That gradual outflows cause prices to decrease gradually is not an obvious result: why would rational agents hold losing stocks when these are expected to drop further in price? This result is key to our explanation of momentum, and we present the intuition later in the Introduction.

In addition to deriving momentum and reversal with rational agents, we contribute to the asset-pricing literature by building a parsimonious model of equilibrium under delegated portfolio management. Delegation, to institutions such as mutual funds and hedge funds, is important in many markets. And while investors let fund managers invest on their behalf, they move across funds, generating flows that are large and linked to the funds' past performance.⁴ Yet, incorporating delegation and fund flows into asset-pricing models is a daunting task: it entails modeling portfolio choice by managers (over assets) and investors (over funds), investor learning about managerial ability (to generate a performance-flow relationship), multiple assets (to study cross-sectional phenomena), all in an equilibrium setting. Our model includes these elements.

Section 2 presents the model. We consider an infinite-horizon continuous-time economy with one riskless and multiple risky assets. We refer to the risky assets as stocks, but they could also be interpreted as industry-level portfolios, asset classes, etc. The economy is populated by a competitive investor and a competitive fund manager. The investor can invest in stocks through an index fund that holds the market portfolio, and through an active fund. The manager determines the active fund's portfolio, and can invest his personal wealth in stocks through that fund. Both agents are infinitely lived and maximize expected utility of intertemporal consumption. In addition, exogenous buy-and-hold investors hold stocks in different proportions than in the market portfolio. We introduce these investors so that the active fund can add value over the index fund: it overweighs "high residual supply" stocks, which are in low demand by buy-and-hold investors and thus underpriced, and underweighs "low residual supply" stocks, which are in high demand and overpriced.⁵ The investor receives the return of the active fund net of an exogenous time-varying cost, which can be interpreted as a managerial perk or a reduced form for low managerial ability.

Section 3 solves the model in the benchmark case of symmetric information, where the investor observes the manager's cost. When the cost increases, the investor reduces her position in the active fund and migrates to the index fund. This amounts to a net sale of stocks in high residual supply (active overweighs), and net purchase of stocks in low residual supply (active underweighs). The manager takes the other side of this transaction through a change in his stake in the active fund.⁶

⁴According to the New York Stock Exchange Factbook, the fraction of stocks held directly by individuals in 2002 was less than 40%. The importance of fund flows and the link to past performance have been documented extensively. See, for example, Chevalier and Ellison (1997) and Sirri and Tufano (1998) for mutual funds, and Ding, Getmansky, Liang and Wermers (2008) and Fund, Hsieh, Naik and Ramadorai (2008) for hedge funds.

⁵Alternatively, we could dispense with buy-and-hold investors and the index fund, and introduce a second active fund. The two active funds could be holding different portfolios because of differences in beliefs between their managers. Heterogeneous beliefs, however, would complicate the model without changing the basic mechanisms.

⁶The manager performs a dual role in our model: select the active portfolio and take the other side of the investor's transactions. We could separate the two roles by ignoring the manager's personal wealth and introducing "smart-money" investors who can hold individual stocks directly and act as counterparty to the fund investor. This would complicate the model without changing the main results. Note that transactions between investor and manager

Because the manager is risk-averse, the investor's flows affect prices: stocks in high residual supply become cheaper and stocks in low residual supply become more expensive. Thus, the mispricing that the active fund exploits becomes more severe.

The investor's flows affect the statistical properties of stock returns. Because they impact stocks in high and stocks in low residual supply in opposite directions, they increase comovement within each group, while reducing comovement across groups. Flows also generate return reversal: for example, when the investor moves out of the active fund, stocks in high residual supply become cheaper, and their expected returns rise because fundamental values do not change.

Section 4 considers the more realistic case of asymmetric information, where the investor does not observe the cost and must infer it from fund performance. We solve the inference problem using recursive (Kalman) filtering, and show that the investor learns about the cost by benchmarking the return of the active fund to that of the index. Because of learning, fund returns trigger flows by the investor, and, in turn, flows feed back into returns generating amplification. Suppose, for example, that a stock in low residual supply is hit by a positive cashflow shock. This raises the return of the active fund, but the index return rises even more because the stock's index weight exceeds the active weight. As a result, the investor infers that the cost has increased and migrates to the index fund. This generates net demand for the stock and amplifies the effect of the original shock.

The analysis of asymmetric information bears surprising similarities with symmetric information. For example, in both cases the covariance matrix of returns is the sum of a fundamental component, driven purely by cashflows, and a non-fundamental one, driven by fund flows. The former is identical across the two cases, while the latter is identical up to a multiplicative scalar. Thus, flows increase comovement between two stocks under asymmetric information if they do so under symmetric information. The multiplicative scalar is larger under asymmetric information, implying that stocks' non-fundamental volatility and comovement are larger. This is because of the amplification effect, present only under asymmetric information. Thus, asymmetric information at the micro level impacts volatility and comovement at the macro level. Under asymmetric information, flows generate return reversal for the same reasons as under symmetric information.

Section 5 extends the analysis of asymmetric information to the case where the investor can adjust only gradually her position in the active fund (because of, e.g., institutional constraints).⁷

take place because the cost reduces only the investor's return from the active fund and not the manager's. This is consistent with the interpretation of the cost as a managerial perk. It is also consistent with the setting where the manager's personal wealth is ignored, and the investor's counterparty are smart-money investors who do not invest through the active fund. We discuss these issues in more detail in Section 2.

⁷Gradual adjustment can be studied under symmetric or asymmetric information. We focus on asymmetric

When adjustment is gradual, reversal is preceded by momentum. Suppose, for example, that a stock in high residual supply is hit by a negative cashflow shock. The investor infers that the cost has increased and migrates gradually over time to the index fund. Because outflows are anticipated and result in net sales of the stock, they trigger an immediate price drop, rendering the stock cheap relative to fundamental value. Surprisingly, however, the drop is expected to continue, leading to momentum. The intuition why the manager holds the stock despite expecting it to underperform is as follows. Because the stock is undervalued, it will eventually overperform, “guaranteeing” the manager a high long-horizon expected return. The manager could earn an even higher expected return by not holding the stock during the period of expected underperformance and buying when that period ends. This, however, is a gamble because the high expected return could disappear if the stock overperforms during that period, approaching again fundamental value.⁸

Our model delivers a number of additional predictions. For example, momentum and reversal are stronger for stocks with high idiosyncratic risk. They also become stronger when the manager’s risk aversion increases relative to the investor’s, and when there is more uncertainty about the manager’s cost. In the latter two cases, momentum also strengthens relative to reversal. Finally, flows generate cross-asset predictability (lead-lag effects). For example, a negative return by a stock in high residual supply predicts not only negative returns by the same stock in the short run and positive in the long run, but also negative (positive) returns by other stocks in high (low) residual supply in the short run and conversely in the long run.

Momentum and reversal have mainly been derived in behavioral models.⁹ In Barberis, Shleifer and Vishny (1998), momentum arises because investors view random-walk earnings as mean-reverting and under-react to news. In Hong and Stein (1999), prices under-react to news because information diffuses slowly across investors and those last to receive it do not infer it from prices. In Daniel, Hirshleifer and Subrahmanyam (1998), overconfident investors over-react to news because they underestimate the noise in their signals. Over-reaction builds up over time, leading to momentum, because the self-attribution bias makes investors gradually more overconfident.

Barberis and Shleifer (2003) is the behavioral model closest to our work. They assume that stocks belong in styles and are traded between switchers, who over-extrapolate performance trends,

information because the additional complexity is small but there is a gain in realism and additional results.

⁸The following three-period example illustrates the point. A stock is expected to pay off at 100 in Period 2. The stock price is 91 in Period 0, and 85 or 95 in Period 1 with equal probabilities. Buying the stock in Period 0 earns the manager a two-period expected capital gain of 9. Buying in Period 1 earns an expected capital gain of 15 if the price is 85 and 5 if the price is 95. A risk-averse manager might prefer earning 9 rather than 15 or 5 with equal probabilities, even though the expected capital gain between Period 0 and 1 is negative.

⁹See, however, Albuquerque and Miao (2008) for an asymmetric-information model.

and fundamental investors. Following a stock's bad performance, switchers become pessimistic about the future performance of the corresponding style, and switch to other styles. Because the extrapolation rule involves lags, switching is gradual and leads to momentum. Crucial to momentum is that fundamental investors are also irrational and do not anticipate the switchers' flows. In addition to providing a rational explanation, our model emphasizes the role of delegated portfolio management and the flows between investment funds.

Our emphasis on fund flows as generators of comovement and momentum is consistent with recent empirical findings. Coval and Stafford (2007) find that mutual funds experiencing large outflows engage in distressed selling of their stock portfolios. Anton and Polk (2008) show that comovement between stocks is larger when these are held by many mutual funds in common, controlling for style characteristics. Lou (2008) predicts flows into mutual funds by the funds' past performance, and imputes flows into individual stocks according to stocks' weight in funds' portfolios. He shows that flows into stocks can explain up to 50% of stock-level momentum, especially for large stocks and in recent data where mutual funds are more prevalent.

The equilibrium implications of delegated portfolio management are the subject of a growing literature. A central theme in that literature is that when poor performance triggers outflows because of an exogenous performance-flow relationship (Shleifer and Vishny 1997, Vayanos 2004), a lower bound on the equity stake of fund managers (He and Krishnamurthy 2008ab), or learning about managerial ability (Dasgupta and Prat 2008, Dasgupta, Prat and Verardo 2008, Guerrerri and Kondor 2008, Malliaris and Yan 2008), the effects of exogenous shocks on prices are amplified.¹⁰ Besides addressing momentum and reversal, we contribute to that literature methodologically by bringing the analysis of delegation within a tractable normal-linear framework. Our framework delivers explicit solutions for asset prices, unlike in many previous papers, while also adding new features such as multiple assets and costs of adjustment.

2 Model

Time t is continuous and goes from zero to infinity. There are N risky assets and a riskless asset. We refer to the risky assets as stocks, but they could also be interpreted as industry-level portfolios, asset classes, etc. The riskless asset has an exogenous, continuously compounded return r . The stocks pay dividends over time. We denote by D_{nt} the cumulative dividend of stock $n = 1, \dots, N$,

¹⁰Equilibrium models of delegated portfolio management that do not emphasize the performance-flow relationship include Cuoco and Kaniel (2008) and Petajisto (2008).

and assume that the vector $D_t \equiv (D_{1t}, \dots, D_{Nt})'$ follows the process

$$dD_t = F_t dt + \sigma dB_t^D, \quad (1)$$

where $F_t \equiv (F_{1t}, \dots, F_{Nt})'$ is a time-varying drift equal to the instantaneous expected dividend, σ is a constant matrix of diffusion coefficients, B_t^D is a d -dimensional Brownian motion, and v' denotes the transpose of the vector v . The expected dividend F_t follows the process

$$dF_t = \kappa(\bar{F} - F_t)dt + \phi\sigma dB_t^F \quad (2)$$

where κ is a mean-reversion parameter, \bar{F} is a long-run mean, ϕ is a positive scalar, and B_t^F is a d -dimensional Brownian motion. The Brownian motions B_t^D and B_t^F are independent of each other. The diffusion matrices for D_t and F_t are proportional for simplicity. Each stock is in supply of one share—a normalization since we can redefine F_t and σ . We denote by $S_t \equiv (S_{1t}, \dots, S_{Nt})'$ the vector of stock prices.

Part of each stock's supply is held by an exogenous set of agents who do not trade. These agents could be the firm's managers or founding families, or unmodeled investors.¹¹ We denote by $1 - \theta_n$ the number of shares of stock n held by these agents, and refer to $\theta \equiv (\theta_1, \dots, \theta_N)$ as the residual-supply portfolio. While all stocks are in equal supply of one share, they can enter the residual-supply portfolio in different proportions. Therefore, the residual-supply portfolio can differ from the market portfolio $\mathbf{1} \equiv (1, \dots, 1)$, in which all stocks enter according to their supply. Allowing the two portfolios to differ is important for our analysis because it generates a benefit of investing in an active fund over an index fund that holds the market portfolio. We assume that θ remains constant over time.

The remaining agents are a competitive investor and a competitive fund manager. The investor can invest in the riskless asset and in two funds that hold only stocks: an index fund that holds the market portfolio, and an active fund. We normalize one share of the index fund to equal the market portfolio, i.e., one share of each stock, and refer to the market portfolio as the index portfolio. The portfolio of the active fund is determined by the manager. The investor determines optimally how to allocate her wealth between the riskless asset, the index fund, and the active fund. She maximizes expected utility of intertemporal consumption. Utility is exponential, i.e.,

$$-E \int_0^\infty \exp(-\alpha c_t - \beta t) dt, \quad (3)$$

¹¹Investors could select specific stocks because of, e.g., tax reasons.

where α is the coefficient of absolute risk aversion, c_t is consumption, and β is a discount factor. The investor cannot invest in individual stocks directly. We denote by x_t and y_t the number of shares of the index and active fund, respectively, held by the investor.

The manager determines the portfolio of the active fund. This portfolio is accessible to the investor, but also to the manager who can invest his personal wealth in the fund. The manager determines optimally the active fund's portfolio and the allocation of his wealth between the riskless asset and the fund. He maximizes expected utility of intertemporal consumption. Utility is exponential, i.e.,

$$-E \int_0^{\infty} \exp(-\bar{\alpha}\bar{c}_t - \beta t) dt, \quad (4)$$

where $\bar{\alpha}$ is the coefficient of absolute risk aversion, \bar{c}_t is consumption, and β is the same discount factor as for the investor. The manager can invest in stocks only through the active fund. We denote by \bar{y}_t the number of shares of the active fund held by the manager.

Allowing the manager to invest his personal wealth in the active fund is a convenient modeling assumption. It generates a well-defined objective that the manager maximizes when choosing the fund's portfolio. Moreover, the manager acts as trading counterparty to the investor: when the investor reduces her holdings of the active fund, effectively selling the stocks held by the fund, the manager takes the other side by raising his stake in the fund. Under the alternative assumption that the manager's personal wealth is invested in the riskless asset (or is ignored), we would need to specify the manager's objective. We would also need to introduce additional "smart-money" investors, who would hold individual stocks directly and act as counterparty to the fund investor. This would complicate the model without changing the main results.

The investor incurs a cost of investing in the active fund. This cost can be interpreted as a managerial perk or a reduced form for low managerial ability.¹² We consider the benchmark case where the cost is observable, but mainly focus on the case where the investor does not observe the cost and must infer it from the fund's performance. Such inference generates a positive relationship between performance and fund flows, which is at the heart of our analysis. We model the cost as a flow (i.e., the cost between t and $t + dt$ is of order dt), and assume that the flow cost is proportional

¹²An example of a managerial perk is late trading, whereby managers use their privileged access to the fund to buy or sell fund shares at stale prices. Late trading was common in many funds and led to the 2003 mutual-fund scandal. A related example is soft-dollar commissions, whereby funds inflate their brokerage commissions to pay for services that mainly benefit managers, e.g., promote the fund to new investors, or facilitate managers' late trading. Managerial ability could concern the quality of trade execution or of stock-picking. Stock-picking could be modeled explicitly in a setting where the active fund trades with other investors (e.g., smart-money). Indeed, in such a setting, the active fund's performance would depend on the quality of the manager's information.

to the number of shares y_t that the investor holds in the fund. We denote the coefficient of proportionality by C_t and assume that it follows the process

$$dC_t = \kappa(\bar{C} - C_t)dt + sdB_t^C, \quad (5)$$

where the mean-reversion parameter κ is the same as for F_t for simplicity, \bar{C} is a long-run mean, and B_t^C is a one-dimensional Brownian motion independent of B_t^D and B_t^F .

The manager does not incur the cost C_t on his personal investment in the active fund. This is consistent with the interpretation of C_t as a managerial perk. It is also consistent with the alternative setting where the manager's personal wealth is ignored, and the investor's counterparty are smart-money investors who hold individual stocks directly and are therefore not affected by C_t .

The gross return of the active fund is made of the dividends and capital gains of the stocks held by the fund. The net return that accrues to the investor is the gross return minus the cost. The gross and net returns of the index fund are computed similarly and coincide because the index fund entails no cost. The investor observes the returns and share prices of the index and active funds. The manager observes this information, as well as the price and return of each individual stock, and the expected dividend F_t .

We allow the manager to derive a benefit from the investor's participation in the active fund. This benefit can be interpreted as a perk or a fee. Because of the benefit, the manager cares about fund inflows and outflows. We model the benefit as a flow (i.e., the benefit between t and $t + dt$ is of order dt), and assume that the flow benefit is proportional to the number of shares y_t that the investor holds in the fund. We denote the coefficient of proportionality by B and assume for simplicity that it is constant over time.

Introducing the benefit B allows for a flexible specification of the manager's objective. When $B = 0$, the manager cares about fund performance only through his personal investment in the fund, and his objective is similar to the fund investor's.¹³ When $B > 0$, the manager is also concerned about the "commercial risk" that the investor might reduce her participation in the fund. In later sections we show that B affects only the average mispricing, but is not a determinant of momentum,

¹³The two objectives are not identical: the manager, who can invest in stocks only through the active fund, prefers the active portfolio to be well-diversified, while the investor can achieve diversification through the index fund. Restricting the manager to the active portfolio eliminates the indeterminacy problem that the active portfolio is unique only up to combinations with the index portfolio. Indeed, the manager could mix the active portfolio with the index, and offer that portfolio to the investor, while achieving the same personal portfolio through an offsetting short position in the index. The notion that active-fund managers over-diversify is empirically plausible. For example, Cohen, Polk and Silli (2008) find that managers maintain well-diversified portfolios even though the stocks receiving the largest weights in these portfolios (the managers' "best ideas") have large positive alphas.

reversal and comovement.¹⁴

The investor's cost and manager's benefit are assumed proportional to y_t for analytical convenience. At the same time, these variables are sensitive to how shares of the active fund are defined (e.g., they change with a stock split). We define one share of the fund by the requirement that its market value equals the equilibrium market value of the entire fund. Under this definition, the number of fund shares held by the investor and the manager in equilibrium sum to one, i.e.,

$$y_t + \bar{y}_t = 1. \quad (6)$$

In equilibrium, the active fund's portfolio is equal to the residual-supply portfolio θ minus the investor's holdings $x_t \mathbf{1}$ of the index fund. Therefore, the equilibrium market value of the fund is $(\theta - x_t \mathbf{1})S_t$. If the manager deviates from equilibrium and chooses one share to be $z_t \equiv (z_{1t}, \dots, z_{Nt})$, then z_t must satisfy

$$z_t S_t = (\theta - x_t \mathbf{1})S_t. \quad (7)$$

3 Symmetric Information

We start with the benchmark case where the cost C_t is observable by both the investor and the manager. We look for an equilibrium in which stock prices take the form

$$S_t = \frac{\bar{F}}{r} + \frac{F_t - \bar{F}}{r + \kappa} - (a_0 + a_1 C_t), \quad (8)$$

where (a_0, a_1) are two constant vectors. The first two terms are the present value of expected dividends, discounted at the riskless rate r , and the last term is a risk premium linear in the cost C_t . The investor's holdings of the active fund in our conjectured equilibrium are

$$y_t = b_0 - b_1 C_t, \quad (9)$$

where (b_0, b_1) are constants. We expect b_1 to be positive, i.e., the investor reduces her investment in the fund when C_t is high. We refer to an equilibrium satisfying (8) and (9) as linear.

¹⁴If the benefit B is a perk that the manager extracts from investors by imposing the cost C_t , a constant B implies that perk extraction is less efficient when C_t is high. Our analysis can be extended to the case where perk extraction is efficient ($B = C_t$), and more generally to the case where B is a linear function of C_t . Our focus is not to examine the effects of time-variation in B , but rather how the manager's concern with commercial risk ($B > 0$) affects equilibrium prices.

3.1 Manager's Optimization

The manager chooses the active fund's portfolio z_t , the number \bar{y}_t of fund shares that he owns, and consumption \bar{c}_t . The manager's budget constraint is

$$dW_t = rW_t dt + \bar{y}_t z_t (dD_t + dS_t - rS_t dt) + By_t dt - \bar{c}_t dt. \quad (10)$$

The first term is the return from the riskless asset, the second term is the return from the active fund in excess of the riskless asset, the third term is the manager's benefit from the investor's participation in the fund, and the fourth term is consumption. To compute the return from the active fund, we note that if one share of the fund corresponds to z_t shares of the stocks, the manager's effective stock holdings are $\bar{y}_t z_t$ shares. These holdings are multiplied by the vector $dR_t \equiv dD_t + dS_t - rS_t dt$ of the stocks' excess returns per share (referred to as returns, for simplicity). Using (1), (2), (5) and (8), we can write the vector of returns as

$$dR_t = [ra_0 + (r + \kappa)a_1 C_t - \kappa a_1 \bar{C}] dt + \sigma \left(dB_{Dt} + \frac{\phi dB_{Ft}}{r + \kappa} \right) - sa_1 dB_{Ct}. \quad (11)$$

Returns depend only on the cost C_t , and not on the expected dividend F_t . The covariance matrix of returns is

$$Cov_t(dR_t, dR'_t) = (f\Sigma + s^2 a_1 a'_1) dt, \quad (12)$$

where $f \equiv 1 + \phi^2/(r + \kappa)^2$ and $\Sigma \equiv \sigma\sigma'$. The matrix $f\Sigma$ represents the covariance driven purely by dividend (i.e., cashflow) news, and we refer to it as fundamental covariance. The matrix $s^2 a_1 a'_1$ represents the additional covariance introduced by flows into the index and active funds, and we refer to it as non-fundamental covariance.

The manager's optimization problem is to choose controls $(\bar{c}_t, \bar{y}_t, z_t)$ to maximize the expected utility (4) subject to the budget constraint (10), the normalization (7), and the investor's holding policy (9). We conjecture that the manager's value function is

$$\bar{V}(W_t, C_t) \equiv -\exp \left[- \left(r\bar{\alpha}W_t + \bar{q}_0 + \bar{q}_1 C_t + \frac{1}{2} \bar{q}_{11} C_t^2 \right) \right], \quad (13)$$

where $(\bar{q}_0, \bar{q}_1, \bar{q}_{11})$ are constants. The Bellman equation is

$$\max_{\bar{c}_t, \bar{y}_t, z_t} [-\exp(-\bar{\alpha}\bar{c}_t) + \mathcal{D}\bar{V} - \beta\bar{V}] = 0, \quad (14)$$

where $\mathcal{D}\bar{V}$ is the drift of the process \bar{V} under the controls $(\bar{c}_t, \bar{y}_t, z_t)$. Proposition 1 shows that the value function (13) satisfies the Bellman equation if $(\bar{q}_0, \bar{q}_1, \bar{q}_{11})$ satisfy a system of three scalar equations. We derive these equations explicitly after imposing market clearing, in the proof of Proposition 3

Proposition 1 *The value function (13) satisfies the Bellman equation (14) if $(\bar{q}_0, \bar{q}_1, \bar{q}_{11})$ satisfy a system of three scalar equations.*

In the proof of Proposition 1 we show that the optimization over $(\bar{c}_t, \bar{y}_t, z_t)$ can be reduced to optimization over the manager's consumption \bar{c}_t and effective stock holdings $\hat{z}_t \equiv \bar{y}_t z_t$. Given \hat{z}_t , the decomposition between \bar{y}_t and z_t is determined by the normalization (7). The first-order condition with respect to \hat{z}_t is

$$E_t(dR_t) = r\bar{\alpha}Cov_t(dR_t, \hat{z}_t dR_t) + (\bar{q}_1 + \bar{q}_{11}C_t)Cov_t(dR_t, dC_t). \quad (15)$$

Eq. (15) can be viewed as a pricing equation. Expected stock returns must compensate the manager for risk. A stock's risk is determined by the contribution to the manager's portfolio variance (first term in the right-hand side), and by the covariance with C_t (second term in the right-hand side).

3.2 Investor's Optimization

The investor chooses a number of shares x_t in the index fund and y_t in the active fund, and consumption c_t . The investor's budget constraint is

$$dW_t = rW_t dt + x_t \mathbf{1} dR_t + y_t (z_t dR_t - C_t dt) - c_t dt. \quad (16)$$

The first three terms are the returns from the riskless asset, the index fund, and the active fund (net of the cost C_t), and the fourth term is consumption. The investor's optimization problem is to choose controls (c_t, x_t, y_t) to maximize the expected utility (3) subject to the budget constraint (16). The investor takes the active fund's portfolio z_t as given and equal to its equilibrium value $\theta - x_t \mathbf{1}$. We conjecture that the investor's value function is

$$V(W_t, C_t) \equiv -\exp \left[- \left(r\alpha W_t + q_0 + q_1 C_t + \frac{1}{2} q_{11} C_t^2 \right) \right], \quad (17)$$

where (q_0, q_1, q_{11}) are constants. The Bellman equation is

$$\max_{c_t, x_t, y_t} [-\exp(-\alpha c_t) + \mathcal{D}V - \beta V] = 0, \quad (18)$$

where $\mathcal{D}V$ is the drift of the process V under the controls (c_t, x_t, y_t) . Proposition 2 shows that the value function (17) satisfies the Bellman equation (18) if (q_0, q_1, q_{11}) satisfy a system of three scalar equations. The proposition shows additionally that the optimal control y_t is linear in C_t , as conjectured in (9).

Proposition 2 *The value function (17) satisfies the Bellman equation (18) if (q_0, q_1, q_{11}) satisfy a system of three scalar equations. The optimal control y_t is linear in C_t .*

In the proof of Proposition 2, we show that the first-order conditions with respect to x_t and y_t are

$$E_t(\mathbf{1}dR_t) = r\alpha\text{Cov}_t[\mathbf{1}dR_t, (x_t\mathbf{1} + y_t z_t)dR_t] + (q_1 + q_{11}C_t)\text{Cov}_t(\mathbf{1}dR_t, dC_t), \quad (19)$$

$$E_t(z_t dR_t) - C_t dt = r\alpha\text{Cov}_t[z_t dR_t, (x_t\mathbf{1} + y_t z_t)dR_t] + (q_1 + q_{11}C_t)\text{Cov}_t(z_t dR_t, dC_t), \quad (20)$$

respectively. Eqs. (19) and (20) are analogous to the manager's first-order condition (15) in that they equate expected returns to risk. The difference with (15) is that the investor is constrained to two portfolios rather than N individual stocks. Eq. (15) is a vector equation with N components, while (19) and (20) are scalar equations derived by pre-multiplying expected returns with the vectors $\mathbf{1}$ and z_t of index- and active-fund weights. Note that the investor's expected return from the active fund in (20) is net of the cost C_t .

3.3 Equilibrium

In equilibrium, the active fund's portfolio z_t is equal to $\theta - x_t\mathbf{1}$, and the shares held by the manager and the investor sum to one. Combining these equations with the first-order conditions (15), (19) and (20), and the value-function equations (Propositions 1 and 2), yields a system of equations characterizing a linear equilibrium. Proposition 3 shows that this system has a unique solution. For notational convenience, we set

$$\Delta \equiv \theta\Sigma\theta'\mathbf{1}\Sigma\mathbf{1}' - (\mathbf{1}\Sigma\theta')^2.$$

The constant Δ is positive and becomes zero when the vectors $\mathbf{1}$ and θ are collinear.

Proposition 3 *There exists a unique linear equilibrium. The vector a_1 is given by*

$$a_1 = \gamma_1\Sigma p'_f, \quad (21)$$

where γ_1 is a positive constant and

$$p_f \equiv \theta - \frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'}\mathbf{1}. \quad (22)$$

As a first illustration of Proposition 3, consider the benchmark case where delegation is costless because the investor's cost C_t of investing in the active fund is constant and equal to zero. This case can be derived by setting the long-run mean \bar{C} and the diffusion coefficient s to zero. When $C_t = \bar{C} = s = 0$, the investor holds

$$y_t = b_0 = \frac{\bar{\alpha}}{\alpha + \bar{\alpha}} \quad (23)$$

shares in the active fund and zero shares in the index fund.¹⁵ The investor holds only the active fund because it offers a superior portfolio than the index fund at no cost. The relative shares of the investor and the manager in the active fund are determined by their risk-aversion coefficients. Stocks' expected returns are

$$E_t(dR_t) = ra_0 dt = \frac{r\alpha\bar{\alpha}}{\alpha + \bar{\alpha}} Cov_t(dR_t, \theta dR_t), \quad (24)$$

and are thus determined by the covariance with the residual-supply portfolio θ . The intuition is that because the index fund receives zero investment, the residual-supply portfolio coincides with the active portfolio z_t . Therefore, it also coincides with the manager's portfolio, and prices all stocks. Note that when $C_t = \bar{C} = s = 0$, the covariance matrix of returns $f\Sigma + s^2 a_1 a_1'$ is only cashflow driven and equal to $f\Sigma$.

Consider next the case where C_t varies over time. Following an increase in C_t , the investor reduces her investment y_t in the active fund. At the same time, because investing in the index fund is costless, the investor increases her investment x_t in that fund to maintain a constant overall exposure to the index. The net change to the investor's portfolio is proportional to the *flow portfolio*

$$p_f \equiv \theta - \frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'}\mathbf{1}.$$

The flow portfolio consists of the residual-supply portfolio θ , plus a position in the index that renders the covariance with the index equal to zero.¹⁶ By reducing her investment in the active

¹⁵Eq. (23) follows from (9) and (A.39). Eq. $x_t = 0$ follows from (23) and (A.30). Eq. (24) follows from (11), (12) and (A.40).

¹⁶The covariance between the index and the flow portfolio is $\mathbf{1}(f\Sigma + s^2 a_1 a_1')p_f' dt$. It is equal to zero because of $\mathbf{1}\Sigma p_f' = 0$ and (21).

fund, the investor is effectively selling a slice of the residual-supply portfolio. At the same time, she increases her investment in the index fund, thus buying a slice of the index portfolio. Because the investor maintains a constant overall exposure to the index, the net change in her portfolio is uncorrelated with the index, and is thus proportional to the flow portfolio. We use the term flow portfolio because it refers to a flow of assets from the investor to the manager.

The flow portfolio is a long-short portfolio. Long positions are in stocks whose residual supply is large and whose weight in the active fund exceeds that in the index. Conversely, short positions are in stocks whose residual supply is small and whose active weight is below the index weight.¹⁷ When C_t increases, the investor sells a slice of the flow portfolio, lowering her effective holdings of stocks corresponding to the long positions and raising those corresponding to the short positions.

Eqs. (8) and (21) imply that the effect of C_t on stock prices is

$$\frac{\partial S_t}{\partial C_t} = -\gamma_1 \Sigma p'_f.$$

An increase in C_t lowers the prices of stocks that covary positively with the flow portfolio and raises the prices of stocks covarying negatively.¹⁸ The intuition is that when C_t increases, the manager acquires a slice of the flow portfolio from the investor.¹⁹ As a result, he requires higher expected returns from stocks that covary positively with the flow portfolio, and the price of these stocks decreases. Conversely, the expected returns of stocks that covary negatively with the flow portfolio decrease, and their price increases. Lemma 1 characterizes the covariance between a stock and the flow portfolio in terms of the stock's idiosyncratic risk.

Lemma 1 *The covariance between stock returns and the return of the flow portfolio is*

$$\text{Cov}_t(dR_t, p_f dR_t) = \text{Cov}_t(d\epsilon_t, p_f d\epsilon_t), \quad (25)$$

where $d\epsilon_t \equiv (d\epsilon_{1t}, \dots, d\epsilon_{Nt})'$ denotes the residual from a regression of stock returns dR_t on the index return $\mathbf{1}dR_t$.

¹⁷Stock n receives positive weight in the flow portfolio if

$$\theta_n - \frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'} > 0 \Leftrightarrow \theta_n - x_t - \frac{\mathbf{1}\Sigma(\theta - x_t\mathbf{1})'}{\mathbf{1}\Sigma\mathbf{1}'} > 0 \Leftrightarrow z_{nt} - \frac{\mathbf{1}\Sigma z'_t}{\mathbf{1}\Sigma\mathbf{1}'} > 0 \Leftrightarrow \frac{z_{nt}}{\mathbf{1}\Sigma z'_t} > \frac{1}{\mathbf{1}\Sigma\mathbf{1}'}.$$

The left-hand side can be interpreted as stock n 's weight in the active fund, and the right-hand side as the stock's weight in the index fund. Note that these are not standard portfolio weights because the denominator involves the matrix Σ rather than the identity matrix.

¹⁸The covariance between stocks and the flow portfolio is $(f\Sigma + s^2 a_1 a'_1) p'_f dt$. Eq. (A.43) implies that this vector is proportional to $\Sigma p'_f$.

¹⁹This is accomplished through an increase in the manager's personal stake in the fund, and through a change in the fund's composition.

Lemma 1 implies that a stock covaries positively with the flow portfolio if its idiosyncratic movement $d\epsilon_{nt}$ (i.e., the part of its return that is orthogonal to the index) covaries positively with the idiosyncratic movement of the flow portfolio. This is likely to occur when the stock is in large residual supply, because it then receives positive weight in the flow portfolio, or when the stock's idiosyncratic movement covaries positively with other stocks in large residual supply.

While residual supply determines the sign of a stock's covariance with the flow portfolio, idiosyncratic risk determines the magnitude. A stock carrying no idiosyncratic risk is uncorrelated with the flow portfolio, regardless of its residual supply. Therefore, it is unaffected by changes in C_t , and carries no commercial risk from the manager's viewpoint. The intuition is that while changes in C_t prompt the investor to reallocate wealth between the active and the index fund, they do not affect her overall willingness to carry market risk. Changes in C_t thus have no effect on the index price, or on any other portfolio carrying only market risk; they affect only the cross-sectional dispersion of stock prices around the index.²⁰

The response of stock prices to C_t is reflected in expected returns. Corollary 1 shows that expected returns are given by a two-factor model, with the factors being the index and the flow portfolio.

Corollary 1 *Stocks' expected returns are given by*

$$E_t(dR_t) = \frac{r\alpha\bar{\alpha}}{\alpha + \bar{\alpha}} \frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'} Cov_t(dR_t, \mathbf{1}dR_t) + \Lambda_t Cov_t(dR_t, p_f dR_t), \quad (26)$$

where

$$\Lambda_t \equiv \frac{r\alpha\bar{\alpha}}{\alpha + \bar{\alpha}} + \frac{\gamma_1}{f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}} \left[(r + \kappa)C_t - \frac{s^2(\alpha\bar{q}_1 + \bar{\alpha}q_1)}{\alpha + \bar{\alpha}} \right]. \quad (27)$$

The factor risk premium Λ_t is increasing in C_t and B , and is positive for $C_t \geq \kappa\bar{C}/(r + \kappa)$.

The presence of the flow portfolio as a priced factor can be viewed as a mispricing relative to the CAPM in which the only factor is the market index. The factor risk premium Λ_t measures the severity of the mispricing. If $\Lambda_t > 0$, then stocks covarying positively with the flow portfolio are underpriced and earn abnormally high expected returns, while stocks covarying negatively are overpriced and earn abnormally low returns.

²⁰If the universe of assets includes more classes than only stocks, changes in C_t would affect the stock index.

That the market CAPM does not price stocks correctly in our model is not surprising because the index portfolio differs from the residual-supply portfolio. In fact, mispricing exists even when delegation is costless. Indeed, setting $C_t = \bar{C} = s = 0$ in (27), we find that the factor risk premium Λ_t is equal to $r\alpha\bar{\alpha}/(\alpha + \bar{\alpha})$. Since $\Lambda_t > 0$, stocks covarying positively with the flow portfolio are underpriced, while stocks covarying negatively are overpriced. It is because of this mispricing that the investor holds only the active fund under costless delegation.

Our main results concern not the existence of a mispricing per se, but the variation of the mispricing with parameters characterizing the delegation of portfolio management. Corollary 1 confirms that mispricing becomes more severe (Λ_t increases) when C_t increases. The corollary shows additionally that mispricing increases in B , the manager's concern with commercial risk. While the effect of C_t is through investor outflows from the active fund, the effect of B is through the manager's attempt to hedge against outflows. Hedging requires holding stocks that perform well when outflows occur, i.e., when C_t increases. Such stocks are those covarying negatively with the flow portfolio, e.g., because they are in small residual supply. The manager's concern with commercial risk adds to the demand for these stocks, exacerbating their overpricing, and lowering their expected returns.²¹

We next turn to the comovement between stocks. Eqs. (12) and (21) imply that the covariance matrix of returns is

$$\text{Cov}_t(dR_t, dR'_t) = (f\Sigma + s^2\gamma_1^2\Sigma p'_f p_f \Sigma) dt, \quad (28)$$

where the first term in parentheses is the fundamental covariance, driven purely by asset cashflows, and the second is the non-fundamental covariance, introduced by fund flows. The non-fundamental covariance can be positive or negative, depending on how stocks covary with the flow portfolio. Consider, for example, two stocks that are in large residual supply and covary positively with the flow portfolio. Following outflows from the active fund, both stocks drop in price, and this generates positive non-fundamental covariance. At the same time, the non-fundamental covariance between these stocks and a third stock that is in small residual supply is negative because the latter stock's price increases following outflows.

We finally determine the autocorrelation of stock returns. As in the rest of our analysis, returns are evaluated over an infinitesimal time period, and therefore concern a single point in time. We

²¹The manager has a hedging motive even when $B = 0$, but the sign is opposite that when B is large. Recall that when $B = 0$, the manager cares about fund performance only through his personal investment in the fund. When C_t is high, mispricing is severe and the fund has profitable investment opportunities. The manager can hedge against a reduction in these opportunities by holding stocks that perform well when C_t decreases. This consideration plays an important role in our analysis of momentum in Section 5.

compute the covariance between the vector of returns at time t and the same vector at time $t' > t$. The diagonal elements of the autocovariance matrix correspond to the autocovariance of individual stocks. The non-diagonal elements correspond to lead-lag effects, i.e., whether the past return of one stock predicts the future return of another. Corollary 2 shows that the autocovariance matrix is equal to the non-fundamental (contemporaneous) covariance times a negative scalar. The matrices are proportional because they are both driven by fund flows, a point that we explain below in the case of the autocovariance matrix.

Corollary 2 *The covariance between stock returns at time t and those at time $t' > t$ is*

$$Cov_t(dR_t, dR_{t'}) = -s^2(r + \kappa)\gamma_1^2 e^{-\kappa(t'-t)} \Sigma p'_f p_f \Sigma (dt)^2. \quad (29)$$

A first implication of Corollary 2 is that stocks exhibit negative autocovariance, i.e., return reversal. Consider, for example, a stock that is in large residual supply and covaries positively with the flow portfolio. The stock's return can be negative because of a negative cashflow shock or because outflows from the active fund generate selling pressure. While cashflow shocks do not affect future expected returns, outflows do, and they cause the stock's expected return to rise. This implies negative autocovariance.

A second implication of Corollary 2 is that lead-lag effects are negative for stock pairs whose covariance with the flow portfolio has the same sign. Consider, for example, two stocks that are in large residual supply and covary positively with the flow portfolio. A negative return of one stock can be caused by cashflows or by outflows from the active fund. Cashflows generate no predictability, but outflows cause the expected return of the other stock to rise. This implies a negative lead-lag effect: a negative return of one stock predicts that the other stock will rise. The negative return also predicts that stocks in small residual supply will drop, a positive lead-lag effect.

4 Asymmetric Information

We next assume that the investor does not observe the cost C_t , and must infer it from fund returns and prices. The manager observes C_t because he observes the return of each stock, and therefore knows the active fund's gross return in addition to the net return.

We look for an equilibrium with the following characteristics. The investor's conditional distribution of C_t is normal with mean \hat{C}_t . The variance of the conditional distribution is, in general, a

deterministic function of time, but we focus on a steady state where it is constant.²² The investor's holdings of the active fund are

$$y_t = b_0 - b_1 \hat{C}_t, \quad (30)$$

where (b_0, b_1) are constants. Eq. (30) has the same form as under symmetric information, except that C_t is replaced by its expectation \hat{C}_t . Stock prices are

$$S_t = \frac{\bar{F}}{r} + \frac{F_t - \bar{F}}{r + \kappa} - (a_0 + a_1 \hat{C}_t + a_2 C_t), \quad (31)$$

where (a_0, a_1, a_2) are three constant vectors. Eq. (31) is more complicated than under symmetric information because prices depend on both \hat{C}_t and C_t : the investor's expectation \hat{C}_t determines her active-fund holdings, and the true value C_t (known to the manager) forecasts the investor's future holdings. Under symmetric information, the effect of C_t depends on the covariance between returns and the flow portfolio. We conjecture that the same is true under asymmetric information, i.e., there exist constants (γ_1, γ_2) such that for $i = 1, 2$,

$$a_i = \gamma_i \Sigma p'_f. \quad (32)$$

We refer to an equilibrium satisfying (30) and (31) as linear.

4.1 Investor's Inference

The investor updates her beliefs about the cost C_t using information on fund returns and prices. She observes the share prices $z_t S_t$ of the active fund and $\mathbf{1} S_t$ of the index fund. These are informative because C_t affects the vector of stock prices S_t . Stock prices, however, do not reveal C_t perfectly because they also depend on the time-varying expected dividend F_t that the investor does not observe.

In addition to prices, the investor observes the net-of-cost return of the active fund, $z_t dR_t - C_t dt$, and the return of the index fund, $\mathbf{1} dR_t$. Because the investor observes prices, she also observes capital gains, and therefore can deduce net dividends (i.e., dividends minus C_t). Net dividends are the incremental information that returns provide to the investor, and we use them instead of returns when solving the investor's inference problem.

²²The steady state is reached in the limit when time t becomes large.

In equilibrium, the active fund's portfolio z_t is equal to $\theta - x_t \mathbf{1}$. Since the investor knows x_t , observing the price and net dividends of the index and active funds is informationally equivalent to observing the price and net dividends of the index fund and of a hypothetical fund holding the residual-supply portfolio θ . Therefore, we can take the investor's information to be the net dividends of the residual-supply portfolio $\theta dD_t - C_t dt$, the dividends of the index fund $\mathbf{1} dD_t$, the price of the residual-supply portfolio θS_t , and the price of the index fund $\mathbf{1} S_t$. We solve the investor's inference problem using recursive (Kalman) filtering.

Proposition 4 *The mean \hat{C}_t of the investor's conditional distribution of C_t evolves according to the process*

$$d\hat{C}_t = \kappa(\bar{C} - \hat{C}_t)dt - \beta_1 \left\{ p_f [dD_t - E_t(dD_t)] - (C_t - \hat{C}_t)dt \right\} - \beta_2 p_f \left[dS_t + a_1 d\hat{C}_t - E_t(dS_t + a_1 d\hat{C}_t) \right], \quad (33)$$

where

$$\beta_1 \equiv \hat{s}^2 \left[1 - (r + k) \frac{\gamma_2 \Delta}{\mathbf{1} \Sigma \mathbf{1}'} \right] \frac{\mathbf{1} \Sigma \mathbf{1}'}{\Delta}, \quad (34)$$

$$\beta_2 \equiv \frac{s^2 \gamma_2}{\frac{\phi^2}{(r + \kappa)^2} + \frac{s^2 \gamma_2^2 \Delta}{\mathbf{1} \Sigma \mathbf{1}'}} \quad (35)$$

and \hat{s}^2 denotes the distribution's steady-state variance. The variance \hat{s}^2 is the unique positive solution of the quadratic equation

$$\hat{s}^4 \left[1 - (r + \kappa) \frac{\gamma_2 \Delta}{\mathbf{1} \Sigma \mathbf{1}'} \right]^2 \frac{\mathbf{1} \Sigma \mathbf{1}'}{\Delta} + 2\kappa \hat{s}^2 - \frac{\frac{s^2 \phi^2}{(r + \kappa)^2}}{\frac{\phi^2}{(r + \kappa)^2} + \frac{s^2 \gamma_2^2 \Delta}{\mathbf{1} \Sigma \mathbf{1}'}} = 0. \quad (36)$$

The term in β_1 in (33) represents the investor's learning from net dividends. The investor lowers her estimate of the cost C_t if the active fund's net dividends are above expectations. Of course, net dividends can be high not only because C_t is low, but also because gross dividends are high. The investor attempts to adjust for this by comparing with the dividends of the index fund. The adjustment corresponds to the term $\frac{\mathbf{1} \Sigma \theta'}{\mathbf{1} \Sigma \mathbf{1}'} \mathbf{1}$, which enters (33) through the flow portfolio p_f .

The term in β_2 in (33) represents the investor's learning from prices. The investor lowers her estimate of C_t if the active fund's price is above expectations. Indeed, the price could be high

because the manager knows privately that C_t is low, and anticipates that the investor will increase her participation in the fund, causing the price to rise, as she learns about C_t . As with dividends, the investor needs to account for the fact that the price of the active fund can be high not only because C_t is low, but also because the manager expects future dividends to be high (F_t small). She attempts to adjust for this by comparing with the price of the index fund. The coefficient β_2 measures the strength of learning from prices. Eq. (34) confirms that there is no learning from prices ($\beta_2 = 0$) if these do not reveal information about C_t ($\gamma_2 = 0$), or if it is impossible to extract this information because the variance of the expected dividend F_t is large ($\phi = \infty$).

Because the investor compares the performance of the active fund to that of the index fund, she is effectively using the index as a benchmark. Note that benchmarking is not part of an explicit contract tying the manager's compensation to the index. Compensation is tied to the index only implicitly: if the active fund outperforms the index, the investor infers that C_t is low and increases her participation in the fund.

We next examine how an increase in a stock's dividend or price affects \hat{C}_t , the investor's expectation of C_t . Eq. (33) shows that \hat{C}_t decreases if the stock receives positive weight in the flow portfolio p_f , and increases if the weight is negative.²³ Intuitively, stocks that receive positive weight are those whose residual supply is large and whose weight in the active fund exceeds that in the index. Good news about these stocks raise the performance of the active fund more than the index, and reduce \hat{C}_t . Conversely, good news about stocks that receive negative weight in the flow portfolio raise \hat{C}_t because the performance of the active fund rises less than the index.

Our analysis implies that a single portfolio—the flow portfolio—characterizes two distinct phenomena: the investor's flow into each stock in response to changes in her beliefs about the cost (Section 3), and the informational flow from stock performance to investor beliefs about the cost (Proposition 4). The dual role of the flow portfolio implies an amplification effect that is central to our results. Suppose that a stock has a positive cashflow shock. If the stock is in large residual supply (i.e., weight in the active fund exceeds that in the index), then the investor becomes more optimistic about the active fund and increases her participation. The investor's inflow, in turn, generates demand for the stock and pushes its price up. Conversely, if the stock is in small residual supply, then the investor reduces her participation in the active fund and migrates to the index fund. The investor's outflow again generates demand for the stock (because the stock's index weight exceeds its active weight) and pushes its price up. In both cases, cashflow shocks amplify their way

²³The gradient of $d\hat{C}_t$ with respect to dD_t is $-\beta_1 p_f$ and with respect to dS_t is $-\beta_2 p_f$.

to prices through fund inflows and outflows.

The intuition why the exact same portfolio characterizes asset flows and information flows is as follows. Recall from Section 3 that the investor maintains a constant exposure to the index, and therefore her flows between active and index fund constitute a portfolio with zero index beta. A zero-beta portfolio also characterizes learning because the investor adjusts the performance of the active fund by that of the index. The adjustment coefficient is derived by regressing the return of the residual-supply portfolio on the index, and the residual of this regression is the return of a zero-beta portfolio.

4.2 Optimization

The manager chooses controls $(\bar{c}_t, \bar{y}_t, z_t)$ to maximize the expected utility (4) subject to the budget constraint (10), the normalization (7), and the investor's holding policy (30). Since stock prices depend on both \hat{C}_t and C_t , the same is true for the manager's value function. We conjecture that the value function is

$$\bar{V}(W_t, \hat{C}_t, C_t) \equiv -\exp\left[-\left(r\bar{\alpha}W_t + \bar{q}_0 + (\bar{q}_1, \bar{q}_2)\bar{X}_t + \frac{1}{2}\bar{X}_t'\bar{Q}\bar{X}_t\right)\right], \quad (37)$$

where $\bar{X}_t \equiv (\hat{C}_t, C_t)'$, $(\bar{q}_0, \bar{q}_1, \bar{q}_2)$ are constants, and \bar{Q} is a constant symmetric 2×2 matrix.

Proposition 5 *The value function (37) satisfies the Bellman equation (14) if $(\bar{q}_0, \bar{q}_1, \bar{q}_2, \bar{Q})$ satisfy a system of six scalar equations.*

The investor chooses controls (c_t, x_t, y_t) to maximize the expected utility (3) subject to the budget constraint (16) and the manager's portfolio policy $z_t = \theta - x_t\mathbf{1}$. Unlike the manager, the investor does not observe C_t , and so her value function depends only on (W_t, \hat{C}_t) . We conjecture that the value function is

$$V(W_t, C_t) \equiv -\exp\left[-\left(r\alpha W_t + q_0 + q_1\hat{C}_t + \frac{1}{2}q_{11}\hat{C}_t^2\right)\right], \quad (38)$$

where (q_0, q_1, q_{11}) are constants.

Proposition 6 *The value function (38) satisfies the Bellman equation (18) if (q_0, q_1, q_{11}) satisfy a system of three scalar equations. The optimal control y_t is linear in \hat{C}_t .*

4.3 Equilibrium

Proposition 7 shows that the system of equations characterizing a linear equilibrium has a solution.

Proposition 7 *There exists a linear equilibrium. The vectors (a_1, a_2) are given by (32), and the constants (γ_1, γ_2) are positive.*

Since $\gamma_1 > 0$, an increase in the investor's expectation of C_t lowers the prices of stocks that covary positively with the flow portfolio and raises the prices of stocks covarying negatively. The intuition is as with symmetric information: if the investor believes that C_t has increased, she reduces her stake in the active fund and invests in the index fund. This amounts to selling a slice of the flow portfolio, i.e., selling stocks in large residual supply and buying stocks in small residual supply. The manager must take the other side of the transaction, and is induced to do so if stocks covarying positively with the flow portfolio drop and stocks covarying negatively rise. Since $\gamma_2 > 0$, the same price movements occur when the increase concerns C_t instead of the expectation \hat{C}_t . In that case, the investor's current stake in the active fund does not change, but prices move in anticipation of the outflows that the investor will generate as she learns about C_t .

We next examine how the price response to \hat{C}_t and C_t is reflected in expected returns. Corollary 3 shows that the same two factors (index and flow portfolio) that price stocks under symmetric information do so also under asymmetric information.

Corollary 3 *Stocks' expected returns are given by*

$$E_t(dR_t) = \frac{r\alpha\bar{\alpha}}{\alpha + \bar{\alpha}} \frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'} Cov_t(dR_t, \mathbf{1}dR_t) + \Lambda_t Cov_t(dR_t, p_f dR_t), \quad (39)$$

where

$$\Lambda_t \equiv \frac{r\alpha\bar{\alpha}}{\alpha + \bar{\alpha}} + \frac{1}{f + \frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'}} \left[g_1 \hat{C}_t + g_2 C_t - \frac{\alpha(k_1 \bar{q}_1 + k_2 \bar{q}_2) + \bar{\alpha} k_1 q_1}{\alpha + \bar{\alpha}} \right], \quad (40)$$

and the constants $g_1 > 0$, $g_2 < 0$, $k > 0$, $k_1 > 0$ and $k_2 > 0$, are defined by (B.8), (B.9), (B.15), (B.16) and (B.17), respectively.

The factor risk premium Λ_t associated to the flow portfolio characterizes mispricing relative to the market CAPM. Since $g_1 > 0$, Λ_t increases in \hat{C}_t , a result consistent with the price response.

Indeed, consider a stock that is in large residual supply and covaries positively with the flow portfolio. Following an increase in \hat{C}_t , the investor sells the stock and the price drops. Corollary 3 implies that the stock's expected return rises, a result consistent with the stock becoming cheaper relative to fundamental value.

Consider next the effect of C_t holding \hat{C}_t constant. Since $g_2 < 0$, Λ_t decreases in C_t , a result that seems at odds with the price response. Indeed, consider again a stock in large residual supply. Following an increase in C_t , the price drops in anticipation of the investor's future outflows. But while the stock becomes cheaper relative to fundamental value, Corollary 3 implies that its expected return declines. The explanation for the apparent inconsistency lies in the time-path of the (instantaneous) expected return. While expected return declines in the short run, it rises in the long run, as the outflows occur. It is the long-run rise in expected return that causes the price to drop. The intuition for the short-run decline underlies our results on momentum and reversal, and is deferred until Section 5.

We next turn to the comovement between stocks. Corollary 4 shows that the instantaneous covariance matrix of returns resembles closely its symmetric-information counterpart (12). Both matrices are the sum of $f\Sigma$, the fundamental covariance driven purely by asset cashflows, and a scalar multiple of $\Sigma p'_f p_f \Sigma$, the non-fundamental covariance introduced by fund flows.

Corollary 4 *The covariance matrix of stock returns is*

$$\text{Cov}_t(dR_t, dR'_t) = (f\Sigma + k\Sigma p'_f p_f \Sigma) dt. \quad (41)$$

The resemblance between covariance matrices obscures an important qualitative difference. Under symmetric information, fund flows depend only on the cost C_t , and are independent of cashflows. Cashflows thus affect only the fundamental covariance $f\Sigma$. Under asymmetric information, fund flows depend on cashflows because the latter affect \hat{C}_t . Comovement between cashflows and fund flows is included in the non-fundamental covariance $k\Sigma p'_f p_f \Sigma$. But while that matrix includes cashflow effects, it is proportional to its symmetric-information counterpart $s^2 \gamma_1^2 \Sigma p'_f p_f \Sigma$.

The explanation for the proportionality lies in the dual role of the flow portfolio. Consider two stocks that are in large residual supply and covary positively with the flow portfolio. Both stocks drop in price following outflows from the active fund—an event triggered by increases in C_t under symmetric information, and in \hat{C}_t under asymmetric information. The resulting positive covariance

between the two stocks is driven purely by the price impact of fund flows. Asymmetric information induces an additional positive covariance, between cashflows and fund flows. Because the two stocks are in large residual supply, negative cashflow news of one stock trigger outflows from the active fund, and this lowers the price of the other stock. Therefore, comovement between cashflows and fund flows works in the same direction as that driven purely by fund flows. This explains why the two types of comovement (which jointly constitute the non-fundamental covariance) are described by proportional matrices.

Comparing the non-fundamental covariance under symmetric and asymmetric information amounts to comparing the scalars multiplying the matrix $\Sigma p'_f p_f \Sigma$. A natural conjecture is that the non-fundamental covariance is larger under asymmetric information because in that case it includes the covariance between cashflows and fund flows. A countervailing effect, however, is that the covariance driven purely by fund flows is smaller under asymmetric information because the investor's expectation \hat{C}_t varies less than the true value C_t . Nevertheless, Proposition 8 confirms the conjecture.

Proposition 8 *The non-fundamental covariance is larger under asymmetric than under symmetric information ($k > s^2 \gamma_1^2$).*

Considering the diagonal terms of the covariance matrix yields the immediate corollary that the volatility of individual stocks is larger under asymmetric information. This corollary is closely related to the amplification effect described in Section 4.1: under asymmetric information, cashflow shocks amplify their way to prices through fund flows, and this makes stocks more volatile.

Corollary 5 *The volatility of individual stocks is larger under asymmetric than under symmetric information.*

We finally determine the autocorrelation of stock returns. As in the case of symmetric information, the autocovariance matrix is equal to the non-fundamental covariance times a negative scalar. Therefore, stocks exhibit return reversal, and lead-lag effects are negative for stock pairs whose covariance with the flow portfolio has the same sign.

Corollary 6 *The covariance between stock returns at time t and those at time $t' > t$ is*

$$Cov_t(dR_t, dR_{t'}) = \left[\chi_1 e^{-(\kappa+\rho)(t'-t)} + \chi_2 e^{-\kappa(t'-t)} \right] \Sigma p'_f p_f \Sigma (dt)^2, \quad (42)$$

where the constants $\rho > 0$, $\chi_1 < 0$ and $\chi_2 < 0$ are defined by (B.10), (B.63) and (B.64), respectively.

While reversal also occurs under symmetric information, the underlying mechanisms are more subtle when information is asymmetric. Consider a stock that is in large residual supply and covaries positively with the flow portfolio. Under symmetric information, a negative return of the stock has predictive power for subsequent returns when it is generated by outflows from the active fund. Under asymmetric information, predictive power exists even when the return is generated by a negative cashflow shock since this triggers outflows. Note that a negative return can also be generated by the expectation of future outflows (i.e., increase in C_t rather than \hat{C}_t). This, however, yields the prediction that returns in the short run will be low, which is in the opposite direction than in the two previous cases. Corollary 6 shows that the combined effects of cashflows and \hat{C}_t dominate that of C_t , and the autocovariance of individual stocks is negative at any horizon.

An additional implication of Corollary 6 concerns the rate at which the autocovariance decays over time. Corollary 2 shows that the rate under symmetric information is κ , the mean-reversion parameter of the cost C_t . Under asymmetric information, autocovariance decays at a mixture of the rates κ and $\kappa + \rho$. The latter rate characterizes the mean-reversion of the investor's expectational error $\hat{C}_t - C_t$, and is larger than κ because of learning. Therefore, reversal occurs faster under asymmetric information, a result consistent with the larger non-fundamental volatility of individual stocks (Proposition 8).

5 Asymmetric Information and Gradual Adjustment

We next extend the analysis of asymmetric information to the case where the investor can adjust only gradually her investment in the active fund. Gradual adjustment can result from contractual restrictions or institutional decision lags.²⁴ We model these frictions as a flow cost $\psi(dy_t/dt)^2/2$ that the investor incurs when changing the number y_t of active-fund shares that she owns.²⁵

We look for an equilibrium with the following new features relative to Section 4. Since y_t cannot be set instantaneously to its optimal level, it becomes a state variable and affects prices.

²⁴An example of contractual restrictions is lock-up periods, often imposed by hedge funds, which require investors not to withdraw capital for a pre-specified time period. Institutional decision lags can arise for investors such as pension funds, foundations or endowments, where decisions are made by boards of trustees that meet infrequently.

²⁵The advantage of the quadratic cost over other formulations (such as an upper bound on $|dy_t/dt|$) is that it preserves the linearity of the model.

Stock prices are

$$S_t = \frac{\bar{F}}{r} + \frac{F_t - \bar{F}}{r + \kappa} - (a_0 + a_1 \hat{C}_t + a_2 C_t + a_3 y_t), \quad (43)$$

where a_0 is a constant vector and

$$a_i = \gamma_i \Sigma p'_f, \quad (44)$$

for constants $(\gamma_1, \gamma_2, \gamma_3)$ and $i = 1, 2, 3$. The investor's speed of adjustment $dy_t/dt \equiv v_t$ is

$$v_t = b_0 - b_1 \hat{C}_t - b_2 y_t, \quad (45)$$

where (b_0, b_1, b_2) are constants. We expect (b_1, b_2) to be positive, i.e., the investor reduces her investment in the active fund faster when she estimates C_t to be large or when her investment is large. We refer to an equilibrium satisfying (43) and (45) as linear. Inference in equilibrium is as in Section 4.1, and Proposition 4 continues to hold.

5.1 Optimization

The manager chooses controls $(\bar{c}_t, \bar{y}_t, z_t)$ to maximize the expected utility (4) subject to the budget constraint (10), the normalization (7), and the investor's holding policy (45). Since stock prices depend on (\hat{C}_t, C_t, y_t) , the same is true for the manager's value function. We conjecture that the value function is

$$\bar{V}(W_t, \hat{C}_t, C_t, y_t) \equiv - \exp \left[- \left(r \bar{\alpha} W_t + \bar{q}_0 + (\bar{q}_1, \bar{q}_2, \bar{q}_3) \bar{X}_t + \frac{1}{2} \bar{X}'_t \bar{Q} \bar{X}_t \right) \right], \quad (46)$$

where $\bar{X}_t \equiv (\hat{C}_t, C_t, y_t)'$, $(\bar{q}_0, \bar{q}_1, \bar{q}_2, \bar{q}_3)$ are constants, and \bar{Q} is a constant symmetric 3×3 matrix.

Proposition 9 *The value function (37) satisfies the Bellman equation (14) if $(\bar{q}_0, \bar{q}_1, \bar{q}_2, \bar{q}_3, \bar{Q})$ satisfy a system of ten scalar equations.*

The investor chooses controls (c_t, x_t, v_t) to maximize the expected utility (3) subject to the budget constraint (16) and the manager's portfolio policy $z_t = \theta - x_t \mathbf{1}$. Under gradual adjustment, it is convenient to solve the optimization problem in two steps. In a first step, we study optimization

over (c_t, x_t) , assuming that v_t is given by (45). We solve this problem using dynamic programming, and conjecture the value function

$$V(W_t, C_t, y_t) \equiv -\exp\left[-\left(r\alpha W_t + q_0 + (q_1, q_2)X_t + \frac{1}{2}X_t'QX_t\right)\right], \quad (47)$$

where $X_t \equiv (\hat{C}_t, y_t)$, (q_0, q_1, q_2) are constants, and Q is a constant symmetric 2×2 matrix. The Bellman equation is

$$\max_{c_t, x_t} [-\exp(-\alpha c_t) + \mathcal{D}V - \beta V] = 0, \quad (48)$$

where $\mathcal{D}V$ is the drift of the process V under the controls (c_t, x_t) . The second step of the solution method is to derive conditions under which the control v_t given by (45) is optimal.

Proposition 10 *The value function (47) satisfies the Bellman equation (48) if (q_0, q_1, q_2, Q) satisfy a system of six scalar equations. The control v_t given by (45) is optimal if (b_0, b_1, b_2) satisfy a system of three scalar equations.*

5.2 Equilibrium

Proposition 11 shows that the system of equations characterizing a linear equilibrium has a solution.

Proposition 11 *There exists a linear equilibrium. The vectors (a_1, a_2, a_3) are given by (44), the constants (γ_1, γ_2) are positive and the constant γ_3 is negative.*

Since $\gamma_1 > 0$, $\gamma_2 > 0$ and $\gamma_3 < 0$, an increase in \hat{C}_t , an increase in C_t and a decrease in y_t have all the same effect: lower the prices of stocks that covary positively with the flow portfolio and raise the prices of stocks covarying negatively. This is because in all three cases the investor moves out of the active fund. A decrease in y_t is a current outflow, while increases in \hat{C}_t and C_t trigger future outflows. An increase in \hat{C}_t does not trigger an instant outflow because of the adjustment cost.

Corollary 7 shows that expected stock returns are determined by the covariance with the same two factors (index and flow portfolio) as under instantaneous adjustment. The factor risk premium Λ_t associated to the flow portfolio characterizes mispricing relative to the market CAPM. Examining how Λ_t depends on \hat{C}_t , C_t and y_t allows us to trace the dynamic response of prices to shocks.

Corollary 7 *Stocks' expected returns are given by*

$$E_t(dR_t) = \frac{r\alpha\bar{\alpha}}{\alpha + \bar{\alpha}} \frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'} Cov_t(dR_t, \mathbf{1}dR_t) + \Lambda_t Cov_t(dR_t, p_f dR_t), \quad (49)$$

where

$$\Lambda_t \equiv r\bar{\alpha} + \frac{1}{f + \frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'}} \left(\tilde{g}_1 \hat{C}_t + g_2 C_t + g_3 y_t - k_1 \bar{q}_1 - k_2 \bar{q}_2 \right), \quad (50)$$

and the constants $\tilde{g}_1 < 0$, $g_2 < 0$, $g_3 < 0$, $k > 0$, $k_1 > 0$ and $k_2 > 0$, are defined by (C.2), (B.9), (C.3), (B.15), (B.16) and (B.17), respectively.

Consider a stock that is in large residual supply and covaries positively with the flow portfolio. Following an increase in \hat{C}_t , the stock's price drops instantly in anticipation of the investor's future outflows. But while the stock becomes cheap relative to fundamental value, its expected return declines, meaning that underperformance is expected to continue. This follows from Corollary 7 because the factor risk premium Λ_t associated to the flow portfolio decreases in \hat{C}_t ($\tilde{g}_1 < 0$). As time passes and outflows occur, the stock's expected return rises, reflecting the stock's cheapness relative to fundamental value. This follows from Corollary 7 because outflows correspond to a decrease in y_t and therefore an increase in Λ_t ($g_3 < 0$). The same pattern of momentum and reversal occurs following an increase in C_t rather than \hat{C}_t , because Λ_t decreases in C_t ($g_2 < 0$).

Why is a price decline away from fundamental value followed by a decline in expected return? And why is the manager willing to buy a stock that is expected to underperform? The intuition is that the stock will eventually overperform and earn the manager a high expected return over a long horizon. The manager could earn an even higher expected return by not holding the stock during the period of expected underperformance and buying when that period ends. This, however, presents a risk: the stock could overperform during that period, approaching again fundamental value, in which case the manager would earn a low expected return. The manager can hedge against that risk by buying the stock immediately, and it is this hedging demand that drives the price to a level from which underperformance is expected to occur.

The hedging demand can be seen from the manager's first-order condition. In the proof of Proposition 9 we show that the first-order condition is

$$E_t(dR_t) = r\bar{\alpha} Cov_t(dR_t, \hat{z}_t dR_t) + \bar{f}_1(\bar{X}_t) Cov_t(dR_t, d\hat{C}_t) + \bar{f}_2(\bar{X}_t) Cov_t(dR_t, dC_t), \quad (51)$$

where $\bar{f}_1(\bar{X}_t)$ and $\bar{f}_2(\bar{X}_t)$ are functions of $\bar{X}_t \equiv (\hat{C}_t, C_t, y_t)'$. Eq. (51) requires that expected stock returns compensate the manager for risk, determined by the contribution to the manager's portfolio variance (first term in the right-hand side), and by the covariances with \hat{C}_t and C_t (second and third terms in the right-hand side). The second and third terms correspond to the manager's hedging demand. Immediately following an increase in \hat{C}_t , the first term does not change because the investor cannot move instantly out of the active fund. The second and third terms, however, decrease for a stock covarying positively with the flow portfolio. Indeed, when \hat{C}_t is high, mispricing is severe and the active fund has profitable investment opportunities. The manager can hedge against a reduction in these opportunities by holding stocks that perform well when \hat{C}_t or C_t decrease, and these are stocks that covary positively with the flow portfolio.

Since shocks to \hat{C}_t and C_t generate momentum and reversal, the autocorrelation of stock returns should exhibit the same pattern. Corollary 8 shows that the autocorrelation is indeed positive for short lags and negative for long lags. More generally, the autocovariance matrix of returns is proportional to the non-fundamental covariance, with the proportionality coefficient being positive for short lags and negative for long lags.

Corollary 8 *The covariance between stock returns at time t and those at time $t' > t$ is*

$$Cov_t(dR_t, dR_{t'}) = \left[\chi_1 e^{-(\kappa+\rho)(t'-t)} + \chi_2 e^{-\kappa(t'-t)} + \chi_3 e^{-b_2(t'-t)} \right] \Sigma p'_f p_f \Sigma (dt)^2, \quad (52)$$

where the constants $\rho > 0$ and (χ_1, χ_2, χ_3) are defined by (B.10), (C.45), (C.46) and (C.47), respectively. The function $\chi(u) \equiv \chi_1 e^{-(\kappa+\rho)u} + \chi_2 e^{-\kappa u} + \chi_3 e^{-b_2 u}$ is positive when u is below a threshold $\hat{u} > 0$ and is negative when u exceeds \hat{u} .

Corollary 8 has the additional implication that lead-lag effects change sign over time. Suppose that a stock covarying positively with the flow portfolio earns a negative return, in which case the active fund expects outflows. In anticipation of these outflows, stocks covarying positively with the flow portfolio become cheap and stocks covarying negatively become expensive. Prices, however, adjust only partially towards expected post-outflow levels because the manager prefers to exploit current mispricings rather than betting that mispricings will become more attractive. As a result, stocks covarying positively with the flow portfolio are expected to underperform in the short run—a positive lead-lag effect—while stocks covarying negatively are expected to outperform—a negative lead-lag effect. These effects reverse in the long run: stocks covarying positively with the flow portfolio earn a high expected return, reflecting their cheapness relative to fundamental value,

while stocks covarying negatively earn a low expected return. In summary, lead-lag effects for stock pairs whose covariance with the flow portfolio has the same sign are positive in the short run and negative in the long run, while the opposite holds when the signs are different.

Our theory predicts not only that momentum is followed by reversal, but also that the cumulative effect is a reversal. Indeed, poor performance by the active fund generates momentum because prices adjust only partially to reflect the ensuing outflows. But because an adjustment occurs and prices move away from fundamental values, the expected return over a long horizon is a reversal. Corollary 9 confirms that a stock's return at time t is negatively correlated with the stock's long-horizon return. We define stocks' long-horizon returns from time t on as $\lim_{T \rightarrow \infty} e^{-r(T-t)} R_{t,T}$, where $R_{t,T} \equiv \int_t^T dR_{t'} e^{r(T-t')}$ denotes cumulative returns between t and $T > t$.²⁶

Corollary 9 *The covariance between stock returns at time t and long-horizon returns from time t on is*

$$\text{Cov}_t \left(dR_t, \lim_{T \rightarrow \infty} e^{-r(T-t)} R'_{t,T} \right) = \left(\int_0^\infty \chi(u) e^{-ru} du \right) \Sigma p'_f p_f \Sigma dt. \quad (53)$$

Moreover, $\int_0^\infty \chi(u) e^{-ru} du < 0$.

Our analysis so far focuses on the predictive power of past returns. Cashflow shocks have no predictive power under symmetric information because they are independent of fund flows. Under asymmetric information, however, cashflow shocks trigger fund flows and can therefore predict returns. Cashflow shocks at time t are dividends dD_t or innovations to expected dividends dF_t . Corollary 10 shows that the covariance matrix between either shock and returns at $t' > t$ is proportional to the non-fundamental covariance matrix, with the proportionality coefficient being positive for short lags and negative for long lags. Therefore, cashflow shocks generate the same type of predictability as returns.

Corollary 10 *The covariance between cashflow shocks (dD_t, dF_t) at time t and returns at time $t' > t$ is given by*

$$\text{Cov}_t(dD_t, dR'_{t'}) = \frac{\beta_1(r + \kappa) \text{Cov}_t(dF_t, dR'_{t'})}{\beta_2 \phi} = \left[\chi_1^D e^{-(\kappa + \rho)(t'-t)} + \chi_2^D e^{-b_2(t'-t)} \right] \Sigma p'_f p_f \Sigma (dt)^2, \quad (54)$$

²⁶Eq. (10) implies that the contribution of one share of stock n to the change in wealth between between t and T is the n th component of the vector $R_{t,T}$.

where the constants $\rho > 0$ and (χ_1^D, χ_2^D) are defined by (B.10) and (C.52)-(C.53), respectively. The function $\chi^D(u) \equiv \chi_1^D e^{-(\kappa+\rho)u} + \chi_2^D e^{-b_2 u}$ is positive when u is below a threshold $\hat{u}^D > 0$ and is negative when u exceeds \hat{u}^D .

5.3 Comparative Statics

We finally illustrate comparative statics of the model. The exogenous parameters are the interest rate r , the diffusion matrix σ of the dividend process D_t , the relative size ϕ of shocks to expected dividends F_t relative to current dividends D_t , the residual-supply portfolio θ , the risk aversion α of the investor and $\bar{\alpha}$ of the manager, the mean-reversion κ and diffusion coefficient s of the cost process C_t , and the coefficient ψ of the investor's adjustment cost. We examine how these parameters affect the extent of momentum and reversal, as measured by the autocovariance of stock returns. From Corollary 8, the covariance between the return of stock n at time t and the return at $t' > t$ is

$$\text{Cov}_t(dR_{nt}, dR_{nt'}) = \left[\chi_1 e^{-(\kappa+\rho)(t'-t)} + \chi_2 e^{-\kappa(t'-t)} + \chi_3 e^{-b_2(t'-t)} \right] (\Sigma p'_n)^2 (dt)^2. \quad (55)$$

An immediate implication of (55) is that momentum and reversal are stronger for stocks with large idiosyncratic risk. Indeed, the covariance between such stocks and the flow portfolio is large in absolute value (Lemma 1), and this corresponds to large $(\Sigma p'_n)^2$. We next examine how momentum and reversal depend on market-wide parameters, entering through the function $\chi(u) \equiv \chi_1 e^{-(\kappa+\rho)u} + \chi_2 e^{-\kappa u} + \chi_3 e^{-b_2 u}$. This function characterizes the strength of momentum relative to reversal and the lag at which momentum switches to reversal. While these characteristics could, in principle, differ across stocks, they are identical in our model because $\chi(u)$ does not depend on n .

The function $\chi(u)$ depends only on a subset of exogenous parameters. Indeed, the matrix σ and vector θ affect the system of equilibrium equations (and hence $\chi(u)$) only through the scalar $\Delta/(\mathbf{1}\Sigma\mathbf{1}') = \theta\Sigma\theta' - (\mathbf{1}\Sigma\theta')^2/(\mathbf{1}\Sigma\mathbf{1}')$. This scalar is a measure of distance between the residual-supply portfolio θ and the index portfolio $\mathbf{1}$, becoming zero when $(\theta, \mathbf{1})$ are collinear. Furthermore, $\chi(u)$ depends on $(\alpha, \bar{\alpha}, \psi)$ only through $(\bar{\alpha}/\alpha, \psi/\alpha)$. Therefore, we can set $\alpha = 1$ without loss of generality.²⁷

²⁷If $(b_1, b_2, \gamma_1, \gamma_2, \gamma_3, \bar{Q}, Q)$ solve the system of equilibrium equations (C.22)-(C.24), (C.29), (C.30), (C.34) and (C.39) for $(\alpha, \bar{\alpha}, \psi)$, then $(\alpha b_1, b_2, \gamma_1, \gamma_2, \gamma_3/\alpha, \bar{D}_\alpha \bar{Q} \bar{D}_\alpha, D Q D)$ solve the same equations for $(1, \bar{\alpha}/\alpha, \psi/\alpha)$, where \bar{D}_α is a diagonal 3×3 matrix with elements $(1, 1, 1/\alpha)$ and D is a diagonal 2×2 matrix with elements $(1, 1/\alpha)$. The constants $(\rho, \chi_1, \chi_2, \chi_3)$ remain the same because of (B.10) and (C.45)-(C.47).

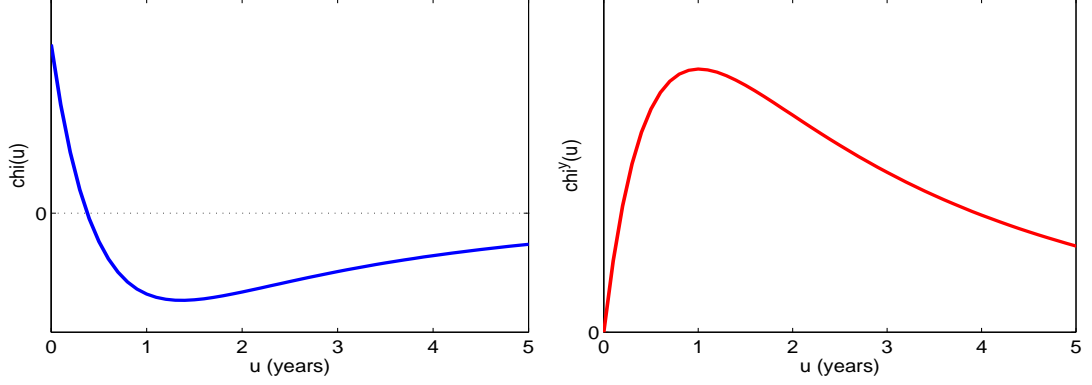


Figure 1: The left panel plots the function $\chi(u)$ that characterizes the autocovariance of returns. The right panel plots the function $\chi^Y(u)$ that characterizes the covariance between lagged returns and the investor's position in the active fund. The lag u in the x -axis takes values from zero to five years. Parameter values are for the base case.

We perform comparative statics relative to the following base case. The interest rate r is 4%. The mean-reversion κ is 30%, implying that shocks to C_t have a half-life of 2.31 years. The risk aversion of the manager is ten, i.e., ten times that of the investor. The coefficients $(\psi, \phi^2, \Delta/(\mathbf{1}\Sigma\mathbf{1}'), s^2)$ are $(0.15, 0.1, 0.1, 0.8)$. The left panel of Figure 1 plots $\chi(u)$ for the base case. Consistent with Corollary 8, momentum appears for short lags (up to four months in the figure) and reversal for long lags. Cumulative momentum is weaker than cumulative reversal, consistent with Corollary 9. But while momentum is weaker cumulatively, it generates the largest predictability per unit time. Indeed, the largest autocovariance in absolute value (i.e., largest $|\chi(u)|$) corresponds to momentum and is for very short lags.²⁸

The right panel of Figure 1 concerns the covariance between the return of stock n at time t and the investor's position in the active fund at $t' > t$. Extending the proof of Corollary 8, we can show that

$$Cov_t(dR_{nt}, y_{t'}) = \left[\chi_1^Y e^{-(\kappa+\rho)(t'-t)} + \chi_2^Y e^{-\kappa(t'-t)} + \chi_3^Y e^{-b_2(t'-t)} \right] (\Sigma p'_f)_n dt,$$

for constants $(\chi_1^Y, \chi_2^Y, \chi_3^Y)$. The function $\chi^Y(u) \equiv \chi_1^Y e^{-(\kappa+\rho)u} + \chi_2^Y e^{-\kappa u} + \chi_3^Y e^{-b_2 u}$ is positive, consistent with the result that high returns of stocks covarying positively with the flow portfolio $((\Sigma p'_f)_n > 0)$ trigger an increase in the investor's position in the active fund. The right panel of Figure 1 plots $\chi^Y(u)$ for the base case. Following a return shock, the investor adjusts her position

²⁸Corollary 8 understates the extent of momentum because it concerns the conditional autocovariance (where conditioning is on the state vector \bar{X}_t). The unconditional autocovariance is derived from its conditional counterpart by adding the autocovariance of expected returns. Since the latter is positive, unconditional momentum is stronger and unconditional reversal is weaker than their conditional counterparts.

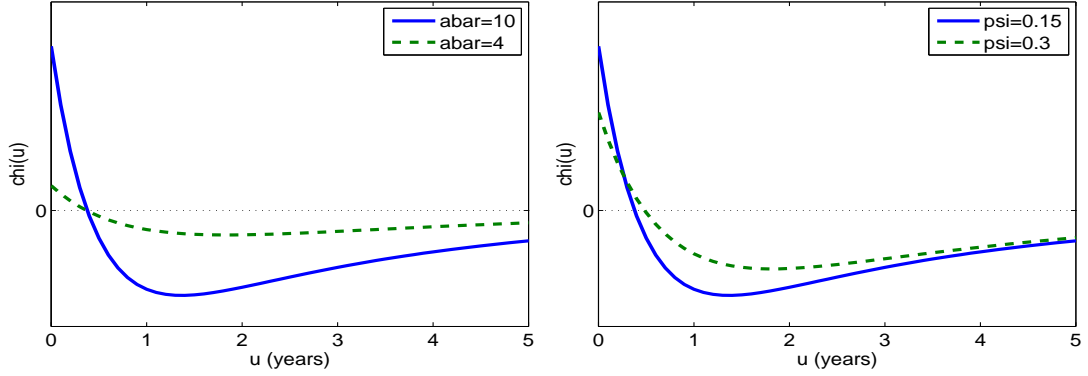


Figure 2: Comparative statics of momentum and reversal. The left panel considers variation in the manager’s risk-aversion $\bar{\alpha}$ from $\bar{\alpha} = 10$ (base case, solid line) to $\bar{\alpha} = 4$ (dashed line). The right panel considers variation in the coefficient ψ of the investor’s adjustment cost from $\psi = 0.15$ (base case, solid line) to $\psi = 0.3$.

fully within twelve months, with the bulk of the adjustment taking place within the first six months.

Figure 2 illustrates comparative statics. The left panel examines how momentum and reversal depend on the manager’s risk aversion $\bar{\alpha}$. When the manager is more risk-averse, he requires a larger price inducement to take the other side of transactions initiated by the investor. Therefore, the non-fundamental price volatility that these transactions generate increases, and amplification effects become stronger.²⁹ Since momentum and reversal are the result of non-fundamental volatility, they also increase. More subtly, momentum increases relative to reversal. The intuition is that in a reversal phase the manager has acquired a position from the investor, and requires an abnormal return as compensation for fundamental risk that he bears. By contrast, in a momentum phase the manager expects to acquire a position in the future. An abnormal return then arises because of the uncertainty that the transaction might not materialize and prices revert back to fundamentals. When the manager is more risk averse, the investor’s transactions generate larger volatility, and therefore the relevant uncertainty in a momentum phase increases relative to that in a reversal phase.

Comparative statics with respect to the diffusion coefficient s of the cost process C_t resemble those with respect to $\bar{\alpha}$: when s is larger, non-fundamental volatility is larger, momentum and reversal increase, and momentum increases relative to reversal. The intuition is that when the investor expects C_t to be more variable, she generates larger fund flows, and this increases non-fundamental price volatility.

²⁹The increase in non-fundamental volatility can be seen from the scalar coefficient k multiplying the non-fundamental covariance matrix (Corollary 4). This coefficient increases from 51.0 for $\bar{\alpha} = 4$ to 137.7 for $\bar{\alpha} = 10$.

The right panel of Figure 2 examines how momentum and reversal depend on the adjustment-cost coefficient ψ . When ψ is small, momentum is limited to shorter lags. Somewhat surprisingly, however, momentum increases. The intuition is that when the investor can adjust more easily her position in the active fund, she generates larger flows, and this increases non-fundamental price volatility.³⁰

³⁰The scalar coefficient k multiplying the non-fundamental covariance matrix increases from 106.9 for $\psi = 0.3$ to 137.7 for $\psi = 0.15$.

Appendix

A Symmetric Information

Proof of Proposition 1: Eqs. (5), (9), (10) and (11) imply that

$$d\left(r\bar{\alpha}W_t + \bar{q}_0 + \bar{q}_1C_t + \frac{1}{2}\bar{q}_{11}C_t^2\right) = \bar{G}dt + r\bar{\alpha}\hat{z}_t\sigma\left(dB_{Dt} + \frac{\phi dB_{Ft}}{r + \kappa}\right) - s(r\bar{\alpha}\hat{z}_ta_1 - \bar{q}_1 - \bar{q}_{11}C_t)dB_{Ct}, \quad (\text{A.1})$$

where

$$\begin{aligned} \bar{G} \equiv & r\bar{\alpha}\{rW_t + \hat{z}_t[r a_0 + (r + \kappa)a_1C_t - \kappa a_1\bar{C}] + B(b_0 - b_1C_t) - \bar{c}_t\} \\ & + \bar{q}_1\kappa(\bar{C} - C_t) + \frac{1}{2}\bar{q}_{11}[2\kappa C_t(\bar{C} - C_t) + s^2]. \end{aligned}$$

Eqs. (13) and (A.1) imply that

$$\mathcal{D}\bar{V} = -\bar{V}\left[\bar{G} - \frac{1}{2}(r\bar{\alpha})^2 f\hat{z}_t\Sigma\hat{z}'_t - \frac{1}{2}s^2(r\bar{\alpha}\hat{z}_ta_1 - \bar{q}_1 - \bar{q}_{11}C_t)^2\right]. \quad (\text{A.2})$$

Substituting (A.2) into (14), we can write the first-order conditions with respect to \bar{c}_t and \hat{z}_t as

$$\bar{\alpha}\exp(-\bar{\alpha}\bar{c}_t) + r\bar{\alpha}\bar{V} = 0, \quad (\text{A.3})$$

$$\bar{h}(C_t) = r\bar{\alpha}(f\Sigma + s^2a_1a'_1)\hat{z}'_t, \quad (\text{A.4})$$

respectively, where

$$\bar{h}(C_t) \equiv r a_0 + (r + \kappa)a_1C_t - \kappa a_1\bar{C} + s^2a_1(\bar{q}_1 + \bar{q}_{11}C_t). \quad (\text{A.5})$$

Eq. (A.4) is equivalent to (15) because of (5), (11) and (12). Using (A.2) and (A.3), we can simplify (14) to

$$\bar{G} - \frac{1}{2}(r\bar{\alpha})^2\hat{z}_t(f\Sigma + s^2a_1a'_1)\hat{z}'_t + r\bar{\alpha}s^2\hat{z}_ta_1(\bar{q}_1 + \bar{q}_{11}C_t) - \frac{1}{2}s^2(\bar{q}_1 + \bar{q}_{11}C_t)^2 + \beta - r = 0. \quad (\text{A.6})$$

Eqs. (A.3) and (13) imply that

$$\bar{c}_t = rW_t + \frac{1}{\bar{\alpha}}\left[\bar{q}_0 + \bar{q}_1C_t + \frac{1}{2}\bar{q}_{11}C_t^2 - \log(r)\right]. \quad (\text{A.7})$$

Substituting (A.7) into (A.6) the terms in W_t cancel, and we are left with

$$\begin{aligned}
& r\bar{\alpha}\hat{z}_t [ra_0 + (r + \kappa)a_1C_t - \kappa a_1\bar{C}] + r\bar{\alpha}B(b_0 - b_1C_t) - r \left(\bar{q}_0 + \bar{q}_1C_t + \frac{1}{2}\bar{q}_{11}C_t^2 \right) \\
& + \bar{q}_1\kappa(\bar{C} - C_t) + \frac{1}{2}\bar{q}_{11} [2\kappa C_t(\bar{C} - C_t) + s^2] + \beta - r + r \log(r) \\
& - \frac{1}{2}(r\bar{\alpha})^2\hat{z}_t(f\Sigma + s^2a_1a_1')\hat{z}_t' + r\bar{\alpha}s^2\hat{z}_ta_1(\bar{q}_1 + \bar{q}_{11}C_t) - \frac{1}{2}s^2(\bar{q}_1 + \bar{q}_{11}C_t)^2 = 0.
\end{aligned} \tag{A.8}$$

The terms in (A.8) that involve \hat{z}_t can be written as

$$\begin{aligned}
& r\bar{\alpha}\hat{z}_t [ra_0 + (r + \kappa)a_1C_t - \kappa a_1\bar{C}] - \frac{1}{2}(r\bar{\alpha})^2\hat{z}_t(f\Sigma + s^2a_1a_1')\hat{z}_t' + r\bar{\alpha}s^2\hat{z}_ta_1(\bar{q}_1 + \bar{q}_{11}C_t) \\
& = r\bar{\alpha}\hat{z}_t\bar{h}(C_t) - \frac{1}{2}(r\bar{\alpha})^2\hat{z}_t(f\Sigma + s^2a_1a_1')\hat{z}_t' \\
& = \frac{1}{2}r\bar{\alpha}\hat{z}_t\bar{h}(C_t) \\
& = \frac{1}{2}\bar{h}(C_t)'(f\Sigma + s^2a_1a_1')^{-1}\bar{h}(C_t),
\end{aligned} \tag{A.9}$$

where the first step follows from (A.5) and the last two from (A.4). Substituting (A.9) into (A.8), we find

$$\begin{aligned}
& \frac{1}{2}\bar{h}(C_t)'(f\Sigma + s^2a_1a_1')^{-1}\bar{h}(C_t) + r\bar{\alpha}B(b_0 - b_1C_t) - r \left(\bar{q}_0 + \bar{q}_1C_t + \frac{1}{2}\bar{q}_{11}C_t^2 \right) \\
& + \bar{q}_1\kappa(\bar{C} - C_t) + \frac{1}{2}\bar{q}_{11} [2\kappa C_t(\bar{C} - C_t) + s^2] + \beta - r + r \log(r) - \frac{1}{2}s^2(\bar{q}_1 + \bar{q}_{11}C_t)^2 = 0.
\end{aligned} \tag{A.10}$$

Eq. (A.10) is quadratic in C_t . Identifying terms in C_t^2 , C_t , and constants, yields three scalar equations in $(\bar{q}_0, \bar{q}_1, \bar{q}_{11})$. ■

Proof of Proposition 2: Eqs. (5), (11) and (16) imply that

$$\begin{aligned}
d \left(r\alpha W_t + q_0 + q_1C_t + \frac{1}{2}q_{11}C_t^2 \right) = & Gdt + r\alpha(x_t\mathbf{1} + y_tz_t)\sigma \left(dB_{Dt} + \frac{\phi dB_{Ft}}{r + \kappa} \right) \\
& - s [r\alpha(x_t\mathbf{1} + y_tz_t)a_1 - q_1 - q_{11}C_t] dB_{Ct},
\end{aligned} \tag{A.11}$$

where

$$\begin{aligned}
G \equiv & r\alpha \{ rW_t + (x_t\mathbf{1} + y_tz_t) [ra_0 + (r + \kappa)a_1C_t - \kappa a_1\bar{C}] - y_tC_t - c_t \} \\
& + q_1\kappa(\bar{C} - C_t) + \frac{1}{2}q_{11} [2\kappa C_t(\bar{C} - C_t) + s^2].
\end{aligned}$$

Eqs. (17) and (A.11) imply that

$$\mathcal{D}V = -V \left\{ G - \frac{1}{2}(r\alpha)^2 f(x_t \mathbf{1} + y_t z_t) \Sigma (x_t \mathbf{1} + y_t z_t)' - \frac{1}{2} s^2 [r\alpha(x_t \mathbf{1} + y_t z_t) a_1 - q_1 - q_{11} C_t]^2 \right\}. \quad (\text{A.12})$$

Substituting (A.12) into (18), we can write the first-order conditions with respect to c_t , x_t and y_t as

$$\alpha \exp(-\alpha c_t) + r\alpha V = 0, \quad (\text{A.13})$$

$$\mathbf{1}h(C_t) = r\alpha \mathbf{1}(f\Sigma + s^2 a_1 a_1')(x_t \mathbf{1} + y_t z_t)', \quad (\text{A.14})$$

$$z_t h(C_t) - C_t = r\alpha z_t (f\Sigma + s^2 a_1 a_1')(x_t \mathbf{1} + y_t z_t)', \quad (\text{A.15})$$

respectively, where

$$h(C_t) \equiv ra_0 + (r + \kappa)a_1 C_t - \kappa a_1 \bar{C} + s^2 a_1 (q_1 + q_{11} C_t). \quad (\text{A.16})$$

Eqs. (A.14) and (A.15) are equivalent to (19) and (20) because of (5), (11) and (12). Solving for c_t , and proceeding as in the proof of Proposition 1, we can simplify (18) to

$$\begin{aligned} & r\alpha(x_t \mathbf{1} + y_t z_t) [ra_0 + (r + \kappa)a_1 C_t - \kappa a_1 \bar{C}] - r\alpha y_t C_t - r \left(q_0 + q_1 C_t + \frac{1}{2} q_{11} C_t^2 \right) \\ & + q_1 \kappa (\bar{C} - C_t) + \frac{1}{2} q_{11} [2\kappa C_t (\bar{C} - C_t) + s^2] + \beta - r + r \log(r) \\ & - \frac{1}{2} (r\alpha)^2 (x_t \mathbf{1} + y_t z_t) (f\Sigma + s^2 a_1 a_1')(x_t \mathbf{1} + y_t z_t)' \\ & + r\alpha s^2 (x_t \mathbf{1} + y_t z_t) a_1 (q_1 + q_{11} C_t) - \frac{1}{2} s^2 (q_1 + q_{11} C_t)^2 = 0. \end{aligned} \quad (\text{A.17})$$

Eq. (A.17) is the counterpart of (A.8) for the investor. The terms in (A.17) that involve (x_t, y_t) can be written as

$$\begin{aligned} & r\alpha(x_t \mathbf{1} + y_t z_t) [ra_0 + (r + \kappa)a_1 C_t - \kappa a_1 \bar{C}] - r\alpha y_t C_t \\ & - \frac{1}{2} (r\alpha)^2 (x_t \mathbf{1} + y_t z_t) (f\Sigma + s^2 a_1 a_1')(x_t \mathbf{1} + y_t z_t)' + r\alpha s^2 (x_t \mathbf{1} + y_t z_t) a_1 (q_1 + q_{11} C_t) \\ & = r\alpha(x_t \mathbf{1} + y_t z_t) h(C_t) - r\alpha y_t C_t - \frac{1}{2} (r\alpha)^2 (x_t \mathbf{1} + y_t z_t) (f\Sigma + s^2 a_1 a_1')(x_t \mathbf{1} + y_t z_t)' \\ & = \frac{1}{2} r\alpha(x_t \mathbf{1} + y_t z_t) h(C_t) - \frac{1}{2} r\alpha y_t C_t, \end{aligned} \quad (\text{A.18})$$

where the first step follows from (A.16) and the second from

$$(x_t \mathbf{1} + y_t z_t) h(C_t) - y_t C_t = r\alpha(x_t \mathbf{1} + y_t z_t)(f\Sigma + s^2 a_1 a_1')(x_t \mathbf{1} + y_t z_t)', \quad (\text{A.19})$$

which in turn follows by multiplying (A.14) by x_t , (A.15) by y_t , and adding up. To eliminate x_t and y_t in (A.18), we use (A.14) and (A.15). Noting that in equilibrium $z_t = \theta - x_t \mathbf{1}$, we can write (A.14) as

$$\mathbf{1}h(C_t) = r\alpha \mathbf{1}(f\Sigma + s^2 a_1 a_1') [x_t \mathbf{1} + y_t(\theta - x_t \mathbf{1})]'. \quad (\text{A.20})$$

Multiplying (A.14) by x_t and adding to (A.15), we similarly find

$$\theta h(C_t) - C_t = r\alpha \theta (f\Sigma + s^2 a_1 a_1') [x_t \mathbf{1} + y_t(\theta - x_t \mathbf{1})]'. \quad (\text{A.21})$$

Eqs. (A.20) and (A.21) form a linear system in $x_t(1 - y_t)$ and y_t . Solving the system, we find

$$x_t(1 - y_t) = \frac{1}{r\alpha D} \{ \mathbf{1}h(C_t)\theta(f\Sigma + s^2 a_1 a_1')\theta' - [\theta h(C_t) - C_t] \mathbf{1}(f\Sigma + s^2 a_1 a_1')\theta' \}, \quad (\text{A.22})$$

$$y_t = \frac{1}{r\alpha D} \{ [\theta h(C_t) - C_t] \mathbf{1}(f\Sigma + s^2 a_1 a_1')\mathbf{1}' - \mathbf{1}h(C_t)\mathbf{1}(f\Sigma + s^2 a_1 a_1')\theta' \}, \quad (\text{A.23})$$

where

$$D \equiv \theta(f\Sigma + s^2 a_1 a_1')\theta' \mathbf{1}(f\Sigma + s^2 a_1 a_1')\mathbf{1}' - [\mathbf{1}(f\Sigma + s^2 a_1 a_1')\theta']^2.$$

Eq. (A.23) implies that the optimal control y_t is linear in C_t . Using (A.22) and (A.23), we can write (A.18) as

$$\begin{aligned} & \frac{1}{2}r\alpha(x_t \mathbf{1} + y_t z_t)h(C_t) - \frac{1}{2}r\alpha y_t C_t - \frac{1}{2}s^2(q_1 + q_{11}C_t)^2 \\ &= \frac{1}{2}r\alpha [x_t \mathbf{1} + y_t(\theta - x_t \mathbf{1})] h(C_t) - \frac{1}{2}r\alpha y_t C_t - \frac{1}{2}s^2(q_1 + q_{11}C_t)^2 \\ &= \frac{1}{2D} \left\{ [\mathbf{1}h(C_t)]^2 \theta(f\Sigma + s^2 a_1 a_1')\theta' - 2[\theta h(C_t) - C_t] \mathbf{1}h(C_t)\mathbf{1}(f\Sigma + s^2 a_1 a_1')\theta' \right. \\ & \quad \left. + [\theta h(C_t) - C_t]^2 \mathbf{1}(f\Sigma + s^2 a_1 a_1')\mathbf{1}' \right\} - \frac{1}{2}s^2(q_1 + q_{11}C_t)^2. \end{aligned} \quad (\text{A.24})$$

Substituting (A.24) into (A.17), we find

$$\begin{aligned} & \frac{1}{2D} \left\{ [\mathbf{1}h(C_t)]^2 \theta(f\Sigma + s^2 a_1 a_1')\theta' - 2[\theta h(C_t) - C_t] \mathbf{1}h(C_t)\mathbf{1}(f\Sigma + s^2 a_1 a_1')\theta' \right. \\ & \quad \left. + [\theta h(C_t) - C_t]^2 \mathbf{1}(f\Sigma + s^2 a_1 a_1')\mathbf{1}' \right\} - r \left(q_0 + q_1 C_t + \frac{1}{2}q_{11}C_t^2 \right) \\ & \quad + q_1 \kappa(\bar{C} - C_t) + \frac{1}{2}q_{11} [2\kappa C_t(\bar{C} - C_t) + s^2] + \beta - r + r \log(r) - \frac{1}{2}s^2(q_1 + q_{11}C_t)^2 = 0. \end{aligned} \quad (\text{A.25})$$

Eq. (A.25) is quadratic in C_t . Identifying terms in C_t^2 , C_t , and constants, yields three scalar equations in (q_0, q_1, q_{11}) . \blacksquare

Proof of Proposition 3: We first impose market clearing and derive the constants $(a_0, a_1, b_0, b_1, \gamma_1)$ as functions of $(\bar{q}_1, \bar{q}_{11}, q_1, q_{11})$. Setting $z_t = \theta - x_t \mathbf{1}$ and $\bar{y}_t = 1 - y_t$, we can write (A.4) as

$$\bar{h}(C_t) = r\bar{\alpha}(f\Sigma + s^2 a_1 a_1')(1 - y_t)(\theta - x_t \mathbf{1})'. \quad (\text{A.26})$$

Premultiplying (A.26) by $\mathbf{1}$, dividing by $r\bar{\alpha}$, and adding to (A.20) divided by $r\alpha$, we find

$$\mathbf{1} \left[\frac{h(C_t)}{r\alpha} + \frac{\bar{h}(C_t)}{r\bar{\alpha}} \right] = \mathbf{1}(f\Sigma + s^2 a_1 a_1')\theta'. \quad (\text{A.27})$$

Eq. (A.27) is linear in C_t . Identifying terms in C_t , we find

$$\left(\frac{r + \kappa + s^2 q_{11}}{r\alpha} + \frac{r + \kappa + s^2 \bar{q}_{11}}{r\bar{\alpha}} \right) \mathbf{1} a_1 = 0 \Rightarrow \mathbf{1} a_1 = 0. \quad (\text{A.28})$$

Identifying constant terms, and using (A.28), we find

$$\left(\frac{r}{r\alpha} + \frac{r}{r\bar{\alpha}} \right) \mathbf{1} a_0 = g \mathbf{1} \Sigma \theta' \Rightarrow \mathbf{1} a_0 = \frac{\alpha \bar{\alpha} f}{\alpha + \bar{\alpha}} \mathbf{1} \Sigma \theta'. \quad (\text{A.29})$$

Substituting (A.28) and (A.29) into (A.20), we find

$$\frac{r\alpha \bar{\alpha} f}{\alpha + \bar{\alpha}} \mathbf{1} \Sigma \theta' = r\alpha g \mathbf{1} \Sigma [x_t \mathbf{1} + y_t(\theta - x_t \mathbf{1})]' \Rightarrow x_t = \frac{\bar{\alpha}}{\alpha + \bar{\alpha}} - y_t \frac{\mathbf{1} \Sigma \theta'}{\mathbf{1} \Sigma \mathbf{1}'}. \quad (\text{A.30})$$

Substituting (A.30) into (A.26), we find

$$\begin{aligned} \bar{h}(C_t) &= r\bar{\alpha}(f\Sigma + s^2 a_1 a_1') \left[\frac{\alpha}{\alpha + \bar{\alpha}} \frac{\mathbf{1} \Sigma \theta'}{\mathbf{1} \Sigma \mathbf{1}'} \mathbf{1} + (1 - y_t) p_f \right]' \\ &= r\bar{\alpha}(f\Sigma + s^2 a_1 a_1') \left[\frac{\alpha}{\alpha + \bar{\alpha}} \frac{\mathbf{1} \Sigma \theta'}{\mathbf{1} \Sigma \mathbf{1}'} \mathbf{1} + (1 - b_0 + b_1 C_t) p_f \right]', \end{aligned} \quad (\text{A.31})$$

where the second step follows from (9). Eq. (A.31) is linear in C_t . Identifying terms in C_t , we find

$$(r + \kappa + s^2 \bar{q}_{11}) a_1 = r\bar{\alpha} b_1 (f \Sigma p_f' + s^2 a_1' p_f' a_1). \quad (\text{A.32})$$

Therefore, a_1 is collinear to the vector $\Sigma p_f'$, as in (21). Substituting (21) into (A.32), we find

$$(r + \kappa + s^2 \bar{q}_{11}) \gamma_1 = r\bar{\alpha} b_1 \left(f + \frac{s^2 \gamma_1^2 \Delta}{\mathbf{1} \Sigma \mathbf{1}'} \right). \quad (\text{A.33})$$

Identifying constant terms in (A.31), and using (21), we find

$$a_0 = \frac{\alpha\bar{\alpha}f}{\alpha + \bar{\alpha}} \frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'} + \left[\frac{\gamma_1(\kappa\bar{C} - s^2\bar{q}_1)}{r} + \bar{\alpha}(1 - b_0) \left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) \right] \Sigma p'_f. \quad (\text{A.34})$$

Premultiplying (A.26) by θ , dividing by $r\bar{\alpha}$, and adding to (A.21) divided by $r\alpha$, we find

$$\theta \left[\frac{h(C_t)}{r\alpha} + \frac{\bar{h}(C_t)}{r\bar{\alpha}} \right] - \frac{C_t}{r\alpha} = \theta(f\Sigma + s^2a_1a'_1)\theta'. \quad (\text{A.35})$$

Eq. (A.35) is linear in C_t . Identifying terms in C_t , we find

$$\left(\frac{r + \kappa + s^2q_{11}}{r\alpha} + \frac{r + \kappa + s^2\bar{q}_{11}}{r\bar{\alpha}} \right) \theta a_1 = \frac{1}{r\alpha}. \quad (\text{A.36})$$

Substituting (21) into (A.36), we find

$$\left(\frac{r + \kappa + s^2q_{11}}{r\alpha} + \frac{r + \kappa + s^2\bar{q}_{11}}{r\bar{\alpha}} \right) \frac{\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'} = \frac{1}{r\alpha}. \quad (\text{A.37})$$

Identifying constant terms in (A.35), we find

$$\left(\frac{r}{r\alpha} + \frac{r}{r\bar{\alpha}} \right) \theta a_0 - \left(\frac{\kappa\bar{C} - s^2q_1}{r\alpha} + \frac{\kappa\bar{C} - s^2\bar{q}_1}{r\bar{\alpha}} \right) \theta a_1 = \theta(f\Sigma + s^2a_1a'_1)\theta'. \quad (\text{A.38})$$

Using (21) and (A.34), we can write (A.38) as

$$b_0 = \frac{\bar{\alpha}}{\alpha + \bar{\alpha}} + \frac{s^2\gamma_1(q_1 - \bar{q}_1)}{r(\alpha + \bar{\alpha}) \left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right)}. \quad (\text{A.39})$$

Substituting b_0 from (A.39) into (A.34), we find

$$a_0 = \frac{\alpha\bar{\alpha}f}{\alpha + \bar{\alpha}} \Sigma\theta' + \gamma_1 \left[\frac{\kappa\bar{C}}{r} - \frac{s^2(\alpha\bar{q}_1 + \bar{\alpha}q_1)}{r(\alpha + \bar{\alpha})} + \frac{\alpha\bar{\alpha}s^2\gamma_1\Delta}{(\alpha + \bar{\alpha})\mathbf{1}\Sigma\mathbf{1}'} \right] \Sigma p'_f. \quad (\text{A.40})$$

The system of equations characterizing equilibrium is as follows. The endogenous variables are $(a_0, a_1, b_0, b_1, \gamma_1, \bar{q}_0, \bar{q}_1, \bar{q}_{11}, q_0, q_1, q_{11})$. The equations linking them are (21), (A.33), (A.37), (A.39), (A.40), the three equations derived from (A.10) by identifying terms in C_t^2 , C_t , and constants, and the three equations derived from (A.25) through the same procedure. To simplify the system,

we note that the variables (\bar{q}_0, q_0) enter only in the equations derived from (A.10) and (A.25) by identifying constants. Therefore they can be determined separately, and we need to consider only the equations derived from (A.10) and (A.25) by identifying linear and quadratic terms. We next simplify these equations, using implications of market clearing.

Using (21) and (A.33), we can write (A.31) as

$$\bar{h}(C_t) = \frac{r\alpha\bar{\alpha}f}{\alpha + \bar{\alpha}} \frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'} \Sigma\mathbf{1}' + \bar{h}_1(C_t)\Sigma p'_f, \quad (\text{A.41})$$

where

$$\bar{h}_1(C_t) \equiv \frac{r\alpha\bar{\alpha}}{\alpha + \bar{\alpha}} \left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) + \frac{\bar{\alpha}s^2\gamma_1(\bar{q}_1 - q_1)}{\alpha + \bar{\alpha}} + r\bar{\alpha}b_1 \left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) C_t.$$

Eq. (21) implies that

$$(f\Sigma + s^2a_1a'_1)\mathbf{1}' = f\Sigma\mathbf{1}' \Rightarrow \mathbf{1}\Sigma(f\Sigma + s^2a_1a'_1)^{-1} = \frac{\mathbf{1}}{f}, \quad (\text{A.42})$$

$$(f\Sigma + s^2a_1a'_1)p'_f = \left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) \Sigma p'_f \Rightarrow p_f\Sigma(f\Sigma + s^2a_1a'_1)^{-1} = \frac{p_f}{f + \frac{\kappa\Delta}{\mathbf{1}\Sigma\mathbf{1}'}}. \quad (\text{A.43})$$

Using (A.41)-(A.43), we find

$$\frac{1}{2}\bar{h}(C_t)'(f\Sigma + s^2a_1a'_1)^{-1}\bar{h}(C_t) = \frac{r^2\alpha^2\bar{\alpha}^2f(\mathbf{1}\Sigma\theta')^2}{2(\alpha + \bar{\alpha})^2\mathbf{1}\Sigma\mathbf{1}'} + \frac{\bar{h}_1(C_t)^2\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'}}{2\left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}}\right)}. \quad (\text{A.44})$$

We next substitute (A.44) into (A.10), and identify terms. Identifying terms in C_t^2 , we find

$$(r + 2\kappa)\bar{q}_{11} + s^2\bar{q}_{11}^2 - r^2\bar{\alpha}^2b_1^2 \left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) \frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} = 0. \quad (\text{A.45})$$

Identifying terms in C_t , we find

$$(r + \kappa)\bar{q}_1 + s^2\bar{q}_1\bar{q}_{11} - r\bar{\alpha}b_1 \left[\frac{r\alpha\bar{\alpha}}{\alpha + \bar{\alpha}} \left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) + \frac{\bar{\alpha}s^2\gamma_1(\bar{q}_1 - q_1)}{\alpha + \bar{\alpha}} \right] \frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} - \kappa\bar{C}\bar{q}_{11} + r\bar{\alpha}Bb_1 = 0. \quad (\text{A.46})$$

Substituting $\bar{h}(C_t)$ from (A.41) into (A.27), and using (21), we find

$$\mathbf{1}h(C_t) = \frac{r\alpha\bar{\alpha}f}{\alpha + \bar{\alpha}} \mathbf{1}\Sigma\theta'. \quad (\text{A.47})$$

Following the same procedure for (A.35) instead of (A.27), we find

$$\theta h(C_t) - C_t = \frac{r\alpha\bar{\alpha}f(\mathbf{1}\Sigma\theta')^2}{\alpha + \bar{\alpha}} + \left[r\alpha \left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) - \frac{\alpha}{\bar{\alpha}}\bar{h}_1(C_t) \right] \frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \quad (\text{A.48})$$

Eq. (21) implies that the denominator D in (A.25) is

$$D = f\Delta \left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right). \quad (\text{A.49})$$

Using (21) and (A.47)-(A.49), we find that the equation derived from (A.25) by identifying terms in C_t^2 is

$$(r + 2\kappa)q_{11} + s^2q_{11}^2 - r^2\alpha^2b_1^2 \left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) \frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} = 0, \quad (\text{A.50})$$

and that derived by identifying terms in C_t is

$$(r + \kappa)q_1 + s^2q_1q_{11} + r\alpha b_1 \left[\frac{r\alpha\bar{\alpha} \left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right)}{\alpha + \bar{\alpha}} + \frac{\alpha s^2\gamma_1(q_1 - \bar{q}_1)}{\alpha + \bar{\alpha}} \right] \frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} - \kappa\bar{C}q_{11} = 0. \quad (\text{A.51})$$

Solving for equilibrium amounts to solving the system of (21), (A.33), (A.37), (A.39), (A.40), (A.45), (A.46), (A.50) and (A.51) in the unknowns $(a_0, a_1, b_0, b_1, \gamma_1, \bar{q}_1, \bar{q}_{11}, q_1, q_{11})$. This reduces to solving the system of (A.33), (A.37), (A.45) and (A.50) in the unknowns $(b_1, \gamma_1, \bar{q}_{11}, q_{11})$: given $(b_1, \gamma_1, \bar{q}_{11}, q_{11})$, a_1 can be determined from (21), (\bar{q}_1, q_1) from the linear system of (A.46) and (A.51), and (a_0, b_0) from (A.40) and (A.39). Replacing the unknown b_1 by

$$\hat{b}_1 \equiv r\bar{\alpha}b_1 \sqrt{f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}} ,$$

we can write the latter system as

$$(r + \kappa + s^2\bar{q}_{11})\gamma_1 = \hat{b}_1 \sqrt{f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}} , \quad (\text{A.52})$$

$$\left(\frac{r + \kappa + s^2q_{11}}{r\alpha} + \frac{r + \kappa + s^2\bar{q}_{11}}{r\bar{\alpha}} \right) \frac{\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'} = \frac{1}{r\alpha}, \quad (\text{A.53})$$

$$(r + 2\kappa)\bar{q}_{11} + s^2\bar{q}_{11}^2 - \frac{\hat{b}_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} = 0, \quad (\text{A.54})$$

$$(r + 2\kappa)q_{11} + s^2q_{11}^2 - \frac{\alpha^2\hat{b}_1^2\Delta}{\bar{\alpha}^2\mathbf{1}\Sigma\mathbf{1}'} = 0. \quad (\text{A.55})$$

Eq. (A.54) is quadratic in \bar{q}_{11} and has a unique positive solution. Likewise, (A.55) is quadratic in q_{11} and has a unique positive solution. In both cases, the solution is increasing in \hat{b}_1 , is equal to zero for $\hat{b}_1 = 0$, and to ∞ for $\hat{b}_1 = \infty$. Treating (\bar{q}_{11}, q_{11}) as implicit functions of \hat{b}_1 , we next reduce the system of (A.52)-(A.55) to that of (A.52) and (A.53) in the unknowns (\hat{b}_1, γ_1) . Since (\bar{q}_{11}, q_{11}) are increasing in \hat{b}_1 , (A.53) has a unique solution $\hat{b}_1 > 0$ for $\gamma_1 \in (0, \bar{\gamma}_1)$, where $\bar{\gamma}_1 \equiv \frac{\bar{\alpha}\mathbf{1}\Sigma\mathbf{1}'}{(\alpha+\bar{\alpha})\Delta(r+\kappa)}$. This solution is decreasing in γ_1 , is equal to ∞ for $\gamma_1 = 0$, and to zero for $\gamma_1 = \bar{\gamma}_1$. Treating \hat{b}_1 as implicit function of γ_1 , we next reduce the system of (A.52) and (A.53) to the single equation (A.53) in the unknown γ_1 . Using (A.54), we can write this equation as

$$\frac{\gamma_1^2}{f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}} = \frac{[(r + 2\kappa)\bar{q}_{11} + s^2\bar{q}_{11}^2] \mathbf{1}\Sigma\mathbf{1}'}{(r + \kappa + s^2\bar{q}_{11})^2\Delta}. \quad (\text{A.56})$$

The left-hand side of (A.56) is increasing in γ_1 , is equal to zero for $\gamma_1 = 0$, and to a positive value for $\gamma_1 = \bar{\gamma}_1$. The right-hand side of (A.56) is increasing in \bar{q}_{11} , and therefore decreasing in γ_1 . It is equal to a positive value for $\gamma_1 = 0$ and to zero for $\gamma_1 = \bar{\gamma}_1$. Therefore, (A.56) has a unique positive solution γ_1 , and there exists a unique linear equilibrium. ■

Proof of Lemma 1: Denoting by β the regression coefficient of dR_t on $\mathbf{1}dR_t$, we have

$$\begin{aligned} Cov_t(dR_t, p_f dR_t) &= Cov_t(dR_t - \beta\mathbf{1}dR_t, p_f dR_t) \\ &= Cov_t(d\epsilon_t, p_f dR_t) \\ &= Cov_t[d\epsilon_t, p_f(dR_t - \beta\mathbf{1}dR_t)] \\ &= Cov_t(d\epsilon_t, p_f d\epsilon_t), \end{aligned}$$

where the first step follows because the index and the flow portfolio are independent (see Footnote 16), the second and fourth steps follow from the definition of $d\epsilon_t$, and the third step follows because $d\epsilon_t$ is independent of $\mathbf{1}dR_t$. ■

Proof of Corollary 1: Stocks' expected returns are

$$\begin{aligned} E_t(dR_t) &= [ra_0 + (r + \kappa)a_1C_t - \kappa a_1\bar{C}] dt \\ &= \left\{ \frac{r\alpha\bar{\alpha}f}{\alpha + \bar{\alpha}} \frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'} \Sigma\mathbf{1}' + \left[(r + \kappa)\gamma_1 C_t - \frac{s^2\gamma_1(\alpha\bar{q}_1 + \bar{\alpha}q_1)}{\alpha + \bar{\alpha}} + \frac{r\alpha\bar{\alpha} \left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right)}{\alpha + \bar{\alpha}} \right] \Sigma p_f' \right\} dt, \end{aligned} \quad (\text{A.57})$$

where the first step follows from (11), and the second from (21), (22) and (A.40). Using (A.42) and (A.43), we can write (A.57) as

$$E_t(dR_t) = \left[\frac{r\alpha\bar{\alpha}}{\alpha + \bar{\alpha}} \frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'} (f\Sigma + s^2 a_1 a_1') \mathbf{1}' + \Lambda_t (f\Sigma + s^2 a_1 a_1') p_f' \right] dt. \quad (\text{A.58})$$

Eq. (A.58) is equivalent to (26) because of (12).

The factor risk premium Λ_t is increasing in C_t because $\gamma_1 > 0$. To show that Λ_t is increasing in B , we compute (\bar{q}_1, q_1) , solving the linear system of (A.46) and (A.51). The system yields

$$\alpha\bar{q}_1 + \bar{\alpha}q_1 = \frac{Y_0 + Y_1\kappa\bar{C} - Y_2B}{Z}, \quad (\text{A.59})$$

where

$$\begin{aligned} Y_0 &\equiv \frac{r^2\alpha^2\bar{\alpha}^2s^2b_1 \left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) \Delta}{(\alpha + \bar{\alpha})\mathbf{1}\Sigma\mathbf{1}'} \left[q_{11} - \bar{q}_{11} + \frac{r(\alpha + \bar{\alpha})b_1\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right], \\ Y_1 &\equiv \alpha\bar{q}_{11} \left(r + \kappa + s^2q_{11} + \frac{r\alpha s^2b_1\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) + \bar{\alpha}q_{11} \left(r + \kappa + s^2\bar{q}_{11} - \frac{r\bar{\alpha}s^2b_1\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right), \\ Y_2 &\equiv r\alpha\bar{\alpha}b_1 \left(r + \kappa + s^2q_{11} + \frac{r\alpha^2s^2b_1\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right), \\ Z &\equiv (r + \kappa + s^2\bar{q}_{11})(r + \kappa + s^2q_{11}) + \frac{r\alpha^2s^2b_1\gamma_1\Delta(r + \kappa + s^2\bar{q}_{11})}{(\alpha + \bar{\alpha})\mathbf{1}\Sigma\mathbf{1}'} - \frac{r\bar{\alpha}^2s^2b_1\gamma_1\Delta(r + \kappa + s^2q_{11})}{(\alpha + \bar{\alpha})\mathbf{1}\Sigma\mathbf{1}'} \end{aligned}$$

Using (A.33) and the positivity of $(b_1, \gamma_1, \bar{q}_{11}, q_{11})$, we find $Y_2 > 0$ and $Z > 0$. Therefore, Λ_t is increasing in B . Moreover, $\Lambda_t > 0$ if this holds for $B = 0$. Eqs. (27) and (A.59) imply that $\Lambda_t > 0$ for $B = 0$ if

$$\frac{r\alpha\bar{\alpha}}{\alpha + \bar{\alpha}} \left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) + \gamma_1(r + \kappa)C_t - \frac{s^2\gamma_1(Y_0 + Y_1\kappa\bar{C})}{(\alpha + \bar{\alpha})Z} > 0. \quad (\text{A.60})$$

Using (A.33) and the positivity of $(b_1, \gamma_1, \bar{q}_{11}, q_{11})$, we find

$$\begin{aligned} \frac{r\alpha\bar{\alpha}s^2\gamma_1^2\Delta}{(\alpha + \bar{\alpha})\mathbf{1}\Sigma\mathbf{1}'} &> \frac{s^2\gamma_1Y_0}{(\alpha + \bar{\alpha})Z}, \\ \gamma_1 &> \frac{s^2\gamma_1Y_1}{(\alpha + \bar{\alpha})Z}. \end{aligned}$$

Therefore, (A.60) holds for $C_t \geq \kappa\bar{C}/(r + \kappa)$. ■

Proof of Corollary 2: The autocovariance matrix is

$$\begin{aligned}
& Cov_t(dR_t, dR_{t'}) \\
&= Cov_t \left\{ \sigma \left(dB_{Dt} + \frac{\phi dB_{Ft}}{r + \kappa} \right) - sa_1 dB_{Ct}, \left[(r + \kappa) a_1 C_{t'} dt + \sigma \left(dB_{Dt'} + \frac{\phi dB_{Ft'}}{r + \kappa} \right) - sa_1 dB_{Ct'} \right]' \right\} \\
&= Cov_t \left[\sigma \left(dB_{Dt} + \frac{\phi dB_{Ft}}{r + \kappa} \right) - sa_1 dB_{Ct}, (r + \kappa) a_1' C_{t'} dt \right] \\
&= Cov_t \left[-sa_1 dB_{Ct}, (r + \kappa) a_1' C_{t'} dt \right] \\
&= -s(r + \kappa) \gamma_1^2 Cov_t(dB_{Ct}, C_{t'}) \Sigma p_f p_f' \Sigma dt, \tag{A.61}
\end{aligned}$$

where the first step follows by using (11) and omitting quantities known at time t , the second step follows because the increments $(dB_{t'}^D, dB_{t'}^F, dB_{t'}^C)$ are independent of information up to time t' , the third step follows because B_t^C is independent of (B_t^D, B_t^F) , and the fourth step follows from (21). Eq. (5) implies that

$$C_{t'} = e^{-\kappa(t'-t)} C_t + \left[1 - e^{-\kappa(t'-t)} \right] \bar{C} + s \int_t^{t'} e^{-\kappa(t'-u)} dB_{Cu}. \tag{A.62}$$

Substituting (A.62) into (A.61), we find (29). ■

B Asymmetric Information

Proof of Proposition 4: We use Theorem 10.3 of Liptser and Shiryaev (LS 2000). The investor learns about C_t , which follows the process (5). She observes the following information:

- The net dividends of the residual-supply portfolio $\theta D_t - C_t dt$. This corresponds to the process $\xi_{1t} \equiv \theta D_t - \int_0^t C_s ds$.
- The dividends of the index fund $\mathbf{1} dD_t$. This corresponds to the process $\xi_{2t} \equiv \mathbf{1} D_t$.
- The price of the residual-supply portfolio θS_t . Given the conjecture (31) for stock prices, this is equivalent to observing the process $\xi_{3t} \equiv \theta(S_t + a_1 \hat{C}_t)$.
- The price of the index portfolio $\mathbf{1} S_t$. This is equivalent to observing the process $\xi_{4t} \equiv \mathbf{1}(S_t + a_1 \hat{C}_t)$.

The dynamics of ξ_{1t} are

$$\begin{aligned}
d\xi_{1t} &= \theta(F_t dt + \sigma dB_t^D) - C_t dt \\
&= \left[(r + \kappa)\theta a_0 - \frac{\kappa\theta\bar{F}}{r} + (r + \kappa)\xi_{3t} + (r + \kappa)\theta a_2 C_t - C_t \right] dt + \theta\sigma dB_t^D \\
&= \left[(r + \kappa)\theta a_0 - \frac{\kappa\theta\bar{F}}{r} + (r + \kappa)\xi_{3t} - \left(1 - \frac{(r + \kappa)\gamma_2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) C_t \right] dt + \theta\sigma dB_t^D, \tag{B.1}
\end{aligned}$$

where the first step follows from (1), the second from (31), and the third from (32). Likewise, the dynamics of ξ_{2t} are

$$d\xi_{2t} = \left[(r + \kappa)\mathbf{1}a_0 - \frac{\kappa\mathbf{1}\bar{F}}{r} + (r + \kappa)\xi_{4t} \right] dt + \mathbf{1}\sigma dB_t^D. \tag{B.2}$$

The dynamics of ξ_{3t} are

$$\begin{aligned}
d\xi_{3t} &= d \left\{ \theta \left[\frac{\bar{F}}{r} + \frac{F_t - \bar{F}}{r + \kappa} - (a_0 + a_2 C_t) \right] \right\} \\
&= \theta \left[\frac{\kappa(\bar{F} - F_t)dt + \phi\sigma dB_t^F}{r + \kappa} - a_2 [\kappa(\bar{C} - C_t)dt + s dB_t^C] \right] \\
&= \kappa \left[\theta \left(\frac{\bar{F}}{r} - a_0 - a_2 \bar{C} \right) - \xi_{3t} \right] dt + \frac{\phi\theta\sigma dB_t^F}{r + \kappa} - s\theta a_2 dB_t^C \\
&= \kappa \left(\frac{\theta\bar{F}}{r} - \theta a_0 - \frac{\gamma_2\Delta\bar{C}}{\mathbf{1}\Sigma\mathbf{1}'} - \xi_{3t} \right) dt + \frac{\phi\theta\sigma dB_t^F}{r + \kappa} - \frac{s\gamma_2\Delta dB_t^C}{\mathbf{1}\Sigma\mathbf{1}'}, \tag{B.3}
\end{aligned}$$

where the first step follows from (31), the second from (2) and (5), and the fourth from (32). Likewise, the dynamics of ξ_{4t} are

$$d\xi_{4t} = \kappa \left(\frac{\mathbf{1}\bar{F}}{r} - \mathbf{1}a_0 - \xi_{4t} \right) dt + \frac{\phi\mathbf{1}\sigma dB_t^F}{r + \kappa}. \tag{B.4}$$

The dynamics (5) and (B.1)-(B.4) map into the dynamics (10.62) and (10.63) of LS by setting $\theta_t \equiv C_t$, $\xi_t \equiv (\xi_{1t}, \xi_{2t}, \xi_{3t}, \xi_{4t})'$, $W_{1t} \equiv \begin{pmatrix} B_t^D \\ B_t^F \end{pmatrix}$, $W_{2t} \equiv B_t^C$, $a_0(t) \equiv \kappa\bar{C}$, $a_1(t) \equiv -\kappa$, $a_2(t) \equiv 0$, $b_1(t) \equiv 0$, $b_2(t) \equiv s$, $\gamma_t \equiv \hat{s}^2$,

$$A_0(t) \equiv \begin{pmatrix} (r + \kappa)\theta a_0 - \frac{\kappa\theta\bar{F}}{r} \\ (r + \kappa)\mathbf{1}a_0 - \frac{\kappa\mathbf{1}\bar{F}}{r} \\ \kappa \left(\frac{\theta\bar{F}}{r} - \theta a_0 - \frac{\gamma_2\Delta\bar{C}}{\mathbf{1}\Sigma\mathbf{1}'} \right) \\ \kappa \left(\frac{\mathbf{1}\bar{F}}{r} - \mathbf{1}a_0 \right) \end{pmatrix},$$

$$A_1(t) \equiv - \begin{pmatrix} 1 - \frac{(r+\kappa)\gamma_2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$A_2(t) \equiv \begin{pmatrix} 0 & 0 & r + \kappa & 0 \\ 0 & 0 & 0 & r + \kappa \\ 0 & 0 & -\kappa & 0 \\ 0 & 0 & 0 & -\kappa \end{pmatrix},$$

$$B_1(t) \equiv \begin{pmatrix} \theta\sigma & 0 \\ \mathbf{1}\sigma & 0 \\ 0 & \frac{\phi\theta\sigma}{r+\kappa} \\ 0 & \frac{\phi\mathbf{1}\sigma}{r+\kappa} \end{pmatrix},$$

$$B_2(t) \equiv - \begin{pmatrix} 0 \\ 0 \\ \frac{s\gamma_2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \\ 0 \end{pmatrix}.$$

The quantities $(b \circ b)(t)$, $(b \circ B)(t)$, and $(B \circ B)(t)$, defined in LS (10.80) are

$$(b \circ b)(t) = s^2,$$

$$(b \circ B)(t) = - \begin{pmatrix} 0 & 0 & \frac{s^2\gamma_2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} & 0 \end{pmatrix},$$

$$(B \circ B)(t) = \begin{pmatrix} \theta\Sigma\theta' & \mathbf{1}\Sigma\theta' & 0 & 0 \\ \mathbf{1}\Sigma\theta' & \mathbf{1}\Sigma\mathbf{1}' & 0 & 0 \\ 0 & 0 & \frac{\phi^2\theta\Sigma\theta'}{(r+\kappa)^2} + \frac{s^2\gamma_2^2\Delta^2}{(\mathbf{1}\Sigma\mathbf{1}')^2} & \frac{\phi^2\mathbf{1}\Sigma\theta'}{(r+\kappa)^2} \\ 0 & 0 & \frac{\phi^2\mathbf{1}\Sigma\theta'}{(r+\kappa)^2} & \frac{\phi^2\mathbf{1}\Sigma\mathbf{1}'}{(r+\kappa)^2} \end{pmatrix}.$$

Theorem 10.3 of LS (first subequation of (10.81)) implies that

$$\begin{aligned} d\hat{C}_t = & \kappa(\bar{C} - \hat{C}_t)dt - \beta_1 \left\{ d\xi_{1t} - \left[(r + \kappa)\theta a_0 - \frac{\kappa\theta\bar{F}}{r} + (r + \kappa)\xi_{3t} - \left(1 - \frac{(r + \kappa)\gamma_2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) \hat{C}_t \right] dt \right. \\ & \left. - \frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'} \left[d\xi_{2t} - \left[(r + \kappa)\mathbf{1}a_0 - \frac{\kappa\mathbf{1}\bar{F}}{r} + (r + \kappa)\xi_{4t} \right] dt \right] \right\} \\ & - \beta_2 \left\{ d\xi_{3t} - \kappa \left(\frac{\theta\bar{F}}{r} - \theta a_0 - \frac{\gamma_2\Delta\bar{C}}{\mathbf{1}\Sigma\mathbf{1}'} - \xi_{3t} \right) dt \right. \\ & \left. - \frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'} \left[d\xi_{4t} - \kappa \left(\frac{\mathbf{1}\bar{F}}{r} - \mathbf{1}a_0 - \xi_{4t} \right) dt \right] \right\} \end{aligned} \quad (\text{B.5})$$

Eq. (33) follows from (B.5) by noting that the term in dt after each $d\xi_{it}$, $i = 1, 2, 3, 4$, is $E_t(d\xi_{it})$. In subsequent proofs we use a different form of (33), where we replace each $d\xi_{it}$, $i = 1, 2, 3, 4$, by its value in (B.1)-(B.4):

$$d\hat{C}_t = \kappa(\bar{C} - \hat{C}_t)dt - \beta_1 \left[p_f \sigma dB_t^D - \left(1 - \frac{(r + \kappa)\gamma_2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) (C_t - \hat{C}_t)dt \right] - \beta_2 \left(\frac{\phi p_f \sigma dB_t^F}{r + \kappa} - \frac{s\gamma_2\Delta dB_t^C}{\mathbf{1}\Sigma\mathbf{1}'} \right). \quad (\text{B.6})$$

Eq. (36) follows from Theorem 10.3 of LS (second subequation of (10.81)). ■

Proof of Proposition 5: Eqs. (1), (2), (5), (31), (32) and (B.6) imply that stock returns are

$$\begin{aligned} dR_t = & \left\{ ra_0 + \left[g_1 \hat{C}_t + g_2 C_t - \kappa(\gamma_1 + \gamma_2)\bar{C} \right] \Sigma p_f' \right\} dt + (\sigma + \beta_1 \gamma_1 \Sigma p_f' p_f \sigma) dB_t^D \\ & + \frac{\phi}{r + \kappa} (\sigma + \beta_2 \gamma_1 \Sigma p_f' p_f \sigma) dB_t^F - s\gamma_2 \left(1 + \frac{\beta_2 \gamma_1 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) \Sigma p_f' dB_t^C. \end{aligned} \quad (\text{B.7})$$

where

$$g_1 \equiv (r + \kappa + \rho)\gamma_1, \quad (\text{B.8})$$

$$g_2 \equiv (r + \kappa)\gamma_2 - \rho\gamma_1, \quad (\text{B.9})$$

$$\rho \equiv \beta_1 \left(1 - \frac{(r + \kappa)\gamma_2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right). \quad (\text{B.10})$$

Eqs. (5), (10), (30), (B.6) and (B.7) imply that

$$\begin{aligned} & d \left(r\bar{\alpha}W_t + \bar{q}_0 + (\bar{q}_1, \bar{q}_2)\bar{X}_t + \frac{1}{2}\bar{X}_t'\bar{Q}\bar{X}_t \right) \\ & = \bar{G}dt + [r\bar{\alpha}\hat{z}_t (\sigma + \beta_1 \gamma_1 \Sigma p_f' p_f \sigma) - \beta_1 \bar{f}_1(\bar{X}_t)p_f \sigma] dB_t^D \\ & + \frac{\phi}{r + \kappa} [r\bar{\alpha}\hat{z}_t (\sigma + \beta_2 \gamma_1 \Sigma p_f' p_f \sigma) - \beta_2 \bar{f}_1(\bar{X}_t)p_f \sigma] dB_t^F \\ & - s \left[r\bar{\alpha}\gamma_2 \left(1 + \frac{\beta_2 \gamma_1 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) \hat{z}_t \Sigma p_f' - \frac{\beta_2 \gamma_2 \Delta \bar{f}_1(\bar{X}_t)}{\mathbf{1}\Sigma\mathbf{1}'} - \bar{f}_2(\bar{X}_t) \right] dB_t^C, \end{aligned} \quad (\text{B.11})$$

where

$$\bar{f}_1(\bar{X}_t) \equiv \bar{q}_1 + \bar{q}_{11}\hat{C}_t + \bar{q}_{12}C_t,$$

$$\bar{f}_2(\bar{X}_t) \equiv \bar{q}_2 + \bar{q}_{12}\hat{C}_t + \bar{q}_{22}C_t,$$

$$\begin{aligned}
\bar{G} \equiv & r\bar{\alpha} \left(rW_t + \hat{z}_t \left\{ ra_0 + \left[g_1\hat{C}_t + g_2C_t - \kappa(\gamma_1 + \gamma_2)\bar{C} \right] \Sigma p'_f \right\} + B(b_0 - b_1\hat{C}_t) - \bar{c}_t \right) \\
& + \bar{f}_1(\bar{X}_t) \left[\kappa(\bar{C} - \hat{C}_t) + \rho(C_t - \hat{C}_t) \right] + \bar{f}_2(\bar{X}_t)\kappa(\bar{C} - C_t) \\
& + \frac{1}{2} \left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right] \frac{\Delta\bar{q}_{11}}{\mathbf{1}\Sigma\mathbf{1}'} + \frac{s^2\beta_2\gamma_2\Delta\bar{q}_{12}}{\mathbf{1}\Sigma\mathbf{1}'} + \frac{1}{2}s^2\bar{q}_{22},
\end{aligned}$$

and \bar{q}_{ij} denotes the element (i, j) of the symmetric 2×2 matrix \bar{Q} . Eqs. (37) and (B.11) imply that

$$\begin{aligned}
\mathcal{D}\bar{V} = & -\bar{V} \left\{ \bar{G} - \frac{1}{2}(r\bar{\alpha})^2 f \hat{z}_t \Sigma \hat{z}'_t \right. \\
& - \frac{1}{2}\beta_1 \left[r\bar{\alpha}\gamma_1 \hat{z}_t \Sigma p'_f - \bar{f}_1(\bar{X}_t) \right] \left[r\bar{\alpha} \left(2 + \frac{\beta_1\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) \hat{z}_t \Sigma p'_f - \frac{\beta_1\Delta\bar{f}_1(\bar{X}_t)}{\mathbf{1}\Sigma\mathbf{1}'} \right] \\
& - \frac{1}{2} \frac{\phi^2\beta_2}{(r+\kappa)^2} \left[r\bar{\alpha}\gamma_1 \hat{z}_t \Sigma p'_f - \bar{f}_1(\bar{X}_t) \right] \left[r\bar{\alpha} \left(2 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) \hat{z}_t \Sigma p'_f - \frac{\beta_2\Delta\bar{f}_1(\bar{X}_t)}{\mathbf{1}\Sigma\mathbf{1}'} \right] \\
& \left. - \frac{1}{2}s^2 \left[r\bar{\alpha}\gamma_2 \left(1 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) \hat{z}_t \Sigma p'_f - \frac{\beta_2\gamma_2\Delta\bar{f}_1(\bar{X}_t)}{\mathbf{1}\Sigma\mathbf{1}'} - \bar{f}_2(\bar{X}_t) \right]^2 \right\}. \tag{B.12}
\end{aligned}$$

Substituting (B.12) into the Bellman equation (14), we can write the first-order conditions with respect to \bar{c}_t and \hat{z}_t as (A.3) and

$$\bar{h}(\bar{X}_t) = r\bar{\alpha}(f\Sigma + k\Sigma p'_f p_f \Sigma) \hat{z}'_t, \tag{B.13}$$

respectively, where

$$\bar{h}(\bar{X}_t) \equiv ra_0 + \left[g_1\hat{C}_t + g_2C_t - \kappa(\gamma_1 + \gamma_2)\bar{C} + k_1\bar{f}_1(\bar{X}_t) + k_2\bar{f}_2(\bar{X}_t) \right] \Sigma p'_f, \tag{B.14}$$

$$k \equiv \beta_1\gamma_1 \left(2 + \frac{\beta_1\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) + \frac{\phi^2\beta_2\gamma_1}{(r+\kappa)^2} \left(2 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) + s^2\gamma_2^2 \left(1 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right)^2, \tag{B.15}$$

$$k_1 \equiv \beta_1 \left(1 + \frac{\beta_1\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) + \frac{\phi^2\beta_2}{(r+\kappa)^2} \left(1 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) + \frac{s^2\beta_2\gamma_2^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \left(1 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right), \tag{B.16}$$

$$k_2 \equiv s^2\gamma_2 \left(1 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right). \tag{B.17}$$

Proceeding as in the case of symmetric information, we can derive counterparts to (A.8) and (A.9), and write the Bellman equation (14) in the equivalent form

$$\begin{aligned}
& \frac{1}{2} \bar{h}(\bar{X}_t)' (f\Sigma + k\Sigma p_f' p_f \Sigma)^{-1} \bar{h}(\bar{X}_t) + r\bar{\alpha}B(b_0 - b_1\hat{C}_t) - r \left[\bar{q}_0 + (\bar{q}_1, \bar{q}_2)\bar{X}_t + \frac{1}{2}\bar{X}_t'\bar{Q}\bar{X}_t \right] \\
& + \bar{f}_1(\bar{X}_t) \left[\kappa(\bar{C} - \hat{C}_t) + \rho(C_t - \hat{C}_t) \right] + \bar{f}_2(\bar{X}_t)\kappa(\bar{C} - C_t) \\
& + \frac{1}{2} \left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right] \frac{\Delta\bar{q}_{11}}{\mathbf{1}\Sigma\mathbf{1}'} + \frac{s^2\beta_2\gamma_2\Delta\bar{q}_{12}}{\mathbf{1}\Sigma\mathbf{1}'} + \frac{1}{2}s^2\bar{q}_{22} + \beta - r + r\log(r) \\
& - \frac{1}{2} \left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2} \right] \frac{\Delta\bar{f}_1(\bar{X}_t)^2}{\mathbf{1}\Sigma\mathbf{1}'} - \frac{1}{2}s^2 \left[\frac{\beta_2\gamma_2\Delta\bar{f}_1(\bar{X}_t)}{\mathbf{1}\Sigma\mathbf{1}'} + \bar{f}_2(\bar{X}_t) \right]^2 = 0. \tag{B.18}
\end{aligned}$$

Eq. (B.18) is quadratic in \bar{X}_t . Identifying quadratic, linear and constant terms, yields six scalar equations in $(\bar{q}_0, \bar{q}_1, \bar{q}_2, \bar{Q})$. ■

Proof of Proposition 6: Dynamics under the investor's filtration can be deduced from those under the manager's by replacing C_t by the investor's expectation \hat{C}_t . Eq. (B.6) implies that the dynamics of \hat{C}_t are

$$d\hat{C}_t = \kappa(\bar{C} - \hat{C}_t)dt - \beta_1 p_f \sigma d\hat{B}_t^D - \beta_2 \left(\frac{\phi p_f \sigma dB_t^F}{r + \kappa} - \frac{s\gamma_2 \Delta dB_t^C}{\mathbf{1}\Sigma\mathbf{1}'} \right), \tag{B.19}$$

where \hat{B}_t^D is a Brownian motion under the investor's filtration. Eq. (B.7) implies that the net-of-cost return of the active fund is

$$\begin{aligned}
z_t dR_t - C_t dt = & z_t \left\{ ra_0 + \left[(g_1 + g_2)\hat{C}_t - \kappa(\gamma_1 + \gamma_2)\bar{C} \right] \Sigma p_f' \right\} dt - \hat{C}_t dt + z_t (\sigma + \beta_1 \gamma_1 \Sigma p_f' p_f \sigma) d\hat{B}_t^D \\
& + z_t \frac{\phi}{r + \kappa} (\sigma + \beta_2 \gamma_1 \Sigma p_f' p_f \sigma) dB_t^F - s\gamma_2 \left(1 + \frac{\beta_2 \gamma_1 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) z_t \Sigma p_f' dB_t^C, \tag{B.20}
\end{aligned}$$

and the return of the index fund is

$$\begin{aligned}
\mathbf{1}dR_t = & \mathbf{1} \left\{ ra_0 + \left[(g_1 + g_2)\hat{C}_t - \kappa(\gamma_1 + \gamma_2)\bar{C} \right] \Sigma p_f' \right\} dt + \mathbf{1} (\sigma + \beta_1 \gamma_1 \Sigma p_f' p_f \sigma) d\hat{B}_t^D \\
& + \mathbf{1} \frac{\phi}{r + \kappa} (\sigma + \beta_2 \gamma_1 \Sigma p_f' p_f \sigma) dB_t^F - s\gamma_2 \left(1 + \frac{\beta_2 \gamma_1 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) \mathbf{1} \Sigma p_f' dB_t^C. \tag{B.21}
\end{aligned}$$

Eqs. (16), (B.19), (B.20) and (B.21) imply that

$$\begin{aligned}
& d\left(r\alpha W_t + q_0 + q_1 \hat{C}_t + \frac{1}{2}q_{11}\hat{C}_t^2\right) \\
&= Gdt + \left[r\alpha(x_t\mathbf{1} + y_t z_t)(\sigma + \beta_1\gamma_1\Sigma p'_f p_f \sigma) - \beta_1 f(\hat{C}_t)p_f \sigma\right] d\hat{B}_t^D \\
&+ \frac{\phi}{r + \kappa} \left[r\alpha(x_t\mathbf{1} + y_t z_t)(\sigma + \beta_2\gamma_1\Sigma p'_f p_f \sigma) - \beta_2 f(\hat{C}_t)p_f \sigma\right] dB_t^F \\
&- s \left[r\bar{\alpha}\gamma_2 \left(1 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)(x_t\mathbf{1} + y_t z_t)\Sigma p'_f - \frac{\beta_2\gamma_2\Delta f(\hat{C}_t)}{\mathbf{1}\Sigma\mathbf{1}'}\right] dB_t^C, \tag{B.22}
\end{aligned}$$

where

$$f(\hat{C}_t) \equiv q_1 + q_{11}\hat{C}_t$$

and

$$\begin{aligned}
G \equiv & r\alpha \left(rW_t + (x_t\mathbf{1} + y_t z_t) \left\{ra_0 + \left[(g_1 + g_2)\hat{C}_t - \kappa(\gamma_1 + \gamma_2)\bar{C}\right] \Sigma p'_f\right\} - y_t \hat{C}_t - c_t\right) \\
&+ f(\hat{C}_t)\kappa(\bar{C} - \hat{C}_t) + \frac{1}{2} \left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r + \kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right] \frac{\Delta q_{11}}{\mathbf{1}\Sigma\mathbf{1}'}.
\end{aligned}$$

Eqs. (37) and (B.11) imply that

$$\begin{aligned}
\mathcal{D}V = & -V \left\{G - \frac{1}{2}(r\alpha)^2 f(x_t\mathbf{1} + y_t z_t)\Sigma(x_t\mathbf{1} + y_t z_t)'\right. \\
&- \frac{1}{2}\beta_1 \left[r\alpha\gamma_1(x_t\mathbf{1} + y_t z_t)\Sigma p'_f - f(\hat{C}_t)\right] \left[r\alpha \left(2 + \frac{\beta_1\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)(x_t\mathbf{1} + y_t z_t)\Sigma p'_f - \frac{\beta_1\Delta f(\hat{C}_t)}{\mathbf{1}\Sigma\mathbf{1}'}\right] \\
&- \frac{1}{2} \frac{\phi^2\beta_2}{(r + \kappa)^2} \left[r\alpha\gamma_1(x_t\mathbf{1} + y_t z_t)\Sigma p'_f - f(\hat{C}_t)\right] \left[r\alpha \left(2 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)(x_t\mathbf{1} + y_t z_t)\Sigma p'_f - \frac{\beta_2\Delta f(\hat{C}_t)}{\mathbf{1}\Sigma\mathbf{1}'}\right] \\
&\left. - \frac{1}{2}s^2 \left[r\alpha\gamma_2 \left(1 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)(x_t\mathbf{1} + y_t z_t)\Sigma p'_f - \frac{\beta_2\gamma_2\Delta f(\hat{C}_t)}{\mathbf{1}\Sigma\mathbf{1}'}\right]^2\right\}. \tag{B.23}
\end{aligned}$$

Substituting (B.12) into the Bellman equation (14), we can write the first-order conditions with respect to c_t , x_t and y_t as (A.13),

$$\mathbf{1}h(\hat{C}_t) = r\alpha\mathbf{1}(f\Sigma + k\Sigma p'_f p_f \Sigma)(x_t\mathbf{1} + y_t z_t)', \tag{B.24}$$

$$z_t h(\hat{C}_t) - \hat{C}_t = r\alpha z_t(f\Sigma + k\Sigma p'_f p_f \Sigma)(x_t\mathbf{1} + y_t z_t)', \tag{B.25}$$

respectively, where

$$h(\hat{C}_t) \equiv ra_0 + \left[(g_1 + g_2)\hat{C}_t - \kappa(\gamma_1 + \gamma_2)\bar{C} + k_1 f(\hat{C}_t) \right] \Sigma p'_f. \quad (\text{B.26})$$

Proceeding as in the case of symmetric information, we can derive counterparts to (A.17), (A.18), (A.24), and write the Bellman equation (18) in the equivalent form

$$\begin{aligned} & \frac{1}{2D} \left\{ \left[\mathbf{1}h(\hat{C}_t) \right]^2 \theta(f\Sigma + k\Sigma p'_f p_f \Sigma) \theta' - 2 \left[\theta h(\hat{C}_t) - \hat{C}_t \right] \mathbf{1}h(\hat{C}_t) \mathbf{1}(f\Sigma + k\Sigma p'_f p_f \Sigma) \theta' \right. \\ & \left. + \left[\theta h(\hat{C}_t) - \hat{C}_t \right]^2 \mathbf{1}(f\Sigma + k\Sigma p'_f p_f \Sigma) \mathbf{1}' \right\} - r \left(q_0 + q_1 \hat{C}_t + \frac{1}{2} q_{11} \hat{C}_t^2 \right) \\ & + f(\hat{C}_t) \kappa(\bar{C} - \hat{C}_t) + \frac{1}{2} \left[\beta_1^2 + \frac{\phi^2 \beta_2^2}{(r + \kappa)^2} + \frac{s^2 \beta_2^2 \gamma_2^2 \Delta}{\mathbf{1}\Sigma \mathbf{1}'} \right] \frac{\Delta[q_{11} - f(\hat{C}_t)^2]}{\mathbf{1}\Sigma \mathbf{1}'} + \beta - r + r \log(r) = 0, \end{aligned} \quad (\text{B.27})$$

where

$$D \equiv \theta(f\Sigma + k\Sigma p'_f p_f \Sigma) \theta' \mathbf{1}(f\Sigma + k\Sigma p'_f p_f \Sigma) \mathbf{1}' - \left[\mathbf{1}(f\Sigma + k\Sigma p'_f p_f \Sigma) \theta' \right]^2.$$

Eq. (B.27) is quadratic in \hat{C}_t . Identifying terms in \hat{C}_t^2 , \hat{C}_t , and constants, yields three scalar equations in (q_0, q_1, q_{11}) . The proof that the optimal control y_t is linear in \hat{C}_t is as in the case of symmetric information. \blacksquare

Proof of Proposition 7: We first impose market clearing and derive the constants $(a_0, b_0, b_1, \gamma_1, \gamma_2)$ as functions of $(\bar{q}_1, \bar{q}_2, \bar{Q}, q_1, q_{11})$. Setting $z_t = \theta - x_t \mathbf{1}$ and $\bar{y}_t = 1 - y_t$, we can write (B.13) and (B.24) as

$$\bar{h}(\bar{X}_t) = r\bar{\alpha}(f\Sigma + k\Sigma p'_f p_f \Sigma)(1 - y_t)(\theta - x_t \mathbf{1})', \quad (\text{B.28})$$

$$\mathbf{1}h(\hat{C}_t) = r\alpha \mathbf{1}(f\Sigma + k\Sigma p'_f p_f \Sigma) [x_t \mathbf{1} + y_t(\theta - x_t \mathbf{1})]', \quad (\text{B.29})$$

respectively. Multiplying (B.24) by x_t and adding to (B.25), we similarly find

$$\theta h(\hat{C}_t) - \hat{C}_t = r\alpha \theta (f\Sigma + k\Sigma p'_f p_f \Sigma) [x_t \mathbf{1} + y_t(\theta - x_t \mathbf{1})]'. \quad (\text{B.30})$$

Premultiplying (B.28) by $\mathbf{1}$, dividing by $r\bar{\alpha}$, and adding to (B.29) divided by $r\alpha$, we find

$$\mathbf{1} \left[\frac{h(\hat{C}_t)}{r\alpha} + \frac{\bar{h}(\bar{X}_t)}{r\bar{\alpha}} \right] = \mathbf{1}(f\Sigma + k\Sigma p'_f p_f \Sigma) \theta'. \quad (\text{B.31})$$

Eq. (B.31) is linear in \bar{X}_t . The terms in \hat{C}_t and C_t are zero because $\mathbf{1}\Sigma p'_f = 0$. Identifying constant terms, we find (A.29). Substituting (A.29) into (B.29), we find (A.30).

Substituting (A.30) into (B.28), and using (30), we find

$$\bar{h}(\bar{X}_t) = r\bar{\alpha}(f\Sigma + k\Sigma p'_f p_f \Sigma) \left[\frac{\alpha}{\alpha + \bar{\alpha}} \frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'} \mathbf{1} + (1 - b_0 + b_1 \hat{C}_t) p_f \right]'. \quad (\text{B.32})$$

Eq. (B.32) is linear in \bar{X}_t . Identifying terms in \hat{C}_t , we find

$$\begin{aligned} g_1 \Sigma p'_f + (k_1 \bar{q}_{11} + k_2 \bar{q}_{12}) \Sigma p'_f &= r\bar{\alpha} b_1 (f\Sigma + k\Sigma p'_f p_f \Sigma) p'_f \\ \Rightarrow g_1 + k_1 \bar{q}_{11} + k_2 \bar{q}_{12} &= r\bar{\alpha} b_1 \left(f + \frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right). \end{aligned} \quad (\text{B.33})$$

Identifying terms in C_t , we find

$$g_2 \Sigma p'_f + (k_1 \bar{q}_{12} + k_2 \bar{q}_{22}) \Sigma p'_f = 0 \Rightarrow g_2 + k_1 \bar{q}_{12} + k_2 \bar{q}_{22} = 0. \quad (\text{B.34})$$

Identifying constant terms, we find

$$a_0 = \frac{\alpha\bar{\alpha}f}{\alpha + \bar{\alpha}} \frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'} \Sigma\mathbf{1}' + \left[\frac{\kappa(\gamma_1 + \gamma_2)\bar{C} - k_1 \bar{q}_1 - k_2 \bar{q}_2}{r} + \bar{\alpha}(1 - b_0) \left(f + \frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) \right] \Sigma p'_f. \quad (\text{B.35})$$

Premultiplying (B.28) by θ , dividing by $r\bar{\alpha}$, and adding to (B.30) divided by $r\alpha$, we find

$$\theta \left[\frac{h(\hat{C}_t)}{r\alpha} + \frac{\bar{h}(\bar{X}_t)}{r\bar{\alpha}} \right] - \frac{\hat{C}_t}{r\alpha} = \theta(f\Sigma + k\Sigma p'_f p_f \Sigma)\theta'. \quad (\text{B.36})$$

Eq. (B.36) is linear in \bar{X}_t . The terms in C_t cancel because of (B.34). Identifying terms in \hat{C}_t , we find

$$\frac{g_1 + g_2 + k_1 q_{11}}{r\alpha} + \frac{g_1 + k_1 \bar{q}_{11} + k_2 \bar{q}_{12}}{r\bar{\alpha}} = \frac{\mathbf{1}\Sigma\mathbf{1}'}{r\alpha\Delta}. \quad (\text{B.37})$$

Identifying constant terms, we find

$$\left(\frac{r}{r\alpha} + \frac{r}{r\bar{\alpha}} \right) \theta a_0 - \left[\frac{\kappa(\gamma_1 + \gamma_2)\bar{C} - k_1 q_1}{r\alpha} + \frac{\kappa(\gamma_1 + \gamma_2)\bar{C} - k_1 \bar{q}_1 - k_2 \bar{q}_2}{r\bar{\alpha}} \right] \frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} = \theta(f\Sigma + k\Sigma p'_f p_f \Sigma)\theta'. \quad (\text{B.38})$$

Using (B.35), we can write (B.38) as

$$b_0 = \frac{\bar{\alpha}}{\alpha + \bar{\alpha}} + \frac{k_1 q_1 - (k_1 \bar{q}_1 + k_2 \bar{q}_2)}{r(\alpha + \bar{\alpha}) \left(f + \frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)}. \quad (\text{B.39})$$

Substituting b_0 from (B.39) into (B.35), we find

$$a_0 = \frac{\alpha \bar{\alpha} f}{\alpha + \bar{\alpha}} \Sigma \theta' + \left[\frac{\kappa(\gamma_1 + \gamma_2) \bar{C}}{r} - \frac{\alpha(k_1 \bar{q}_1 + k_2 \bar{q}_2) + \bar{\alpha} k_1 q_1}{r(\alpha + \bar{\alpha})} + \frac{\alpha \bar{\alpha} k \Delta}{(\alpha + \bar{\alpha}) \mathbf{1}\Sigma\mathbf{1}'} \right] \Sigma p'_f. \quad (\text{B.40})$$

The system of equations characterizing equilibrium is as follows. The endogenous variables are $(a_0, b_0, b_1, \gamma_1, \gamma_2, \beta_1, \beta_2, \hat{s}^2, \bar{q}_1, \bar{q}_2, \bar{Q}, q_1, q_{11})$. (As in Proposition 3, we can drop (\bar{q}_0, q_0) .) The equations linking them are (34)-(36), (B.33), (B.34), (B.37), (B.39), (B.40), the five equations derived from (B.18) by identifying linear and quadratic terms, and the two equations derived from (B.27) by identifying linear and quadratic terms. We next simplify the latter two sets of equations, using implications of market clearing.

Consider the equations derived from (B.18). Using (B.39), we can write (B.32) as

$$\bar{h}(\bar{X}_t) = \frac{r\alpha\bar{\alpha}f}{\alpha + \bar{\alpha}} \frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'} \Sigma \mathbf{1}' + \bar{h}_1(\hat{C}_t) \Sigma p'_f, \quad (\text{B.41})$$

where

$$\bar{h}_1(\hat{C}_t) \equiv \frac{r\alpha\bar{\alpha}}{\alpha + \bar{\alpha}} \left(f + \frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right) + \frac{\bar{\alpha}(k_1 \bar{q}_1 + k_2 \bar{q}_2 - k_1 q_1)}{\alpha + \bar{\alpha}} + r\bar{\alpha}b_1 \left(f + \frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right) \hat{C}_t.$$

Noting that

$$(f\Sigma + k\Sigma p'_f p_f \Sigma) \mathbf{1}' = f\Sigma \mathbf{1}' \Rightarrow \mathbf{1}\Sigma (f\Sigma + k\Sigma p'_f p_f \Sigma)^{-1} = \frac{\mathbf{1}}{f}, \quad (\text{B.42})$$

$$(f\Sigma + k\Sigma p'_f p_f \Sigma) p'_f = \left(f + \frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right) \Sigma p'_f \Rightarrow p_f \Sigma (f\Sigma + k\Sigma p'_f p_f \Sigma)^{-1} = \frac{p_f}{f + \frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'}}}, \quad (\text{B.43})$$

and using (B.41), we find

$$\frac{1}{2} \bar{h}(\bar{X}_t)' (f\Sigma + k\Sigma p'_f p_f \Sigma)^{-1} \bar{h}(\bar{X}_t) = \frac{r^2 \alpha^2 \bar{\alpha}^2 f (\mathbf{1}\Sigma\theta')^2}{2(\alpha + \bar{\alpha})^2 \mathbf{1}\Sigma\mathbf{1}'} + \frac{\bar{h}_1(\hat{C}_t)^2 \frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'}}{2 \left(f + \frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)}. \quad (\text{B.44})$$

We next substitute (B.44) into (B.18), and identify terms. Identifying terms in \hat{C}_t^2 , $\hat{C}_t C_t$ and C_t^2 , we find

$$\frac{1}{2} \bar{X}'_t (\bar{Q} \bar{\mathcal{R}}_2 \bar{Q} + \bar{Q} \bar{\mathcal{R}}_1 + \bar{\mathcal{R}}'_1 \bar{Q} - \bar{\mathcal{R}}_0) \bar{X}_t = 0, \quad (\text{B.45})$$

where

$$\begin{aligned}\bar{\mathcal{R}}_2 &\equiv \left(\begin{array}{cc} \left[\beta_1^2 + \frac{\phi^2 \beta_2^2}{(r+\kappa)^2} + \frac{s^2 \beta_2^2 \gamma_2^2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right] \frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} & \frac{s^2 \beta_2 \gamma_2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \\ \frac{s^2 \beta_2 \gamma_2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} & s^2 \end{array} \right), \\ \bar{\mathcal{R}}_1 &\equiv \left(\begin{array}{cc} \frac{r}{2} + \kappa + \rho & -\rho \\ 0 & \frac{r}{2} + \kappa \end{array} \right), \\ \bar{\mathcal{R}}_0 &\equiv \left(\begin{array}{cc} r^2 \bar{\alpha}^2 b_1^2 \left(f + \frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) \frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} & 0 \\ 0 & 0 \end{array} \right).\end{aligned}$$

Eq. (B.45) must hold for all \bar{X}_t . Since the square matrix in (B.45) is symmetric, it must equal zero, and this yields the algebraic Riccati equation

$$\bar{Q}\bar{\mathcal{R}}_2\bar{Q} + \bar{Q}\bar{\mathcal{R}}_1 + \bar{\mathcal{R}}_1'\bar{Q} - \bar{\mathcal{R}}_0 = 0. \quad (\text{B.46})$$

We next identify terms in \hat{C}_t and C_t . Terms in \hat{C}_t yield

$$\begin{aligned}(r + \kappa + \rho) \bar{q}_1 + \left[\beta_1^2 + \frac{\phi^2 \beta_2^2}{(r + \kappa)^2} \right] \frac{\Delta \bar{q}_1 \bar{q}_{11}}{\mathbf{1}\Sigma\mathbf{1}'} + s^2 \left(\frac{\beta_2 \gamma_2 \Delta \bar{q}_1}{\mathbf{1}\Sigma\mathbf{1}'} + \bar{q}_2 \right) \left(\frac{\beta_2 \gamma_2 \Delta \bar{q}_{11}}{\mathbf{1}\Sigma\mathbf{1}'} + \bar{q}_{12} \right) \\ - r \bar{\alpha} b_1 \left[\frac{r \alpha \bar{\alpha} \left(f + \frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right)}{\alpha + \bar{\alpha}} + \frac{\bar{\alpha} (k_1 \bar{q}_1 + k_2 \bar{q}_2 - k_1 q_1)}{\alpha + \bar{\alpha}} \right] - \kappa \bar{C} (\bar{q}_{11} + \bar{q}_{12}) + r \bar{\alpha} B b_1 = 0, \quad (\text{B.47})\end{aligned}$$

and terms in C_t yield

$$\begin{aligned}(r + \kappa) \bar{q}_2 - \rho \bar{q}_1 + \left[\beta_1^2 + \frac{\phi^2 \beta_2^2}{(r + \kappa)^2} \right] \frac{\Delta \bar{q}_1 \bar{q}_{12}}{\mathbf{1}\Sigma\mathbf{1}'} + s^2 \left(\frac{\beta_2 \gamma_2 \Delta \bar{q}_1}{\mathbf{1}\Sigma\mathbf{1}'} + \bar{q}_2 \right) \left(\frac{\beta_2 \gamma_2 \Delta \bar{q}_{12}}{\mathbf{1}\Sigma\mathbf{1}'} + \bar{q}_{22} \right) \\ - \kappa \bar{C} (\bar{q}_{12} + \bar{q}_{22}) = 0. \quad (\text{B.48})\end{aligned}$$

Consider next the equations derived from (B.27). Substituting $\bar{h}(\bar{X}_t)$ from (B.41) into (B.31), we find

$$\mathbf{1}h(\hat{C}_t) = \frac{r\alpha\bar{\alpha}f}{\alpha + \bar{\alpha}} \mathbf{1}\Sigma\theta'. \quad (\text{B.49})$$

Following the same procedure for (B.36) instead of (B.31), we find

$$\theta h(\hat{C}_t) - C_t = \frac{r\alpha\bar{\alpha}f}{\alpha + \bar{\alpha}} \frac{(\mathbf{1}\Sigma\theta')^2}{\mathbf{1}\Sigma\mathbf{1}'} + \left[r\alpha \left(f + \frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) - \frac{\alpha}{\bar{\alpha}} \bar{h}_1(C_t) \right] \frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'}.\quad (\text{B.50})$$

The denominator D in (B.27) is

$$D = f\Delta \left(f + \frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right). \quad (\text{B.51})$$

Using (B.49), (B.50) and (B.51), we find that the equation derived from (B.27) by identifying terms in \hat{C}_t^2 is

$$(r + 2\kappa)q_{11} + \left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r + \kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right] \frac{\Delta q_{11}^2}{\mathbf{1}\Sigma\mathbf{1}'} - r^2\alpha^2b_1^2 \left(f + \frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) \frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} = 0, \quad (\text{B.52})$$

and the equation derived from (B.27) by identifying terms in \hat{C}_t is

$$(r + \kappa)q_1 + \left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r + \kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right] \frac{\Delta q_1 q_{11}}{\mathbf{1}\Sigma\mathbf{1}'} + r\alpha b_1 \left\{ \frac{r\alpha\bar{\alpha} \left(f + \frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right)}{\alpha + \bar{\alpha}} + \frac{\alpha[k_1q_1 - (k_1\bar{q}_1 + k_2\bar{q}_2)]}{\alpha + \bar{\alpha}} \right\} \frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} - \kappa\bar{C}q_{11} = 0. \quad (\text{B.53})$$

Solving for equilibrium amounts to solving the system of (34)-(36), (B.33), (B.34), (B.37), (B.39), (B.40), (B.46)-(B.48), (B.52) and (B.53) in the unknowns $(a_0, b_0, b_1, \gamma_1, \gamma_2, \beta_1, \beta_2, \hat{s}^2, \bar{q}_1, \bar{q}_2, \bar{Q}, q_1, q_{11})$. This reduces to solving the system of (B.33), (B.34), (B.37), (B.46) and (B.52) in the unknowns $(b_1, \gamma_1, \gamma_2, \bar{Q}, q_{11})$: given $(b_1, \gamma_1, \gamma_2, \bar{Q}, q_{11})$, $(\beta_1, \beta_2, \hat{s}^2)$ can be determined from (34)-(36), $(\bar{q}_1, \bar{q}_2, q_1)$ from the linear system of (B.47), (B.48) and (B.53), and (a_0, b_0) from (B.39) and (B.40). **[[[To be completed.]]]** ■

Proof of Corollary 3: Stocks' expected returns are

$$\begin{aligned} E_t(dR_t) &= \left\{ ra_0 + \left[g_1\hat{C}_t + g_2C_t - \kappa(\gamma_1 + \gamma_2)\bar{C} \right] \Sigma p'_f \right\} dt \\ &= \left\{ \frac{r\alpha\bar{\alpha}f}{\alpha + \bar{\alpha}} \frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'} \Sigma\mathbf{1}' + \left[g_1\hat{C}_t + g_2C_t - \frac{\alpha(k_1\bar{q}_1 + k_2\bar{q}_2) + \bar{\alpha}k_1q_1}{\alpha + \bar{\alpha}} + \frac{r\alpha\bar{\alpha} \left(f + \frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right)}{\alpha + \bar{\alpha}} \right] \Sigma p'_f \right\} dt, \end{aligned} \quad (\text{B.54})$$

where the first step follows from (B.7) and the second from (B.40). Using (B.42) and (B.43), we can write (B.54) as

$$E_t(dR_t) = \left[\frac{r\alpha\bar{\alpha}}{\alpha + \bar{\alpha}} \frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'} (f\Sigma + k\Sigma p'_f p_f \Sigma) \mathbf{1}' + \Lambda_t (f\Sigma + k\Sigma p'_f p_f \Sigma) p'_f \right] dt. \quad (\text{B.55})$$

Eq. (B.55) is equivalent to (39) because of (41). ■

Proof of Corollary 4: The corollary follows from (B.7). ■

Proof of Proposition 8: Rearranging (B.15), we find

$$k = 2 \left\{ \beta_1 + \beta_2 \left[\frac{\phi^2}{(r + \kappa)^2} + \frac{s^2 \gamma_2^2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right] \right\} \gamma_1 + s^2 \gamma_2^2 + \left[\beta_1^2 + \frac{\phi^2 \beta_2^2}{(r + \kappa)^2} + \frac{s^2 \beta_2^2 \gamma_2^2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right] \frac{\Delta \gamma_1^2}{\mathbf{1}\Sigma\mathbf{1}'}. \quad (\text{B.56})$$

Rearranging (35), we find

$$\beta_2 \left[\frac{\phi^2}{(r + \kappa)^2} + \frac{s^2 \gamma_2^2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right] = s^2 \gamma_2, \quad (\text{B.57})$$

and rearranging (36), we find

$$\begin{aligned} \hat{s}^4 \left[1 - (r + k) \frac{\gamma_2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right]^2 \frac{\mathbf{1}\Sigma\mathbf{1}'}{\Delta} + \frac{\frac{s^4 \gamma_2^2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'}}{\frac{\phi^2}{(r + \kappa)^2} + \frac{s^2 \gamma_2^2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'}} &= s^2 - 2\kappa \hat{s}^2 \\ \Rightarrow \left[\beta_1^2 + \frac{\phi^2 \beta_2^2}{(r + \kappa)^2} + \frac{s^2 \beta_2^2 \gamma_2^2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right] \frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} &= s^2 - 2\kappa \hat{s}^2, \end{aligned} \quad (\text{B.58})$$

where the second step follows from (34) and (35). Substituting (B.57) and (B.58) into (B.56), we find

$$\begin{aligned} k &= 2\beta_1 \gamma_1 + s^2 (\gamma_1 + \gamma_2)^2 - 2\kappa \hat{s}^2 \gamma_1^2 \\ &= s^2 (\gamma_1 + \gamma_2)^2 + 2\hat{s}^2 \gamma_1 \left[\frac{\mathbf{1}\Sigma\mathbf{1}'}{\Delta} - \kappa \gamma_1 - (r + \kappa) \gamma_2 \right], \end{aligned} \quad (\text{B.59})$$

where the second step follows from (34). [[[To be completed.]]] ■

Proof of Corollary 5: The corollary follows from Proposition 8 by considering the diagonal elements of the covariance matrix. ■

Proof of Corollary 6: Using (B.7) and proceeding as in the derivation of (A.61), we find

$$\begin{aligned} \text{Cov}_t(dR_t, dR_{t'}) &= \text{Cov}_t \left[(\sigma + \beta_1 \gamma_1 \Sigma p'_f p_f \sigma) dB_t^D + \frac{\phi}{r + \kappa} (\sigma + \beta_2 \gamma_1 \Sigma p'_f p_f \sigma) dB_t^F \right. \\ &\quad \left. - s \gamma_2 \left(1 + \frac{\beta_2 \gamma_1 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) \Sigma p'_f dB_t^C, (g_1 \hat{C}_{t'} + g_2 C_{t'}) p_f \Sigma dt \right]. \end{aligned} \quad (\text{B.60})$$

Using the dynamics (5) and (B.6), we can express $(\hat{C}_{t'}, C_{t'})$ as a function of their time t values and the Brownian shocks (dB_u^D, dB_u^F, dB_u^C) for $u \in [t, t']$. The covariance (B.60) depends only on how

the Brownian shocks at time t impact $(\hat{C}_{t'}, C_{t'})$. (See the proof of Corollary 6.) To compute this impact, we set $(\hat{C}_t, C_t, \bar{C})$ and the Brownian shocks for $u > t$ to zero. This amounts to solving the “impulse-response” dynamics

$$\begin{aligned} dC_t &= -\kappa C_t dt, \\ d\hat{C}_t &= \left[-\kappa \hat{C}_t + \rho(C_t - \hat{C}_t) \right] dt, \end{aligned}$$

with the initial conditions

$$\begin{aligned} C_t &= sdB_t^C, \\ \hat{C}_t &= -\beta_1 p_f \sigma dB_t^D - \beta_2 \left(\frac{\phi p_f \sigma dB_t^F}{r + \kappa} - \frac{s \gamma_2 \Delta dB_t^C}{\mathbf{1}\Sigma\mathbf{1}'} \right). \end{aligned}$$

The solution to these dynamics is

$$C_{t'} = e^{-\kappa(t'-t)} sdB_{t'}^C, \tag{B.61}$$

$$\hat{C}_{t'} = e^{-\kappa(t'-t)} sdB_{t'}^C - e^{-(\kappa+\rho)(t'-t)} \left[\beta_1 p_f \sigma dB_{t'}^D + \frac{\phi \beta_2 p_f \sigma dB_{t'}^F}{r + \kappa} + s \left(1 - \frac{\beta_2 \gamma_2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) dB_{t'}^C \right]. \tag{B.62}$$

Substituting (B.61) and (B.62) into (B.60), we find (42) with

$$\chi_1 \equiv -g_1 \left\{ \beta_1 \left(1 + \frac{\beta_1 \gamma_1 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) + \left[\frac{\phi^2 \beta_2}{(r + \kappa)^2} - s^2 \gamma_2 \left(1 - \frac{\beta_2 \gamma_2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) \right] \left(1 + \frac{\beta_2 \gamma_1 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) \right\}, \tag{B.63}$$

$$\chi_2 \equiv -s^2 (g_1 + g_2) \gamma_2 \left(1 + \frac{\beta_2 \gamma_1 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right). \tag{B.64}$$

Eq. (35) implies that the term in square brackets in (B.63) is zero, and therefore

$$\chi_1 = -g_1 \beta_1 \left(1 + \frac{\beta_1 \gamma_1 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) < 0.$$

Eqs. (B.8) and (B.9) imply that

$$\chi_2 = -s^2 (r + \kappa) (\gamma_1 + \gamma_2) \gamma_2 \left(1 + \frac{\beta_2 \gamma_1 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) < 0.$$

■

C Asymmetric Information and Gradual Adjustment

Proof of Proposition 9: Eq. (B.6), describing the dynamics of \hat{C}_t , remains valid under gradual adjustment. Eq. (B.7) is replaced by

$$dR_t = \left\{ ra_0 + \left[\tilde{g}_1 \hat{C}_t + g_2 C_t + g_3 y_t - \kappa(\gamma_1 + \gamma_2) \bar{C} - b_0 \gamma_3 \right] \Sigma p'_f \right\} dt + (\sigma + \beta_1 \gamma_1 \Sigma p'_f p_f \sigma) dB_t^D \\ + \frac{\phi}{r + \kappa} (\sigma + \beta_2 \gamma_1 \Sigma p'_f p_f \sigma) dB_t^F - s \gamma_2 \left(1 + \frac{\beta_2 \gamma_1 \Delta}{\mathbf{1} \Sigma \mathbf{1}'} \right) \Sigma p'_f dB_t^C, \quad (\text{C.1})$$

where

$$\tilde{g}_1 \equiv g_1 + b_1 \gamma_3, \quad (\text{C.2})$$

$$g_3 \equiv (r + b_2) \gamma_3. \quad (\text{C.3})$$

Eq. (B.12) remains valid, provided that we set

$$\bar{G} \equiv r \bar{\alpha} \left(r W_t + \hat{z}_t \left\{ ra_0 + \left[\tilde{g}_1 \hat{C}_t + g_2 C_t + g_3 y_t - \kappa(\gamma_1 + \gamma_2) \bar{C} - b_0 \gamma_3 \right] \Sigma p'_f \right\} + B y_t - \bar{c}_t \right) \\ + \bar{f}_1(\bar{X}_t) \left[\kappa(\bar{C} - \hat{C}_t) + \rho(C_t - \hat{C}_t) \right] + \bar{f}_2(\bar{X}_t) \kappa(\bar{C} - C_t) + \bar{f}_3(\bar{X}_t) (b_0 - b_1 \hat{C}_t - b_2 y_t) \\ + \frac{1}{2} \left[\beta_1^2 + \frac{\phi^2 \beta_2^2}{(r + \kappa)^2} + \frac{s^2 \beta_2^2 \gamma_2^2 \Delta}{\mathbf{1} \Sigma \mathbf{1}'} \right] \frac{\Delta \bar{q}_{11}}{\mathbf{1} \Sigma \mathbf{1}'} + \frac{s^2 \beta_2 \gamma_2 \Delta \bar{q}_{12}}{\mathbf{1} \Sigma \mathbf{1}'} + \frac{1}{2} s^2 \bar{q}_{22},$$

$$\bar{f}_1(\bar{X}_t) \equiv \bar{q}_1 + \bar{q}_{11} \hat{C}_t + \bar{q}_{12} C_t + \bar{q}_{13} y_t,$$

$$\bar{f}_2(\bar{X}_t) \equiv \bar{q}_2 + \bar{q}_{12} \hat{C}_t + \bar{q}_{22} C_t + \bar{q}_{23} y_t,$$

$$\bar{f}_3(\bar{X}_t) \equiv \bar{q}_3 + \bar{q}_{13} \hat{C}_t + \bar{q}_{23} C_t + \bar{q}_{33} y_t,$$

where \bar{q}_{ij} denotes the element (i, j) of the symmetric 3×3 matrix \bar{Q} . The first-order condition (B.13) remains valid, provided that we set

$$\bar{h}(\bar{X}_t) \equiv ra_0 + \left[\tilde{g}_1 \hat{C}_t + g_2 C_t + g_3 y_t - \kappa(\gamma_1 + \gamma_2) \bar{C} - b_0 \gamma_3 + k_1 \bar{f}_1(\bar{X}_t) + k_2 \bar{f}_2(\bar{X}_t) \right] \Sigma p'_f. \quad (\text{C.4})$$

Eq. (B.18) becomes

$$\begin{aligned}
& \frac{1}{2} \bar{h}(\bar{X}_t)' (f\Sigma + k\Sigma p_f' p_f \Sigma)^{-1} \bar{h}(\bar{X}_t) + r\bar{\alpha} B y_t - r \left[\bar{q}_0 + (\bar{q}_1, \bar{q}_2, \bar{q}_3) \bar{X}_t + \frac{1}{2} \bar{X}_t' \bar{Q} \bar{X}_t \right] \\
& + \bar{f}_1(\bar{X}_t) \left[\kappa(\bar{C} - \hat{C}_t) + \rho(C_t - \hat{C}_t) \right] + \bar{f}_2(\bar{X}_t) \kappa(\bar{C} - C_t) + \bar{f}_3(\bar{X}_t) (b_0 - b_1 \hat{C}_t - b_2 y_t) \\
& + \frac{1}{2} \left[\beta_1^2 + \frac{\phi^2 \beta_2^2}{(r + \kappa)^2} + \frac{s^2 \beta_2^2 \gamma_2^2 \Delta}{\mathbf{1} \Sigma \mathbf{1}'} \right] \frac{\Delta \bar{q}_{11}}{\mathbf{1} \Sigma \mathbf{1}'} + \frac{s^2 \beta_2 \gamma_2 \Delta \bar{q}_{12}}{\mathbf{1} \Sigma \mathbf{1}'} + \frac{1}{2} s^2 \bar{q}_{22} + \beta - r + r \log(r) \\
& - \frac{1}{2} \left[\beta_1^2 + \frac{\phi^2 \beta_2^2}{(r + \kappa)^2} \right] \frac{\Delta \bar{f}_1(\bar{X}_t)^2}{\mathbf{1} \Sigma \mathbf{1}'} - \frac{1}{2} s^2 \left[\frac{\beta_2 \gamma_2 \Delta \bar{f}_1(\bar{X}_t)}{\mathbf{1} \Sigma \mathbf{1}'} + \bar{f}_2(\bar{X}_t) \right]^2 = 0. \tag{C.5}
\end{aligned}$$

Eq. (C.5) is quadratic in \bar{X}_t . Identifying quadratic, linear and constant terms, yields ten scalar equations in $(\bar{q}_0, \bar{q}_1, \bar{q}_2, \bar{q}_3, \bar{Q})$. ■

Proof of Proposition 10: Eq. (B.19), describing the dynamics of \hat{C}_t under the investor's filtration, remains valid under gradual adjustment. Eqs. (B.20) and (B.21) are replaced by

$$\begin{aligned}
z_t dR_t - C_t dt &= z_t \left\{ r a_0 + \left[(\tilde{g}_1 + g_2) \hat{C}_t + g_3 y_t - \kappa(\gamma_1 + \gamma_2) \bar{C} - b_0 \gamma_3 \right] \Sigma p_f' \right\} dt - \hat{C}_t dt \\
&+ z_t (\sigma + \beta_1 \gamma_1 \Sigma p_f' p_f \sigma) d\hat{B}_t^D + z_t \frac{\phi}{r + \kappa} (\sigma + \beta_2 \gamma_1 \Sigma p_f' p_f \sigma) dB_t^F - s \gamma_2 \left(1 + \frac{\beta_2 \gamma_1 \Delta}{\mathbf{1} \Sigma \mathbf{1}'} \right) z_t \Sigma p_f' dB_t^C, \tag{C.6}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{1} dR_t &= \mathbf{1} \left\{ r a_0 + \left[(\tilde{g}_1 + g_2) \hat{C}_t + g_3 y_t - \kappa(\gamma_1 + \gamma_2) \bar{C} - b_0 \gamma_3 \right] \Sigma p_f' \right\} dt \\
&+ \mathbf{1} (\sigma + \beta_1 \gamma_1 \Sigma p_f' p_f \sigma) d\hat{B}_t^D + \mathbf{1} \frac{\phi}{r + \kappa} (\sigma + \beta_2 \gamma_1 \Sigma p_f' p_f \sigma) dB_t^F - s \gamma_2 \left(1 + \frac{\beta_2 \gamma_1 \Delta}{\mathbf{1} \Sigma \mathbf{1}'} \right) \mathbf{1} \Sigma p_f' dB_t^C, \tag{C.7}
\end{aligned}$$

respectively. Suppose that the investor optimizes over (c_t, x_t) but follows the control v_t given by (45). Eq. (B.23) remains valid, provided that we set

$$\begin{aligned}
G &\equiv r\alpha \left[rW_t + (x_t \mathbf{1} + y_t z_t) \left\{ r a_0 + \left[(\tilde{g}_1 + g_2) \hat{C}_t + g_3 y_t - \kappa(\gamma_1 + \gamma_2) \bar{C} - b_0 \gamma_3 \right] \Sigma p_f' \right\} - y_t \hat{C}_t - \frac{\psi v_t^2}{2} - c_t \right] \\
&+ f_1(X_t) \kappa(\bar{C} - \hat{C}_t) + f_2(X_t) v_t + \frac{1}{2} \left[\beta_1^2 + \frac{\phi^2 \beta_2^2}{(r + \kappa)^2} + \frac{s^2 \beta_2^2 \gamma_2^2 \Delta}{\mathbf{1} \Sigma \mathbf{1}'} \right] \frac{\Delta q_{11}}{\mathbf{1} \Sigma \mathbf{1}'},
\end{aligned}$$

$$f_1(X_t) \equiv q_1 + q_{11} \hat{C}_t + q_{12} y_t,$$

$$f_2(X_t) \equiv q_2 + q_{12} \hat{C}_t + q_{22} y_t,$$

and replace $f(\hat{C}_t)$ by $f_1(X_t)$, where q_{ij} denotes the element (i, j) of the symmetric 2×2 matrix Q . The first-order condition (B.24) is replaced by

$$\mathbf{1}h(X_t) = r\alpha\mathbf{1}(f\Sigma + k\Sigma p'_f p_f \Sigma)(x_t\mathbf{1} + y_t z_t)', \quad (\text{C.8})$$

where

$$h(X_t) \equiv ra_0 + \left[(\tilde{g}_1 + g_2)\hat{C}_t + g_3 y_t - \kappa(\gamma_1 + \gamma_2)\bar{C} - b_0\gamma_3 + k_1 f_1(X_t) \right] \Sigma p'_f. \quad (\text{C.9})$$

The counterpart to (A.17) is

$$\begin{aligned} & r\alpha(x_t\mathbf{1} + y_t z_t) \left\{ ra_0 + \left[(\tilde{g}_1 + g_2)\hat{C}_t + g_3 y_t - \kappa(\gamma_1 + \gamma_2)\bar{C} - b_0\gamma_3 \right] \Sigma p'_f \right\} - r\alpha y_t \hat{C}_t - r\alpha \frac{\psi v_t^2}{2} \\ & - r \left[q_0 + (q_1, q_2)X_t + \frac{1}{2}X'_t Q X_t \right] + f_1(X_t)\kappa(\bar{C} - \hat{C}_t) + f_2(X_t)v_t \\ & + \frac{1}{2} \left[\beta_1^2 + \frac{\phi^2 \beta_2^2}{(r + \kappa)^2} + \frac{s^2 \beta_2^2 \gamma_2^2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right] \frac{\Delta[q_{11} - f_1(X_t)^2]}{\mathbf{1}\Sigma\mathbf{1}'} + \beta - r + r \log(r) \\ & - \frac{1}{2}(r\alpha)^2(x_t\mathbf{1} + y_t z_t)(f\Sigma + k\Sigma p'_f p_f \Sigma)(x_t\mathbf{1} + y_t z_t)' + r\alpha k_1 f_1(X_t)(x_t\mathbf{1} + y_t z_t)\Sigma p'_f = 0. \quad (\text{C.10}) \end{aligned}$$

The terms in (C.10) that involve $x_t\mathbf{1} + y_t z_t$ can be written as

$$\begin{aligned} & r\alpha(x_t\mathbf{1} + y_t z_t) \left\{ ra_0 + \left[(\tilde{g}_1 + g_2)\hat{C}_t + g_3 y_t - \kappa(\gamma_1 + \gamma_2)\bar{C} - b_0\gamma_3 \right] \Sigma p'_f \right\} \\ & - \frac{1}{2}(r\alpha)^2(x_t\mathbf{1} + y_t z_t)(f\Sigma + k\Sigma p'_f p_f \Sigma)(x_t\mathbf{1} + y_t z_t)' + r\alpha k_1 f_1(X_t)(x_t\mathbf{1} + y_t z_t)\Sigma p'_f \\ & = r\alpha(x_t\mathbf{1} + y_t z_t)h(X_t) - \frac{1}{2}(r\alpha)^2(x_t\mathbf{1} + y_t z_t)(f\Sigma + k\Sigma p'_f p_f \Sigma)(x_t\mathbf{1} + y_t z_t)', \\ & = r\alpha y_t \theta h(X_t) - \frac{1}{2}(r\alpha)^2 y_t^2 \theta (f\Sigma + k\Sigma p'_f p_f \Sigma) \theta' \\ & + r\alpha x_t(1 - y_t) \left\{ \mathbf{1}h(X_t) - \frac{1}{2}r\alpha\mathbf{1}(f\Sigma + k\Sigma p'_f p_f \Sigma) [x_t(1 - y_t)\mathbf{1} + 2y_t\theta]' \right\}', \quad (\text{C.11}) \end{aligned}$$

where the first step follows from (C.9) and the second from the equilibrium condition $z_t = \theta - x_t\mathbf{1}$.

Using $z_t = \theta - x_t\mathbf{1}$, we can write (C.8) as

$$\mathbf{1}h(X_t) = r\alpha\mathbf{1}(f\Sigma + k\Sigma p'_f p_f \Sigma) [x_t(1 - y_t)\mathbf{1} + y_t\theta]' \quad (\text{C.12})$$

$$\Rightarrow x_t(1 - y_t) = \frac{\mathbf{1}h(X_t) - r\alpha y_t f\mathbf{1}\Sigma\theta'}{r\alpha f\mathbf{1}\Sigma\mathbf{1}'}. \quad (\text{C.13})$$

Eqs. (C.12) and (C.13) imply that

$$\begin{aligned}
& r\alpha x_t(1-y_t) \left\{ \mathbf{1}h(X_t) - \frac{1}{2}r\alpha\mathbf{1}(f\Sigma + k\Sigma p'_f p_f \Sigma) [x_t(1-y_t)\mathbf{1} + 2y_t\theta]' \right\}' \\
&= \frac{1}{2}[r\alpha x_t(1-y_t)]^2 \mathbf{1}(f\Sigma + k\Sigma p'_f p_f \Sigma)\mathbf{1}' \\
&= \frac{[\mathbf{1}h(X_t) - r\alpha f y_t \mathbf{1}\Sigma\theta']^2}{2f\mathbf{1}\Sigma\mathbf{1}'}. \tag{C.14}
\end{aligned}$$

Substituting (C.11) and (C.14) into (C.10), we find

$$\begin{aligned}
& r\alpha y_t \theta h(X_t) - \frac{1}{2}(r\alpha)^2 y_t^2 \theta (f\Sigma + k\Sigma p'_f p_f \Sigma)\theta' + \frac{[\mathbf{1}h(X_t) - r\alpha f y_t \mathbf{1}\Sigma\theta']^2}{2f\mathbf{1}\Sigma\mathbf{1}'} - r\alpha y_t \hat{C}_t - r\alpha \frac{\psi v_t^2}{2} \\
& - r \left[q_0 + (q_1, q_2)X_t + \frac{1}{2}X'_t Q X_t \right] + f_1(X_t)\kappa(\bar{C} - \hat{C}_t) + f_2(X_t)v_t \\
& + \frac{1}{2} \left[\beta_1^2 + \frac{\phi^2 \beta_2^2}{(r+\kappa)^2} + \frac{s^2 \beta_2^2 \gamma_2^2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right] \frac{\Delta[q_{11} - f_1(X_t)^2]}{\mathbf{1}\Sigma\mathbf{1}'} + \beta - r + r \log(r) = 0. \tag{C.15}
\end{aligned}$$

Since v_t in (45) is linear in X_t , (C.15) is quadratic in X_t . Identifying quadratic, linear and constant terms, yields six scalar equations in (q_0, q_1, q_2, Q) .

We next consider the investor's optimization over v_t and derive a first-order condition under which the control (45) is optimal. Suppose that the investor deviates from the control (45), adding $\omega d\epsilon$ over the interval $[t, t + d\epsilon]$ and subtracting $\omega d\epsilon$ over the interval $[t + dt - d\epsilon, t + dt]$, where the infinitesimal $d\epsilon > 0$ is $o(dt)$. The increase in adjustment cost over the first interval is $\psi v_t \omega (d\epsilon)^2$ and over the second interval is $-\psi v_{t+dt} \omega (d\epsilon)^2$. These changes reduce the investor's wealth at time $t + dt$ by

$$\begin{aligned}
& \psi v_t \omega (d\epsilon)^2 (1 + rdt) - \psi v_{t+dt} \omega (d\epsilon)^2 \\
&= \psi \omega (d\epsilon)^2 (r v_t dt - d v_t) \\
&= \psi \omega (d\epsilon)^2 (r v_t dt + b_1 d\hat{C}_t + b_2 dy_t) \\
&= \psi \omega (d\epsilon)^2 \left[(r + b_2)v_t dt + b_1 d\hat{C}_t \right] \\
&= \psi \omega (d\epsilon)^2 \left\{ (r + b_2)(b_0 - b_1 \hat{C}_t - b_2 y_t) dt \right. \\
& \quad \left. + b_1 \left[\kappa(\bar{C} - \hat{C}_t) dt - \beta_1 p_f \sigma d\hat{B}_t^D - \beta_2 \left(\frac{\phi p_f \sigma dB_t^F}{r + \kappa} - \frac{s \gamma_2 \Delta dB_t^C}{\mathbf{1}\Sigma\mathbf{1}'} \right) \right] \right\}, \tag{C.16}
\end{aligned}$$

where the third step follows from (45), the fourth from the definition of v_t , and the fifth from (45) and (B.19). The change in the investor's wealth between t and $t + dt$ is derived from (16), (C.6) and (C.7), by subtracting (C.16) and replacing y_t in the term $(x_t \mathbf{1} + y_t z_t)$ by $y_t + \omega(d\epsilon)^2$:

$$\begin{aligned} dW_t = & G_\omega dt - \psi\omega(d\epsilon)^2 b_1 \left[\kappa(\bar{C} - \hat{C}_t)dt - \beta_1 p_f \sigma d\hat{B}_t^D - \beta_2 \left(\frac{\phi p_f \sigma dB_t^F}{r + \kappa} - \frac{s\gamma_2 \Delta dB_t^C}{\mathbf{1}\Sigma\mathbf{1}'} \right) \right] \\ & \{x_t \mathbf{1} + [y_t + \omega(d\epsilon)^2] z_t\} \left[(\sigma + \beta_1 \gamma_1 \Sigma p_f' p_f \sigma) d\hat{B}_t^D + \frac{\phi}{r + \kappa} (\sigma + \beta_2 \gamma_1 \Sigma p_f' p_f \sigma) dB_t^F \right. \\ & \left. - s\gamma_2 \left(1 + \frac{\beta_2 \gamma_1 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) \Sigma p_f' dB_t^C \right], \end{aligned} \quad (\text{C.17})$$

where

$$\begin{aligned} G_\omega \equiv & rW_t + \{x_t \mathbf{1} + [y_t + \omega(d\epsilon)^2] z_t\} \left\{ ra_0 + [(\tilde{g}_1 + g_2)\hat{C}_t + g_3 y_t - \kappa(\gamma_1 + \gamma_2)\bar{C} - b_0 \gamma_3] \Sigma p_f' \right\} \\ & - [y_t + \omega(d\epsilon)^2] \hat{C}_t - \frac{\psi v_t^2}{2} - c_t - \psi\omega(d\epsilon)^2 (r + b_2)(b_0 - b_1 \hat{C}_t - b_2 y_t). \end{aligned}$$

The investor's position in the active fund at $t + dt$ is the same under the deviation as under no deviation. Therefore, the investor's expected utility at $t + dt$ is given by the value function (47) with the wealth W_{t+dt} being determined by (C.17). The drift $\mathcal{D}V$ corresponding to the change in the value function between t and $t + dt$ is given by (B.23), provided that we set

$$G \equiv r\alpha G_\omega + f_{1\omega}(X_t)\kappa(\bar{C} - \hat{C}_t) + f_2(X_t)v_t + \frac{1}{2} \left[\beta_1^2 + \frac{\phi^2 \beta_2^2}{(r + \kappa)^2} + \frac{s^2 \beta_2^2 \gamma_2^2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right] \frac{\Delta q_{11}}{\mathbf{1}\Sigma\mathbf{1}'},$$

$$f_{1\omega}(X_t) \equiv f_1(X_t) - r\alpha\psi\omega(d\epsilon)^2 b_1,$$

and replace $f(\hat{C}_t)$ by $f_{1\omega}(X_t)$. The drift is maximum for $\omega = 0$, and this yields the first-order condition

$$z_t [h(X_t) - r\alpha\psi b_1 k_1 y_t \Sigma p_f'] - \hat{C}_t = r\alpha z_t (f\Sigma + k\Sigma p_f' p_f \Sigma)(x_t \mathbf{1} + y_t z_t)' + \psi h_\psi(X_t), \quad (\text{C.18})$$

where

$$h_\psi(X_t) \equiv (r + b_2)(b_0 - b_1 \hat{C}_t - b_2 y_t) + b_1 \kappa(\bar{C} - \hat{C}_t) - b_1 \left[\beta_1^2 + \frac{\phi^2 \beta_2^2}{(r + \kappa)^2} + \frac{s^2 \beta_2^2 \gamma_2^2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right] \frac{\Delta f_1(X_t)}{\mathbf{1}\Sigma\mathbf{1}'}$$

Using (C.8) and the equilibrium condition $z_t = \theta - x_t \mathbf{1}$, we can write (C.18) as

$$\theta [h(X_t) - r\alpha\psi b_1 k_1 y_t \Sigma p_f'] - \hat{C}_t = r\alpha\theta (f\Sigma + k\Sigma p_f' p_f \Sigma)(x_t \mathbf{1} + y_t z_t)' + \psi h_\psi(X_t). \quad (\text{C.19})$$

Using (C.13) and $z_t = \theta - x_t \mathbf{1}$, we can write (C.19) as

$$\theta [h(X_t) - r\alpha\psi b_1 k_1 y_t \Sigma p'_f] - \hat{C}_t = r\alpha\theta(f\Sigma + k\Sigma p'_f p_f \Sigma) \left[y_t \theta + \frac{\mathbf{1}h(X_t) - r\alpha y_t f \mathbf{1}\Sigma\theta'}{r\alpha f \mathbf{1}\Sigma\mathbf{1}'} \mathbf{1} \right]' + \psi h_\psi(X_t). \quad (\text{C.20})$$

Eq. (C.20) is linear in X_t . Identifying linear and constant terms, yields three scalar equations in (b_0, b_1, b_2) . ■

Proof of Proposition 11: We first impose market clearing and derive the constants $(a_0, b_0, b_1, b_2, \gamma_1, \gamma_2, \gamma_3)$ as functions of $(\bar{q}_1, \bar{q}_2, \bar{q}_3, \bar{Q}, q_1, q_2, Q)$. Proceeding as in the proof of Proposition 7, we can write the manager's first-order condition (B.13) as

$$\bar{h}(\bar{X}_t) = r\bar{\alpha}(f\Sigma + k\Sigma p'_f p_f \Sigma) \left[\frac{\alpha}{\alpha + \bar{\alpha}} \frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'} \mathbf{1} + (1 - y_t)p_f \right]'. \quad (\text{C.21})$$

Eq. (B.28) is linear in \bar{X}_t . Identifying terms in \hat{C}_t , C_t , and y_t , we find, respectively,

$$\tilde{g}_1 + k_1 \bar{q}_{11} + k_2 \bar{q}_{12} = 0, \quad (\text{C.22})$$

$$g_2 + k_1 \bar{q}_{12} + k_2 \bar{q}_{22} = 0, \quad (\text{C.23})$$

$$g_3 + k_1 \bar{q}_{13} + k_2 \bar{q}_{23} = -r\bar{\alpha} \left(f + \frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right). \quad (\text{C.24})$$

Identifying constant terms, we find

$$a_0 = \frac{\alpha\bar{\alpha}f}{\alpha + \bar{\alpha}} \frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'} \Sigma\mathbf{1}' + \left[\frac{\kappa(\gamma_1 + \gamma_2)\bar{C} + b_0\gamma_3 - k_1\bar{q}_1 - k_2\bar{q}_2}{r} + \bar{\alpha} \left(f + \frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) \right] \Sigma p'_f. \quad (\text{C.25})$$

Eqs. (C.9) and (C.25) imply that

$$\mathbf{1}h(X_t) = \frac{r\alpha\bar{\alpha}f}{\alpha + \bar{\alpha}} \mathbf{1}\Sigma\theta'. \quad (\text{C.26})$$

Using (C.26), we can write (C.20) as

$$\theta [h(X_t) - r\alpha\psi b_1 k_1 y_t \Sigma p'_f] - \hat{C}_t = r\alpha\theta(f\Sigma + k\Sigma p'_f p_f \Sigma) \left[\frac{\bar{\alpha}}{\alpha + \bar{\alpha}} \frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'} \mathbf{1} + y_t p_f \right]' + \psi h_\psi(X_t). \quad (\text{C.27})$$

Premultiplying (C.21) by θ , dividing by $r\bar{\alpha}$, and adding to (C.27) divided by $r\alpha$, we find

$$\theta \left[\frac{h(X_t)}{r\alpha} + \frac{\bar{h}(\bar{X}_t)}{r\bar{\alpha}} - \psi b_1 k_1 y_t \Sigma p'_f \right] - \frac{\hat{C}_t}{r\alpha} = \theta(f\Sigma + k\Sigma p'_f p_f \Sigma)\theta' + \frac{\psi h_\psi(X_t)}{r\alpha}. \quad (\text{C.28})$$

Eq. (C.28) is linear in \bar{X}_t . The terms in C_t cancel because of (C.23). Identifying terms in \hat{C}_t , we find

$$\frac{\tilde{g}_1 + g_2 + k_1 q_{11}}{r\alpha} + \frac{\tilde{g}_1 + k_1 \bar{q}_{11} + k_2 \bar{q}_{12}}{r\bar{\alpha}} = \frac{\mathbf{1}\Sigma\mathbf{1}'}{r\alpha\Delta} - \frac{\psi b_1 \left\{ \frac{(r+\kappa+b_2)\mathbf{1}\Sigma\mathbf{1}'}{\Delta} + \left[\beta_1^2 + \frac{\phi^2 \beta_2^2}{(r+\kappa)^2} + \frac{s^2 \beta_2^2 \gamma_2^2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right] q_{11} \right\}}{r\alpha}. \quad (\text{C.29})$$

Identifying terms in y_t , we find

$$\frac{g_3 + k_1 q_{12}}{r\alpha} + \frac{g_3 + k_1 \bar{q}_{13} + k_2 \bar{q}_{23}}{r\bar{\alpha}} - \psi b_1 k_1 = - \frac{\psi \left\{ \frac{b_2(r+b_2)\mathbf{1}\Sigma\mathbf{1}'}{\Delta} + b_1 \left[\beta_1^2 + \frac{\phi^2 \beta_2^2}{(r+\kappa)^2} + \frac{s^2 \beta_2^2 \gamma_2^2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right] q_{12} \right\}}{r\alpha}. \quad (\text{C.30})$$

Identifying constant terms, we find

$$\begin{aligned} & \left(\frac{r}{r\alpha} + \frac{r}{r\bar{\alpha}} \right) \theta a_0 - \left[\frac{\kappa(\gamma_1 + \gamma_2)\bar{C} + b_0\gamma_3 - k_1 q_1}{r\alpha} + \frac{\kappa(\gamma_1 + \gamma_2)\bar{C} + b_0\gamma_3 - k_1 \bar{q}_1 - k_2 \bar{q}_2}{r\bar{\alpha}} \right] \frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \\ & = \theta(f\Sigma + k\Sigma p'_f p_f \Sigma) \theta' + \frac{\psi \left\{ b_0(r+b_2) + b_1 \kappa \bar{C} - b_1 \left[\beta_1^2 + \frac{\phi^2 \beta_2^2}{(r+\kappa)^2} + \frac{s^2 \beta_2^2 \gamma_2^2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right] \frac{\Delta q_1}{\mathbf{1}\Sigma\mathbf{1}'} \right\}}{r\alpha}. \end{aligned} \quad (\text{C.31})$$

The system of equations characterizing equilibrium is as follows. The endogenous variables are $(a_0, b_0, b_1, b_2, \gamma_1, \gamma_2, \gamma_3, \beta_1, \beta_2, \hat{s}^2, \bar{q}_1, \bar{q}_2, \bar{q}_3, \bar{Q}, q_1, q_2, Q)$. (As in Propositions 3 and 7, we can drop (\bar{q}_0, q_0) .) The equations linking them are (34)-(36), (C.22)-(C.25), (C.29)-(C.31), the nine equations derived from (C.5) by identifying linear and quadratic terms, and the five equations derived from (C.15) by identifying linear and quadratic terms. We next simplify the latter two sets of equations, using implications of market clearing.

Consider the equations derived from (C.5). Eq. (C.21) implies that

$$\bar{h}(\bar{X}_t) = \frac{r\alpha\bar{\alpha}f}{\alpha + \bar{\alpha}} \frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'} \Sigma\mathbf{1}' + r\bar{\alpha} \left(f + \frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) (1 - y_t) \Sigma p'_f. \quad (\text{C.32})$$

Using (B.42), (B.43) and (C.32), we find

$$\frac{1}{2} \bar{h}(\bar{X}_t)' (f\Sigma + k\Sigma p'_f p_f \Sigma)^{-1} \bar{h}(\bar{X}_t) = \frac{r^2 \alpha^2 \bar{\alpha}^2 f (\mathbf{1}\Sigma\theta')^2}{2(\alpha + \bar{\alpha})^2 \mathbf{1}\Sigma\mathbf{1}'} + \frac{r^2 \bar{\alpha}^2 \left(f + \frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) (1 - y_t)^2 \Delta}{2 \mathbf{1}\Sigma\mathbf{1}'}. \quad (\text{C.33})$$

We next substitute (C.33) into (C.5), and identify linear and quadratic terms. Quadratic terms yield the algebraic Riccati equation

$$\bar{Q}\bar{\mathcal{R}}_2\bar{Q} + \bar{Q}\bar{\mathcal{R}}_1 + \bar{\mathcal{R}}_1'\bar{Q} - \bar{\mathcal{R}}_0 = 0, \quad (\text{C.34})$$

where

$$\begin{aligned}\bar{\mathcal{R}}_2 &\equiv \begin{pmatrix} \left[\beta_1^2 + \frac{\phi^2 \beta_2^2}{(r+\kappa)^2} + \frac{s^2 \beta_2^2 \gamma_2^2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right] \frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} & \frac{s^2 \beta_2 \gamma_2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} & 0 \\ \frac{s^2 \beta_2 \gamma_2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} & s^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \bar{\mathcal{R}}_1 &\equiv \begin{pmatrix} \frac{r}{2} + \kappa + \rho & -\rho & 0 \\ 0 & \frac{r}{2} + \kappa & 0 \\ b_1 & 0 & \frac{r}{2} + b_2 \end{pmatrix}, \\ \bar{\mathcal{R}}_0 &\equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & r^2 \bar{\alpha}^2 \left(f + \frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) \frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \end{pmatrix}.\end{aligned}$$

Terms in \hat{C}_t yield

$$\begin{aligned}(r + \kappa + \rho) \bar{q}_1 + b_1 \bar{q}_3 + \left[\beta_1^2 + \frac{\phi^2 \beta_2^2}{(r + \kappa)^2} \right] \frac{\Delta \bar{q}_1 \bar{q}_{11}}{\mathbf{1}\Sigma\mathbf{1}'} + s^2 \left(\frac{\beta_2 \gamma_2 \Delta \bar{q}_1}{\mathbf{1}\Sigma\mathbf{1}'} + \bar{q}_2 \right) \left(\frac{\beta_2 \gamma_2 \Delta \bar{q}_{11}}{\mathbf{1}\Sigma\mathbf{1}'} + \bar{q}_{12} \right) \\ - \kappa \bar{C} (\bar{q}_{11} + \bar{q}_{12}) - b_0 \bar{q}_{13} = 0,\end{aligned}\tag{C.35}$$

terms in C_t yield

$$\begin{aligned}(r + \kappa) \bar{q}_2 - \rho \bar{q}_1 + \left[\beta_1^2 + \frac{\phi^2 \beta_2^2}{(r + \kappa)^2} \right] \frac{\Delta \bar{q}_1 \bar{q}_{12}}{\mathbf{1}\Sigma\mathbf{1}'} + s^2 \left(\frac{\beta_2 \gamma_2 \Delta \bar{q}_1}{\mathbf{1}\Sigma\mathbf{1}'} + \bar{q}_2 \right) \left(\frac{\beta_2 \gamma_2 \Delta \bar{q}_{12}}{\mathbf{1}\Sigma\mathbf{1}'} + \bar{q}_{22} \right) \\ - \kappa \bar{C} (\bar{q}_{12} + \bar{q}_{22}) - b_0 \bar{q}_{23} = 0,\end{aligned}\tag{C.36}$$

and terms in y_t yield

$$\begin{aligned}(r + b_2) \bar{q}_3 + \left[\beta_1^2 + \frac{\phi^2 \beta_2^2}{(r + \kappa)^2} \right] \frac{\Delta \bar{q}_1 \bar{q}_{13}}{\mathbf{1}\Sigma\mathbf{1}'} + s^2 \left(\frac{\beta_2 \gamma_2 \Delta \bar{q}_1}{\mathbf{1}\Sigma\mathbf{1}'} + \bar{q}_2 \right) \left(\frac{\beta_2 \gamma_2 \Delta \bar{q}_{13}}{\mathbf{1}\Sigma\mathbf{1}'} + \bar{q}_{23} \right) \\ + r^2 \bar{\alpha}^2 \left(f + \frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) \frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} - \kappa \bar{C} (\bar{q}_{13} + \bar{q}_{23}) - b_0 \bar{q}_{33} = 0.\end{aligned}\tag{C.37}$$

Consider next the equations derived from (C.15). Using (C.26) and (C.27), we can write (C.15) as

$$\begin{aligned}\frac{r^2 \alpha^2 \bar{\alpha}^2 f (\mathbf{1}\Sigma\theta')^2}{2(\alpha + \bar{\alpha})^2 \mathbf{1}\Sigma\mathbf{1}'} + \frac{r^2 \alpha^2 y_t^2 \left(f + \frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) \Delta}{2 \mathbf{1}\Sigma\mathbf{1}'} + r \alpha \psi y_t \left[h_\psi(X_t) + \frac{r \alpha b_1 k_1 y_t \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right] - r \alpha \frac{\psi v_t^2}{2} \\ - r \left[q_0 + (q_1, q_2) X_t + \frac{1}{2} X_t' Q X_t \right] + f_1(X_t) \kappa (\bar{C} - \hat{C}_t) + f_2(X_t) v_t \\ + \frac{1}{2} \left[\beta_1^2 + \frac{\phi^2 \beta_2^2}{(r + \kappa)^2} + \frac{s^2 \beta_2^2 \gamma_2^2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right] \frac{\Delta [q_{11} - f_1(X_t)^2]}{\mathbf{1}\Sigma\mathbf{1}'} + \beta - r + r \log(r) = 0.\end{aligned}\tag{C.38}$$

We next substitute (45) into (C.38), and identify linear and quadratic terms. Quadratic terms yield the algebraic Riccati equation

$$Q\mathcal{R}_2Q + Q\mathcal{R}_1 + \mathcal{R}'_1Q - \mathcal{R}_0 = 0, \quad (\text{C.39})$$

where

$$\begin{aligned} \mathcal{R}_2 &\equiv \begin{pmatrix} \left[\beta_1^2 + \frac{\phi^2 \beta_2^2}{(r+\kappa)^2} + \frac{s^2 \beta_2^2 \gamma_2^2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right] \frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} & 0 \\ 0 & 0 \end{pmatrix}, \\ \mathcal{R}_1 &\equiv \begin{pmatrix} \frac{r}{2} + \kappa & r\alpha\psi b_1 \left[\beta_1^2 + \frac{\phi^2 \beta_2^2}{(r+\kappa)^2} + \frac{s^2 \beta_2^2 \gamma_2^2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right] \frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \\ b_1 & \frac{r}{2} + b_2 \end{pmatrix}, \\ \mathcal{R}_0 &\equiv \begin{pmatrix} -r\alpha\psi b_1^2 & -r\alpha\psi b_1(r + \kappa + 2b_2) \\ -r\alpha\psi b_1(r + \kappa + 2b_2) & r^2\alpha^2 \left(f + \frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) \frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} + \frac{r^2\alpha^2\psi b_1 k_1 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} - r\alpha\psi b_2(2r + 3b_2) \end{pmatrix}. \end{aligned}$$

Terms in \hat{C}_t yield

$$(r + \kappa)q_1 + b_1q_2 + \left[\beta_1^2 + \frac{\phi^2 \beta_2^2}{(r + \kappa)^2} + \frac{s^2 \beta_2^2 \gamma_2^2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right] \frac{\Delta q_1 q_{11}}{\mathbf{1}\Sigma\mathbf{1}'} - \kappa \bar{C} q_{11} - b_0 q_{12} - r\alpha\psi b_0 b_2 = 0, \quad (\text{C.40})$$

and terms in y_t yield

$$\begin{aligned} (r + b_2)q_2 + \left[\beta_1^2 + \frac{\phi^2 \beta_2^2}{(r + \kappa)^2} + \frac{s^2 \beta_2^2 \gamma_2^2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right] \frac{\Delta(q_2 q_{12} + r\alpha\psi b_1 q_1)}{\mathbf{1}\Sigma\mathbf{1}'} - \kappa \bar{C} q_{12} - b_0 q_{22} \\ - r\alpha\psi [b_0(r + 2b_2) + b_1 \kappa \bar{C}] = 0. \end{aligned} \quad (\text{C.41})$$

Solving for equilibrium amounts to solving the system of (34)-(36), (C.22)-(C.25), (C.29)-(C.31), (C.34)-(C.37), (C.39)-(C.41) in the unknowns $(a_0, b_0, b_1, b_2, \gamma_1, \gamma_2, \gamma_3, \beta_1, \beta_2, \hat{s}^2, \bar{q}_1, \bar{q}_2, \bar{q}_3, \bar{Q}, q_1, q_2, Q)$. This reduces to solving the system of (C.22)-(C.24), (C.29), (C.30), (C.34), (C.39) in the unknowns $(b_1, b_2, \gamma_1, \gamma_2, \gamma_3, \bar{Q}, Q)$: given $(b_1, b_2, \gamma_1, \gamma_2, \gamma_3, \bar{Q}, Q)$, $(\beta_1, \beta_2, \hat{s}^2)$ can be determined from (34)-(36), $(\bar{q}_1, \bar{q}_2, \bar{q}_3, q_1, q_2)$ from the linear system of (C.35)-(C.37), (C.40) and (C.41), and (a_0, b_0) from (C.25) and (C.31). **[[[To be completed.]]]** ■

Proof of Corollary 7: Stocks' expected returns are

$$\begin{aligned} E_t(dR_t) &= \left\{ ra_0 + \left[\tilde{g}_1 \hat{C}_t + g_2 C_t + g_3 y_t - \kappa(\gamma_1 + \gamma_2) \bar{C} - b_0 \gamma_3 \right] \Sigma p'_f \right\} dt \\ &= \left\{ \frac{r\alpha \bar{\alpha} f}{\alpha + \bar{\alpha}} \frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'} \Sigma\mathbf{1}' + \left[\tilde{g}_1 \hat{C}_t + g_2 C_t + g_3 y_t - k_1 \bar{q}_1 - k_2 \bar{q}_2 + r\bar{\alpha} \left(f + \frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right) \right] \Sigma p'_f \right\} dt, \end{aligned} \quad (\text{C.42})$$

where the first step follows from (C.1) and the second from (C.25). Eq. (49) follows from (C.42) by the same arguments as in the proof of Corollary 3. \blacksquare

Proof of Corollary 8: Under gradual adjustment, (B.60) is replaced by

$$\begin{aligned} Cov_t(dR_t, dR'_t) = Cov_t \left[(\sigma + \beta_1 \gamma_1 \Sigma p'_f p_f \sigma) dB_t^D + \frac{\phi}{r + \kappa} (\sigma + \beta_2 \gamma_1 \Sigma p'_f p_f \sigma) dB_t^F \right. \\ \left. - s \gamma_2 \left(1 + \frac{\beta_2 \gamma_1 \Delta}{\mathbf{1} \Sigma \mathbf{1}'} \right) \Sigma p'_f dB_t^C, \left(\tilde{g}_1 \hat{C}_t + g_2 C_t + g_3 y_t \right) p_f \Sigma dt \right]. \end{aligned} \quad (C.43)$$

To compute the covariance (C.43), we proceed as in the proof of Corollary 6 and consider the impulse-response dynamics

$$\begin{aligned} dC_t &= -\kappa C_t dt, \\ d\hat{C}_t &= \left[-\kappa \hat{C}_t + \rho(C_t - \hat{C}_t) \right] dt, \\ dy_t &= -\left(b_1 \hat{C}_t + b_2 y_t \right) dt, \end{aligned}$$

with the initial conditions

$$\begin{aligned} C_t &= s dB_t^C, \\ \hat{C}_t &= -\beta_1 p_f \sigma dB_t^D - \beta_2 \left(\frac{\phi p_f \sigma dB_t^F}{r + \kappa} - \frac{s \gamma_2 \Delta dB_t^C}{\mathbf{1} \Sigma \mathbf{1}'} \right), \\ y_t &= 0. \end{aligned}$$

The solution to these dynamics is (B.61), (B.62) and

$$\begin{aligned} y_{t'} &= -\frac{b_1}{b_2 - \kappa} \left[e^{-\kappa(t'-t)} - e^{-b_2(t'-t)} \right] s dB_t^C \\ &+ \frac{b_1}{b_2 - \kappa - \rho} \left[e^{-(\kappa + \rho)(t'-t)} - e^{-b_2(t'-t)} \right] \left[\beta_1 p_f \sigma dB_t^D + \frac{\phi \beta_2 p_f \sigma dB_t^F}{r + \kappa} + s \left(1 - \frac{\beta_2 \gamma_2 \Delta}{\mathbf{1} \Sigma \mathbf{1}'} \right) dB_t^C \right]. \end{aligned} \quad (C.44)$$

Substituting (B.61), (B.62) and (C.44) into (C.43), we find (52) with

$$\chi_1 \equiv \left(\frac{g_3 b_1}{b_2 - \kappa - \rho} - \tilde{g}_1 \right) \beta_1 \left(1 + \frac{\beta_1 \gamma_1 \Delta}{\mathbf{1} \Sigma \mathbf{1}'} \right), \quad (C.45)$$

$$\chi_2 \equiv s^2 \left(\frac{g_3 b_1}{b_2 - \kappa} - \tilde{g}_1 - g_2 \right) \gamma_2 \left(1 + \frac{\beta_2 \gamma_1 \Delta}{\mathbf{1} \Sigma \mathbf{1}'} \right), \quad (C.46)$$

$$\chi_3 \equiv -\frac{g_3 b_1 \beta_1}{b_2 - \kappa - \rho} \left(1 + \frac{\beta_1 \gamma_1 \Delta}{\mathbf{1} \Sigma \mathbf{1}'} \right) - \frac{s^2 g_3 b_1 \gamma_2}{b_2 - \kappa} \left(1 + \frac{\beta_2 \gamma_1 \Delta}{\mathbf{1} \Sigma \mathbf{1}'} \right). \quad (C.47)$$

The function $\chi(u) \equiv \chi_1 e^{-(\kappa+\rho)u} + \chi_2 e^{-\kappa u} + \chi_3 e^{-b_2 u}$ has the same sign as $\hat{\chi}(u) \equiv \chi_1 e^{-\rho u} + \chi_2 + \chi_3 e^{-(b_2-\kappa)u}$. When $u = 0$,

$$\hat{\chi}(0) = \chi_1 + \chi_2 + \chi_3 = -\tilde{g}_1 \beta_1 \left(1 + \frac{\beta_1 \gamma_1 \Delta}{\mathbf{1} \Sigma \mathbf{1}'}\right) - s^2 (\tilde{g}_1 + g_2) \gamma_2 \left(1 + \frac{\beta_2 \gamma_1 \Delta}{\mathbf{1} \Sigma \mathbf{1}'}\right) > 0.$$

When u goes to ∞ and $b_2 > \kappa$, $\hat{\chi}(u)$ converges to χ_2 . Eqs. (B.8), (B.9), (C.2) and (C.3) imply that

$$\chi_2 = s^2 (r + \kappa) \left(\frac{b_1 \gamma_3}{b_2 - \kappa} - \gamma_1 - \gamma_2\right) \gamma_2 \left(1 + \frac{\beta_2 \gamma_1 \Delta}{\mathbf{1} \Sigma \mathbf{1}'}\right) < 0.$$

When u goes to ∞ and $b_2 < \kappa$, $\hat{\chi}(u)$ is asymptotically equal to $\chi_3 e^{-(b_2-\kappa)u} < 0$. Therefore, in both cases $\hat{\chi}(u) < 0$ for large u . The derivative of $\hat{\chi}(u)$ is $\hat{\chi}'(u) = -\chi_1 \rho e^{-\rho u} - \chi_3 (b_2 - \kappa) e^{-(b_2-\kappa)u}$. When $u = 0$,

$$\hat{\chi}'(0) = -\chi_1 \rho - \chi_3 (b_2 - \kappa) = (\tilde{g}_1 \rho + g_3 b_1) \beta_1 \left(1 + \frac{\beta_1 \gamma_1 \Delta}{\mathbf{1} \Sigma \mathbf{1}'}\right) + s^2 g_3 b_1 \gamma_2 \left(1 + \frac{\beta_2 \gamma_1 \Delta}{\mathbf{1} \Sigma \mathbf{1}'}\right) < 0.$$

Since $\hat{\chi}'(u)$ can change sign at most once, it is either negative, or negative and then positive. Therefore, $\hat{\chi}(u)$ is positive and then negative, and the same is true for $\chi(u)$. ■

Proof of Corollary 9: Since the long-horizon return from time t on is $\int_t^\infty dR_{t'} e^{-r(t'-t)}$, the covariance in (53) is

$$\begin{aligned} \text{Cov}_t \left(dR_t, \int_t^\infty dR_{t'} e^{-r(t'-t)} \right) &= \int_t^\infty \text{Cov}_t(dR_t, dR_{t'}) e^{-r(t'-t)} \\ &= \left(\int_0^\infty \chi(u) e^{-ru} du \right) \Sigma p'_f p_f \Sigma dt, \end{aligned} \quad (\text{C.48})$$

where the second step follows from Corollary 8. Using the definition of $\chi(u)$, we find

$$\int_0^\infty \chi(u) e^{-ru} du = \frac{\chi_1}{r + \kappa + \rho} + \frac{\chi_2}{r + \kappa} + \frac{\chi_3}{r + b_2}. \quad (\text{C.49})$$

Substituting (χ_1, χ_2, χ_3) from (C.45)-(C.47) into (C.49), and using (B.8), (B.9), (C.2) and (C.3), we find

$$\int_0^\infty \chi(u) e^{-ru} du = -\beta_1 \gamma_1 \left(1 + \frac{\beta_1 \gamma_1 \Delta}{\mathbf{1} \Sigma \mathbf{1}'}\right) - s^2 \gamma_2 (\gamma_1 + \gamma_2) \left(1 + \frac{\beta_2 \gamma_1 \Delta}{\mathbf{1} \Sigma \mathbf{1}'}\right) < 0. \quad \blacksquare$$

Proof of Corollary 10: When dR_t is replaced by dD_t , (C.43) is replaced by

$$\text{Cov}_t(dD_t, dR'_t) = \text{Cov}_t \left[\sigma dB_t^D, \left(\tilde{g}_1 \hat{C}_{t'} + g_2 C_{t'} + g_3 y_{t'} \right) p_f \Sigma dt \right], \quad (\text{C.50})$$

and when dR_t is replaced by dF_t , (C.43) is replaced by

$$\text{Cov}_t(dF_t, dR'_t) = \text{Cov}_t \left[\frac{\phi \sigma}{r + \kappa} dB_t^F, \left(\tilde{g}_1 \hat{C}_{t'} + g_2 C_{t'} + g_3 y_{t'} \right) p_f \Sigma dt \right]. \quad (\text{C.51})$$

Substituting (B.61), (B.62) and (C.44) into (C.50) and (C.51), we find (54) with

$$\chi_1^D \equiv \left(\frac{g_3 b_1}{b_2 - \kappa - \rho} - \tilde{g}_1 \right) \beta_1, \quad (\text{C.52})$$

$$\chi_2^D \equiv -\frac{g_3 b_1 \beta_1}{b_2 - \kappa - \rho}. \quad (\text{C.53})$$

The function $\chi^D(u) \equiv \chi_1^D e^{-(\kappa+\rho)u} + \chi_2^D e^{-b_2 u}$ is equal to $-\tilde{g}_1 \beta_1 > 0$ when $u = 0$. When u goes to ∞ and $b_2 > \kappa + \rho$, $\chi^D(u)$ is asymptotically equal to $\chi_1^D e^{-(\kappa+\rho)u}$. Eqs. (B.8), (B.9), (C.2) and (C.3) imply that

$$\chi_1^D = (r + \kappa + \rho) \left(\frac{\gamma_3 b_1}{b_2 - \kappa - \rho} - \gamma_1 \right) < 0.$$

When u goes to ∞ and $b_2 < \kappa + \rho$, $\chi^D(u)$ is asymptotically equal to $\chi_2^D e^{-b_2 u} < 0$. Since $\chi^D(u)$ can change sign at most once, it is positive and then negative. ■

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