Risk-sharing or risk-taking? Financial innovation, margin requirements and incentives*

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PRELIMINARY

Abstract

We analyze the tradeoff between the benefits of financial innovation in terms of enhanced risk-sharing and its costs in terms of financial instability. We model the process of financial innovation as the design of a derivative contract between a protection buyer, seeking to hedge his risk exposure, and a protection seller. If the latter learns that her derivative position is likely to be loss-making, this amounts to an off-balance sheet liability for her. It undermines her incentives to exert monitoring effort to reduce default risk for her other holdings. In turn, such risk-taking incentives generate counterparty risk. Hence, financial innovation can create systemic risk, by inducing contagion between otherwise independent asset classes. In this context, margins and capital requirements can improve incentives and mitigate risk.

*The views expressed do not necessarily reflect those of the European Central Bank or the Eurosystem.

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1 Introduction

We analyze the trade-off between the benefit of financial innovation in terms of gains from trade and its cost in terms of financial instability.[1] The benefit arises since financial innovation enhances hedging opportunities between agents with different risk-bearing capacities. The cost arises when hedging creates *hidden leverage* that increases risk-taking incentives. We build on this trade-off to develop an incentive-based rationale for margins and their substitutability or complementarity with capital requirements.[2]

We model financial innovation as the design of an optimal contract between a risk-averse buyer of insurance who seeks protection against a risk exposure and a risk-neutral seller of insurance who provides the protection. An important example of such trade is offered by credit default swaps (CDS).

Financial institutions selling protection have their own risky assets and liabilities, i.e., they are exposed to balance sheet risk. Controlling balance sheet risk is costly. For example, financial institutions must devote resources to scrutinize the default risk of their borrowers and to manage their maturity mismatch.[3] Not controlling balance sheet risk (risk-taking) can lead to the failure of the protection seller and to the default on his contractual obligations. Protection buyers are therefore exposed to counterparty risk. For example, Lehman Brothers and Bear Stearns defaulted on their CDS derivative obligations because of losses incurred on their other investments, in particular sub-prime mortgages.

Our main assumption is that the care with which financial institutions manage their balance sheet risk is unobservable to outsiders and that financial institutions are protected by limited liability. This creates a moral hazard problem between the buyer and seller of

[1] Rajan (2006) notes that while innovation enhances risk-sharing opportunities it can also create new risk-taking venues.

[2] Our focus on the trade-off between benefits and costs of financial innovation is in line with Allen and Carletti (2006), Parlour and Plantin (2008) and Parlour and Winton (2008). But the economic mechanisms we analyze are entirely different from those studied in these papers.

[3] For example, in the wake of the 2007-09 crisis many financial institutions financed themselves through short-term debt. While such financing was relatively easy to establish, it left these institutions exposed to the risk of not being able to qucikly roll-over their liabilities (Brunnermeier and Oehmke, 2009; Acharya et al., 2009).

protection. In this context, we show that financial innovation designed to hedge risk creates hidden leverage: Ex-ante, when entering the position, the hedge trade is neither an asset nor a liability for the protection seller. For example, the seller of a credit default swap pays the buyer in case of credit events (default, restructuring) but collects an insurance premium otherwise. On average, for the seller to be willing to enter the position, she must at least break even. But, if the protection seller observes negative information about the hedged risk after entering the deal, then the position becomes an off-balance sheet liability. For instance, after bad news about the future solvency of firms, the seller of a CDS is more likely to pay out the insurance than after good news. This liability undermines the protection seller's incentive to control her balance sheet risk.[4] This is because she bears the full cost of risk control while the benefit accrues in part to the protection buyer.[5]

Given the incentives of the protection seller, the buyer faces a trade-off between risk-sharing and risk-taking. If he wants to curb risk-taking incentives, he must reduce the hidden leverage by accepting an incomplete hedge. Since such under-insurance is costly, he may instead opt for a complete hedge, recognizing that it will encourage risk-taking and lead to counterparty risk.

Our analysis thus identifies a channel through which financial innovation together with asymmetric information can lead to systemic risk. In the absence of a moral hazard problem, the risk the protection buyer is hedging and the balance sheet risk of the protection seller are independent. The counterparty risk that arises endogenously when the hidden leverage leads to risk-taking is a form of contagion. Advance negative news about the risk of the protection buyer propagates to the protection seller whose default risk increases.[6]

We use the model to develop an incentive-based theory of margins. When a market

[4] Holmström and Tirole (1998) show how *liquidity* shocks can weaken incentives to exert effort. In contrast, in the present analysis, we show how *information* shocks can weaken incentives.

[5] This is in line the debt-overhang effect identified by Myers (1977). Note, however, that instead of exogenous debt, we have endogenous liabilities emerging from optimal contracting.

[6] Our approach differs from other models of systemic risk, see, e.g., Freixas, Parigi and Rochet (2000), Cifuentes, Shin and Ferrucci (2005), and Allen and Carletti (2006), since in our analysis contagion arises because of incentive problems.

infrastructure such as a central counterparty (CCP) exists, the buyer and seller of protection can agree as part of their hedging transaction that the seller deposit cash upfront with the CCP as an initial margin. This is costly since she must liquidate some of her assets to deposit them with the CCP, where they will earn a lower rate of return. The benefit is that the cash deposited is ring-fenced from risk-taking. The initial margin thus reduces the size of the moral hazard problem between the protection buyer and seller. It therefore makes a more complete hedge incentive-compatible. However, margins have a potential downside. By insuring against counterparty risk, the protection buyer is more willing to accept risk-taking by the protection seller. Our analysis thus provides some support, but also shows some of the limitations, of the view that market infrastructures could reduce the vulnerability of the financial system to systemic risk.[7]

When the failure of financial institutions affects third parties, privately optimal hedging arrangements that entail risk-taking, including those with initial margins, are not socially optimal. In that case, one way to mitigate systemic risk is to impose capital requirements. Requiring a financial institution to hold capital in proportion to its hedging activities counters the hidden leverage embedded in these activities. Extra capital strengthens the balance sheet of the protection seller for a given amount of hedging. Having more to lose in case of default ("skin in the game") reduces her risk-taking incentives. By extension, financial institutions that opt out of such capital requirements should not be allowed to undertake hedging activities.

The remainder of the paper is organized as follows. In Section 2, we describe the model setup. In Section 3, we analyze the benchmark case in which effort is observable and there is no moral hazard. In Section 4, we analyze the optimal contract when effort is unobservable. We characterize when the two counterparties choose a contract with risk-taking following bad news about the hedged risk. In Section 5, we discuss the implementation of the optimal contract. In Section 6, we provide an incentive-based theory of margin requirements. In

[7]See, for example, Pirrong (2009).

Section 7, we examine regulatory interventions when there is a wedge between private and social welfare. Section 8 concludes. Proofs are in the appendix.

2 The model

There are three dates, $t = 0, 1, 2$, and two agents, the protection buyer and the protection seller, who can enter a hedging contract at $t = 0$.

Protection buyer. The protection buyer is risk-averse with twice differentiable concave utility function, denoted by u. At $t = 0$ he is endowed with a risky exposure of size I whose per unit return is $\tilde{\theta}$. The return is realized at $t = 2$. It can take on two values: $\bar{\theta}$ with probability π and $\underline{\theta}$ with probability $1 - \pi$. Moreover, the buyer has an amount C of cash which has zero net return. He seeks insurance to hedge the risk $\tilde{\theta}$.

Protection seller. The protection seller is risk-neutral. At time $t = 0$ she has an amount K of assets in place which have an uncertain per unit return $\tilde{R}$ at $t = 2$ (balance sheet risk).

At $t = 1$ the protection seller has to exert costly unobservable effort e to manage the risk of her assets. To capture the moral hazard problem in the simplest possible way, we assume that the protection seller can choose between effort, $e = 1$, and no effort, $e = 0$. If she exerts effort, we assume that $K\tilde{R}(e = 1) = KR > K$. If she does not exert effort, then $K\tilde{R}(e = 0) = KR$ with probability p and $K\tilde{R}(e = 0) = 0$ with probability $1 - p$.[8] That is, if the seller does not manage risk, her assets are wiped out with probability $1 - p$. In this case, the seller is protected by the limited liability and she defaults on her obligations (counterparty risk). If the seller does not exert effort, she obtains a private benefit B per unit of assets. Note that the impact of the seller's effort on $\tilde{R}$ does not depend on the return of the buyer's asset $\tilde{\theta}$.

We assume that the opportunity cost of not exerting effort is higher than the private

[8] In Section 7, we consider a case when losses $L \geq 0$ are incurred if the protection seller does not exert effort.

benefit: $(1-p)\,R > B$. Hence, the protection seller prefers effort to no effort if she is solely concerned with managing the risk of her assets.

Advance information. Information about the risk $\tilde{\theta}$ underlying the hedge is publicly revealed at $t = 0.5$, before the seller makes her effort decision at $t = 1$. Specifically, a signal $\tilde{s}$ about the return $\tilde{\theta}$ is observed. Let λ be the probability of a correct signal:

$$\lambda = \text{prob}[\bar{s}|\bar{\theta}] = \text{prob}[\underline{s}|\underline{\theta}]$$

The probability π is updated to $\bar{\pi}$ upon observing $\bar{s}$ and to $\underline{\pi}$ upon observing $\underline{s}$, where

$$\begin{aligned}\bar{\pi} &= \text{prob}[\bar{\theta}|\bar{s}] = \frac{\text{prob}[\bar{s}|\bar{\theta}]\text{prob}[\bar{\theta}]}{\text{prob}[\bar{s}]} = \frac{\lambda\pi}{\lambda\pi + (1-\lambda)(1-\pi)} \\ \underline{\pi} &= \text{prob}[\bar{\theta}|\underline{s}] = \frac{\text{prob}[\underline{s}|\bar{\theta}]\text{prob}[\bar{\theta}]}{\text{prob}[\underline{s}]} = \frac{(1-\lambda)\pi}{(1-\lambda)\pi + \lambda(1-\pi)}\end{aligned}$$

according to Bayes' Law.

We assume that $\lambda \geq \frac{1}{2}$. If $\lambda = \frac{1}{2}$, $\bar{\pi} = \pi = \underline{\pi}$ and the signal is completely uninformative. It is as if there was no advance information about the hedged risk. For $\lambda > \frac{1}{2}$, $\bar{\pi} > \pi > \underline{\pi}$, observing $\bar{s}$ increases the probability of $\tilde{\theta} = \bar{\theta}$ (good signal) whereas observing $\underline{s}$ decreases the probability of $\tilde{\theta} = \bar{\theta}$ (bad signal). If $\lambda = 1$, the signal is perfectly informative and it is as if the realization of $\tilde{\theta}$ was already observed at $t = 0.5$.

Contract. The contract specifies a transfer τ from the protection seller to the protection, conditional on all contractible information (in case $\tau < 0$, the buyer pays the seller). For simplicity we assume the realization of $\tilde{\theta}$, the return on the seller's assets $\tilde{R}$ and the advance signal $\tilde{s}$ are all publicly observable and contractible. Hence, the contract is given by $\tau = \tau(\tilde{\theta}, \tilde{R}, \tilde{s})$. The contract must also be consistent with the limited liability of the protection seller, so that $K\tilde{R} > \tau(\tilde{\theta}, \tilde{R}, \tilde{s})$. We assume $KR > I\Delta\theta$, which implies that as long as the agent exerts effort, the limited liability constraint does not bind, as shown below.

The sequence of events is summarized in Figure 1 below.

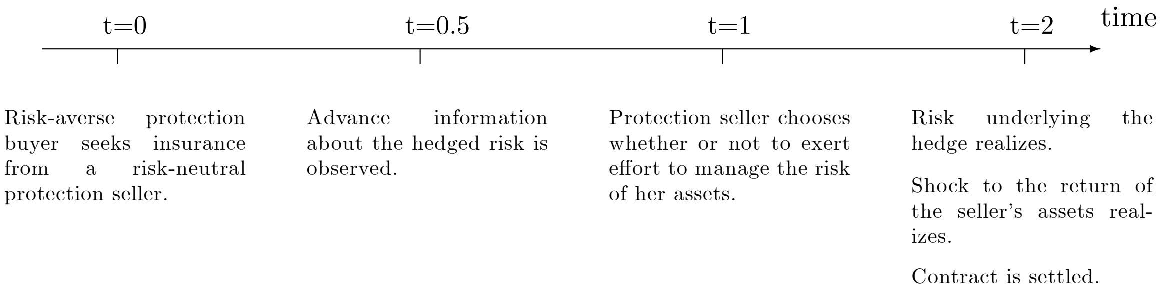

Figure 1: The timing of events

3 First-best: observable effort

In this section we consider the case where the protection buyer can observe the effort level of the protection seller so that there is no moral hazard problem. While implausible, this benchmark case will enable us to identify the inefficiencies generated by moral hazard.

Consider the case where the protection buyer instructs the protection seller to exert effort after both a good and a bad signal. In that case the seller's assets always return $\tilde{R}(1) = R$. Hence we don't need explicitly write $\tilde{R}$ when writing the variables upon which τ is contingent. Also, as will be clear below, under our assumption that $KR > I\Delta\theta$, the limited liability constraint of the agent does not bind when he exerts effort. Hence, for simplicity we neglect that constraint.

The protection buyer solves

$$\max_{\tau(\bar{\theta},\bar{s}),\tau(\underline{\theta},\bar{s}),\tau(\bar{\theta},\underline{s}),\tau(\underline{\theta},\underline{s})} \pi\lambda u(C + I\bar{\theta} + \tau(\bar{\theta},\bar{s})) + (1-\pi)(1-\lambda)u(C + I\underline{\theta} + \tau(\underline{\theta},\bar{s})) \tag{1}$$
$$+\pi(1-\lambda)u(C + I\bar{\theta} + \tau(\bar{\theta},\underline{s})) + (1-\pi)\lambda u(C + I\underline{\theta} + \tau(\underline{\theta},\underline{s}))$$

subject to the seller's participation constraint

$$\pi\lambda[KR - \tau(\bar{\theta},\bar{s})] + \pi(1-\lambda)[KR - \tau(\bar{\theta},\underline{s})] + (1-\pi)\lambda[KR - \tau(\underline{\theta},\underline{s})]$$
$$+ (1-\pi)(1-\lambda)[KR - \tau(\underline{\theta},\bar{s})] \geq KR$$

The expression on the right-hand side is seller's payoff if she does not enter the hedge. It is given by the return on her capital, KR.

The participation constraint can be written as

$$0 \geq \pi \left[\lambda \tau(\bar{\theta}, \bar{s}) + (1-\lambda)\tau(\bar{\theta}, \underline{s})\right] + (1-\pi)\left[\lambda \tau(\underline{\theta}, \underline{s}) + (1-\lambda)\,\tau(\underline{\theta}, \bar{s})\right] = E[\tau] \tag{2}$$

where the expectation is over $\tilde{\theta}$ and $\tilde{s}$. The protection seller agrees to the contract as long as the average payment to the buyer is non-positive. The proof of Proposition 1 in appendix gives the solution of this maximization problem. It is easy to show that the corresponding value function is greater than what would be obtained if effort was not always requested. Thus, we can state our first result.

Proposition 1 (First-best contract) *When effort is observable, the optimal contract entails effort after both signals, provides full insurance, and is actuarially fair. The transfers are given by:*

$$\tau^{FB}(\bar{\theta}, \bar{s}) \;=\; \tau^{FB}(\bar{\theta}, \underline{s}) = -(1-\pi)I\Delta\theta = I(E[\tilde{\theta}] - \bar{\theta}) < 0 \tag{3}$$

$$\tau^{FB}(\underline{\theta}, \bar{s}) \;=\; \tau^{FB}(\underline{\theta}, \underline{s}) = \pi I\Delta\theta = I(E[\tilde{\theta}] - \underline{\theta}) > 0 \tag{4}$$

In the first-best contract, the consumption of the protection buyer is equalized across states (full insurance). The contract does not react to the signal. Expected transfers are zero (the contract is actuarially fair) and there are no rents to the protection seller. The seller pays the buyer if $\tilde{\theta} = \underline{\theta}$ and vice versa if $\tilde{\theta} = \bar{\theta}$. The payments are proportional to the size of the hedged position I and to its riskiness, measured by $\Delta\theta$.

It is optimal for the protection buyer to demand effort after both signals. He is fully insured and the seller's assets are safe so there is no counterparty risk. If there was no effort, the buyer would be exposed to counterparty risk and full insurance would no longer be possible.

Finally, note that the values of the transfers given in the proposition confirms our initial claim that, under our assumption that $KR > I\Delta\theta$, the limited liability condition does not bind.

4 Second-best: unobservable effort

4.1 Effort after both signals

We now turn to the case where the effort level of the protection seller is not observable by the buyer. We first characterize the optimal contract inducing effort of the seller after both a good and a bad signal.

As the protection buyer expects the seller to always exert effort, he expects that $\tilde{R}$ is always equal to R. From his perspective, when writing the objective, there is no need to account for variability in $\tilde{R}$, which for simplicity can be omitted from the contract. Thus, the protection buyer solves (1) subject to (2) and the seller's incentive compatibility constraints. Since the signal about the hedged risk is observed before the effort decision is made, the incentive constraints are conditional on the realization of the signal.

Suppose a good signal, $\tilde{s} = \bar{s}$, is observed. Then, the incentive-compatibility constraint is given by

$$\bar{\pi}[KR - \tau(\bar{\theta}, \bar{s})] + (1 - \bar{\pi})[KR - \tau(\underline{\theta}, \bar{s})] \geq$$
$$\bar{\pi}[p(KR - \tau(\bar{\theta}, \bar{s}))] + (1 - \bar{\pi})[p(KR - \tau(\underline{\theta}, \bar{s}))] + BK$$

The expression on the right-hand side is seller's (out–of–equilibrium) expected payoff if she does not exert effort. With probability $1 - p$, the seller defaults. In this case, the protection seller cannot make any positive payment to the protection buyer and the latter has no interest in making payment to the former since such transfers would be lost to both parties. Hence $\tau(\tilde{\theta}, \tilde{s} \mid \text{default}) = 0$. The incentive-compatibility constraint after a bad signal, $\tilde{s} = \underline{s}$, is derived analogously.

Simplifying the incentive constraint for each realization of the signal, we get:

$$\mathcal{P} \geq \bar{\pi}\tau(\bar{\theta}, \bar{s}) + (1-\bar{\pi})\tau(\underline{\theta}, \bar{s}) = E[\tau|\bar{s}] \equiv \bar{\tau} \tag{5}$$

$$\mathcal{P} \geq \underline{\pi}\tau(\bar{\theta}, \underline{s}) + (1-\underline{\pi})\tau(\underline{\theta}, \underline{s}) = E[\tau|\underline{s}] \equiv \underline{\tau} \tag{6}$$

where

$$\mathcal{P} = K\left(R - \frac{B}{1-p}\right) \tag{7}$$

denotes "pledgeable income" (or incentive-compatible hedging capital). Available pledgeable income puts an upper bound on the expected transfer to the protection buyer, conditional on the observed signal. Note that $\mathcal{P} > 0$ since we assumed $(1-p)R > B$.

When the signal is informative, $\lambda > \frac{1}{2}$, we have the following result.

Lemma 1 (First-best attainable) *When effort is not observable and the signal is informative, the first-best can be achieved if and only if the protection seller has enough pledgeable income, i.e., for* $\mathcal{P} > (\pi - \underline{\pi})I\Delta\theta = I(E[\tilde{\theta}] - E[\tilde{\theta}|\underline{s}])$.

For sufficiently high pledgeable income levels, incentive-compatibility constraints are not binding and the first-best allocation can be reached even when effort is not observable. The threshold level of pledgeable income beyond which the first-best is attainable, $(\pi - \underline{\pi})I\Delta\theta$, is proportional to the size of the hedged position I, to its riskiness $\Delta\theta$, and to the informativeness of the signal λ (which induces a higher wedge between the prior and the updated probability). We can state the following corollary.

Corollary 1 *When the signal is uninformative, the first-best is always reached:* $\mathcal{P} > (\pi - \underline{\pi})I\Delta\theta = 0$.

Consider the case when the signal is informative and the pledgeable income is small enough so that the first-best is not attainable. To ensure that the protection seller always exerts effort, the optimal contract must satisfy two incentive-compatibility constraints. The next lemma states that only one of them will be binding.

Lemma 2 (Incentives given the signal) *When effort is not observable and the first-best is not attainable, $\mathcal{P} < (\pi - \underline{\pi}) I \Delta\theta$, the incentive constraint after a good signal is slack whereas the incentive constraint after a bad signal is binding.*

Ex ante, before the signal is observed, the hedging position is neither an asset nor a liability for the protection seller. After observing a good signal about the underlying risk, the hedge is more likely to be an asset for the seller. He is more likely to be paid by the buyer than the other way around. Thus, good news do not generate incentive problems. Negative news, on the other hand, make it more likely that the hedge moves against the seller. Now it is the seller who is more likely to pay the buyer. For $\mathcal{P} < (\pi - \underline{\pi}) I \Delta\theta$, this undermines her incentives to exert effort. She has to bear the full cost of effort while the benefit accrues in part to the protection buyer. This is reminiscent of the debt-overhang effect (Myers, 1977). The hedge contains *hidden leverage* that affects seller's incentives to control her balance sheet risk when she has limited pledgeable income.

The following proposition characterizes the second-best contract with effort after both signals.

Proposition 2 (Second-best contract with effort) *When effort is not observable and the first-best is not attainable, $\mathcal{P} < (\pi - \underline{\pi}) I \Delta\theta$, the optimal contract that induces effort after both signals provides full insurance conditional on the signal and is actuarially fair. The transfers are given by:*

$$
\begin{aligned}
\tau^{SB,e=1}(\bar{\theta}, \bar{s}) &= -(1-\bar{\pi}) I \Delta\theta - \frac{prob[\underline{s}]}{prob[\bar{s}]}\mathcal{P} = I(E[\tilde{\theta}|\bar{s}] - \bar{\theta}) - \frac{prob[\underline{s}]}{prob[\bar{s}]}\mathcal{P} < 0 \\
\tau^{SB,e=1}(\underline{\theta}, \bar{s}) &= \bar{\pi} I \Delta\theta - \frac{prob[\underline{s}]}{prob[\bar{s}]}\mathcal{P} = I(E[\tilde{\theta}|\bar{s}] - \underline{\theta}) - \frac{prob[\underline{s}]}{prob[\bar{s}]}\mathcal{P} > 0 \\
\tau^{SB,e=1}(\bar{\theta}, \underline{s}) &= -(1-\underline{\pi}) I \Delta\theta + \mathcal{P} = I(E[\tilde{\theta}|\underline{s}] - \bar{\theta}) + \mathcal{P} < 0 \\
\tau^{SB,e=1}(\underline{\theta}, \underline{s}) &= \underline{\pi} I \Delta\theta + \mathcal{P} = I(E[\tilde{\theta}|\underline{s}] - \underline{\theta}) + \mathcal{P} > 0
\end{aligned}
$$

As in the first-best contract, expected transfers are zero (the contract is actuarially fair) and there are no rents to the protection seller (the participation constraint is binding). The protection seller pays the protection buyer if $\tilde{\theta} = \underline{\theta}$, while the reverse holds true when $\tilde{\theta} = \bar{\theta}$:

$$\tau(\underline{\theta}, \tilde{s}) > 0 > \tau(\bar{\theta}, \tilde{s})$$

The key difference between the first-best contract and the second-best contract with effort is that the former does not depend on the signal, while the latter does:

$$\tau^{SB,e=1}(\tilde{\theta}, \underline{s}) < \tau^{FB}(\tilde{\theta}, \underline{s}) = \tau^{FB}(\tilde{\theta}, \bar{s}) < \tau^{SB,e=1}(\tilde{\theta}, \bar{s})$$

To preserve the seller's incentives to exert effort, the buyer must reduce the hidden leverage by accepting that the hedge does not provide full insurance. In particular, the incentive-compatible amount of insurance is smaller following a bad signal. Hence, the protection buyer must bear signal risk. Correspondingly, the protection seller must be left with some rent after a bad signal in order to induce effort. The protection buyer "reclaims" this rent after a good signal so that the expected rent to the seller is zero. Conditional on the signal, the second-best contract provides full insurance against the underlying risk $\tilde{\theta}$:

$$\tau(\underline{\theta}, \bar{s}) - \tau(\bar{\theta}, \bar{s}) = \tau(\underline{\theta}, \underline{s}) - \tau(\bar{\theta}, \underline{s}) = I\Delta\theta > 0$$

In sum, when pledgeable income is below a certain threshold, the non-observability of effort leads to *limited risk-sharing*. Since such under-insurance is costly for the risk-averse protection buyer, he may instead opt to not induce seller's effort all the time. If the seller does not exert effort, the buyer is exposed to counterparty risk. Hence, the choice between a contract with and without effort entails a trade-off between signal and counterparty risk.

We next investigate the properties of the contract that does not always induce effort.

4.2 No effort after a bad signal

The protection buyer may find the reduced risk-sharing in the contract with effort after both signals too costly. He may instead choose to accept risk-taking by the protection seller in exchange for a more complete hedge. Since it is always in the interest of the seller to exert effort after a good signal, risk-taking can only occur after a bad signal. In this subsection, we characterize the optimal contract with effort after good news and no effort after bad news.

The objective function of the protection buyer is then given by:

$$\max_{\tau(\bar{\theta},\bar{s}),\tau(\underline{\theta},\bar{s}),\tau(\bar{\theta},\underline{s}),\tau(\underline{\theta},\underline{s})} \pi\lambda u(C + I\bar{\theta} + \tau(\bar{\theta},\bar{s})) + (1-\pi)(1-\lambda)u(C + I\underline{\theta} + \tau(\underline{\theta},\bar{s})) \tag{8}$$
$$+\pi(1-\lambda)[pu(C + I\bar{\theta} + \tau(\bar{\theta},\underline{s})) + (1-p)u(C + I\bar{\theta})]$$
$$+(1-\pi)\lambda[pu(C + I\underline{\theta} + \tau(\underline{\theta},\underline{s})) + (1-p)u(C + I\underline{\theta})]$$

The contract entails risk-taking following a bad signal. With probability $1-p$ the seller may default. Since the seller's default is a contractible event, it is privately optimal to set the transfers equal to zero. The transfer can only be from the buyer to the seller and the buyer is better off not making any transfer to the seller.[9]

The incentive-compatibility constraints are given by

$$\mathcal{P} \geq \bar{\tau} \tag{9}$$
$$\mathcal{P} < \underline{\tau} \tag{10}$$

The seller exerts effort after a good signal. Following a bad signal, she prefers to run the risk of default when the expected transfers to the buyer are sufficiently high.

[9] Note, however, that if the default of the seller entails costs to third parties, a positive transfer from the buyer in the default state may be optimal from a social point of view. Such a transfer induces the buyer to internalize the costs stemming from the risk-taking by the seller. We return to this issue in Section 7.

The seller's participation constraint is

$$\text{prob}[\bar{s}](KR - \bar{\tau}) + \text{prob}[\underline{s}][p(KR - \underline{\tau}) + KB] \geq KR$$

or, equivalently,

$$-\text{prob}[\underline{s}](1-p)\mathcal{P} \geq \text{prob}[\bar{s}]\bar{\tau} + \text{prob}[\underline{s}]p\underline{\tau} \tag{11}$$

The expected transfer from the seller to the buyer (right-hand side) is negative. If seller enters the hedge, she must be compensated for the potential loss of pledgeable income due to the lack of effort after bad news (left-hand side). Note that higher pledgeable income makes it more difficult for a protection seller to accept a contract with no effort. Higher returns on seller's assets KR increase the outside opportunity of the seller, and they may not materialize in the hedging contract. Similarly, a smaller private benefit B reduces the value of the contract by reducing the benefit of not exerting effort after a bad signal.

The following proposition characterizes the second-best contract with effort after a good signal and no effort after a bad signal.

Proposition 3 (Second-best contract with risk-taking) *When effort is not observable and the first-best is not attainable, $\mathcal{P} < (\pi - \underline{\pi})I\Delta\theta$, the optimal contract with risk-taking after a bad signal provides full insurance conditional on no default and is actuarially unfair. The transfers are given by:*

$$\begin{aligned}
\tau^{e=1,e=0}(\bar{\theta}, \bar{s}) &= \tau^{e=1,e=0}(\bar{\theta}, \underline{s}) = -(1-\pi)\, I\Delta\theta \frac{1 - prob[\underline{s}|\underline{\theta}](1-p)}{1 - prob[\underline{s}](1-p)} - \frac{\mathcal{P} prob[\underline{s}](1-p)}{1 - prob[\underline{s}](1-p)} < 0 \\
\tau^{e=1,e=0}(\underline{\theta}, \bar{s}) &= \tau^{e=1,e=0}(\underline{\theta}, \underline{s}) = \pi I\Delta\theta \frac{1 - prob[\underline{s}|\bar{\theta}](1-p)}{1 - prob[\underline{s}](1-p)} - \frac{\mathcal{P} prob[\underline{s}](1-p)}{1 - prob[\underline{s}](1-p)} > 0
\end{aligned}$$

As in the first- and second-best contract with effort, there are no rents to the protection seller (the participation constraint is binding). Again, the seller pays the buyer if $\tilde{\theta} = \underline{\theta}$ and vice versa if $\tilde{\theta} = \bar{\theta}$:

$$\tau^{e=1,e=0}(\underline{\theta}, \tilde{s}) > 0 > \tau^{e=1,e=0}(\bar{\theta}, \tilde{s})$$

There are three differences between the second-best contract with effort and the contract with risk-taking after bad news. First, the contract with risk-taking does not react to the signal:

$$\tau^{e=1,e=0}(\tilde{\theta}, \bar{s}) = \tau^{e=1,e=0}(\tilde{\theta}, \underline{s})$$

As long the protection seller does not default, the consumption of the buyer is equalized across states (as in the first-best contract). Second, unlike in the contract with effort, the buyer is now exposed to counterparty risk. He is completely unhedged in the default state. Third, the contract with no effort after a bad signal is not actuarially fair (the expected transfers from the seller to the buyer are negative, see equation (11)).

In sum, in the second-best contract with risk-taking, the protection buyer can get more risk-sharing from the hedge by accepting counterparty risk.

4.3 Risk-sharing and risk-taking

The contract with effort after both signals entails limited risk-sharing but no risk-taking, while the contract with no effort after a bad signal entails full risk-sharing but allows risk-taking after bad news. In this section, we examine under what conditions it is privately optimal to allow risk-taking.

Proposition 4 (Endogenous counterparty risk) *Suppose effort is not observable and the first-best is not attainable, $\mathcal{P} < (\pi - \underline{\pi})I\Delta\theta$. If the probability of default is sufficiently small, the contract with no effort after a bad signal is optimal for low levels of pledgeable income $\mathcal{P}$.*

The key factor in the choice between the second-best contract with and without risk-taking is whether counterparty or signal risk is more costly for the protection buyer. For low levels of pledgeable income, the moral hazard problem is severe. Providing incentives to avoid risk-taking after a bad signal, requires a considerable reduction in hidden leverage. The buyer then has to bear a lot of signal risk. If, at the same time, default is unlikely (p

is high), the counterparty risk under the risk-taking contract is small. It is then optimal for the protection buyer to allow risk-taking by the protection seller.

Counterparty risk thus arises endogenously due to moral hazard. "Fat" tails are generated through the incentive structure rather than by assumption. Note that pledgeable income $\mathcal{P}$ is increasing in the return of seller's assets, R. Hence, privately optimal hedging contracts are more likely to allow risk-taking in an environment of low returns (endogenous "search for yield").

In sum, under the conditions in the proposition, the privately optimal contract entails no effort after a bad signal. For higher levels of pledgeable income, the moral hazard problem diminishes allowing for more risk-sharing under effort. The second-best contract with effort becomes optimal. For $\mathcal{P} \geq (\pi - \underline{\pi}) I \Delta\theta$, the first-best is reached.

5 Implementation

Suppose pledgeable income $\mathcal{P}$ is sufficiently high so that the first-best is attainable, $\mathcal{P} \geq (\pi - \underline{\pi}) I \Delta\theta$. Then, the optimal contract can be implemented with a forward contract. Recall that if the signal is completely uninformative, the condition above is satisfied and hence a forward always implements the optimal contract.

For pledgeable income levels such that $0 < \mathcal{P} < \hat{\mathcal{P}}$, where $\hat{\mathcal{P}}$ denotes the threshold level of pledgeable income below which the optimal contract allows risk-taking, a forward contract with the added feature of freeing the protection buyer from the obligation to honor the forward if the seller defaults implements the optimal contract.

For pledgeable income levels such that $\hat{\mathcal{P}} \leq \mathcal{P} < (\pi - \underline{\pi}) I \Delta\theta$, a simple forward contract does not implement the optimal contract. The transfers under a "plain vanilla" forward do not depend on advance information about the hedged risk. However, the optimal contract does react to such information to preserve the incentives of protection sellers to control their balance sheet risk. Furthermore, the participation constraint with the optimal contract always binds. It eliminates rents by enabling cross-subsidization across signals. With a simple forward, such cross-subsidization is not feasible and protection sellers will obtain rents.

6 Initial margins

Suppose a market infrastructure exists that enables implementation of initial margins, i.e., a requirement for the protection seller to deposit some cash with a central counterparty (CCP) when the contract is signed. Such an infrastructure changes the technology that agents have access to. It makes some choices of the seller observable and contractible. In this section, we analyze the incentive effects of initial margins. In case of the contract with effort, in which risk-sharing is limited by the incentive constraint of the seller, we examine whether initial margins can help increase the amount of insurance for the protection buyer. In case of the contract with risk-taking, in which the buyer is exposed to counterparty risk, we examine the role of initial margins in providing insurance against the seller's default.

Consider a contract with initial margins. It specifies a set of transfers, $\tau(\tilde{\theta}, \tilde{s})$, and a fraction α of assets to be deposited as cash with the CCP.

If the seller exerts effort, her participation constraint is given by:

$$\alpha K + (1 - \alpha) KR - E[\tau] \geq KR$$

or, equivalently,

$$E[\tau] \leq \alpha K (1 - R) \tag{12}$$

The expression on the right-hand side is negative and represents the opportunity cost of depositing cash with the CCP. The seller forgoes the net return of assets over cash, $R - 1$. since expected transfers are no longer equal to zero, a contract with margins will be actuarially unfair. Placing a higher initial margin α makes it more difficult for the protection seller to accept the contract.

The incentive-compatibility constraint after a bad signal is given by:

$$\alpha K + (1 - \alpha) KR - \underline{\tau} \geq p [\alpha K + (1 - \alpha) KR - \underline{\tau}] + (1 - \alpha) BK$$

If the seller does not exert effort (right-hand side), she earns the private benefit B only on the assets she still controls. There is no private benefit associated with the cash deposited with the clearing house. Higher margins thus reduce the private benefit of risk-taking: the cash is ring-fenced from moral hazard. In case of default, the seller loses the cash deposited as it is transferred to the buyer. We can re-write the incentive constraint as

$$\alpha K + (1 - \alpha) \mathcal{P} \geq \underline{\tau} \tag{13}$$

where $\mathcal{P}$ denotes, as before, the pledgeable income. For $K > \mathcal{P}$, the initial margin relaxes the incentive constraint. As for the incentive constraint after a good signal, we know from our previous analysis that it will not bind.

Let g denote the per unit size of pledgeable income, $g \equiv \frac{\mathcal{P}}{K}$. The next lemma states the conditions when margins are not used, i.e., when $\alpha^* = 0$ is optimal.

Lemma 3 (No margins) *When the first-best is attainable, $g \geq (\pi - \underline{\pi}) \frac{I}{K} \Delta\theta$, or when the pledgeable income is higher than the assets in place, $g \geq 1$, margins are not used.*

Margins are costly (cash a has lower return than other assets) and tighten the participation constraint (12). They can, however, relax the incentive-compatibility constraint (13) by reducing the benefit from risk-taking. When the first-best is attainable, the incentive constraint does not bind. When $g \geq 1$, margins do not relax the incentive constraint. In either case, initial margins will not be used. Hence, they can only be beneficial if

$$g < \min \left\{ (\pi - \underline{\pi}) \frac{I}{K} \Delta\theta, 1 \right\}$$

We first characterize the optimal contract with margins and effort after both signals. We then investigate the optimal contract with margins and risk-taking.

6.1 Margins and effort

We know that the optimal contract will provide full insurance to the protection buyer conditional on the signal. Thus, we can state his objective function in terms of the expected transfers conditional on the signal. When the protection seller exerts effort, the objective function of the protection buyer is given by

$$\max_{\alpha,\bar{\tau},\underline{\tau}} \text{prob}[\bar{s}]u(C + IE[\tilde{\theta}|\bar{s}] + \bar{\tau}) + \text{prob}[\underline{s}]u(C + IE[\tilde{\theta}|\underline{s}] + \underline{\tau})$$

The seller's participation and incentive constraints are given by (12) and (13), respectively.

The following proposition characterizes the optimal contract with margins and effort after both signals.

Proposition 5 (Optimal margins with effort) *Let $\bar{u}'(\bar{\tau})$ and $\underline{u}'(\underline{\tau})$ denote the buyer's marginal utilities conditional on the good and the bad signal, respectively. Margins are used, $\alpha^* > 0$, if and only if:*

$$\frac{\underline{u}'(\underline{\tau}(\alpha^*))}{\bar{u}'(\bar{\tau}(\alpha^*))} \geq 1 + \frac{R-1}{prob[\underline{s}]\,(1-g)} \tag{14}$$

with equality for $0 < \alpha^* < 1$. Margins are not used, $\alpha^* = 0$, if the reverse inequality holds in (14) at $\alpha = 0$. The expected transfers are given by:

$$\begin{aligned} \underline{\tau} &= \alpha^* K + (1-\alpha^*)\,\mathcal{P} \\ \bar{\tau} &= -\alpha^* \frac{K\,(R-1) + prob[\underline{s}]\,(K-\mathcal{P})}{prob[\bar{s}]} - \mathcal{P}\frac{prob[\underline{s}]}{prob[\bar{s}]}. \end{aligned}$$

The benefit of margins is improved risk-sharing via the transfers $\bar{\tau}(\alpha^*)$ and $\underline{\tau}(\alpha^*)$. The margin itself is never paid to the the protection buyer since the protection seller does not default when she exerts effort. The first-best would be obtained when $\frac{\underline{u}'(\underline{\tau})}{\bar{u}'(\bar{\tau})} = 1$ so that there is full insurance against signal risk. In the first-best,

$$\underline{\tau} - \bar{\tau} = I(E[\tilde{\theta}|\bar{s}] - E[\tilde{\theta}|\underline{s}]) \tag{15}$$

But to preserve the seller's incentives to exert effort when the first-best is not attainable, the protection buyer must bear signal risk and the left-hand side of (15) is bigger than the right-hand side (due to (14)). Since $\frac{\partial \bar{\tau}}{\partial \alpha^*} < 0$ and $\frac{\partial \underline{\tau}}{\partial \alpha^*} > 0$, higher margins reduce the left-hand side, moving the transfers closer to the full insurance.

The cost of margins is that they make the contract with effort actuarially unfair. The optimal margin balances enhanced insurance against signal risk and actuarial fairness. The right-hand side of (14) gives the rate at which the trade-off occurs. The numerator of the fraction, $R-1$, is the opportunity cost of foregone asset return and a measure of actuarial unfairness of the contract. The denominator, $1-g$, represents the effect of improved incentives. It gives the extent to which margins relax the incentive constraint after a bad signal.

6.2 Margins and risk-taking

If the protection seller does not exert effort after a bad signal and engages in risk-taking, she defaults with probability $1-p$. If she defaults, the margin is transferred to the protection buyer.

The participation constraint of the protection seller is now given by

$$-(1-p)\text{prob}[\underline{s}](\alpha K + (1-\alpha)\mathcal{P}) + \alpha K\,(1-R) \geq \text{prob}[\bar{s}]\bar{\tau} + \text{prob}[\underline{s}]p\underline{\tau} \tag{16}$$

The left-hand side is the sum of the loss conditional on default and the ex-ante opportunity cost due to the foregone asset return $R-1$. The right-hand side is the expected transfer from the protection seller to the buyer. It is the same as in the case without margins (see (11)).

The objective function of the protection buyer is given by

$$\max_{\alpha,\bar{\tau},\underline{\tau}} \text{prob}[\bar{s}]u(C+IE[\tilde{\theta}|\bar{s}]+\bar{\tau})+\text{prob}[\underline{s}]\left[pu(C+IE[\tilde{\theta}|\underline{s}]+\underline{\tau})+(1-p)E[u(C+I\tilde{\theta}+\alpha K)|\underline{s}]\right]$$

In case the protection seller defaults, the protection buyer obtains the cash deposit, αK.

The following proposition characterizes the optimal contract with margins and risk-taking after a bad signal.

Proposition 6 (Optimal margins with risk-taking) *Let u'_d and u'_{nd} denote the buyer's marginal utilities when the seller defaults and when she does not, respectively. Margins are used, $\alpha^* > 0$, if and only if:*

$$prob[\underline{s}](1-p)\frac{E[u'_d(\alpha^*)|\underline{s}]}{u'_{nd}(\alpha^*)} \geq R - 1 + prob[\underline{s}](1-p)(1-g) \tag{17}$$

with equality for $0 < \alpha^* < 1$. Margins are not used, $\alpha^* = 0$, if the reverse inequality holds in (17) at $\alpha = 0$. The expected transfers are given by (15) and

$$\underline{\tau} = -\alpha^* \frac{K(R-1) + prob[\underline{s}](1-p)(K-\mathcal{P})}{1 - prob[\underline{s}](1-p)} + \frac{prob[\bar{s}]I(E[\tilde{\theta}|\bar{s}] - E[\tilde{\theta}|\underline{s}]) - prob[\underline{s}](1-p)\mathcal{P}}{1 - prob[\underline{s}](1-p)} \tag{18}$$

As in the risk-taking contract without margins, the protection buyer gets full insurance when the seller does not default. In this case, his consumption is given by $C + I\underline{\theta} + \underline{\tau}(\alpha^*)$ where the latter is given by (18). He is, however, exposed to counterparty risk since the seller may default with probability $1-p$ after a bad signal.

The benefit of a margin under the risk-taking contract is the insurance it provides against counterparty risk (left-hand side of (17)). The wedge between the marginal utilities under default and no default is reduced. Margins increase the buyer's expected consumption if the seller defaults, which happens with probability prob$[\underline{s}](1-p)$. At the same time, they reduce his consumption when there is no default since $\frac{\partial \underline{\tau}}{\partial \alpha^*} < 0$. The protection buyer requires a smaller transfer after a bad signal since this is the state in which the margin may be paid to him.

The cost of margins has two components. First, there is the ex-ante opportunity cost, $R - 1$. Second, there is the loss of income in case of default. The optimal margin under risk-taking balances these costs with the benefit of protecting the buyer from counterparty risk.

6.3 Margins, risk-sharing and risk-taking

If a market infrastructure exists that enables the buyer and the seller to use initial margins (CCP), it is privately optimal to do so whenever $\alpha^* > 0$. There is no need to force participation. When the contract with margins entails no risk-taking, the margin acts as a commitment device for the protection seller not to take risks once she observes negative news about hedged risks. When the contract entails risk-taking, the margin protects the buyer against the default of the seller.

The choice between the contract with margins and effort and the contract with margins and risk-taking depends again on whether counterparty or signal risk is more costly for the protection buyer. As in Section 4.3, the contract with risk-taking may be chosen when pledgeable income is low and the moral hazard problem is severe.

The overall effect of margins on risk-taking, and hence counterparty risk, is ambiguous. On the one hand, margins reduce the signal risk faced by the buyer and make risk-control by the seller more attractive. On the other hand, margins protect the buyer from counterparty risk and make risk-taking by the seller more attractive. If the latter effect is small, then margins reduce the risk-taking effect of financial innovation. If the buyer benefits a lot from the insurance against counterparty risk, then margins lead to more risk-taking.

7 Social optimality and regulation

Whenever the privately optimal hedging contract entails risk control by the protection seller, her assets are safe and there is no default. When the level of pledgeable income is low, the privately optimal contract, however, entails risk-taking and counterparty risk (see Proposition 4). Risk-taking by financial institutions entails costs for third parties, e.g., bankruptcy costs or disruptions in payment systems and interbank markets. To examine this possibility, suppose that the default of the seller leads to losses $L \geq 0$. The losses L are a measure of the externality the default of a protection seller imposes on the financial system, i.e., L measures the systemic importance of a financial institution. Since the seller is protected by limited liability, she does not internalize its systemic relevance.

Social (utilitarian) welfare is given by the sum of the buyer's utility and the seller's profits net of losses L. Social welfare decreases with losses and there exists a threshold level of L, L^*, such that for losses larger than L^*, social optimality requires the avoidance of risk-taking even though it may be privately optimal. The conflict between private and social optimality opens up the scope for regulating systemically relevant financial institutions that are engaged in financial innovation.

One way to mitigate risk-taking incentives is to impose capital requirements on protection sellers. For a given amount of hedging activities, extra capital strengthens their balance sheets by increasing pledgeable income. This counters the hidden leverage embedded in financial innovation. Thus, requiring financial institutions to hold capital in proportion to their derivative exposures reduces risk-taking. By extension, (systemically relevant) financial institutions that opt out of such capital requirements should not be allowed to undertake hedging activities.

Our analysis shows that financial market infrastructures have an important effect on systemic risk. For example, the clearing of derivative contracts by a central counterparty (CCP) makes margin requirements possible. As long as margins are privately optimal, the clearing need not be mandated. As for the effect of margin requirements on risk-taking, we showed that it is ambiguous. Margins make risk-taking by the protection seller less attractive but they introduce complacency of the buyer by insuring him against seller default. If the former effect dominates, margins and capital requirements are substitutes. Market infrastructures then allow to economize on the use of costly capital. If the latter effect dominates, margins and capital requirements are complements. In that case, market infrastructures must be supported by additional regulation.

Mandating the clearing of derivatives through a CCP can also improve social welfare by penalizing the protection buyer for his complacency about seller default. We showed that it is privately optimal to have zero transfers in case the seller defaults (see the discussion following equation (8)). When the losses L incurred by third parties are high, it is socially optimal to have the buyer internalize the consequences of being complacent about the seller's risk-taking. One way to internalize the losses is to have the CCP collect a payment from the buyer even if his counterparty is no longer around.

8 Conclusion

We consider a financial innovation, whereby a protection seller offers a hedge to a protection buyer. We show how this innovation, designed to facilitate risk-sharing, can generate incentives for risk-taking. When the position of the protection seller becomes loss-making, this creates hidden leverage, discouraging the mitigation of risks in the seller's core business. Such elevated moral hazard raises the default risk of the protection seller and, correspondingly, the counterparty risk for the protection buyer. Thus innovation can lead to systemic risk, in the form of contagion from positions in innovative products to the traditional business of financial institutions.

We show that for well-capitalized institutions, margin requirements mitigate this problem, by reducing the severity of the moral hazard problem. Therefore, the establishment of CCPs can be part of an appropriate regulatory response. But, our analysis implies that poorly capitalized firms should be banned from the sale of such protection, even in markets with CCPs and margin requirements.

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Appendix

Proof of Proposition 1

Let μ denote the Lagrange multiplier on the participation constraint 2. The first-order conditions with respect to transfers $\tau(\bar{\theta},\bar{s})$, $\tau(\underline{\theta},\bar{s})$, $\tau(\bar{\theta},\underline{s})$ and $\tau(\underline{\theta},\underline{s})$ are given by:

$$\begin{aligned}
\pi\lambda u'(C+I\bar{\theta}+\tau(\bar{\theta},\bar{s}))-\mu\pi\lambda &= 0\\
(1-\pi)(1-\lambda)u'(C+I\underline{\theta}+\tau(\underline{\theta},\bar{s}))-\mu(1-\pi)(1-\lambda) &= 0\\
\pi(1-\lambda)u'(C+I\bar{\theta}+\tau(\bar{\theta},\underline{s}))-\mu\pi(1-\lambda) &= 0\\
(1-\pi)\lambda u'(C+I\underline{\theta}+\tau(\underline{\theta},\underline{s}))-\mu(1-\pi)\lambda &= 0
\end{aligned}$$

It follows that the marginal utility of the buyer of insurance is equalized across $(\tilde{\theta},\tilde{s})$ states (full insurance) and that the participation constraint is binding:

$$\bar{u}'(\tau(\bar{\theta},\bar{s}))=\bar{u}'(\tau(\bar{\theta},\underline{s}))=\underline{u}'(\tau(\underline{\theta},\underline{s}))=\underline{u}'(\tau(\underline{\theta},\bar{s}))=\mu>0 \tag{A.1}$$

The optimal transfers are obtained by using the fact that the participation constraint is binding and that consumption is the same across $(\tilde{\theta},\tilde{s})$ states.

Proof of Lemma 1

Let $\mu_{\bar{s}}$ and $\mu_{\underline{s}}$ denote the Lagrange multipliers on the incentive compatibility constraints (5) and (6), respectively (μ again denotes the multiplier on the participation constraint (2)). The first-order conditions with respect to transfers $\tau(\bar{\theta},\bar{s})$, $\tau(\underline{\theta},\bar{s})$, $\tau(\bar{\theta},\underline{s})$ and $\tau(\underline{\theta},\underline{s})$ are given by:

$$\begin{aligned}
\pi\lambda u'(C+I\bar{\theta}+\tau(\bar{\theta},\bar{s}))-\mu_{\bar{s}}\bar{\pi}-\mu\pi\lambda &= 0\\
(1-\pi)(1-\lambda)u'(C+I\underline{\theta}+\tau(\underline{\theta},\bar{s}))-\mu_{\bar{s}}(1-\bar{\pi})-\mu(1-\pi)(1-\lambda) &= 0\\
\pi(1-\lambda)u'(C+I\bar{\theta}+\tau(\bar{\theta},\underline{s}))-\mu_{\underline{s}}\underline{\pi}-\mu\pi(1-\lambda) &= 0\\
(1-\pi)\lambda u'(C+I\underline{\theta}+\tau(\underline{\theta},\underline{s}))-\mu_{\underline{s}}(1-\underline{\pi})-\mu(1-\pi)\lambda &= 0
\end{aligned}$$

We re-write the first-order conditions as

$$\bar{u}'(\tau(\bar{\theta},\bar{s})) = \mu+\mu_{\bar{s}}\frac{\bar{\pi}}{\pi\lambda} \tag{A.2}$$

$$\underline{u}'(\tau(\underline{\theta},\bar{s})) = \mu+\mu_{\bar{s}}\frac{1-\bar{\pi}}{(1-\pi)(1-\lambda)} \tag{A.3}$$

$$\bar{u}'(\tau(\bar{\theta},\underline{s})) = \mu+\mu_{\underline{s}}\frac{\underline{\pi}}{\pi(1-\lambda)} \tag{A.4}$$

$$\underline{u}'(\tau(\underline{\theta},\underline{s})) = \mu+\mu_{\underline{s}}\frac{1-\underline{\pi}}{(1-\pi)\lambda} \tag{A.5}$$

where we use a shorthand $\bar{u}'(\tau(\bar{\theta},\tilde{s}))$ to denote marginal utility in state $\bar{\theta}$ conditional on the signal $\tilde{s}$ and, similarly, $\underline{u}'(\tau(\underline{\theta},\tilde{s}))$ to denote marginal utility in state $\underline{\theta}$ conditional on the signal $\tilde{s}$.

Since

$$
\begin{aligned}
\frac{\bar{\pi}}{\pi\lambda} &= \frac{\text{prob}[\bar{\theta}|\bar{s}]}{\text{prob}[\bar{\theta}\cap\bar{s}]} = \frac{1}{\text{prob}[\bar{s}]} \\
\frac{1-\bar{\pi}}{(1-\pi)(1-\lambda)} &= \frac{\text{prob}[\underline{\theta}|\bar{s}]}{\text{prob}[\underline{\theta}\cap\bar{s}]} = \frac{1}{\text{prob}[\bar{s}]} \\
\frac{\underline{\pi}}{\pi(1-\lambda)} &= \frac{\text{prob}[\bar{\theta}|\underline{s}]}{\text{prob}[\bar{\theta}\cap\underline{s}]} = \frac{1}{\text{prob}[\underline{s}]} \\
\frac{1-\underline{\pi}}{(1-\pi)\lambda} &= \frac{\text{prob}[\underline{\theta}|\underline{s}]}{\text{prob}[\underline{\theta}\cap\underline{s}]} = \frac{1}{\text{prob}[\underline{s}]}
\end{aligned}
$$

holds, it follows that there is full risk-sharing conditional on the signal:

$$
\begin{aligned}
\bar{u}'(\tau(\bar{\theta},\bar{s})) &= \underline{u}'(\tau(\underline{\theta},\bar{s})) \\
\bar{u}'(\tau(\bar{\theta},\underline{s})) &= \underline{u}'(\tau(\underline{\theta},\underline{s}))
\end{aligned}
$$

As in the first-best case, we therefore have

$$
\tau(\underline{\theta},\bar{s}) - \tau(\bar{\theta},\bar{s}) = \tau(\underline{\theta},\underline{s}) - \tau(\bar{\theta},\underline{s}) = I\Delta\theta > 0 \tag{A.6}
$$

It follows that, conditional on the signal, the transfer to the buyer when the asset return is low is higher than when the asset return is high, $\tau(\underline{\theta},\tilde{s}) > \tau(\bar{\theta},\tilde{s})$.

Next, we show that the participation constraint must bind. Suppose not, i.e. $\mu = 0$. Then, equations (A.2) and (A.3) imply that $\mu_{\bar{s}} > 0$. Similarly, (A.4) and (A.5) imply that $\mu_{\underline{s}} > 0$. Both incentive constraints bind so that $\mathcal{P} = \bar{\tau} = \underline{\tau}$. Since the participation constraint is slack, it must be that

$$
\begin{aligned}
0 &> E[\tau] \equiv \text{prob}[\bar{s}]\bar{\tau} + \text{prob}[\underline{s}]\underline{\tau} \\
&= \mathcal{P}\left(\text{prob}[\bar{s}] + \text{prob}[\underline{s}]\right) \\
&= \mathcal{P}
\end{aligned}
$$

which contradicts $\mathcal{P} > 0$. Hence, the participation constraint binds, $E[\tau] = 0$.

It follows that at least one incentive constraint must be slack. If not, then $\bar{\tau} = \underline{\tau} = \mathcal{P} > 0$, which contradicts $E[\tau] = 0$.

Suppose both incentive constraints are slack, $\mu_{\bar{s}} = \mu_{\underline{s}} = 0$. Then, we obtain full insurance as in (A.1) and the contract is given by proposition 1 (first-best). The conditions under which the incentive constraints are indeed slack are given by:

$$
\begin{aligned}
\mathcal{P} &> \bar{\pi}\tau^{FB}(\bar{\theta},\bar{s}) + (1-\bar{\pi})\tau^{FB}(\underline{\theta},\bar{s}) = (\pi - \bar{\pi})I\Delta\theta \\
\mathcal{P} &> \underline{\pi}\tau^{FB}(\bar{\theta},\underline{s}) + (1-\underline{\pi})\tau^{FB}(\underline{\theta},\underline{s}) = (\pi - \underline{\pi})I\Delta\theta
\end{aligned}
$$

When the signal is informative, $\lambda > \frac{1}{2}$, we have $\bar{\pi} > \pi > \underline{\pi}$. The result in the lemma follows.

Proof of Lemma 2

We have shown above that at least one incentive constraint must be slack. They cannot both be slack since we assume that $\mathcal{P} < (\pi - \underline{\pi})I\Delta\theta$. We now show that it is the incentive constraint following a *bad* signal that is binding. Suppose not, so that $\mathcal{P} = \bar{\tau} > 0 > \underline{\tau}$ where the last inequality follows from $E[\tau] = 0$. Then, $\mu_{\underline{s}} = 0$ and $\mu_{\bar{s}} \geq 0$ and equations (A.2) through (A.5) yield

$$\bar{u}'(\tau(\bar{\theta}, \underline{s})) = \underline{u}'(\tau(\underline{\theta}, \underline{s})) = \mu \leq \bar{u}'(\tau(\bar{\theta}, \bar{s})) = \underline{u}'(\tau(\underline{\theta}, \bar{s}))$$

Comparing the first with the third term and the second with the fourth term yields

$$\begin{aligned} \tau(\bar{\theta}, \underline{s}) &\geq \tau(\bar{\theta}, \bar{s}) \\ \tau(\underline{\theta}, \underline{s}) &\geq \tau(\underline{\theta}, \bar{s}) \end{aligned}$$

Using $\tau(\underline{\theta}, \tilde{s}) > \tau(\bar{\theta}, \tilde{s})$ (equation (A.6)) and $\bar{\pi} > \underline{\pi}$, we can write

$$\begin{aligned} 0 &< \bar{\tau} \equiv \bar{\pi}\tau(\bar{\theta}, \bar{s}) + (1 - \bar{\pi})\tau(\underline{\theta}, \bar{s}) \\ &< \underline{\pi}\tau(\bar{\theta}, \bar{s}) + (1 - \underline{\pi})\tau(\underline{\theta}, \bar{s}) \\ &\leq \underline{\pi}\tau(\bar{\theta}, \underline{s}) + (1 - \underline{\pi})\tau(\underline{\theta}, \underline{s}) \equiv \underline{\tau} \end{aligned}$$

But $\underline{\tau} < 0$, a contradiction. Hence, only the incentive constraint after a bad signal binds.

Proof of Proposition 2

The optimal contract is given by the binding incentive constraint following a bad signal:

$$\mathcal{P} = \underline{\tau},$$

the binding participation constraint

$$\text{prob}[\bar{s}]\bar{\tau} + \text{prob}[\underline{s}]\underline{\tau} = 0,$$

and full risk-sharing conditional on the signal (A.6).

Proof of Proposition 3

Let $\mu_{\bar{s}}$ denote the Lagrange multiplier on the incentive constraint following a good signal and let μ denote the multiplier on the participation constraint (11). The first-order conditions with respect to transfers $\tau(\bar{\theta}, \bar{s})$, $\tau(\underline{\theta}, \bar{s})$, $\tau(\bar{\theta}, \underline{s})$ and $\tau(\underline{\theta}, \underline{s})$ are:

$$\bar{u}'(\tau(\bar{\theta},\bar{s})) = \mu + \frac{\mu_{\bar{s}}}{\text{prob}[\bar{s}]} \quad \text{(A.7)}$$

$$\underline{u}'(\tau(\underline{\theta},\bar{s})) = \mu + \frac{\mu_{\bar{s}}}{\text{prob}[\bar{s}]} \quad \text{(A.8)}$$

$$\bar{u}'(\tau(\bar{\theta},\underline{s})) = \mu \quad \text{(A.9)}$$

$$\underline{u}'(\tau(\underline{\theta},\underline{s})) = \mu \quad \text{(A.10)}$$

The last two conditions imply that the participation constraint binds as $\mu > 0$. Moreover, we again have full sharing of the $\tilde{\theta}$ risk conditional on the signal, except for a default state:

$$\begin{aligned} \bar{u}'(\tau(\bar{\theta},\bar{s})) &= \underline{u}'(\tau(\underline{\theta},\bar{s})) \\ \bar{u}'(\tau(\bar{\theta},\underline{s})) &= \underline{u}'(\tau(\underline{\theta},\underline{s})) \end{aligned}$$

and hence

$$\tau(\underline{\theta},\tilde{s}) - \tau(\bar{\theta},\tilde{s}) = I\Delta\theta > 0 \quad \text{(A.11)}$$

Next, we show that the incentive constraint after a good signal (9) is slack, implying $\mu_{\bar{s}} = 0$. Suppose that the constraint is not slack and $\mathcal{P} = \bar{\tau} < \underline{\tau}$. Since $\mathcal{P} > 0$, both expected transfers are positive, which violates the participation constraint.

Since incentive constraints are slack, the first-order conditions become

$$\bar{u}'(\tau(\bar{\theta},\bar{s})) = \bar{u}'(\tau(\bar{\theta},\underline{s})) = \underline{u}'(\tau(\underline{\theta},\underline{s})) = \underline{u}'(\tau(\underline{\theta},\bar{s})) = \mu \quad \text{(A.12)}$$

Conditional on no default, there is full insurance, as in the first-best case (A.1):

$$\tau(\tilde{\theta},\bar{s}) = \tau(\tilde{\theta},\underline{s}) \quad \text{(A.13)}$$

The buyer is, however, exposed to counterparty risk.

The optimal contract with no effort after a bad signal is given by (A.12) and the binding participation constraint. We now check under what condition the incentive constraint following a bad signal (10) is indeed slack. Starting with the binding participation constraint and using (A.11) and (A.13), we get

$$\begin{aligned} -\text{prob}[\underline{s}](1-p)\mathcal{P} &= \text{prob}[\bar{s}][\bar{\pi}\tau(\bar{\theta},\bar{s}) + (1-\bar{\pi})\tau(\underline{\theta},\bar{s})] + \text{prob}[\underline{s}]p[\underline{\pi}\tau(\bar{\theta},\underline{s}) + (1-\underline{\pi})\tau(\underline{\theta},\underline{s})] \\ &= \text{prob}[\bar{s}][\tau(\underline{\theta},\bar{s}) - \bar{\pi}I\Delta\theta] + \text{prob}[\underline{s}]p[\tau(\underline{\theta},\underline{s}) - \underline{\pi}I\Delta\theta] \\ &= \tau(\underline{\theta},\underline{s})[\text{prob}[\bar{s}] + \text{prob}[\underline{s}]p] - I\Delta\theta[\text{prob}[\bar{s}]\bar{\pi} + \text{prob}[\underline{s}]p\underline{\pi}] \end{aligned}$$

Using Bayes' Rule and simplifying, we arrive at

$$-\text{prob}[\underline{s}](1-p)\mathcal{P} = \tau(\underline{\theta},\underline{s})[1-\text{prob}[\underline{s}](1-p)] - I\Delta\theta\pi[1-\text{prob}[\underline{s}|\bar{\theta}](1-p)]$$

Hence,

$$\tau(\underline{\theta},\underline{s}) = \pi I\Delta\theta\frac{1-\text{prob}[\underline{s}|\bar{\theta}](1-p)}{1-\text{prob}[\underline{s}](1-p)} - \mathcal{P}\frac{\text{prob}[\underline{s}](1-p)}{1-\text{prob}[\underline{s}](1-p)} \quad \text{(A.14)}$$

Since $\underline{\tau} = \tau(\underline{\theta}, \underline{s}) - \underline{\pi} I \Delta\theta$, for the incentive constraint after a bad signal to be slack, it must be that

$$\mathcal{P} < \tau(\underline{\theta}, \underline{s}) - \underline{\pi} I \Delta\theta$$

Substituting for $\tau(\underline{\theta}, \underline{s})$ and simplifying yields

$$\mathcal{P} < (\pi - \underline{\pi}) I \Delta\theta - (1-p) I \Delta\theta [\pi \text{prob}[\underline{s}|\bar{\theta}] - \underline{\pi} \text{prob}[\underline{s}]]$$

or, equivalently,

$$\mathcal{P} < (\pi - \underline{\pi}) I \Delta\theta$$

This is the same condition as in Lemma 1. The incentive constraint after a bad signal is slack when the first-best is not attainable.

Proof of Proposition 4

The proof proceeds in three steps. First, we show that the expected utility of the contract with effort after both signals is increasing in $\mathcal{P}$:

$$\begin{aligned}
\frac{\partial EU^{e=1}}{\partial \mathcal{P}} &= -\frac{\text{prob}[\underline{s}]}{\text{prob}[\bar{s}]} \left[\pi\lambda \bar{u}'(\tau(\bar{\theta}, \bar{s})) + (1-\pi)(1-\lambda)\underline{u}'(\tau(\underline{\theta}, \bar{s}))\right] \\
&\quad + \pi(1-\lambda)\bar{u}'(\tau(\bar{\theta}, \underline{s})) + (1-\pi)\lambda \underline{u}'(\tau(\underline{\theta}, \underline{s})) \\
&= \text{prob}[\underline{s}] \left[\bar{u}'(\tau(\bar{\theta}, \underline{s})) - \bar{u}'(\tau(\bar{\theta}, \bar{s}))\right] > 0
\end{aligned}$$

since $\tau(\bar{\theta}, \underline{s}) < \tau(\bar{\theta}, \bar{s})$ due to the signal risk.

Second, we show that the expected utility of the contract with no effort following a bad signal is decreasing in $\mathcal{P}$:

$$\begin{aligned}
\frac{\partial EU^{e=1,e=0}}{\partial \mathcal{P}} &= -\frac{\text{prob}[\underline{s}](1-p)}{1 - \text{prob}[\underline{s}](1-p)} \left[\pi\lambda \bar{u}'(\tau(\bar{\theta}, \bar{s})) + (1-\pi)(1-\lambda)\underline{u}'(\tau(\underline{\theta}, \bar{s}))\right. \\
&\quad \left. + \pi(1-\lambda) p \bar{u}'(\tau(\bar{\theta}, \underline{s})) + (1-\pi)\lambda p \underline{u}'(\tau(\underline{\theta}, \underline{s}))\right] \\
&= -\frac{\text{prob}[\underline{s}](1-p)}{1 - \text{prob}[\underline{s}](1-p)} \left[\pi(\lambda + p(1-\lambda))\bar{u}'(\tau(\bar{\theta}, \bar{s}))\right. \\
&\quad \left. + (1-\pi)((1-\lambda) + p\lambda)\underline{u}'(\tau(\underline{\theta}, \bar{s}))\right] < 0
\end{aligned}$$

Third, we provide sufficient condition for $EU^{e=1}(\mathcal{P} = 0) < EU^{e=1,e=0}(\mathcal{P} = 0)$ so that no effort after a bad signal is optimal for low $\mathcal{P}$.

We have

$$\begin{aligned}
EU^{e=1}(\mathcal{P} = 0) &= [\pi\lambda + (1-\pi)(1-\lambda)] u(C + I\underline{\theta} + \bar{\pi} I \Delta\theta) + [\pi(1-\lambda) + (1-\pi)\lambda] \times \\
&\quad u(C + I\underline{\theta} + \underline{\pi} I \Delta\theta) \\
&= \text{prob}[\bar{s}] u(C + I\underline{\theta} + \bar{\pi} I \Delta\theta) + \text{prob}[\underline{s}] u(C + I\underline{\theta} + \underline{\pi} I \Delta\theta) \\
&= \text{prob}[\bar{s}] u(C + IE[\tilde{\theta}|\bar{s}]) + \text{prob}[\underline{s}] u(C + IE[\tilde{\theta}|\underline{s}]) \qquad \text{(A.15)}
\end{aligned}$$

and

$$\begin{aligned}
EU^{e=1,e=0}(\mathcal{P}=0) &= (\text{prob}[\bar{s}] + p\text{prob}[\underline{s}])\, u\left(C + I\underline{\theta} + \pi\frac{1-\text{prob}[\underline{s}|\bar{\theta}](1-p)}{1-\text{prob}[\underline{s}](1-p)} I\Delta\theta\right) \\
&\quad + (1-p)\left[\pi(1-\lambda)u(C+I\bar{\theta}) + (1-\pi)\lambda u(C+I\underline{\theta})\right] \\
&= (\text{prob}[\bar{s}] + \text{prob}[\underline{s}]p)\, u\left(C + I\hat{E}[\tilde{\theta}]\right) \\
&\quad + \text{prob}[\underline{s}](1-p)\left[\underline{\pi} u(C+I\bar{\theta}) + (1-\underline{\pi})u(C+I\underline{\theta})\right] \qquad \text{(A.16)}
\end{aligned}$$

where

$$\hat{E}[\tilde{\theta}] = \hat{\pi}\bar{\theta} + (1-\hat{\pi})\underline{\theta}$$

and

$$\hat{\pi} = \pi\frac{1-\text{prob}[\underline{s}|\bar{\theta}](1-p)}{1-\text{prob}[\underline{s}](1-p)}$$

Note that

$$\bar{\pi} > \hat{\pi} > \pi > \underline{\pi} \qquad \text{(A.17)}$$

for $p \in (0,1)$. Note that $\bar{\pi} = \hat{\pi}$ for $p = 0$ and that $\hat{\pi} = \pi$ for $p = 1$. The first two inequalities follow from the fact that prob$[\underline{s}]$ >prob$[\underline{s}|\bar{\theta}]$ for $\lambda > \frac{1}{2}$ (informative signal). Hence,

$$\frac{1-\text{prob}[\underline{s}|\bar{\theta}](1-p)}{1-\text{prob}[\underline{s}](1-p)} \geq 1$$

Combining (A.15) and (A.16), we have that no effort after a bad signal dominates effort (when $\mathcal{P} = 0$) if and only if

$$\begin{aligned}
&\text{prob}[\bar{s}]u(C+IE[\tilde{\theta}|\bar{s}]) + \text{prob}[\underline{s}]u(C+IE[\tilde{\theta}|\underline{s}]) \\
&\qquad < (\text{prob}[\bar{s}] + \text{prob}[\underline{s}]p)\, u(C+I\hat{E}[\tilde{\theta}]) + \text{prob}[\underline{s}](1-p)EU\left((C+I\tilde{\theta})|\underline{s}\right)
\end{aligned}$$

where

$$EU\left((C+I\tilde{\theta})|\underline{s}\right) = \underline{\pi}u(C+I\bar{\theta}) + (1-\underline{\pi})u(C+I\underline{\theta})$$

After collecting terms, we have

$$\begin{aligned}
&\text{prob}[\bar{s}]\left[u(C+IE[\tilde{\theta}|\bar{s}]) - u(C+I\hat{E}[\tilde{\theta}])\right] + \text{prob}[\underline{s}]\left[u(C+IE[\tilde{\theta}|\underline{s}]) - EU\left((C+I\tilde{\theta})|\underline{s}\right)\right] \\
&\qquad < \text{prob}[\underline{s}]p\left[u(C+I\hat{E}[\tilde{\theta}]) - EU\left((C+I\tilde{\theta})|\underline{s}\right)\right]
\end{aligned}$$

All the differences in the square brackets are positive. The first one due to (A.17), the second one due to the concavity of u, and the third one due to both the concavity of u and (A.17).

Rearranging, we arrive at

$$\frac{\text{prob}[\bar{s}]}{\text{prob}[\underline{s}]}\frac{u(C+IE[\tilde{\theta}|\bar{s}]) - u(C+I\hat{E}[\tilde{\theta}])}{u(C+I\hat{E}[\tilde{\theta}]) - EU\left((C+I\tilde{\theta})|\underline{s}\right)} + \frac{u(C+IE[\tilde{\theta}|\underline{s}]) - EU\left((C+I\tilde{\theta})|\underline{s}\right)}{u(C+I\hat{E}[\tilde{\theta}]) - EU\left((C+I\tilde{\theta})|\underline{s}\right)} < p \qquad \text{(A.18)}$$

It is clear that the left-hand side is strictly positive so that seller's effort dominates when p is small. The left-hand is, however, also strictly smaller than one so that no effort after a bad signal dominates when p is large.[10]

The condition

$$\frac{\text{prob}[\bar{s}]}{\text{prob}[\underline{s}]}\frac{u(C+IE[\tilde{\theta}|\bar{s}])-u(C+I\hat{E}[\tilde{\theta}])}{u(C+I\hat{E}[\tilde{\theta}])-EU\left((C+I\tilde{\theta})|\underline{s}\right)}+\frac{u(C+IE[\tilde{\theta}|\underline{s}])-EU\left((C+I\tilde{\theta})|\underline{s}\right)}{u(C+I\hat{E}[\tilde{\theta}])-EU\left((C+I\tilde{\theta})|\underline{s}\right)}<1$$

simplifies to

$$\text{prob}[\bar{s}]u(C+IE[\tilde{\theta}|\bar{s}])+\text{prob}[\underline{s}]u(C+IE[\tilde{\theta}|\underline{s}])<u(C+I\hat{E}[\tilde{\theta}])$$

By concavity,

$$\text{prob}[\bar{s}]u(C+IE[\tilde{\theta}|\bar{s}])+\text{prob}[\underline{s}]u(C+IE[\tilde{\theta}|\underline{s}])<u(C+IE[\tilde{\theta}])$$

and so the condition holds when

$$u(C+IE[\tilde{\theta}])\leq u(C+I\hat{E}[\tilde{\theta}])$$

which is always the case due to (A.17).

Hence, whenever $EU^{e=1}(\mathcal{P}=0)<EU^{e=1,e=0}(\mathcal{P}=0)$ holds, the privately optimal contract entails no effort after a bad signal for low levels of pledgeable income $\mathcal{P}$. For levels of $\mathcal{P}\geq\hat{\mathcal{P}}$ where $\hat{\mathcal{P}}$ is given by $EU^{e=1}\left(\hat{\mathcal{P}}\right)=EU^{e=1,e=0}\left(\hat{\mathcal{P}}\right)$ and $\hat{\mathcal{P}}<(\pi-\underline{\pi})I\Delta\theta$, the optimal contract is the second-best contract with effort. For $\mathcal{P}>(\pi-\underline{\pi})I\Delta\theta$, the first-best is reached.

Proof of Lemma 3

Let g denote the per unit size of pledgeable income so that $g\equiv\frac{\mathcal{P}}{K}$. Recall that first-best is attainable for $\mathcal{P}\geq(\pi-\underline{\pi})I\Delta\theta$ (lemma 1). Hence, if $g\geq(\pi-\underline{\pi})\frac{I}{K}\Delta\theta$, margins would only imply the opportunity cost of forgoing a profitable investment opportunity. It is thus optimal to set $\alpha^*=0$.

For $g\geq 1$, we have that $\mathcal{P}\geq K$. In this case, the incentive constraint in (13) cannot be relaxed using initial margins. Hence, they will not be used.

In sum, for

$$g\geq\min\left\{(\pi-\underline{\pi})\frac{I}{K}\Delta\theta,1\right\}$$

it is not optimal to use initial margins and $\alpha^*=0$.

[10] Note that this inequality is evaluated at $\mathcal{P}=0$ and $\mathcal{P}$ is a function of p. There is, however, an open set of parameters for which no effort after a bad signal dominates.

Proof of Proposition 5

Let μ and $\mu_{\underline{s}}$ denote the Lagrange multipliers on the participation and incentive-compatibility constraints (12) and (13), respectively. Furthermore, let μ_0 and μ_1 be the Lagrange multipliers on the feasibility constraints $\alpha \geq 0$ and $\alpha \leq 1$. The first-order conditions with respect to expected transfers $\bar{\tau}$, $\underline{\tau}$ and initial margin α are:

$$\bar{u}'(\bar{\tau}) = \mu \tag{A.19}$$

$$\underline{u}'(\underline{\tau}) = \mu + \frac{\mu_{\bar{s}}}{\text{prob}[\underline{s}]} \tag{A.20}$$

$$\mu_{\bar{s}}(K - \mathcal{P}) + \mu_0 = \mu K (R-1) + \mu_1 \tag{A.21}$$

where $\bar{u}'(\bar{\tau})$ and $\underline{u}'(\underline{\tau})$ denote the marginal utility conditional on the good and the bad signal, respectively.

The first condition implies that $\mu > 0$ and the participation constraint binds. Substituting (A.19) and (A.21) into (A.20), we arrive at:

$$\frac{\underline{u}'(\underline{\tau})}{\bar{u}'(\bar{\tau})} = 1 + \frac{K(R-1)}{\text{prob}[\underline{s}](K-\mathcal{P})} + \frac{\mu_1 - \mu_0}{\bar{u}'(\bar{\tau})\text{prob}[\underline{s}](K-\mathcal{P})}$$

When margins are not used, $\mu_1 = 0$. If they are used, then $\mu_0 = 0$ and equation (A.21) implies that the incentive-compatibility constraint binds, $\mu_{\bar{s}} > 0$ (since $R > 1$ and $K > \mathcal{P}$). Then,

$$\underline{\tau} = \alpha K + (1-\alpha)\mathcal{P}$$

and

$$\bar{\tau} = -\alpha \frac{K(R-1) + \text{prob}[\underline{s}](K-\mathcal{P})}{\text{prob}[\bar{s}]} - \mathcal{P}\frac{\text{prob}[\underline{s}]}{\text{prob}[\bar{s}]}$$

Proof of Proposition 6

Let μ denote the Lagrange multiplier on the participation constraint (16). Furthermore, let μ_0 and μ_1 be the Lagrange multipliers on the feasibility constraints $\alpha \geq 0$ and $\alpha \leq 1$. The first-order conditions with respect to expected transfers $\bar{\tau}$, $\underline{\tau}$ and initial margin α are:

$$\begin{aligned} \bar{u}'(\bar{\tau}) &= \mu \\ \underline{u}'(\underline{\tau}) &= \mu \\ \text{prob}[\underline{s}](1-p)KE[u_d'|\underline{s}] + \mu_0 &= \mu\left[K(R-1) + \text{prob}[\underline{s}](1-p)(K-\mathcal{P})\right] + \mu_1 \end{aligned}$$

where $E[u_d'|\underline{s}]$ denotes the expected marginal utility in case the protection seller defaults:

$$E[u_d'|\underline{s}] \equiv \underline{\pi} u'(C + I\bar{\theta} + \alpha K) + (1-\underline{\pi})u'(C + I\underline{\theta} + \alpha K)$$

The first two first-order conditions yield $\bar{u}'(\bar{\tau}) = \underline{u}'(\underline{\tau}) \equiv u_{nd}'$. Plugging into the third first-order condition gives (17). Since the contract does not depend on the signal, the transfers satisfy (15). Combining (15) and the participation constraint (16) yields (18).