OPTIMAL SURE PORTFOLIO PLANS†

by

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Abstract This paper is a sequel to [2], where a model of optimal accumulation of capital and portfolio choice over an infinite horizon in continuous time was considered in which the vector process representing returns to investment is a general semimartingale with independent increments and the welfare functional has the 'discounted constant relative risk aversion' (CRRA) form. A problem of optimal choice of a sure (i.e. non–random) portfolio plan can be defined in such a way that solutions of this problem correspond to solutions of the problem of optimal choice of a portfolio–cum–saving plan, provided that the distant future is sufficiently discounted. This has been proved in [2], and is in part proved again here by different methods. Using the canonical representation of a PII–semimartingale, a formula of Lévy–Khinchin type is derived for the Bilateral Laplace Transform of the compound interest process generated by a sure portfolio plan. With its aid, the existence of an optimal sure portfolio plan is proved under suitable conditions, and various causes of non–existence are identified. Programming conditions characterising an optimal sure portfolio plan are also obtained.

Key Words Investment, portfolios, semimartingales, processes with independent increments, random measures, optimisation.

AMS *subject classifications.* 49A60, 49B60, 90A09, 93E20

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1. INTRODUCTION

The portfolio problem considered here may be stated briefly as follows. An investor holds divisible assets which can be traded at market prices without transaction costs. Time is continuous and the planning horizon infinite. There is given a vector semimartingale $X = (x^1,\ldots,x^\Lambda)$, representing logarithms of asset returns or prices, which is a process with independent increments (PII) with respect to a given filtration defining the investor's information structure. The investor chooses a portfolio plan π, defined as a predictable vector process without oscillatory discontinuities, left continuous except perhaps at fixed discontinuities of X, specifying the proportions π_t^λ of capital assigned to the available assets. These proportions are in general constrained to be non–negative, but if the X–process is continuous there is a variant of the model in which short sales are allowed; these assumptions ensure that capital always remains positive. A portfolio plan π is called *sure* if it is non–random, *invariable* if in addition it is constant over time; sure portfolio plans are sometimes also called *portfolio functions.* Each portfolio plan π generates a portfolio log–returns or compound interest process x^π, and x^π is a PII if π is sure, a process with stationary independent increments (PSII) if π is invariable. The investor's preferences are characterised by a constant $b > 0$, called the coefficient of constant relative risk aversion (CRRA), and for $b \neq 1$ the objective is to choose (if possible) a sure π^* such that the function

$$\Psi(\pi,T) = (1-b)^{-1}\ell n\, Ee^{(1-b)x^\pi(T)} \qquad \ldots(1.1)$$

attains a (finite) maximum among all sure π at $\pi = \pi^*$, simultaneously for all $T \in \mathscr{T} = [0,\infty)$. Such a π^* is called an *optimal sure portfolio plan.* If $b = 1$, Ψ is replaced by

$$\Psi^\ell(\pi,T) = Ex^\pi(T), \qquad \ldots(1.2)$$

with $\Psi^\ell(\pi,T) = -\infty$ if $Ex^\pi(T)$ is undefined (i.e. has the form $\infty - \infty$). If X is a PSII, the objective stated above may be replaced by that of finding an optimal invariable portfolio plan, but we shall touch on this case only briefly.

The above formulation may seem arbitrary, as regards both the restriction to sure solutions and the definition of the objective. In fact, restricting the search for a maximum of (1) or (2) to sure portfolio plans involves no essential loss. Moreover, it follows from the results of Foldes (1991a) that, subject to minor conditions, an optimal sure portfolio plan as defined here exists iff an optimal portfolio–cum–saving (PS) plan exists in a (more or less) standard continuous–time, infinite–horizon neo–classical model where the X–process is a PII and the investor's utility function has the 'discounted CRRA' form – see (6) and (7) below. There is also a correspondence between conditions for optimality in the two problems. These results are summarised more fully below; they establish the economic respectability, if not necessarily the empirical realism, of the present formulation.

In the present paper, we shall be concerned mainly with the existence (or non-existence) and properties of an optimal sure portfolio plan π^*, including the derivation of programming conditions from which the vectors π_t^* could be calculated if the characteristics of the X–process were known. The purpose is partly to extend known results, particularly to cases where X has non–stationary increments and jumps (which may be neither bounded nor bounded away from zero), partly to explore and explain the use of the canonical representation of a semimartingale as a technique for analysing portfolio problems. There now follow some remarks on proof procedures, the type of results to be obtained, the economic relevance of the portfolio problem formulated here, the choice of definitions and the connection with related work – in particular with Foldes (1991a), to which this paper is a sequel.

The main mathematical argument may be summarised as follows. We assume $b \neq 1$ unless otherwise stated. Given a portfolio plan π, a formula expressing x^π in terms of the x^λ and π^λ has been derived in Foldes (1990), see also (2.5–6) below, and using the canonical representation of X this formula can be written as a sum of integrals with respect to the ('local') characteristics of X. If π is sure, the expression $Ee^{(1-b)x^\pi(T)}$ (if finite) is a value of the bilateral Laplace Transform of a PII, and the 'cumulant generating function' $\Psi(\pi,T)$ can be written as a sum of integrals, related to the Lévy–Khinchin formula for the characteristic function of a PII, involving π and the characteristics of X – see below, Theorem 5, also eqs.(3.4) and (7.7–10). One application of this formula is an alternative proof of a Certainty Equivalence (CE) Theorem appearing in Foldes (1991a); we return to this point below. If the characteristics of X satisfy some assumptions of non-degeneracy and smoothness (including the absence of fixed discontinuities) and some bounds, one can further write

$$\Psi(\pi,T) = \int_0^T \psi(\pi_t,t)dt, \qquad ...(1.3)$$

the function ψ being defined for pairs $(\check{\pi},t)$ where $\check{\pi}$ is an admissible portfolio vector. If the restriction $\pi \geq 0$ (no short sales) is in force, the domain of $\psi(\cdot,t)$ is a unit simplex

$$\mathscr{S} = \{\check{\pi}\in\Re^{\Lambda} : \check{\pi} \geq 0, \Sigma_{\lambda}\check{\pi}^{\lambda} = 1\}; \qquad ...(1.4)$$

if short sales are not restricted, the domain is $\Re^{\Lambda}$. It is then shown that a sure π^* is optimal sure iff for each t the vector π^*_t maximises $\psi(\cdot,t)$ on its domain. If $\pi \geq 0$ applies, the assumptions imply that $\psi(\cdot,t)$ is for each t a strictly concave function on $\mathscr{S}$ and that $\psi(\cdot,\cdot)$ is continuous. The existence of a maximum of $\psi(\cdot,t)$ on $\mathscr{S}$ at some unique $\pi^*(t)$ and the continuity of $\pi^*(\cdot)$ and $\psi(\pi^*(\cdot),\cdot)$ then follow directly; moreover the conditions characterising $\pi^*(t)$ are obtained by elementary concave programming – see esp. Theorems 4 and 6. This argument does not apply in the case of continuous X with short sales permitted, but existence and characterisation of an optimum are obtained very simply from conditions for a maximum of Ψ or of ψ if the covariance matrix of $X_T - X_S$ for $S < T$ is always positive definite – see Theorem 3.

If X has a set $\mathscr{J} = (\tau_m)$ of fixed discontinuities, (3) is replaced by

$$\Psi(\pi,T) = \int_0^T \psi(\pi_t,t)dt + \Sigma_{\tau_m \leq T}\, \psi_m(\pi_{\tau_m}), \qquad ...(1.5)$$

each $\psi_m(\cdot)$ being defined and (under assumptions) continuous and strictly concave on $\mathscr{S}$. Then a sure π^* is optimal sure iff π^*_t maximises $\psi(\cdot,t)$ for every $t \notin \mathscr{J}$ and maximises $\psi_m(\cdot)$ for every $t = \tau_m \in \mathscr{J}$. The results on existence and uniqueness and the programming conditions extend in a fairly straightforward way, but in general an optimal π^* will not be left continuous at points of $\mathscr{J}$.

It will be useful to review briefly the theory in Foldes (1991a) in order to provide some background for the present work. The welfare (or criterion) functional considered there has the form

$$(1-b)^{-1} \, E \int_0^\infty \bar{c}_t^{\,1-b} \cdot q_t \, dt \qquad \text{if } b \neq 1, \qquad \ldots(1.6)$$

$$E \int_0^\infty \ell n \, \bar{c}_t \cdot q_t \, dt \qquad \text{if } b = 1, \qquad \ldots(1.7)$$

where the discount density q is a positive function of finite variation on compacts of $\mathcal{T}$ which is continuous on the right with limits on the left (*corlol*). The process $\bar{c}$, representing consumption in 'natural' units, may be chosen from a class of positive, adapted, corlol processes satisfying the condition that the corresponding capital process $\bar{k}$ is positive. Given a portfolio plan π, the substitution $c_t = \bar{c}_t e^{-x^\pi(t)}$ defines a consumption process in 'π–standardised units', and it is found that the class of admissible c–processes is the same for each π, namely the class $\mathcal{C}$ of positive, adapted, corlol processes satisfying $\int_0^\infty c(t)dt \leq K_0$ a.s., where K_0 is the investor's initial capital. For details see [2] S.2, also Foldes (1991a), S.2, also Foldes (1990) S.2. A PS plan may then be defined as a pair (c,π), where c and π may be chosen separately, and the welfare functional may be written

$$\varphi(c,\pi) = (1-b)^{-1} \, E \int_0^\infty c_t^{1-b} \, e^{(1-b)x^\pi(t)} \, q_t \, dt \qquad \text{if } b \neq 1, \qquad \ldots(1.6a)$$

$$\varphi(c,\pi) = \int_0^\infty E(\ell n \, c_t) q_t \, dt + \int_0^\infty (E x_t^\pi) q_t \, dt \qquad \text{if } b = 1, \qquad \ldots(1.7a)$$

(assuming suitable conditions of integrability). The supremum of the welfare functional is denoted φ^*, also $\varphi^*(b)$ for $b \neq 1$, and a PS plan (c^*,π^*) is called *optimal* if $\varphi(c^*,\pi^*) = \varphi^*$ and φ^* is finite. Further, one can define the capital process in π–standardised units corresponding to c by $k_T = K_0 - \int_0^T c(t)dt$ and the consumption ratio process θ by $\theta_t = c_t/k_t = \bar{c}_t/\bar{k}_t$, so that a plan may also be specified as a pair (θ,π). A consumption plan is called *sure* if c, or equivalently θ, is non–random, and it is called *invariable* if in addition θ_t is constant on $\mathcal{T}$; then sure and invariable plans (c,π) are defined in the obvious way.

The following results concerning certainty equivalence were proved in Foldes (1991a), under conditions stated below, for the PS model with discounted CRRA utility and X a PII.

THEOREM 1: *First Certainty Equivalence Theorem.*

An optimal sure plan is optimal (i.e. a plan which is optimal in the class of all sure plans is optimal in the class of all plans).

THEOREM 2: *Second Certainty Equivalence Theorem.*

If an optimal plan exists, then a sure optimal plan exists (i.e. a plan exists which is both optimal and sure).

If X is a PSII, it is assumed that $q(t) \propto e^{-rt}$ for some r, and then 'sure' may be replaced by 'invariable' in Theorems 1 and 2.

The CE Theorems were obtained in Foldes (1991a) as corollaries of the Complete Class Theorem (CCT) for sure plans, which asserts that, for every plan with finite welfare, there is a sure plan whose welfare is at least as great (and finite). The CCT was proved for $b \neq 1$ under two conditions additional to those assumed throughout the model, namely that the probability space has some properties of a function space and, for $b < 1$, that $\varphi^*(\beta) < \infty$ for some $\beta \in (0,b)$; but it was shown that Theorem 2 is valid even without the latter assumption. For $b = 1$, the situation is analogous to that for $b < 1$ if short sales are permitted and X is continuous, while all results were proved without both additional conditions in the case $\pi \geq 0$.

It was further shown in Foldes (1991a) S.4, see also Foldes (1978), that for $b \neq 1$ the functional $\varphi(c,\pi)$ has for each *fixed sure* π a finite supremum on $\mathscr{C}$ iff the integral

$$\mathfrak{N}^{\pi} = \mathfrak{N}(\pi,q) = \int_0^{\infty} [Ee^{(1-b)x^{\pi}(t)} q_t]^{1/b} \, dt \qquad \text{...(1.8)}$$

converges, the value of the supremum being

$$(1-b)^{-1} K_0^{1-b}(\mathfrak{N}^{\pi})^b, \qquad \text{...(1.9)}$$

and then this value is attained by precisely one element of $\mathscr{C}$, namely the sure element c^{π} defined by

$$c_t^{\pi} = (K_0/\mathfrak{N}^{\pi})[Ee^{(1-b)x^{\pi}(t)} q_t]^{1/b}, \qquad t \in \mathscr{T} \qquad \text{...(1.10)}$$

Finding a sure PS plan (c^*,π^*) to maximise $\varphi(c,\pi)$ is therefore equivalent to finding a sure π^* to maximise $\varphi(c^{\pi},\pi)$, or equivalently to maximise $(1-b)^{-1}\mathfrak{N}(\pi,q)$. Now, a sure π can be chosen separately on each interval (S,T], and bearing in mind that the corresponding x^{π} will be a PII it is seen that if π^* maximises $(1-b)^{-1}\mathfrak{N}(\pi,q)$ among all sure π it also maximises

$$(1-b)^{-1}\ \ell n\ Ee^{(1-b)[x^{\pi}(T)-x^{\pi}(S)]} = \Psi(\pi,T)-\Psi(\pi,S) \qquad ...(1.11)$$

for each pair $S < T$, which in turn is equivalent to maximising $\Psi(\pi,T)$ for each T, cf. (1) above. Conversely, if a sure π^* exists such that $\Psi(\pi,T)$ attains a (finite) maximum among all sure π at $\pi = \pi^*$ for all T, then the (right continuous) function $Ee^{(1-b)x^*(T)}$ is finite for all T and one can find discount densities q which decrease fast enough far out so that $\mathfrak{N}(\pi^*,q)$ is finite; for each such q there is a c^* such that (c^*,π^*) is an optimal sure (and sure optimal) PS plan.

If $b = 1$, a slightly different procedure shows that, if (c^*,π^*) is an optimal sure PS plan, then for suitable q the function π^* maximises (1.2) among all sure π, for each T simultaneously, or equivalently maximises the increment

$$E[x^{\pi}(T)-x^{\pi}(S)] = \Psi^{\ell}(\pi,T)-\Psi^{\ell}(\pi,S) \qquad ...(1.12)$$

among all sure π for each pair $S < T$. Conversely, a sure π^* having this property defines an optimal sure (c^*,π^*) for suitable q – see Foldes (1991a) S.6.

The upshot of this discussion is that, by virtue of Theorem 1, conditions for the existence of an optimal PS plan, and procedures for constructing such a plan, can (for suitable q) be obtained by considering the corresponding questions for an optimal sure portfolio plan as defined by (1) or (2) above. The role of Theorem 2 is to guarantee that no cases of existence are omitted by reason only of the restriction to sure plans, and in particular to simplify the construction of examples of non–existence. The function–space properties used in proving the CCT are not restrictive in the cases which actually arise. As was mentioned above, it was asserted in Foldes (1991a) that Theorem 1 can be proved under weaker conditions than those assumed there. An alternative proof procedure was outlined, but part of the proof was deferred because it uses the representation formula for

Ψ or Ψ^{ℓ} to be derived here. The points remaining to be verified to complete the proof – see Foldes (1991a) eq.(4.12), also (6.12) – are these. Let π^* be an optimal sure portfolio plan (as defined here). If $b \neq 1$ and the restriction $\pi \geq 0$ applies, it has to be shown that for each asset λ the function

$$\ell n \, Ee^{x^{\lambda}(t)-bx^*(t)} - \ell n \, Ee^{(1-b)x^*(t)} \qquad ...(1.13)$$

is finite and non–decreasing on $\mathcal{T}$. If $b \neq 1$ and short sales are allowed (X continuous), it has to be shown that the functions are constant. For $b = 1$, (13) is replaced by

$$\ell n \, Ee^{x^{\lambda}(t)-x^*(t)} \, . \qquad ...(1.14)$$

These points will be considered below for the various cases which arise.

So far, the study of optimal sure portfolio plans has been motivated by pointing out its connection with the neo-classical PS problem. If one does not insist on a consumption–based theory, but merely requires a portfolio model with a reasonable criterion and tractable properties, one may consider in its own right the problem of choosing a portfolio plan which maximises (1) or (2) simultaneously for all $T \in \mathcal{T}$ on the set Π of *all* admissible portfolio plans. Let us call such an element an *optimal portfolio plan in the wide sense.* It can be shown, by a slight modification of the argument of Foldes (1991a), that under minor conditions the following analogues of the CE and CC Theorems hold:

(i). An optimal sure portfolio plan is optimal in the wide sense.

(ii). If a portfolio plan exists which is optimal in the wide sense, then an optimal sure portfolio plan exists.

(iii). For every portfolio plan π^0 for which $\Psi(\pi^0,T)$ is finite on $\mathcal{T}$, there exists a sure $\pi^{\diamond}$ satisfying $\Psi(\pi^0,T) \leq \Psi(\pi^{\diamond},T) < \infty$ on $\mathcal{T}$.

As above, assertions (i) and (ii) may be obtained as corollaries of (iii), which for $b \neq 1$ can be proved under two special conditions: first, that Ω has the appropriate function space properties; secondly, that for each $T \in \mathcal{T}$ there is some $\beta \in (0,b)$ for which $\sup \{Ee^{(1-\beta)x^{\pi}(T)}: \pi \in \Pi\} < \infty$. (We leave aside the case $b = 1$ for brevity). The second condition is not needed in order to show that a portfolio plan which is the *unique* optimum in the wide sense is sure. Moreover (i) can be proved, for all $b > 0$, independently of (iii) and without the special conditions. These results provide a simpler, but less deep, justification for restricting choice to sure portfolio plans; they will not be discussed further here.

In Foldes (1990) and the main discussion in Foldes (1991a), a portfolio plan π was defined simply as an adapted process which is left continuous with right limits (*collor*); as noted in the Postscript to the latter paper, such a definition gives rise to cases of non-existence of an optimum when X is a PII with fixed discontinuities. The present definition avoids this problem while leaving intact the results from both papers summarised above. It would be possible to work with a wider definition of a portfolio plan, say as a vector process π satisfying $\Sigma_\lambda \pi_t^\lambda = 1$ which is predictable and locally bounded, (non–negative if short sales are forbidden). When referrring to this concept, we speak of *predictable portfolio plans*, the corresponding sure plans being the (Lebesgue) *measurable portfolio functions*. The theory of conditions for optimality in Foldes (1990) goes through for the predictable portfolio plans, as does the alternative proof of Theorem 1; thus many of the results obtained below remain valid with minor changes, usually with simpler proofs. Nevertheless the narrower definition stated at the outset has been preferred here. One reason is that the proofs of the Complete Class Theorem and Theorem 2 given in Foldes (1991a) require a plan to be determined by its values on a countable dense set of $\mathcal{T}$, and so do not work for general predictable portfolio plans. Also, it is of interest to study conditions which allow an optimal solution to be found in a relatively small class of functions. The predictable definition is the more appropriate when the object is to obtain an existence theorem in a situation where X is specified only as a general semimartingale, as in Foldes (1991b).

Theoretical and applied works on optimal saving and on portfolio choice using one version or another of the PII/CRRA specification are legion and I shall not attempt to review them here. As regards *methods*, some novelty may be claimed for the present work. Leaving aside differences from related contributions which have been discussed in my earlier papers, the main technique introduced here, which to my knowledge has not previously been applied to portfolio problems, is the use of the canonical representation of semimartingales and in particular of integrals with respect to jump measures. This technique has been applied here only to the relatively simple case of PII, but should be useful in more general problems also. Brief explanations of relevant concepts and results are given as the discussion proceeds. As regards *results*, comparisons are complicated by the diversity of problems, models, assumptions, techniques and the form of presentation. The relatively straightforward discrete-time case may be left aside. A brief survey of relevant continuous-time literature appears in Foldes (1990), though this focusses on conditions for optimality in PS problems with general semimartingale investments and a general welfare functional. Most results for the PII/CRRA set–up are obtained as applications of more general theorems, and so are somewhat scattered and piecemeal. The models which have been considered most fully are those driven by Brownian motion; in particular, the programming conditions for an optimal portfolio in the CRRA case – see (3.12), (3.15–16) and (3.18–19) below – are well known. We refer to Pagès (1989) and Huang & Pagès (1990) for recent results on existence and characterisation of optimal plans in infinite–horizon PS models driven by Brownian motion. I am not aware of results for infinite–horizon models with CRRA utilities and a *general* PII market process which are readily comparable with those given in Sections 4–8 below; but the programming conditions are what one would expect to get from any maximum principle which happened to be applicable, at least if X has almost surely at most a finite set of discontinuities in each finite interval of time. The present proofs of both programming conditions and existence are remarkably elementary, and the discussion of existence (and non–existence) contains some points which, though quite simple, do not appear to be generally known.

The argument is arranged as follows. We begin in Section 2 by stating some definitions and recalling some formulae from Foldes (1990) and (1991a). Section 3 considers the case where X is continuous. The formula for $\Psi(\pi,T)$ is derived and used to complete the alternative proof of Theorem 1, and results on the existence and characterisation of optimal plans are obtained, distinguishing between the cases with and without short sales. The reason for discussing the continuous case separately is partly to emphasise the simplicity of the proofs, partly to dispose of the case where short sales are allowed. Also, a case–by–case approach shortens the formulae to be considered and involves only a little repetition, because the independence of the components of a PII allows results to be combined by multiplication or addition. Section 4 turns to the case where X has jumps but no fixed discontinuities, and derives a formula for $\Psi(\pi,T)$ as well as some bounds on the integrals appearing in this formula. Some additional assumptions are stated and their implications discussed. With these preliminaries, the rest of the proof of Theorem 1 as well as results on existence and characterisation of optimal sure portfolio plans are quickly obtained in Section 5. The case where X is a PSII is briefly considered in Section 6. The effect of fixed discontinuities is discussed in Section 7. Up to this point it is assumed that $b \neq 1$, and Section 8 reviews the modifications needed in the case of logarithmic utility. Section 9 concludes with a review of examples of non–existence found in the course of the discussion.

2. THE MODEL

As explained in Section 1, the portfolio model considered here is essentially the PS model defined in Foldes (1991a) S.2 with a modified criterion, and the latter is in turn a slightly modified version of the model in Foldes (1990). To keep the exposition more or less self–contained, we set out in this Section the main points needed here.

There is given a time domain $\mathcal{T} = [0,\infty)$, a complete probability space $(\Omega, \mathcal{A}, P)$ with a filtration $\mathfrak{A} = (\mathcal{A}_t;\, t \in \mathcal{T})$ satisfying the 'usual conditions' of right continuity and completeness, where $\mathcal{A} = \mathcal{A}_\infty$ while $\mathcal{A}_0 = \mathcal{A}_{0-}$ is generated by the P–null sets. In the product space $\Omega \times \mathcal{T}$ we define the σ–algebras $\mathcal{O}$ and $\mathcal{P}$ of optional and predictable sets. The following conventions apply to processes and functions unless we state or imply otherwise. Scalar processes take finite real values, while vector processes are $\mathfrak{R}^\Lambda$–valued functions of (ω,t) with a fixed integer $\Lambda \geq 1$. The Euclidian norm in both $\mathfrak{R}$ and $\mathfrak{R}^\Lambda$ is written $|\cdot|$. If processes ζ and ζ' are indistinguishable, we write $\zeta \equiv \zeta'$ and treat them as identical. For a scalar process, $\zeta > 0$ means $\zeta(\omega,t) > 0$ for all (ω,t), $\zeta \geq 0$ means $\zeta(\omega,t) \geq 0$ for all (ω,t), while similar notation for vector processes means that the condition applies to each component. The terms positive, negative, increasing, decreasing have their strict meaning. Processes are assumed, or may easily be shown to be, at least optionally measurable. For processes, *finite variation* means a.s. finite variation of the paths on compacts of $\mathcal{T}$. A process ζ is called *locally integrable* if there is a sequence of stopping times $T_n \uparrow \infty$ a.s. such that $E|\zeta(T_n)| < \infty$ for each n, similarly *integrable on compacts* if $E|\zeta(T)| < \infty$ for each $T \in \mathcal{T}$. Processes are defined only for $t \geq 0$ but are formally regarded as left continuous at $t = 0$, i.e. we set $\zeta(0-) = \zeta(0)$. Semimartingales and their components are by definition finite and corlol on $\mathcal{T}$. Thus for stochastic integrals we have $\int_{[0,T]} = \int_{(0,T]}$, which we write as $\int_0^T$; similarly for other time integrals. We write $\mathcal{E}\zeta$ for the martingale exponential of a semimartingale, see Doléans-Dade (1970), $\mathcal{L} = \mathcal{E}^{-1}$ for the inverse (mart–log) when this exists; in particular, if η is a semimartingale such that η_t and η_{t-} are always positive, we have

$$(\mathscr{L}\eta)_T = \int_0^T (1/\eta_{t-})d\eta_t \qquad \text{for } T \in \mathscr{T}, \text{ a.s.}$$

The concepts of semimartingale and process with independent increments (PII) are always defined relative to $\mathfrak{A}$. All PII considered here are assumed or may be shown to be semimartingales, and we usually say simply 'PII' rather than 'PII–semimartingale'. Definitions and properties of such processes will be broadly as in Jacod (1979) and Jacod & Shiryaev (1987); an excellent survey covering most of the points needed here appears in Shiryaev(1981). Further useful surveys dealing with PSII are Fristed (1974) and Taylor (1973).

As in Foldes (1990) and (1991a), a finite number of assets (or securities) indexed by $\lambda = 1,...,\Lambda$ are assumed to be available at all times. For each λ there is given a semimartingale x^λ with $x^\lambda(\omega,0) = 0$ called the *log–returns* or *compound interest process for* λ, and the formula $z^\lambda = \exp\{x^\lambda\}$ defines a positive semimartingale called the *returns* or *price process for* λ. The vector $X = (x^1,...,x^\Lambda)$ is called the *market log–returns process.* Decompositions of x^λ and X are written variously as

$$x^\lambda = (M^{\lambda c} + V^{\lambda c}) + (M^{\lambda d} + V^{\lambda d}) = x^{\lambda c} + x^{\lambda d},$$

$$X = (M^c + V^c) + (M^d + V^d) = X^c + X^d, \qquad ...(2.1)$$

where $M^{\lambda c}$ and $M^{\lambda d}$ are continuous and compensated–jump martingales respectively, while $V^{\lambda c}$ and $V^{\lambda d}$ are continuous and discontinuous processes of finite variation; all these processes vanish at $t=0$, and in general only $M^{\lambda c}$ is uniquely defined. When some components are missing we sometimes write just M^λ, V^λ, x^λ etc. for those which are present. In this paper, it is further assumed that X is a vector PII, i.e. for each T, the increments $X_t - X_T$ for $t \geq T$ are independent of $\mathscr{A}_T$. The set of fixed times of discontinuity of X is denoted $\mathscr{J}$.

A *portfolio plan* π or π*–plan* is defined as a vector process with components π^λ which is predictable, free of oscillatory discontinuities, left continuous for $t \notin \mathscr{J}$, and satisfies

$$\Sigma_\lambda \pi^\lambda(\omega,t) = 1 \qquad ...(2.2)$$

for all (ω,t). It is convenient to set $\pi(0) = \pi(0+)$; this convention, together with the assumption that X does not jump at zero, will ensure that all important properties assumed or proved below for $t > 0$ extend to $t \geq 0$ if appropriate one–sided limits are taken, and we shall omit further comments on this point. *If* X *has no fixed discontinuities, a* π*–plan is simply an adapted collor process satisfying* (2). We denote by Π^o the set of all portfolio plans and by Π^+ the subset satisfying $\pi \geq 0$, or explicitly

$$0 \leq \pi^\lambda(\omega,t) \leq 1 \qquad \text{...(2.3)}$$

for all (ω,t) and each λ. The set of all π which are *admissible* in a particular problem is denoted by Π. Special importance attaches to sure portfolio plans. We say that π is *sure* if it is indistinguishable from a non–random function on $\mathcal{T}$, and denote by Π^{os}, Π^{+s} the corresponding subsets of Π^o, Π^+, or simply Π^s if it does not matter which case is considered. We say that π is *invariable* if it is sure and there is a vector $\check{\pi}$ such that $\pi_t = \check{\pi}$ for all t.

Given a portfolio plan π, the *portfolio returns process* z^π is defined as the unique semimartingale satisfying the equation

$$z^\pi(T) = 1 + \Sigma_\lambda \int_0^T z^\pi(t-)\pi^\lambda(t)e^{-x^\lambda(t-)}de^{x^\lambda(t)},$$

and it may be checked that z^π_t and z^π_{t-} are positive for all t a.s. iff

$$z^\pi_t/z^\pi_{t-} = \Sigma_\lambda \pi^\lambda_t e^{\Delta x^\lambda(t)} > 0 \qquad \text{...(2.4)}$$

for all t a.s. This condition, which may be interpreted as a requirement of solvency for the investor, is satisfied for all $\pi \in \Pi^o$ if X is continuous, and for all $\pi \in \Pi^+$ (no short sales) with a general X. Accordingly it was assumed in Foldes (1990) and (1991a) that $\Pi = \Pi^+$ in general, while both cases $\Pi = \Pi^o$ and $\Pi = \Pi^+$ were considered for continuous X. We retain these assumptions, which also apply to the summary of previous work in Section 1 above, for the next few paragraphs, but in later Sections we shall be concerned only with sure π, so that Π^+ will be replaced by Π^{+s}, Π^o by Π^{os}.

Since (2.4) will be satisfied for all π considered here, we may write $z^\pi = e^{x^\pi}$ with $x^\pi(t)$ and $x^\pi(t-)$ defined and finite for all t a.s. The process x^π is called the *log–returns* or *compound interest process for* π. The change–of–variables formula yields

$$\int_0^T e^{-x^\lambda(t-)} de^{x^\lambda(t)} = x_T^\lambda + \tfrac{1}{2}\langle x^{\lambda c}\rangle_T + \Sigma_{t\leq T}[e^{\Delta x^\lambda(t)} - 1 - \Delta x_t^\lambda],$$

the sum on the right converging absolutely for all T a.s., and using this formula we may calculate $x^\pi(T)$ explicitly as

$$\begin{aligned} x^\pi(\omega,T) &= x_T^\pi = x_T^{\pi c} + x_T^{\pi d} \\ &= \int \Sigma_\lambda \pi^\lambda \, dM^{\lambda c} \\ &+ \int \Sigma_\lambda \pi^\lambda \, dV^{\lambda c} + \tfrac{1}{2}\int \Sigma_\lambda \pi^\lambda \, d\langle M^{\lambda c}\rangle - \tfrac{1}{2}\int \Sigma_\lambda \Sigma_\ell \pi^\lambda \pi^\ell \, d\langle M^{\lambda c}, M^{\ell c}\rangle \\ &+ \int \Sigma_\lambda \pi^\lambda \, dM^{\lambda d} \\ &+ \Sigma_{t\leq T}[\Delta x^\pi - \Sigma_\lambda \pi^\lambda \, \Delta M^\lambda]; \end{aligned} \qquad ...(2.5)$$

here $\int = \int_0^T$, all variables and angle brackets on the right of the equation should have the subscript t, and

$$\Delta x_t^\pi = \ell n[\Sigma_\lambda \pi_t^\lambda \exp(\Delta x_t^\lambda)] \qquad ...(2.6)$$

– see Foldes (1990) eqs.(2.1–17) for details. The sum over t in the last line of (5) converges absolutely for all T a.s. The symbols $x_T^{\pi c}$, $x_T^{\pi d}$ are just shorthand for the first two lines and the last two lines on the right of (5) respectively, and of course $x^{\pi d} \equiv 0$ when X is continuous. The first and third lines represent local martingales, the second and fourth processes of finite variation, so that (5) gives a decomposition of x^π analogous to that of x^λ in (1). In particular, if the 'single–asset portfolio λ' is defined by

$$\pi^\lambda \equiv 1, \qquad \pi^\ell \equiv 0 \quad \text{for } \ell \neq \lambda, \qquad ...(2.7)$$

the four lines on the right of (5) may be identified with the corresponding terms in (1); thus we may without ambiguity write x^λ instead of x^π, with similar notation for other functions to be introduced later.

For later reference we note that, if X has bounded jumps, the solvency condition (2.4) can be satisfied, and the process x^π defined, even if limited short sales are allowed. More precisely, for an arbitrary vector $\breve{\pi} \in \mathfrak{R}^\Lambda$ with $\Sigma_\lambda \breve{\pi}^\lambda = 1$, let $\breve{\pi}^-$ denote the sum of its negative co–ordinates, and let

$$\Pi^\rho = \{\pi \in \Pi^0 : \pi^-(\omega,t) > -\rho \text{ all } (\omega,t)\}. \qquad ...(2.8)$$

Then, if (say)

$$|\Delta x^\lambda(\omega,t)| \leq 1 \quad \text{all } (\omega,t) \text{ and } \lambda, \qquad ...(2.9)$$

one can find $\rho > 0$ so small that (4) is satisfied for all $\pi \in \Pi^{\rho}$. Indeed, if ξ is an arbitrary vector in $\mathfrak{R}^{\Lambda}$ with $|\xi^{\lambda}| \leq 1$ for each λ, we have

$$\Sigma_{\lambda} \breve{\pi}^{\lambda} e^{\xi^{\lambda}} \geq \breve{\pi}^{-} \cdot e^{\max_{\lambda} \xi^{\lambda}} + (1-\breve{\pi}^{-}) \cdot e^{\min_{\lambda} \xi^{\lambda}} \geq \breve{\pi}^{-} \cdot e + (1-\breve{\pi}^{-}) \cdot e^{-1}, \qquad \text{...(2.10)}$$

which is positive if $\breve{\pi}^{-} > 1/(1-e^{2})$, so that it is enough to take

$$\rho \leq 1/(e^{2}-1). \qquad \text{...(2.11)}$$

It can be checked that, if π is sure, x^{π} is a PII, and if π is invariable then x^{π} is a PSII. Note also that a sure π is bounded on compacts of $\mathcal{T}$ in all cases considered here: by the collor property if $\pi \in \Pi^{os}$ and X is continuous, by (3) if $\pi \in \Pi^{+s}$, (and by definition in the case of measurable portfolio functions). *Since from now on we shall consider only sure* π, we shall often say 'all π', 'optimal π^*' etc. instead of 'all sure π', 'optimal sure π^*' etc.

The definitions of the investor's objective and of the concept of *optimal sure portfolio plan* have been given in the first paragraph of Section 1 – see (1.1) and (1.2), also (1.11) and (1.12) for an alternative formulation – and need not be repeated, but for precision some remarks should be added. Optimality is, of course, always defined relative to some admissible set Π. Also, we speak of an optimal π^* only when the maximum in question is defined and finite. Specifically, if $b \neq 1$, then $Ee^{(1-b)x^{\pi}(T)}$ is defined and positive for all (π,T) since $x^{\pi}(T)$ is finite, so that $(1-b)\Psi(\pi,T) > -\infty$ always. Thus for $b < 1$ an optimal π^* cannot exist unless $\Psi(\pi,T)$ is finite for all (π,T). On the other hand, for $b > 1$ an optimal π^* can exist even if $\Psi(\pi,T) = -\infty$ for some (π,T). If X is continuous, Ψ is in any case always finite – cf. (3.4) below. For brevity we often write

$$\eta^{\pi}(t) = e^{(1-b)x^{\pi}(t)}, \qquad \text{...(2.12)}$$

similarly $\eta^{\pi c}$, $\eta^{\pi d}$ when x^{π} is replaced by $x^{\pi c}$, $x^{\pi d}$ etc.

3. CONTINUOUS X

In this Section, X is a continuous vector PII, so that (2.1) reduces to $X = X^c = M^c + V^c = M + V$. The situation usually considered in portfolio problems with a continuous PII is that where V^c is a linear drift and M^c is a Brownian motion with covariance, but we may as well begin with the general case. Thus M^c is a continuous Gaussian martingale with angle bracket $\langle M^c\rangle$ where $\langle M^c\rangle_t$ is the matrix with elements $\langle M^{\lambda c},M^{\ell c}\rangle_t$, the vector function V^c and the matrix function $\langle M^c\rangle$ being *deterministic* continuous functions of finite variation on compacts, and for each $S < T$ the 'covariance matrix' $\langle M^c\rangle_T - \langle M^c\rangle_S$ is symmetric, non–negative definite – see Jacod (1979) 4.9–10, 4.15, 5.10, also Jacod & Shiryaev (1987) I.4.9–10, II.4–5. To shorten the notation, we sometimes drop the superscript c .

We first complete the alternative proof of Theorem 1 for continuous X. The first step is to calculate $\Psi(\pi,T)$ – which we write here as $\Psi^c(\pi,T)$ – for sure π, where Π may for the moment be Π^{os} or Π^{+s}. We refer to (2.5), write out the formula for $\eta^\pi = e^{(1-b)x^\pi}$ with $x^\pi = x^{\pi c}$ and note that π is bounded on compacts of $\mathscr{T}$, so that

$$\mathscr{E}\{(1-b)\int \Sigma_\lambda \pi^\lambda dM^\lambda\}_T =$$
$$\exp\Big\{(1-b)\int_0^T \Sigma_\lambda \pi^\lambda dM^\lambda - \tfrac{1}{2}(1-b)^2\int_0^T \Sigma_\lambda\Sigma_\ell \pi^\lambda\pi^\ell d\langle M^\lambda,M^\ell\rangle\Big\} \qquad ...(3.1)$$

is a (true) martingale and its expectation is 1. Now $\exp\{(1-b)x_T^\pi\}$ may be written (on adding and subtracting terms) as the product of this martingale and the expression

$$\exp\Big\{(1-b)\int_0^T \Sigma_\lambda \pi^\lambda dV^\lambda + \tfrac{1}{2}(1-b)\int_0^T \Sigma_\lambda \pi^\lambda d\langle M^\lambda\rangle - \tfrac{1}{2}b(1-b)\int_0^T \Sigma_\lambda\Sigma_\ell \pi^\lambda\pi^\ell d\langle M^\lambda,M^\ell\rangle\Big\}. \quad ...(3.2)$$

The expression (2) is deterministic since π is sure; denoting it by ν_T^π, we obtain

$$E\eta_T^{\pi c} = E\exp\{(1-b)x_T^\pi\} = \nu_T^\pi, \qquad ...(3.3)$$

or, taking logarithms, dividing by (1–b) and referring to (1.1),

$$\Psi^c(\pi,T) = \int_0^T \Sigma_\lambda \pi_t^\lambda\Big[dV_t^{\lambda c} + \tfrac{1}{2}d\langle M^{\lambda c}\rangle_t - \tfrac{1}{2}b\Sigma_\ell \pi_t^\ell d\langle M^{\lambda c},M^{\ell c}\rangle_t\Big]. \qquad ...(3.4)$$

If π^* and π are two elements of Π^s, an analogous procedure, using the martingale $\mathscr{E}\{\int \Sigma(\pi^\lambda - b\pi^{*\lambda})dM^\lambda\}$, yields

$$\ell n\, E \exp\{x_T^\pi - b x_T^*\}$$
$$= \int_0^T \Sigma_\lambda(\pi^\lambda - b\pi^{*\lambda})\Big[dV^{\lambda c} + \tfrac{1}{2}d\langle M^{\lambda c}\rangle - \tfrac{1}{2}b\Sigma_\ell \pi^{*\ell} d\langle M^{\lambda c}, M^{\ell c}\rangle\Big]$$
$$- \tfrac{1}{2}b\int_0^T \Sigma_\lambda \Sigma_\ell (\pi^\lambda - \pi^{*\lambda})\pi^{*\ell} d\langle M^{\lambda c}, M^{\ell c}\rangle, \qquad \text{...(3.5)}$$

where $x^* = x^{\pi^*}$, hence

$$\ell n\, E \exp\{x_T^\pi - b x_T^*\} - \ell n\, E \exp\{(1-b)x_T^*\}$$
$$= \int_0^T \Sigma_\lambda(\pi^\lambda - \pi^{*\lambda})\Big[dV^{\lambda c} + \tfrac{1}{2}d\langle M^{\lambda c}\rangle - b\Sigma_\ell \pi^{*\ell} d\langle M^{\lambda c}, M^{\ell c}\rangle\Big]. \qquad \text{...(3.6)}$$

Note that, with X continuous and π, π^* sure and bounded on compacts, the expressions (1–6) are always finite for $T < \infty$.

Now let π^* be optimal sure, so that π^* maximises Ψ^c for all T among all $\pi \in \Pi$. For any $\pi = \pi^* + \delta\pi \in \Pi$, the functions $\pi^* + \alpha\delta\pi$ with $0 < \alpha \leq 1$ are in Π, so that by optimality and (4) we obtain

$$0 \geq (1/\alpha)[\Psi^c(\pi^* + \alpha\delta\pi, T) - \Psi^c(\pi^*, T)]$$
$$= \int_0^T \Sigma_\lambda \delta\pi^\lambda \Big[dV^\lambda + \tfrac{1}{2}d\langle M^\lambda\rangle - b\Sigma_\ell \pi^{*\ell} d\langle M^\lambda, M^\ell\rangle\Big] - \tfrac{1}{2}\alpha b \int_0^T \Sigma_\lambda \Sigma_\ell \delta\pi^\lambda \delta\pi^\ell d\langle M^\lambda, M^\ell\rangle. \qquad \text{...(3.7)}$$

The last term vanishes as $\alpha \downarrow 0$, yielding the result that the expression (6) is non–positive for arbitrary sure π. Given any interval [S,T), one can set π equal to the 'single–asset portfolio λ' on that interval and $\pi = \pi^*$ elsewhere, implying that the function

$$\ell n\, E \exp\{x_T^\lambda - b x_T^*\} - \ell n\, E \exp\{(1-b)x_T^*\} \qquad \text{...(3.8)}$$

is non–increasing on $\mathscr{T}$ – see (1.13). In case $\Pi = \Pi^{OS}$, an arbitrary $\delta\pi$ can be replaced by $-\delta\pi$, and then the function (8) is identically zero on $\mathscr{T}$. Taking into account the discussion of Section 1, this proves Theorem 1 for both cases with X continuous.

Regarding *uniqueness* of an optimal sure plan, this is assured if for each pair $S < T$ the matrix $\langle M^c\rangle_T - \langle M^c\rangle_S$ is *positive* definite. To see this, suppose that π^* and π are optimal sure and note that (6), with $\int_0^T$ replaced by $\int_S^T$, is non–positive. Interchanging π^* and π, adding the resulting inequalities and rearranging one obtains

$$0 \geq b \int_S^T \Sigma_\lambda \Sigma_\ell (\pi^{*\lambda} - \pi^\lambda)(\pi^{*\ell} - \pi^\ell) d\langle M^\lambda, M^\ell\rangle, \qquad \text{...(3.9)}$$

contrary to positive definiteness unless π^* and π coincide on the interval.

Now let $\Pi = \Pi^{OS}$ and consider in more detail the *conditions characterising an optimal sure* π^*. On writing out the condition that (8) vanishes identically on $\mathscr{T}$ one has

$$V_T^\lambda + \tfrac{1}{2}\langle M^\lambda\rangle_T - b\int_0^T \Sigma_\ell \pi^{*\ell} d\langle M^\lambda, M^\ell\rangle$$
$$= \int_0^T \Sigma_L \pi^{*L}\left[dV^L + \tfrac{1}{2}d\langle M^L\rangle - b\Sigma_\ell \pi^{*\ell} d\langle M^L, M^\ell\rangle\right] \qquad \text{...(3.10)}$$

for all T, $\lambda = 1,\dots,\Lambda$. Assuming now that V^c and $\langle M^c\rangle$ are absolutely continuous and introducing the (a.e. defined, measurable) derivatives

$$dV_t^{\lambda c}/dt = v_t^\lambda, \quad d\langle M^{\lambda c}, M^{\ell c}\rangle_t/dt = \sigma_t^{\lambda\ell}, \qquad \text{...(3.11)}$$

(10) may be rewritten in the familiar differential form as

$$v_t^\lambda + \tfrac{1}{2}\sigma_t^{\lambda\lambda} - b\Sigma_\ell \pi_t^{*\ell}\sigma_t^{\lambda\ell} = \Sigma_L \pi_t^{*L}\left[v_t^L + \tfrac{1}{2}\sigma_t^{LL} - b\Sigma_\ell \pi_t^{*\ell}\sigma_t^{L\ell}\right] \qquad \text{...(3.12)}$$

for almost all t, $\lambda = 1,\dots,\Lambda$. If these equations are to be satisfied by a collor vector function π^*, then (apart from special cases) each of the functions v^λ and $\sigma^{\lambda\ell}$ must be assumed to be collor. Unless otherwise stated, we shall for simplicity assume from now on that V^c and $\langle M^c\rangle$ actually have *continuous* derivatives (11) on $\mathscr{T}$, and further that for each t the matrix $[\sigma_t^{\lambda\ell}]$ is *positive* definite. (At $t = 0$, we take the right derivative and set the left derivative equal to it). Then the *necessary* conditions for a collor vector function π^* to be optimal in Π^{OS} are that for each t the vector π_t^* satisfies the equations (12) for each λ, as well as the constraint $\Sigma_\lambda \pi_t^{*\lambda} = 1$. A function π^* satisfying these conditions is continuous. Of course, the concavity of Ψ^c ensures that the stated conditions are also *sufficient* for optimality. To see this formally, evaluate $\Psi^c(\pi,T) - \Psi^c(\pi^*,T)$ from (4) to obtain an expression like that in the second line of (7) with $\alpha = 1$; the first term in this expression vanishes by (12) since $\Sigma\, \delta\pi_t^\lambda = 0$ while the second term is non–positive definite, so $\Psi^c(\pi,T) - \Psi^c(\pi^*,T) \le 0$ as required.

By way of transition to the question of *existence*, we note that under present assumptions the equations (12) characterising π_t^* could also be obtained as follows. Define a function of the vector $\breve{\pi} \in \Re^\Lambda$ and of $t \in \mathscr{T}$ by

$$\psi^c(\breve{\pi},t) = \Sigma_\lambda \breve{\pi}^\lambda\left[v_t^\lambda + \tfrac{1}{2}\sigma_t^{\lambda\lambda} - \tfrac{1}{2}b\Sigma_\ell \breve{\pi}^\ell \sigma_t^{\lambda\ell}\right], \qquad \text{...(3.13)}$$

and note that the function Ψ^c given by (4) has for each $\pi \in \Pi^{OS}$ a time derivative

$$\psi^c(\pi_t,t) = (\partial/\partial t)\Psi^c(\pi,t). \qquad \text{...(3.14)}$$

Consider, for each t separately, the problem of choosing a vector $\breve{\pi}$ to maximise $\psi^c(\cdot,t)$ subject to the constraint

$$\Sigma_\lambda \breve{\pi}^\lambda = 1. \qquad ...(3.15)$$

Introducing a multiplier β_t for the constraint, one obtains *necessary* conditions for a maximum at a point $\breve{\pi} = \pi_t^*$ in the form

$$v_t^\lambda + \tfrac{1}{2}\sigma_t^{\lambda\lambda} - b\Sigma_\ell \breve{\pi}^\ell \sigma_t^{\lambda\ell} = \beta_t, \qquad \lambda = 1,...,\Lambda, \qquad ...(3.16)$$

and on multiplying by $\breve{\pi}^\ell$ and adding up the resulting equations one sees that β_t agrees with the right–hand side of (12). The conditions that (12) holds for a given vector $\breve{\pi} = \pi_t^*$ satisfying (15), or equivalently that (15) and (16) hold for some $\breve{\pi} = \pi_t^*$ and some number β_t, are clearly also *sufficient* for a constrained maximum of $\psi^c(\cdot,t)$ at π_t^*. Now write the system (15–16) as a matrix equation of the form

$$\hat{S}_t \hat{\pi}_t = \hat{v}_t \qquad ...(3.17)$$

where (dropping the subscript t)

$$\hat{S} = \begin{pmatrix} b\sigma^{11} & . & . & b\sigma^{1\Lambda} & 1 \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ b\sigma^{\Lambda 1} & . & . & b\sigma^{\Lambda\Lambda} & 1 \\ 1 & . & . & 1 & 0 \end{pmatrix}, \quad \hat{\pi} = \begin{pmatrix} \breve{\pi}^1 \\ \cdot \\ \cdot \\ \breve{\pi}^\Lambda \\ \beta \end{pmatrix}, \quad \hat{v} = \begin{pmatrix} v^1 + \tfrac{1}{2}\sigma^{11} \\ \cdot \\ \cdot \\ v^\Lambda + \tfrac{1}{2}\sigma^{\Lambda\Lambda} \\ 1 \end{pmatrix}. \qquad ...(3.17a)$$

The equation (17) has a unique solution $\hat{\pi}_t^* = (\pi_t^*, \beta_t)$ if the bordered matrix $\hat{S}_t$ is non–singular, which will be the case if the covariance matrix $S_t = [\sigma_t^{\lambda\ell}]$ is positive definite. (To check this, note that the latter condition implies that the quadratic form $h'(bS_t)h$, $h \in \Re^\Lambda$, is positive definite, hence is also positive definite subject to $\Sigma_\lambda h^\lambda = 0$; but this implies that $\det(\hat{S}) < 0$, see for instance Samuelson (1961) p.378.) It is also evident that the solution $\hat{\pi}_t^*$ will be a continuous function of t if the v^λ and $\sigma^{\lambda\ell}$ are all continuous, as will the functions $\pi^*: t \mapsto \pi_t^*$ and $t \mapsto \psi^c(\pi_t^*,t)$. Thus π^* is in Π^{os}, and since π_t^* maximises $\psi^c(\breve{\pi},t)$ subject to (15) for each t, it follows that π^* maximises $\Psi^c(\pi,T)$ on Π^{os} for each T and so is optimal sure. This completes the existence proof for the case $\Pi = \Pi^{os}$. To sum up so far, we have

THEOREM 3: *Optimality Theorem for* X *Continuous,* $\Pi = \Pi^{os}$.

If continuous derivatives v_t^λ and $\sigma_t^{\lambda\ell}$ exist on $\mathscr{T}$ and the matrix $[\sigma_t^{\lambda\ell}]$ is positive definite for each t, there exists a unique optimal sure portfolio plan π^*. For each t, the vector π_t^* is characterised by the equations (3.16) – or (3.12) – and (3.15). The functions $t \mapsto \pi_t^*$ and $t \mapsto \psi^c(\pi_t^*,t)$ are continuous.

We turn to the case $\Pi = \Pi^{+s}$, assuming again that continuous derivatives v^λ and $\sigma^{\lambda\ell}$ exist and that the matrices $[\sigma_t^{\lambda\ell}]$ are positive definite. A modified version of the approach based on maximising, for each t separately, the function $\psi^c(\check{\pi},t)$ defined in (13) yields a very simple treatment of the *existence and properties of an optimal sure* π. Consider choosing a vector in the simplex $\mathscr{S}$ – see (1.4) – to maximise $\psi^c(\cdot,t)$. This function is continuous, and it is strictly concave on $\mathscr{S}$ since the form $\Sigma\Sigma\check{\pi}^\lambda\check{\pi}^\ell\sigma_t^{\lambda\ell}$ is positive definite; thus it attains a maximum at some unique point π_t^*. Conditions which are necessary and sufficient for a maximum are easily obtained, for example with the aid of Lagrange multipliers for the constraints $\check{\pi}^\lambda \geq 0$ and $\Sigma\,\check{\pi}^\lambda = 1$, and take the form

$$\pi_t^{*\lambda} \geq 0 \text{ each } \lambda, \quad \Sigma_\lambda \pi_t^{*\lambda} = 1; \qquad \gamma_t^{\lambda c} - \gamma_t^{*c} \leq 0, \quad \pi_t^{*\lambda}(\gamma_t^{\lambda c} - \gamma_t^{*c}) = 0, \text{ each } \lambda; \qquad ...(3.18)$$

where

$$\gamma_t^{\lambda c} = v_t^\lambda + \tfrac{1}{2}\sigma_t^{\lambda\lambda} - b\Sigma_\ell \pi_t^{*\ell}\sigma_t^{\lambda\ell}, \qquad \gamma_t^{*c} = \Sigma_\lambda \check{\pi}_t^{*\lambda}\gamma_t^{\lambda c}. \qquad ...(3.19)$$

If t is now varied, it follows directly from the conditions of the problem, or from any one of a number of 'maximum' theorems – see Bank (1983) esp. T.4.3.3 – that continuity of the coefficients v^λ and $\sigma^{\lambda\ell}$ with respect to t and uniqueness of the solution π_t^* for each t implies continuity of the functions $\pi^*: t \mapsto \pi_t^*$ and $t \mapsto \psi^c(\pi_t^*,t)$, (and if multipliers are used, these are continuous also). Thus $\pi^* \in \Pi^{+s}$, and since π_t^* maximises $\psi(\cdot,t)$ for each t it is obvious that π^* is optimal. In case X is a continuous PSII, i.e. a Brownian motion with drift and covariance, the solution π_t^* is of course constant on $\mathscr{T}$. In the next two Sections we shall review the preceding arguments – for the case $\Pi = \Pi^{+s}$ only – and consider in what respects the presence of jumps creates complications. Here we state

THEOREM 4: *Optimality Theorem for* X *Continuous,* $\Pi = \Pi^{+s}$.

If continuous derivatives v_t^λ and $\sigma_t^{\lambda\ell}$ exist on $\mathcal{T}$ and the matrix $[\sigma_t^{\lambda\ell}]$ is positive definite for each t, there exists a unique optimal sure non–negative portfolio plan π^*. For each t, the vector π_t^* is characterised by the conditions (3.18–19). The functions $t \mapsto \pi_t^*$ and $t \mapsto \psi^c(\pi_t^*, t)$ are continuous.

REMARK. In Theorems 3 and 4, 'continuous' may be replaced by 'collor'. If measurable (instead of only collor) portfolio functions are admitted, then – subject to reservations about null sets – analogous results hold with 'continuous' replaced by 'measurable and bounded on compacts' or, in the case of Theorem 4, simply by 'measurable'.

4. MOVING DISCONTINUITIES

Suppose that X with $X(0) = 0$ is a general PII–semimartingale *without fixed times of discontinuity.* We may suppose that X has a 'canonical' representation of the form

$$x_T^\lambda = M_T^{\lambda c} + V_T^{\lambda c} + \int_0^T \int_{|\xi|\le 1} \xi^\lambda \cdot (\mu - F)(d\xi,dt) + \int_0^T \int_{|\xi|>1} \xi^\lambda \cdot \mu(d\xi,dt), \qquad \ldots(4.1)$$

$\lambda = 1,..,\Lambda$, where for the moment $M^{\lambda c}$ and $V^{\lambda c}$ just have the properties specified in the first paragraph of Section 3; the two remaining terms correspond to $M^{\lambda d}$ and $V^{\lambda d}$ in (2.1). Before developing our theory further, we set out some facts about the measures μ and F and the integrals which they define; full details may be found in Jacod (1979), Jacod & Shiryaev (1987), Shiryaev (1981).

We introduce an auxiliary 'space of jumps' Ξ which is a copy of $\mathfrak{R}^\Lambda$, with vectors $\xi = (\xi^1,\ldots,\xi^\Lambda)$ and Euclidian norm $|\xi|$. Denoting by $\mathscr{B}^\Lambda$ the Borel sets in Ξ, we define optional and predictable σ–algebras in $(\Omega\times\mathscr{T})\times\Xi$ by $\mathfrak{O} = \mathscr{O}\times\mathscr{B}^\Lambda$, $\mathfrak{P} = \mathscr{P}\times\mathscr{B}^\Lambda$. Next, $\mu = \mu(\omega;d\xi,dt)$ is a random measure, i.e. for each ω there is a σ–finite measure on the Borel sets of $\Xi\times\mathscr{T}$. Specifically, μ is the (Poisson) measure of jumps associated with the PII X and is defined by

$$\mu(d\xi,dt) = \Sigma_{s>0}\, I_{\{\Delta X_s \neq 0\}}\delta_{(s,\Delta X_s)}(d\xi,dt),$$

i.e. if $\Delta X(\omega,s) = \xi \neq 0$, then μ places a unit mass (δ–function) at the point (ω,ξ,s). If $t\in\mathscr{T}$ and (A_n) is a sequence of pairwise disjoint Borel sets of $\Xi\times(t,\infty)$ with $E\mu(A_n) < \infty$, the random variables $\mu(A_n)$ are independent of one another and of $\mathscr{A}_t$. The deterministic measure $F(d\xi,dt)$ is the compensating or intensity (Lévy) measure for μ, defined for Borel sets A of $\Xi\times\mathscr{T}$ by $F(A) = E\mu(A)$. It is non–negative, σ–finite and atomless with $F(\{0\}\times\mathscr{T}) = 0$, $F(\Xi\times\{0\}) = 0$, and satisfies

$$\int_0^T \int_\Xi (1\wedge|\xi|^2)F(d\xi,dt) < \infty \qquad \text{for each } T\in\mathscr{T}. \qquad \ldots(4.2)$$

F may be factorised (not uniquely) in the form

$$F(d\xi,dt) = f(d\xi,t)dG(t) \qquad \ldots(4.3)$$

where G with $G(0) = 0$ is non–decreasing and finite on $\mathscr{T}$, also continuous because X has no fixed discontinuities, and f is a transition measure, i.e. $f(\cdot,t)$ is a Borel measure on Ξ for each t while $f(A,\cdot)$ is a measurable function of t for each Borel set $A \subset \Xi\backslash\{0\}$.

Let $W = W(\omega,\xi,t)$ denote a $\mathfrak{P}$–measurable function. For real–valued $W \geq 0$, the integral

$$\int_0^T \int_\Xi W \cdot d\mu = \int_0^T \int_\Xi W(\omega,\xi,t)\mu(\omega;d\xi,dt) \qquad \text{...(4.4)}$$

is defined pathwise for $T \leq \infty$ in the usual way, and the definition extends to W of indefinite sign and T such that $\int_0^T \int_\Xi |W| d\mu < \infty$. This integral represents an optional process in the usual sense. For vector–valued W, $\int_0^T \int_\Xi W \cdot d\mu$ denotes the vector process with co–ordinates $\int_0^T \int_\Xi W^\lambda \cdot d\mu$. Similar remarks apply to $\int_0^T \int_\Xi W \cdot dF$, except that this integral represents a predictable process. For $W \geq 0$, the definition of F as the compensating measure for μ leads to

$$E\int_0^T \int_\Xi W \cdot d\mu = E\int_0^T \int_\Xi W \cdot dF \qquad \text{for } T \leq \infty. \qquad \text{...(4.5)}$$

In particular, if W is the indicator of a set $\{|\xi|>1\}\times[0,T]$ with $T \in \mathscr{T}$, (2) implies

$$E\,\mu\{\{|\xi|>1\}\times[0,T]\} = F\{\{|\xi|>1\}\times[0,T]\} < \infty. \qquad \text{...(4.6)}$$

As a further example, if we set $W = \xi^\lambda I_{\{|\xi|>1\}}$, the integral on the left of (4) agrees with the last term in (1); it is, for each ω, a representation of the sum of jumps Δx_t^λ occurring not later than T, restricted to those vectors ΔX_t for which $|\Delta X_t|^2 = \Sigma_\lambda (\Delta x_t^\lambda)^2 > 1$. The number of such jumps is a.s. finite, so that the sum converges absolutely; thus the integral is a right continuous process of finite variation, and it is consistent with our general definitions to write

$$V_T^{\lambda d} = V^{\lambda d}(\omega,T) = \int_0^T \int_{|\xi|>1} \xi^\lambda \mu(d\xi,dt). \qquad \text{...(4.7)}$$

In general, $V^{\lambda d}$ need not be locally integrable.

The situation is different if $\{|\xi|>1\}$ is replaced in the preceding examples by $\{|\xi|\leq 1\}$, because the X–process may have a countable infinity of jumps in every neighbourhood of the zero level during a finite interval $[0,T]$. Thus, in general, the μ and F measures of a set $\{|\xi|\leq 1\}\times[0,T]$ will be only σ–finite, and the integral analogous to (7) may be undefined, i.e. have the form $\infty - \infty$. A more general definition of integrals representing sums of jumps is therefore needed.

Returning to the case of a general real or vector–valued $W \in \mathfrak{P}$ and defining $\int_0^T \int_\Xi |W| d\mu$ and $\int_0^T \int_\Xi |W| dF$ as above, it can be shown that local integrability of these two processes are equivalent conditions. If one of these holds, we may define a new process $\mathcal{L}_T = \int_0^T \int_\Xi W \cdot d(\mu - F)$ by

$$\int_0^T \int_\Xi W \cdot d(\mu - F) = \int_0^T \int_\Xi W \cdot d\mu - \int_0^T \int_\Xi W \cdot dF, \qquad \text{...(4.8)}$$

and this process is a local martingale. If $\int_0^T \int_\Xi |W| d\mu$, or equivalently $\int_0^T \int_\Xi |W| dF$, is integrable on compacts, then (8) is a (true) martingale. More generally, if

$$\int_0^T \int_\Xi \left[|W|^2 \cdot I_{\{|W| \le 1\}} + |W| \cdot I_{\{|W| > 1\}} \right] dF \qquad \text{...(4.9)}$$

is locally integrable,the expression

$$\mathcal{L}(\omega, T) = \int_0^T \int_\Xi W(\omega, \xi, t) \cdot (\mu - F)(\omega; d\xi, dt) \qquad \text{...(4.10)}$$

may be defined as the value at (ω, T) of a compensated–jump local martingale $\mathcal{L}$ with $\mathcal{L}(0) = 0$ such that

$$\Delta \mathcal{L}(\omega, t) = \int_\Xi W(\omega, \xi, t) \mu(\omega; d\xi \times \{t\}). \qquad \text{...(4.11)}$$

If now we set $W(\omega, \xi, t) = \xi^\lambda I_{\{|\xi| \le 1\}}$, it follows from (2) that the (deterministic) integral $\int_0^T \int_{|\xi| \le 1} (\xi^\lambda)^2 \cdot dF$ is defined and finite, so that the criterion (9) is satisfied and the process

$$M_T^{\lambda d} = M^{\lambda d}(\omega, T) = \int_0^T \int_{|\xi| \le 1} \xi^\lambda \cdot (\mu - F)(d\xi, dt) \qquad \text{...(4.12)}$$

is well defined as a compensated–jump local martingale with $M_0^{\lambda d} = 0$. In fact, we have

$$E\, \Sigma_{t \le T} (\Delta M_t^{\lambda d})^2 = E \int_0^T \int_{|\xi| \le 1} (\xi^\lambda)^2 \cdot d\mu = \int_0^T \int_{|\xi| \le 1} (\xi^\lambda)^2 \cdot dF < \infty,$$

from which it follows that $M^{\lambda d}$ is actually a martingale with $E(M_T^{\lambda d})^2 < \infty$ for each $T \in \mathcal{T}$, see Jacod & Shiryaev (1987) I.4.50(c), Jacod (1979) 3.65–66, and of course $E(M_T^{\lambda d}) = 0$.

Reverting now to $W = \xi^\lambda I_{\{|\xi| > 1\}}$, the condition (9) is that $\int_0^T \int_{|\xi| > 1} |\xi^\lambda| d\mu$, or equivalently $\int_0^T \int_{|\xi| > 1} |\xi^\lambda| dF$, be integrable on compacts. In case this condition holds, the process $\int_0^T \int_{|\xi|>1} \xi^\lambda \cdot d(\mu - F)$ is a martingale with zero expectation. Referring to (1), and bearing in mind that $EM_T^{\lambda c} = EM_T^{\lambda d} = 0$ and that $V_T^{\lambda c}$ is deterministic and finite, it is seen that the condition in question is equivalent to Ex_T^λ being defined and finite, and then we obtain

$$x_T^\lambda - Ex_T^\lambda = M_T^{\lambda c} + \int_0^T \int_\Xi \xi^\lambda \cdot d(\mu - F), \qquad \text{...(4.13)}$$

a PII–martingale with zero expectation.

Suppose conversely that $\mathcal{L}$ with $\mathcal{L}(0) = 0$ is a given PII–*local* martingale without fixed discontinuities and with $E\mathcal{L}_T$ defined and finite for each T. We may replace X by $\mathcal{L}$, x^λ by $\mathcal{L}^\lambda$ throughout the preceding discussion – with corresponding changes in the meaning of M^c, V^c, μ, F etc. but without otherwise changing the notation – and in particular obtain (13) with $\mathcal{L}_T^\lambda - E\mathcal{L}_T^\lambda$ on the left. On the other hand, it is known that each $\mathcal{L}_T^\lambda$, considered simply as a local martingale with $\mathcal{L}_0^\lambda = 0$, has a representation given by the right–hand side of (13), see Jacod (1979) 3.77. From this we infer that $\mathcal{L}$ is in fact a *true* martingale with $E\mathcal{L}_t = \mathcal{L}_0 = 0$.

Finally, some simple points which will be needed below. If $\mathcal{L}$ is any PII–semimartingale with finite expectation, it follows from first principles that $\mathcal{L} - E\mathcal{L}$ is a martingale M, which may be assumed right continuous. Then $E\mathcal{L} = \mathcal{L} - M$ is also a semimartingale, which for a deterministic function means that it is right continuous and of finite variation on compacts, Jacod (1979) 2.27, Jacod & Shiryaev (1987) 1.4.29. Similarly, if $\mathcal{L}$ is a positive PII–semimartingale, then $E\mathcal{L}$ is positive, and if $E\mathcal{L}$ is finite then $\mathcal{L}/E\mathcal{L}$ is a positive martingale M and $E\mathcal{L} = \mathcal{L}/M$ is a semimartingale.

* * *

We return to our model, with X as in (1), V^d and M^d as in (7) and (12), $\Pi = \Pi^{+s}$. Applying (7) and (12) to the decomposition of x^π in (2.5) and taking into account (2.6), the last two lines of (2.5) may be written as

$$x_T^{\pi d} = M_T^{\pi d} + x_T^{\pi\mu o} + x_T^{\pi\mu 1}, \qquad \text{...(4.14)}$$

where

$$M_T^{\pi d} = \int_0^T \int_{|\xi|\leq 1} \left[\Sigma_\lambda \pi_t^\lambda \xi^\lambda \right] (\mu - F)(d\xi, dt),$$

$$x_T^{\pi\mu o} = \int_0^T \int_{\{|\xi|\leq 1\}} \left[\ell n(\Sigma_\lambda \pi_t^\lambda e^{\xi\lambda}) - \Sigma_\lambda \pi_t^\lambda \xi^\lambda \right] \mu(d\xi, dt),$$

$$x_T^{\pi\mu 1} = \int_0^T \int_{\{|\xi| > 1\}} \left[\ell n(\Sigma_\lambda \pi_t^\lambda e^{\xi\lambda}) \right] \mu(d\xi, dt). \qquad \text{...(4.15)}$$

For $\pi \epsilon \Pi^{+s}$, the integrands appearing here are deterministic, and those in the first two lines are bounded. The compensated integral in the first line again represents a martingale with zero expectation. The integrals in the second and third lines are a.s. defined and finite for each T and represent processes of finite variation because the corresponding sums converge absolutely. The first two lines of (15) derive only from M^d, the last only from V^d, and it follows from the properties of μ that the third line is independent of the other two. Note also that, with π deterministic, each line of (15) represents a PII, and these processes are stochastically continuous because X has no fixed discontinuities.

Now denote by $x_T^{\pi F o}$, $x_T^{\pi F 1}$ the expressions obtained on replacing μ by F in the second and third lines of (15). Then we have

$$Ex_T^{\pi d} = E(x_T^{\pi\mu o} + x_T^{\pi\mu 1}) = x_T^{\pi F o} + x_T^{\pi F 1} \qquad \text{...(4.16)}$$

provided that the integral $Ex_T^{\pi\mu 1}$ is defined. This is seen as follows. The first line of (15) has zero expectation. The second represents a stochastically continuous PII with jumps bounded in absolute value by a constant, which therefore has finite moments of all orders – see Gihman & Skorohod (1975) IV.1, Lemma 2. Thus (dropping the superscript π for brevity) $Ex_T^{\mu o}$ is defined and finite for each T, and we may replace μ therein by F; but F and the integrand are deterministic, so $Ex_T^{\mu o} = Ex_T^{F o} = x_T^{F o}$. Passing to the third line, we consider separately the processes $x^{\mu 1\pm}$ formed by restricting the integration to positive and negative values of the integrand. For integrands of fixed sign we may replace μ by F when forming the expectation, yielding $Ex_T^{\mu 1\pm} = Ex_T^{F 1\pm} = x_T^{F 1\pm}$. If $Ex_T^{\mu 1}$ is defined, at least one of the terms $x_T^{F 1\pm}$ must be finite, and the result follows. It further follows from the integral representation and the continuity of G in (3) that $x_T^{\pi F o}$ and (when finite) $x_T^{\pi F 1}$ are continuous functions of T.

We now assume until further notice that $b \neq 1$ and consider how to extend the theory of Section 3. The present Section will derive various preliminary results, leaving the discussion of the existence and properties of optimal plans to Section 5. The first task is to derive an explicit expression for $Ee^{(1-b)x^{\pi}(T)}$ with sure π. Since x^{π} is finite, this expectation must always be positive, but now it need no longer be finite. The theory of Section 3 may be regarded as dealing with the case where μ and F vanish, whereas here we shall assume until further notice that M^c and V^c vanish. Referring again to the formula (2.5) for x^{π} and noting that for sure π the second line on the right side is deterministic while the first is independent of the third and fourth, it is clear that the formula for $Ee^{(1-b)x^{\pi}}$ when all terms are present will simply be the product of that obtained in Section 3 and that derived under our new assumptions. The formula for Ψ when all terms are present will be the corresponding sum, which we now write as $\Psi = \Psi^c + \Psi^d$. The conditions for optimality will also combine in a straightforward way. However, some care is needed when making additional assumptions about the 'characteristics' V^c, $\langle M^c \rangle$ and F separately, because a decomposition like (1) is not unique. In this Section and the next we sometimes omit the superscript d.

Fix $\pi \in \Pi^{+s}$, and for the time being write simply $\eta_t = e^{(1-b)x(t)}$, $x = x^{\pi d}$. *We require a formula for* $E\eta_t$. Since η is a semimartingale with η_t and η_{t-} positive, the stochastic integral $\mathscr{L}\eta = \int(1/\eta_-)d\eta$ is well defined. Using the change-of-variables formula, then substituting from (2.5) and (2.6) and simplifying, we have

$$(\mathscr{L}\eta)_T = \int_0^T \frac{d\eta(t)}{\eta(t-)} = (1-b)x_T + \Sigma_{t\leq T}\left[e^{(1-b)\Delta x(t)} - 1 - (1-b)\Delta x(t)\right]$$

$$= (1-b)\int_0^T \Sigma_\lambda \pi_t^\lambda dM_t^{\lambda d} + \Sigma_{t\leq T}\left[(\Sigma_\lambda \pi_t^\lambda e^{\Delta x^\lambda(t)})^{1-b} - 1 - (1-b)\Sigma_\lambda \pi_t^\lambda \Delta M_t^\lambda\right], \qquad ...(4.17)$$

the sums over t converging absolutely for each T, a.s. Arguing as for (14–15) above, this may be written as

$$(\mathscr{L}\eta)_T = (1-b)M_T^{\pi d} + J_T^{\mu o} + J_T^{\mu 1}, \qquad ...(4.18)$$

where

$$J_T^{\mu o} = J^{\mu o}(\pi,T) = \Sigma_{t\leq T}\Big[(\Sigma_\lambda \pi_t^\lambda e^{\Delta M_t^\lambda})^{1-b} - 1 - (1-b)\Sigma_\lambda \pi_t^\lambda \Delta M_t^\lambda\Big]$$
$$= \int_o^T \int_{|\xi|\leq 1}\Big[(\Sigma_\lambda \pi_t^\lambda e^{\xi^\lambda})^{1-b} - 1 - (1-b)\Sigma_\lambda \pi_t^\lambda \xi^\lambda\Big]\mu(d\xi,dt), \qquad \ldots(4.19)$$
$$J_T^{\mu 1} = J^{\mu 1}(\pi,T) = \Sigma_{t\leq T}\, I_{\{|\Delta x_t|>1\}}\Big[(\Sigma_\lambda \pi_t^\lambda e^{\Delta x_t^\lambda})^{1-b} - 1\Big]$$
$$= \int_o^T \int_{|\xi|>1}\Big[(\Sigma_\lambda \pi_t^\lambda e^{\xi^\lambda})^{1-b} - 1\Big]\mu(d\xi,dt). \qquad \ldots(4.20)$$

The expectation of (18) is calculated in much the same way as for (14). The integrands are again deterministic. The term $M^{\pi d}$ has zero expectation. The processes $J^{\mu o}$, $J^{\mu 1}$ are stochastically continuous PII of finite variation, and we define J^{Fo}, J^{F1} as in (19) and (20) with μ replaced by F. Since the jumps of $J^{\mu o}$ are bounded, this process has finite moments, and on forming the expectation we may again replace μ by F; then, since F and the integrand are deterministic, (19) yields

$$EJ_T^{\mu o} = J_T^{Fo} = \int_o^T \int_{|\xi|\leq 1}\Big[(\Sigma_\lambda \pi_t^\lambda e^{\xi^\lambda})^{1-b} - 1 - (1-b)\Sigma_\lambda \pi_t^\lambda \xi^\lambda\Big]F(d\xi,dt). \qquad \ldots(4.21)$$

Turning to (20), we note that by (6) the negative part of $J^{\mu 1}$ has finite expectation for each $T \in \mathscr{T}$. For the positive part, the expectation is defined (but could be $+\infty$), and arguing as above we obtain

$$EJ_T^{\mu 1} = J_T^{F1} = \int_o^T \int_{|\xi|>1}\Big[(\Sigma_\lambda \pi_t^\lambda e^{\xi^\lambda})^{1-b} - 1\Big]F(d\xi,dt). \qquad \ldots(4.22)$$

To sum up so far, we have

$$-\infty < E(\mathscr{L}\eta)_T = J_T^{Fo} + J_T^{F1}\,; \quad |J_T^{Fo}| < \infty; \quad J_T^{F1} > -\infty; \qquad \ldots(4.23)$$
$$E(\mathscr{L}\eta)_T < \infty \quad \text{iff} \quad J_T^{F1} < \infty \quad \text{iff} \quad \int_o^T \int_{|\xi|>1} (\Sigma_\lambda \pi_t^\lambda e^{\xi^\lambda})^{1-b}\, F(d\xi,dt) < \infty. \quad \ldots(4.24)$$

Note also that, since η and $\mathscr{L}\eta$ are PII without fixed discontinuities, it follows as for (16) above that the expectation of each of these processes is continuous as long as it remains finite.

We now want to show that we may write

$$\ell n\, E\eta_T = E(\mathscr{L}\eta)_T \qquad \ldots(4.25)$$

whenever one side or the other is defined and finite for all $T \in \mathscr{T}$.

Suppose first that $E(\mathscr{L}\eta)_T < \infty$ for all T. Define a process L by

$L_t = (\mathscr{L}\eta)_t - E(\mathscr{L}\eta)_t$;

since $\mathscr{L}\eta$ is a PII, L is a PII–martingale with $EL_t = 0$, and $E(\mathscr{L}\eta)_t$ is of finite variation. Since $\mathscr{E}\mathscr{L}\eta = \eta$ and $E(\mathscr{L}\eta)$ is continuous, we have

$(\mathscr{E}L)_T = \eta_T \cdot \exp\{-E(\mathscr{L}\eta)_T\} > 0, \qquad (\mathscr{E}L)_0 = 1.$

Further, if L is a martingale with $EL = 0$, then $\mathscr{E}L$ is a local martingale with $0 < E(\mathscr{E}L)_T \leq 1$, and since $\mathscr{E}L$ is also a PII and has finite expectation it is a martingale; thus (25) follows from

$1 = E(\mathscr{E}L)_T = E\eta_T \cdot \exp\{-E(\mathscr{L}\eta)_T\}.$

Suppose conversely that $E\eta_T < \infty$ for all T, and bear in mind that $E\eta_T > 0$; (25) will follow if we show that $E(\mathscr{L}\eta)_T < \infty$, and according to (22–24) it is enough to show that $EJ_T^{\mu 1} = J_T^{F1}$ is finite for all T. Taking into account the independence properties of μ, it is enough to prove this under the assumption that $X = V^d$. Then (18) reduces to $\mathscr{L}\eta = J^{\mu 1}$, or equivalently

$$\eta_T - 1 = \int_0^T \eta_{t-} dJ_t^{\mu 1}.$$

Taking expectations, we may once again replace $J^{\mu 1}$ by the deterministic function J^{F1}, and then by Fubini's theorem

$$E(\eta_T) - 1 = \int_0^T E(\eta_{t-}) dJ_t^{F1}.$$

Now $E(\eta_{t-}) = (E\eta)_{t-}$; to check this, note that the process Y defined by $Y_t = \eta_t / E\eta_t$ is a PII–martingale, hence $E(Y_{t-}) = E(Y_t) = 1$, see Dellacherie & Meyer (1980) VI.2.4, yielding the result. Since $E(\eta_t) = (E\eta)_t$ is a continuous semimartingale with $(E\eta)_0 = 1$, we can apply the change–of–variables formula to obtain

$$J_T^{F1} = \int_0^T \left[\frac{d(E\eta)_t}{(E\eta_t)_-}\right] = (\mathscr{L}E\eta)_T = \ell n(E\eta_T) . \| \qquad \ldots(4.26)$$

The modifications needed if one side or the other of (25) is finite only on an interval of $\mathscr{T}$ are straightforward, and it is clear that

$$E\eta_T = \infty \quad \text{iff} \quad E(\mathscr{L}\eta)_T = \infty \quad \text{iff} \quad J_T^{F1} = \infty. \qquad \ldots(4.27)$$

It is convenient at this point to introduce some alternative notation. For $\breve{\pi} \in \mathscr{S}$ and $\xi \in \mathfrak{R}^{\Lambda}$ define functions

$$h(\breve{\pi},\xi) = (1-b)^{-1}\Big[(\Sigma_{\lambda}\breve{\pi}^{\lambda}e^{\xi^{\lambda}})^{1-b} - 1 - (1-b)I_{\{|\xi|\leq 1\}}\Sigma\breve{\pi}^{\lambda}\xi^{\lambda}\Big],$$

$$h^{0}(\breve{\pi},\xi) = h(\breve{\pi},\xi)I_{\{|\xi|\leq 1\}}, \qquad h^{1}(\breve{\pi},\xi) = h(\breve{\pi},\xi)I_{\{|\xi|>1\}}, \qquad \text{...(4.28)}$$

and for $i = 0,1$ write

$$\Psi^{i}(\pi,T) = (1-b)^{-1}\, J^{Fi}(\pi,T) = \int_{0}^{T}\int_{\Xi} h^{i}(\pi_t,\xi)F(d\xi,dt). \qquad \text{...(4.29)}$$

Taking into account the independence properties of the various components of X and the results of Section 3, the preceding discussion can be summed up in

THEOREM 5: *'Bilateral Laplace Transform' for Compound Interest Processes.*

Let X be a PII–semimartingale with no fixed times of discontinuity and representation (4.1), and let $X^{c} = M^{c}+V^{c}$, $X^{d} = X-X^{c} = M^{d}+V^{d}$, where M^{d}, V^{d} are defined as in (4.12) and (4.7). Given a fixed sure $\pi \geq 0$, let $x^{\pi} = x^{\pi c} + x^{\pi d}$ be the portfolio log–return for π, where $x^{\pi c}, x^{\pi d}$ are calculated from X^{c}, X^{d} separately, i.e $x^{\pi c}$ is given by the first two lines of (2.5), $x^{\pi d}$ by the last two lines. Then, for $b \neq 1$,

$$\Psi(\pi,T) \stackrel{\text{def}}{=} (1-b)^{-1}\ell n\, E\, e^{(1-b)x^{\pi}(T)} = \Psi^{c}(\pi,T) + \Psi^{d}(\pi,T), \qquad \text{...(4.30)}$$

where Ψ^{c} is given by (3.4) and

$$\Psi^{d}(\pi,T) = \Psi^{0}(\pi,T) + \Psi^{1}(\pi,T) = \int_{0}^{T}\int_{\Xi} h(\pi_t,\xi)F(d\xi,dt). \qquad \text{...(4.31)}$$

We have $(1-b)\Psi(\pi,T) > -\infty$ always, $(1-b)\Psi(\pi,T) < \infty$ iff the integral in (4.24) converges.

REMARKS. (i) There are straightforward extensions of the above statements to cases where the integral in (24), with $\int_{0}^{T}$ replaced by $\int_{S}^{T}$, converges for some pairs $S < T$ but not for others, but we shall usually make assumptions which exclude this possibility.

(ii) As one might expect, the expression for $(1-b)\Psi^{d}$ given by (31) with (28) agrees with the formula which would have been obtained if we had started with the Lévy-Khinchin formula for the characteristic function of X^{d} – see Jacod (1979) 3.55, Jacod & Shiryaev (1987) II.4c – then calculated $E\exp\{iux_{T}^{\pi d}\}$, and finally replaced iu by $1-b$.

(iii) The essential point of the argument following (25) is that $\mathscr{E}L$ is a *true* martingale, given that L is one. The best *general* criterion (without independence of increments) for showing that a discontinuous exponential supermartingale is actually a martingale appears to be that given by Lepingle & Mémin (1978) eq.(3.8). Noting that $\Delta(\mathscr{L}\eta)_t = e^{(1-b)\Delta x(t)} - 1$, where $x = x^{\pi d}$, this criterion applied to the present situation shows that a sufficient condition for $\mathscr{E}L$ to be a martingale is that the expression

$$E\,\Pi_{t\leq T}\Big[\{1+\Delta(\mathscr{L}\eta)_t\}\cdot\exp\{-\Delta(\mathscr{L}\eta)_t/[1+\Delta(\mathscr{L}\eta)_t]\}\Big]$$
$$= E\exp\Sigma_{t\leq T}\Big[e^{(b-1)\Delta x_t} - 1 - (b-1)\Delta x_t\Big] \qquad \text{...(4.32)}$$

be finite for each $T\in\mathscr{T}$. On the other hand, the assumption that (24) converges means – taking into account (21–23) – that

$$E\,\Sigma_{t\leq T}\Big[e^{(1-b)\Delta x_t} - 1 - (1-b)I_{\{|\Delta x_t|\leq 1\}}\Sigma\pi^\lambda_t\Delta x_t\Big] \qquad \text{...(4.33)}$$

is finite, showing that the criterion is far from providing a necessary condition.

* * *

In our discussion of optimal portfolios we shall sometimes consider assumptions additional to those which have been imposed so far; the rest of this Section reviews these assumptions and some of their consequences. We begin with conditions yielding F–integrable bounds on the integrands in the expressions $J^{Fo}(\pi,T)$ and $J^{F1}(\pi,T)$ which for given $T\in\mathscr{T}$ are uniform across all sure $\pi\geq 0$, and so define uniform bounds for the integrals themselves. *Starting with the formula* (22) for J^{F1}_T, it follows from (6) that the term -1 is integrable, so that it is enough to consider the positive part of the integrand. *For* $b<1$ we have

$$(\Sigma\pi^\lambda e^{\xi^\lambda})^{1-b} \leq \Sigma(\pi^\lambda e^{\xi^\lambda})^{1-b} \leq \Sigma e^{(1-b)\xi^\lambda} \qquad \text{...(4.34)}$$

using the inequalities Hardy et al. (1952) T.27 and $0\leq\pi^\lambda\leq 1$, so that for given T the $J^{F1}(\pi,T)$ are uniformly bounded if

$$\int_0^T\int_{|\xi|>1} e^{(1-b)\xi^\lambda}F(d\xi,dt) < \infty \quad \text{for each } \lambda, \qquad \text{...(4.35)}$$

or equivalently if

$$J^{F1}(\lambda,T) < \infty \quad \text{for each } \lambda. \qquad \text{...(4.36)}$$

Now note that, for every $b\neq 1$, Jensen's inequality implies

$$(1-b)^{-1}(\Sigma\pi^\lambda e^{\xi^\lambda})^{1-b} \geq (1-b)^{-1}\Sigma\pi^\lambda e^{(1-b)\xi^\lambda}, \qquad ...(4.37)$$

so that *for* $b > 1$ it is again seen that (35) or (36) is sufficient for a uniform bound.

We next consider the formula (21) for $J^{FO}(\pi,T)$. Note first that

$$\Sigma\, \pi^\lambda e^{\xi^\lambda} \geq e^{\Sigma\pi^\lambda \xi^\lambda} \qquad ...(4.38)$$

by convexity, so that *for* $b < 1$ we have

$$\left[\Sigma\pi^\lambda e^{\xi^\lambda}\right]^{1-b} -1-(1-b)\Sigma\pi^\lambda\xi^\lambda \geq e^{(1-b)\Sigma\pi^\lambda\xi^\lambda} -1-(1-b)\Sigma\pi^\lambda\xi^\lambda \geq 0, \qquad ...(4.39)$$

showing that the integrand in (21) is *non-negative.* A simple *upper bound* (though not the best possible) is obtained from

$$\left[\Sigma\pi^\lambda e^{\xi^\lambda}\right]^{1-b} -1-(1-b)\Sigma\pi^\lambda\xi^\lambda \leq e^{(1-b)\max_\lambda\xi^\lambda} -1-(1-b)\min_\lambda\xi^\lambda, \qquad ...(4.40)$$

showing that for given T the $J^{FO}(\pi,T)$ are uniformly bounded if

$$\int_0^T\int_{|\xi|\leq 1}\left[e^{(1-b)\max_\lambda\xi^\lambda} -1-(1-b)\min_\lambda\xi^\lambda\right]F(d\xi,dt) < \infty. \qquad ...(4.41)$$

In case $b > 1$, the integrand in (21) no longer has definite sign, but using (37) we get

$$\left[\Sigma\pi^\lambda e^{\xi^\lambda}\right]^{1-b} -1-(1-b)\Sigma\,\pi^\lambda\xi^\lambda \leq \Sigma\pi^\lambda\left[e^{(1-b)\xi^\lambda} -1-(1-b)\xi^\lambda\right], \qquad ...(4.42)$$

so that the $J^{FO}(\pi,T)$ are uniformly bounded if

$$J^{FO}(\lambda,T) = \int_0^T\int_{|\xi|\leq 1}\left[e^{(1-b)\xi^\lambda} -1-(1-b)\xi^\lambda\right]F(d\xi,dt) < \infty \quad \text{for each } \lambda. \qquad ...(4.43)$$

We already know that this condition is indeed satisfied – see (23). To get an integrable *lower bound*, we note that

$$\left[\Sigma\pi^\lambda e^{\xi^\lambda}\right]^{1-b} -1-(1-b)\Sigma\,\pi^\lambda\xi^\lambda \geq e^{(1-b)\max_\lambda\xi^\lambda} -1-(1-b)\min_\lambda\xi^\lambda \qquad ...(4.44)$$

for $b > 1$, so that (41) is again sufficient.

Taking into account (28)–(29), we may sum up this discussion of bounds as follows.

PROPOSITION 1. Let $b \neq 1$ and fix $T \in \mathscr{T}$.

(i) The functions $\Psi^O(\pi,T)$, or equivalently the $J^O(\pi,T)$, are finite for all $\pi \in \Pi^{+s}$.

If (41) is satisfied, there is a Borel function h^{O+} on $\mathfrak{R}^\Lambda$, vanishing for $|\xi| > 1$, such that

$$|h^O(\breve{\pi},\xi)| \leq h^{O+}(\xi), \qquad \breve{\pi} \in \mathscr{S}, \quad \xi \in \mathfrak{R}^\Lambda, \qquad ...(4.45)$$

$$\int_0^T\int_\Xi h^{O+}(\xi)F(d\xi,dt) < \infty; \qquad ...(4.46)$$

then the $\Psi^O(\pi,T)$ are uniformly bounded for all $\pi\in\Pi^{+s}$, and bounds may be calculated from (41) and, for $b > 1$, from (43).

(ii) The functions $\Psi^1(\pi,T)$, or equivalently the $J^1(\pi,T)$, are finite for all $\pi \in \Pi^{+s}$ iff (35) is satisfied. Then there is a Borel function h^{1+} on $\mathfrak{R}^\Lambda$, vanishing for $|\xi| \leq 1$, such that

$$|h^1(\breve{\pi},\xi)| \leq h^{1+}(\xi), \qquad \breve{\pi} \in \mathscr{A}, \quad \xi \in \mathfrak{R}^\Lambda, \qquad \text{...(4.47)}$$

$$\int_0^T \int_\Xi h^{1+}(\xi) F(d\xi,dt) < \infty; \qquad \text{...(4.48)}$$

the $\Psi^1(\pi,T)$ are uniformly bounded for all $\pi \in \Pi^{+s}$, and bounds may be calculated from (35) and (6).

(iii) Finiteness of the functions $\Psi(\pi,T)$, $\Psi^d(\pi,T)$, $\Psi^1(\pi,T)$, $J^1(\pi,T)$ are equivalent conditions. Thus the $\Psi(\pi,T)$ are finite for all $\pi \in \Pi^{+s}$ and all $T \in \mathscr{T}$ iff $\Psi(\lambda,T)$ is finite on $\mathscr{T}$ for each single–security portfolio λ.

If a discount density q is given, the result for the PS model corresponding to (iii) is as follows: if the maximum welfare from consumption is finite for each single-asset portfolio, then the same is true for every non-negative sure portfolio plan. If $b < 1$ and the Complete Class Theorem applies, it follows that the same is true for every non-negative portfolio plan; in this case, the condition that (35) holds for each T is harmless in the sense that a single exception would imply the existence of a portfolio-cum-saving plan yielding infinite welfare. For $b > 1$, on the other hand, it appears that (35) does imply substantive restrictions. Without this condition, it is conceivable that $\Psi^d(\lambda,T) = -\infty$ for each λ (for some or all T) – for example, if each asset has a symmetric stable distribution with index < 2 – yet that $\Psi^d(\pi,T)$ is finite for all T for some sure π; (at any rate, I have not shown that this cannot happen).

The condition that $J^{F1}(\pi,T) < \infty$ for all π and T is of course satisfied if $(1-b)\Delta x_t^\lambda$ is bounded above for all λ and t by a constant. Such an assumption may be reasonable in a crude empirical sense, and indeed some markets limit the size of price jumps by regulation; but working with bounded distributions limits statistical modelling, regulated prices are not market–clearing prices, and it is usually difficult to specify realistic (approximately least) upper bounds. Again, the model would be simplified if it were assumed that there is a smallest possible size of jump (in case a jump occurs at all), since then we could set $J^{FO} \equiv 0$ for all π by changing the scale of the X–process; but once again this may be undesirable for modelling reasons.

We next consider assumptions specifying additional smoothness for the measure F. Recalling that $F(d\xi,dt)$ may be written $f(d\xi,t)dG(t)$, *we shall in the rest of this Section assume the following*:

G is absolutely continuous with a (version of its) derivative g which is positive and continuous on $\mathcal{T}$. ...(4.49)

The measures $f_t = f(d\xi,t)$ for $t \geq 0$ are mutually absolutely continuous with Radon-Nikodym derivatives denoted by

$$\theta_{t,\tau}(\xi) = df_t/df_\tau\,, \qquad ...(4.50)$$

and, for each $\tau \geq 0$,

$$\theta_{t,\tau}(\xi) \to 1, \text{ uniformly in } \xi \neq 0, \text{ as } t \to \tau. \qquad ...(4.51)$$

Under (49), each of the integrability conditions (35), (41), (43), (46), (48) implies for *almost* all $t \leq T$ the corresponding conditions with $\int_0^T$ deleted and $F(d\xi,dt)$ replaced by $f(d\xi,t)$; let us call the latter *derivative conditions* and write them as (35)′,...,(48)′. It then follows easily from (50)–(51) that each integrability condition implies the corresponding derivative condition for *all* $t \leq T$. In the same way, it follows from (6) that $f_t\{|\xi|>1\} < \infty$ for each t. Also, if $F(\Xi \times \mathcal{T}) > 0$, then $f_t(\Xi) > 0$ for each t.

We may now write (31) as

$$\Psi^d(\pi,T) = \int_0^T \Big[\int_\Xi h(\pi_t,\xi)f(d\xi,t)\Big] g(t)dt \qquad ...(4.52)$$

where h is defined by (28). Also, by analogy with (3.13–14), we write

$$\psi^d(\breve{\pi},t) = \int_\Xi h(\breve{\pi},\xi)f(d\xi,t)g(t); \qquad \psi^d(\pi_t,t) = (\partial/\partial t)\Psi^d(\pi_t,t);$$
$$\psi(\breve{\pi},t) = \psi^c(\breve{\pi},t) + \psi^d(\breve{\pi},t); \qquad \psi(\pi_t,t) = (\partial/\partial t)\Psi(\pi_t,t); \qquad ...(4.53)$$

the first equation in each line is to be read as the definition of a function on $\mathcal{S} \times \mathcal{T}$, the second as a property. If h is replaced in the first line by h^0 or h^1, the resulting function is written ψ^0 or ψ^1, and of course $\psi^d = \psi^0 + \psi^1$. It follows readily from Proposition 1 that, under (49) and (50–51), the functions ψ^d, ψ^0, ψ^1 are *defined* for all $(\breve{\pi},t)$, and

$\psi^{o}(\check{\pi},t)$ is always *finite*; if (35) is satisfied for all T, then $\psi^{1}(\check{\pi},t)$ is also *finite* for all $(\check{\pi},t)$. Further, since $h(\cdot,\xi)$ is strictly concave on $\mathscr{S}$ for each ξ, these assumptions imply that the integral $\psi^{d}(\cdot,t)$ in *concave* on $\mathscr{S}$ for each t, indeed *strictly concave* if $f_t(\Xi) > 0$; similarly if h, ψ^{d}, f are replaced by h^{o}, ψ^{od}, $f\cdot I_{\{|\xi|\leq 1\}}$, or again by h^{1}, ψ^{1d}, $f\cdot I_{\{|\xi|>1\}}$.

If (35) is satisfied, then $\psi^{1}(\check{\pi},t)$ is *continuous* on $\mathscr{S}\times\mathscr{T}$. To check this, note first that by Proposition 1(ii) the assumption implies (48) for all T, hence also (48)′. Now let $(\check{\pi}_{o},t_{o})$ be given and let $t_n\to t_o$, $\check{\pi}_n\to\check{\pi}_o$; then, using the definition of ψ^1 and (50) we have
$\psi^{1}(\check{\pi}_n,t_n)-\psi^{1}(\check{\pi}_o,t_o)=\int_{\Xi}\Big[h^{1}(\check{\pi}_n,\xi)\theta_{t_n,t_o}(\xi)g(t_n)-h^{1}(\check{\pi}_o,\xi)g(t_o)\Big]f(d\xi,t_o)$,
and this tends to zero as $n\to\infty$ by dominated convergence, taking into account (48)′, (49) and (51).

A similar argument, using part (i) of Proposition 1, shows that ψ^{o} is *continuous* on $\mathscr{S}\times\mathscr{T}$ if (41) is satisfied. However, *this result can be obtained even without* (41), as follows. Note as a first step that, by virtue of the independence properties noted above, we may w.l.o.g. assume that $X=M^{d}$, hence $\psi=\psi^{o}$, $h=h^{o}$. The second step is to show that, for *fixed* τ, the integral

$$\psi^{o}(\check{\pi},\tau)=\int_{\Xi}h^{o}(\check{\pi},\xi)f(d\xi,\tau)g(\tau), \qquad \text{...(4.54)}$$

is continuous on $\mathscr{S}$. Of course, we now have $|\Delta X|\leq 1$, hence $|\Delta x^{\lambda}|\leq 1$, for all (ω,t). As noted in Section 2, this condition implies that, for a given vector $\check{\pi}$, the function $\Sigma_{\lambda}\check{\pi}^{\lambda}e^{\Delta x^{\lambda}(\omega,t)}$ remains positive even if some co–ordinates of $\check{\pi}$ are negative, provided that the sum $\check{\pi}^{-}$ of negative co–ordinates is less in absolute value than some sufficiently small $\rho>0$ – see (2.11). Let

$$\mathscr{S}^{\rho}=\{\check{\pi}\in\mathbb{R}^{\Lambda}:\ \Sigma_{\lambda}\check{\pi}^{\lambda}=1 \text{ and } \check{\pi}^{-}>\rho\}.$$

It follows that $\Sigma_{\lambda}\check{\pi}^{\lambda}e^{\xi^{\lambda}}$ is defined on $\mathscr{S}^{\rho}\times\{|\xi|\leq 1\}$ for small ρ, and is bounded there by a constant depending on b and ρ; the same is therefore true of $h^{o}(\check{\pi},\xi)$ – see (28). For each $\check{\pi}\in\mathscr{S}^{\rho}$, consider the invariable portfolio plan π defined by setting $\pi_t=\check{\pi}$ for each t. This plan generates a certain log–returns process x^{π}, and as in the argument preceding Theorem 5 it is seen that the function $\Psi^{o}(\pi,T)$ is well defined and finite for each T. Consequently, by (49–51), $\psi^0(\breve{\pi},\tau)$ is defined and finite for all $\breve{\pi}\in\mathscr{S}^\rho$ for each fixed τ. Now $\mathscr{S}^\rho$ is convex and contains $\mathscr{S}$ in its interior. A calculation of second derivatives shows that $h^0(.,\xi)$ is a strictly concave function of $\breve{\pi}$ on $\mathscr{S}^\rho$, so that $\psi^0(\cdot,\tau)$ is concave on $\mathscr{S}^\rho$. This implies that $\psi^0(\cdot,\tau)$ is continuous on the interior of $\mathscr{S}^\rho$, in particular on $\mathscr{S}$.

The third step, which will complete the proof, is to show that $\psi^0(\breve{\pi},\cdot)$ is continuous in t and that the continuity is uniform with respect to $\breve{\pi}\in\mathscr{S}$. Suppose first that $b < 1$, so that $h^0 \geq 0$ on $\mathscr{S}$ – see (28) and (39). For given $(\breve{\pi},\tau)$ and $0 < \epsilon < 1$, there exists a $\delta > 0$ such that $|t-\tau| < \delta$ implies $|\theta_{t,\tau}(\xi)-1| < \epsilon$ for all ξ and $1-\epsilon < g_t/g_\tau < 1+\epsilon$ – see (49) and (51). Consequently, using (50),

$$|\psi^0(\breve{\pi},t) - \psi^0(\breve{\pi},\tau)| \leq \int_\Xi h^0(\breve{\pi},\xi)|\theta_{t,\tau}(\xi)g_t-g_\tau|f(d\xi,\tau)$$

$$\leq [(1+\epsilon)^2 - 1]g_\tau\cdot \max\{\int_\Xi h^0(\breve{\pi},\xi)f(d\xi,\tau) : \breve{\pi}\in\mathscr{S}\}. \qquad ...(4.55)$$

The maximum exists because the integral is continuous on $\mathscr{S}$, and the right–hand side tends to zero as $\epsilon \to 0$, yielding the result.

If $b > 1$, the function h^0 does not have definite sign. However, writing

$$h^0(\breve{\pi},\xi) = \{h^0(\breve{\pi},\xi) - \Sigma_\lambda h^0(\lambda,\xi)\} + \{\Sigma_\lambda h^0(\lambda,\xi)\}, \qquad ...(4.56)$$

it follows from the definition of $h^0(\lambda,\xi)$ and (42) that, for $\breve{\pi}\in\mathscr{S}$, each of the terms in braces has definite sign. Moreover, a part of the argument in the second step above, with $\breve{\pi} = \lambda$, shows that each $h^0(\lambda,\xi)$ is integrable with respect to $f(d\xi,\tau)$. The result then follows from inequalities like (55) with $h^0(\breve{\pi},\xi)$ replaced by

$$|h^0(\breve{\pi},\xi) - \Sigma_\lambda h^0(\lambda,\xi)| + |\Sigma_\lambda h^0(\lambda,\xi)|.$$

This discussion may be summed up by

PROPOSITION 2. Let $b \neq 1$. Suppose that $F(\Xi\times\mathscr{T}) > 0$, that F satisfies (4.49) and (4.50–51), and that (4.35) holds for each T. Then $\psi^0(\breve{\pi},t)$, $\psi^1(\breve{\pi},t)$ and $\psi^d = \psi^0 + \psi^1$ are defined, finite and continuous on $\mathscr{S}\times\mathscr{T}$; consequently the functions $\Psi^0(\pi,T)$ and $\Psi^1(\pi,T)$, or equivalently the $J^{F0}(\pi,T)$ and $J^{F1}(\pi,T)$, are uniformly bounded on Π^{+s} for each T. Further, for each T, $\psi^d(\cdot,t)$ is strictly concave on $\mathscr{S}$.

5. OPTIMAL PORTFOLIOS WITH MOVING DISCONTINUITIES

We return to the discussion of optimality and extend the theory of Section 3 to the case where X is a PII with no fixed times of discontinuity and $\Pi = \Pi^{+s}$. Other assumptions stated in earlier Sections will be introduced as required, but are not in force at the outset. We make use of the decomposition $X = X^c + X^d$, referring to Sections 3 and 4 for components of formulae which stem from X^c and X^d respectively.

We begin by completing, in outline, the alternative proof of Theorem 1. It is enough to show that, if π^* is optimal sure, then for each λ the function (1.13) is finite and non–decreasing on $\mathcal{T}$. For brevity, write

$$R_T = R(\pi,\pi^*,T) = \ell n \, E \, e^{x^\pi(T) - bx^*(T)}, \qquad \text{...(5.1)}$$

where initially π and π^* are arbitrary fixed elements of Π^{+s}. Once again, the first step is to derive an explicit formula for this expectation. We proceed as in Sections 3–4, but instead of $\eta_T = \exp\{(1-b)x^\pi_T\}$ we consider the process $\zeta_T = \exp\{x^\pi_T - bx^*_T\}$. A formula for ζ is obtained as before from (2.5–6), and once again we separate the continuous and discontinuous parts, writing ζ^c, ζ^d, R^c, R^d etc. and noting that $\zeta = \zeta^c\zeta^d$, $R = R^c + R^d$. Now R^c_T is always finite and has been calculated in (3.5). An argument like that starting with (4.17), with ζ instead of η, yields – in abridged notation –

$$R^d_T = \int_0^T\int_\Xi \Big[(\Sigma_\lambda \pi^\lambda_t e^{\xi^\lambda})(\Sigma_\lambda \pi^{*\lambda}_t e^{\xi^\lambda})^{-b} - 1 - I_{\{|\xi|\le 1\}}\Sigma_\lambda(\pi^\lambda_t - b\pi^{*\lambda}_t)\xi^\lambda\Big] F(d\xi,dt), \qquad \text{...(5.2)}$$

and referring to (4.28–31) it is seen that $R^d(\pi,\pi,T) = (1-b)\Psi^d(\pi,T)$. We have $R^d_T > -\infty$ always, and – cf. (4.24) –

$$R^d_T < \infty \quad \text{iff} \quad \int_0^T\int_{|\xi|>1} \Big[(\Sigma_\lambda \pi^\lambda_t e^{\xi^\lambda})(\Sigma_\lambda \pi^{*\lambda}_t e^{\xi^\lambda})^{-b}\Big] F(d\xi,dt) < \infty. \qquad \text{...(5.3)}$$

Now let π^* be optimal sure, so that as in (3.7) we have

$$0 \ge (1/\alpha)[\Psi(\pi^*+\alpha\delta\pi,T) - \Psi(\pi^*,T)], \qquad 0 < \alpha \le 1 \qquad \text{...(5.4)}$$

for all $\pi = \pi^*+\delta\pi \in \Pi^{+s}$. Optimality implies that $\Psi(\pi^*,T)$ is finite, and we assume for the time being that $\Psi(\pi^*+\alpha\delta\pi,T)$ is also finite for all $\delta\pi$, α and T. Write $\delta\Psi(\alpha) = \Psi(\pi^*+\alpha\delta\pi,T) - \Psi(\pi^*,T)$, $\Psi = \Psi^c + \Psi^d$, substitute into (4) for Ψ^c from (3.4) and for Ψ^d from (4.31) with (4.28), and in the resulting expression pass to the limit under the integral signs as $\alpha \downarrow 0$. (To justify this, note that $0 = \delta\Psi(0) \geq \delta\Psi(\alpha)$ and that in the formula for $\delta\Psi(\alpha)$ the integrands are concave functions of α vanishing at $\alpha = 0$, so that in the formula for $\delta\Psi(\alpha)/\alpha$ the integrands not decrease as α decreases, and their value at $\alpha = 1$ defines an integrable lower bound.) Writing $R(\pi,\pi^*,T) = R_T^\pi$, $R(\pi^*,\pi^*,T) = R_T^*$, this calculation yields

$$0 \geq \lim_{\alpha \downarrow 0} [\delta\Psi(\alpha)/\alpha] = R_T^\pi - R_T^* \geq \Psi(\pi,T) - \Psi(\pi^*,T), \qquad \text{...(5.5)}$$

and since $R_T^* = (1-b)\Psi(\pi^*,T)$ it follows in particular that R^π is finite. As in Section 3, one can set π equal to the single–asset portfolio λ during any given interval $[S,T)$ and $\pi = \pi^*$ elsewhere, implying

$$0 \geq (R_T^\lambda - R_S^\lambda) - (R_T^* - R_S^*) \qquad \text{...(5.6)}$$

i.e. $R^\lambda - R^*$ is non–increasing.

This completes the proof if $\Psi(\pi^*+\alpha\delta\pi,T)$ is finite for all $\delta\pi$, α and T, but in fact this assumption is inessential. Of course, if $b < 1$ we must have $\Psi < \infty$ always if an optimum is to exist. We also know that finiteness of X implies $(1-b)\Psi > -\infty$. There remains the possibility that $b > 1$ and $\Psi(\pi,T) = -\infty$ for some sure $\pi = \pi^* + \delta\pi$ and some T. Now, the argument in the preceding paragraph requires only that $\Psi(\pi^*+\alpha\delta\pi,T) > -\infty$ for *small* $\alpha > 0$, which in turn requires that

$$r(\alpha) \stackrel{\text{def}}{=} \int_0^T \int_{|\xi|>1} \left[\Sigma_\lambda(\pi_t^{*\lambda}+\alpha\delta\pi_t^\lambda)e^{\xi^\lambda})\right]^{1-b} F(d\xi,dt) < \infty.$$

Since $\pi^{*\lambda}+\alpha\delta\pi^\lambda = (1-\alpha)\pi^{*\lambda}+\alpha\pi^\lambda \geq (1-\alpha)\pi^{*\lambda}$, we have, for $b > 1$,

$$\left[\Sigma_\lambda(\pi_t^{*\lambda}+\alpha\delta\pi_t^\lambda)e^{\xi^\lambda})\right]^{1-b} \leq (1-\alpha)^{1-b}\left[\Sigma_\lambda\pi_t^{*\lambda}e^{\xi^\lambda})\right]^{1-b},$$

and on integrating this gives

$$r(\alpha) \leq (1-\alpha)^{1-b}r(0),$$

which is finite since $\Psi(\pi^*,T)$ is finite.||

Regarding the *uniqueness* of an optimal sure π^*, we need only add to the remarks of Section 3 that the function $h(\breve{\pi},\xi)$ defined in (4.28) is, for fixed ξ, a strictly concave function of the vector $\breve{\pi}$, so that, if one writes $F(d\xi,dt) = f(d\xi,t)dG(t)$ as in (4.3), it is seen from (4.30–31) that π^* must be unique if G is strictly increasing and $f(\Xi,t) > 0$ for all t. Thus it is sufficient for uniqueness if *either* these conditions *or* the conditions assumed in Theorem 4 are satisfied.

We turn to the *existence and characterisation of an optimal sure* π^*, *assuming now that all the assumptions mentioned in Theorem* 4 *and Proposition* 2 *are in force.* Consider the problem of maximising $\psi(\breve{\pi},t)$ on $\mathscr{S}$ for *fixed* $t > 0$. Explicitly, we now have

$$\psi(\breve{\pi},t) = \Sigma_\lambda \breve{\pi}^\lambda(v_t^\lambda + \tfrac{1}{2}\sigma_t^{\lambda\lambda} - \tfrac{1}{2}b\Sigma_\ell \breve{\pi}^\ell \sigma_t^{\lambda\ell})$$
$$+(1-b)^{-1}\int_\Xi \Big[(\Sigma_\lambda \breve{\pi}^\lambda e^{\xi^\lambda})^{1-b} - 1 - (1-b)I_{\{|\xi|\le 1\}}\Sigma\breve{\pi}^\lambda \xi^\lambda\Big] f(d\xi,t)g(t)$$
$$= \psi^c(\breve{\pi},t) + \psi^d(\breve{\pi},t) \qquad \text{...(5.7)}$$

– see (3.13) and (4.53) with (4.28). As shown in Sections 3 and 4, $\psi(\cdot,t)$ is finite and continuous on $\mathscr{S}$ and so is bounded and attains its maximum at some point π_t^*; further, $\psi(\cdot,t)$ is strictly concave, so that π_t^* is the unique maximum. The derivation of programming conditions characterising a maximum is also straightforward. The problem is to maximise (7) subject to the constraints $\breve{\pi}^\lambda \ge 0$ and $\Sigma\breve{\pi}^\lambda = 1$. If π_t^* is a solution, then

$0 \ge (1/\alpha)[\psi(\pi_t^*+\alpha\delta\breve{\pi},t) - \psi(\pi^*,t)], \qquad 0 < \alpha \le 1,$

for all $\breve{\pi} = \pi_t^*+\alpha\delta\breve{\pi} \in \mathscr{S}$. This condition has the same form as (4), and under present assumptions all values of $\psi(\cdot,t)$ are uniformly bounded; the passage to the limit under the integral sign as $\alpha \downarrow 0$ may therefore be justified as in the paragraph following (4), and the limit is non–positive and uniformly bounded for all admissible $\delta\breve{\pi}$. Explicitly, we get

$0 \ge \Sigma_\lambda \delta\breve{\pi}^\lambda \gamma_t^\lambda \ge$ constant $\quad$ for every $\breve{\pi} = \pi_t^* + \delta\breve{\pi} \in \mathscr{S}$, $\quad$ where

$$\gamma_t^\lambda = v_t^\lambda + \tfrac{1}{2}\sigma_t^{\lambda\lambda} - b\Sigma_\ell \pi_t^{*\ell}\sigma_t^{\lambda\ell} + \int_\Xi \Big[e^{\xi^\lambda}(\Sigma_\ell \pi_t^{*\ell} e^{\xi^\ell})^{-b} - I_{\{|\xi|\le 1\}}\xi^\lambda\Big] f(d\xi,t)g(t). \qquad \text{...(5.8)}$$

On choosing for $\breve{\pi}$ the single–asset portfolios one obtains necessary conditions of the form

$$\pi_t^{*\lambda} \ge 0 \quad \text{each } \lambda; \quad \Sigma_\lambda \pi_t^{*\lambda} = 1;$$
$$\gamma_t^\lambda - \gamma_t^* \le 0, \quad \pi_t^{*\lambda}(\gamma_t^\lambda - \gamma_t^*) = 0 \quad \text{each } \lambda; \qquad \gamma_t^* = \Sigma_\lambda \pi_t^{*\lambda}\gamma_t^\lambda\,; \qquad \text{...(5.9)}$$

cf.(3.18–19). It remains to 'assemble' the solutions π_t^* into a function $\pi^*\colon t \mapsto \pi_t^*$ defined for $t \ge 0$. As in Section 3, the continuity of this function, and that of the 'maximum value function' $t \mapsto \psi(\pi_t^*,t)$, follows from the continuity of $(\breve{\pi},t) \mapsto \psi(\breve{\pi},t)$. This can be checked by a direct argument, taking into account the uniqueness of π_t^* for each t, or it can be inferred from a 'maximum theorem', e.g. Bank (1983) T.4.3.3. Thus $\pi^* \in \Pi^{+s}$, and obviously π^* maximises $\Psi(\pi,T)$ for each T. To sum up, we have

THEOREM 6: *Optimality Theorem for* X *with Moving Discontinuities,* $\Pi = \Pi^{+s}$.
Let X have representation (4.1). Suppose that M^c and V^c satisfy the assumptions of Theorem 4, that X has no fixed discontinuities, and that M^d and V^d satisfy the assumptions of Proposition 2. Then there exists a unique optimal sure non–negative portfolio plan π^*. For each t, the vector π_t^* is characterised by the programming conditions (5.8–9). The functions $t \mapsto \pi_t^*$ and
$t \mapsto \psi(\pi_t^*,t) = (1-b)^{-1}(d/dt)\ell n\,[E\,e^{(1-b)x^*(t)}]$ are continuous.

REMARKS. (i) The simplicity of the proof of Theorem 6 is due partly to the assumptions which ensure the uniqueness of the maximising vector π_t^*, and also, for $b > 1$, to the assumption ruling out the possibility that $\psi(\check{\pi},t)$ takes the value $-\infty$ for some $\check{\pi}$, possibly on different subsets of $\mathscr{S}$ for different t. Without these simplifications, a problem of 'continuous selection' arises if a continuous optimal π^* is required – see Bank (1983), Wagner (1989), also Dutta & Mitra (1989); I have not attempted to work out the details.

(ii) For simplicity, conditions have been imposed on the characteristics of X which yield a continuous, rather than merely a collor, optimal sure π^*. If only a collor π^* is required, it is clear that v^λ, $\sigma^{\lambda\ell}$ and g need only be collor, but the choice of a suitable definition of 'left continuity with right limits' for the family (f_t) is less obvious. It is enough (but perhaps too much) to assume in place of (4.50–51) that for each τ the measures $(f_t;t\leq\tau)$ are absolutely continuous with respect to f_τ with $df_t/df_\tau \longrightarrow 1$ uniformly in ξ as $t \uparrow \tau$, and further that there exists a measure $f_{\tau+}$, absolutely continuous with respect to each f_t with $t > \tau$, such that $df_{\tau+}/df_t \longrightarrow 1$ uniformly as $t \downarrow \tau$.

(iii) If non–negative measurable portfolio functions are admitted, a result analogous to Theorem 6 holds with 'continuous' replaced by 'measurable', provided that due allowance is made for null sets. In this case, 'continuous' may be replaced by 'measurable' in the assumptions of Theorem 4. The assumption in (4.49) that g is continuous may be omitted, and conditions (4.50–51) are not needed. The only new point of substance to be verified is that the functions $t \mapsto \pi_t^*$ and $t \mapsto \psi(\pi_t^*,t)$ are measurable; this may be obtained (for example) as a special case of the implicit function lemma in Beneš (1970).

(iv) We conclude this Section with an important, if rather obvious, general point. Under present assumptions, the analytic properties of an optimal portfolio plan, in particular the properties of continuity, reflect corresponding properties of the functions v^λ, $\sigma^{\lambda\ell}$, f and g which determine the characteristics of the market process X, not the properties of the sample paths of X. In particular, jumps of X at movable (totally inaccessible) times do not as such give rise to portfolio discontinuities.

6. STATIONARY INCREMENTS

Only a few words need be added here about the special case where X is a PSII–semimartingale with $X(0) = 0$ and $q(t) = e^{-rt}$ with some real r. These assumptions imply that the functions v_t^λ and $\sigma_t^{\lambda\ell}$ exist and have constant values v^λ and $\sigma^{\lambda\ell}$, and that for all t one can set $dG(t)/dt \equiv g(t) \equiv 1$, $f(.,t) \equiv f(.)$ with some measure f, so that $F(d\xi,dt)$ may be replaced throughout by $f(d\xi)dt$. Of course, X has no fixed discontinuities. For brevity, we consider only the case $\Pi = \Pi^{+s}$, but the discussion of the case $\Pi = \Pi^{os}$ with X continuous in Section 3 could be extended along similar lines. We set $K_0 = 1$.

The following is now an immediate corollary of Theorem 5.

PROPOSITION 3. If $\pi \in \Pi^{+s}$ is such that $\Psi(\pi,T)$ is finite, then

$$\Psi(\pi,T) = \int_0^T \psi(\pi_t)dt, \qquad ...(6.1)$$

where ψ is defined as in (5.7) with the variable t omitted and $g \equiv 1$. Similarly, (1) holds if Ψ and ψ are replaced by Ψ^c and ψ^c with ψ^c defined as in (3.13), or again by Ψ^d and ψ^d with ψ^d as in (4.53). If π is invariable, i.e. if $\pi_t = \pi_1$ for all t, where π_1 is some fixed vector in $\mathscr{S}$, then x^π is a PSII and

$$\Psi(\pi,T) = T\psi(\pi_1). \qquad ...(6.2)$$

If $\Psi(\pi,T)$ is finite on $\mathscr{T}$ for all $\pi \in \Pi^{+s}$, then $\psi(\breve{\pi})$ is defined and finite on $\mathscr{S}$ and conversely. In this case ψ is concave and continuous on $\mathscr{S}$. If in addition either $[\sigma^{\lambda\ell}]$ is positive definite or $f(\Xi) > 0$, then ψ is strictly concave on $\mathscr{S}$, and an optimal sure π exists and is unique and invariable. The programming conditions characterising an optimal vector π_1^* are obtained immediately from (5.8–9). Formula (1) and the strict concavity have been used (with a promise of proof to come) in Foldes (1991a) S.5, and the additional points to be made about the present special case have been set out there.

7. FIXED DISCONTINUITIES

The consequences of allowing X to have fixed discontinuities will be discussed rather briefly, omitting arguments analogous to those given in earlier Sections. Thus we shall not set out the extension of the alternative proof of Theorem 1 to this case; it follows the same lines as in Section 5, but with a more complicated formula for Ψ – see (7–10) below.

If X is a PII–semimartingale with a set $\mathcal{J}$ of fixed times of discontinuity, one can represent X as the sum $\bar{X} + \bar{\bar{X}}$ of two independent PII–semimartingales, where $\bar{\bar{X}}$ has no fixed discontinuities and $\bar{X}$ has no continuous part and no moving discontinuities. We set $\bar{X}_0 = \bar{\bar{X}}_0 = 0$. For simplicity, we assume from the outset that $\bar{\bar{X}}$ has the properties assumed for X in Theorem 6. [1] When X is replaced by $\bar{\bar{X}}$ in the discussion of previous Sections, we write $\bar{\bar{\mu}}, \bar{\bar{F}}, \bar{\bar{M}}^d, \bar{\bar{V}}^d$ in place of μ, F, M^d, V^d; in particular, $\bar{\bar{X}}$ has a representation like (4.1), and $\bar{\bar{M}}^d, \bar{\bar{V}}^d$ are defined as in (4.12) and (4.7) with appropriate replacements.

Let (τ_m) be an enumeration of the points of $\mathcal{J}$, and let $\Delta\bar{X}_m$ with components $\Delta\bar{X}_m^\lambda$ be the corresponding random variables representing jumps of $\bar{X}$ at τ_m. In general, a finite interval of $\mathcal{T}$ can contain an infinite subset of $\mathcal{J}$, so that the τ_m may not be in ascending order. A canonical representation of $\bar{X}$ may be written

$$\begin{aligned}\bar{X}_T^\lambda &= \Sigma_{\tau_m \le T} \int_{|\xi|\le 1} \xi^\lambda \cdot (\bar{\mu}-\bar{F})(d\xi \times \{\tau_m\}) \\ &+ \Sigma_{\tau_m \le T} \int_{|\xi|\le 1} \xi^\lambda \cdot \bar{F}(d\xi\times\{\tau_m\}) + \Sigma_{\tau_m\le T}\int_{|\xi|>1} \xi^\lambda \cdot \bar{\mu}(d\xi\times\{\tau_m\}); \qquad \ldots(7.1)\end{aligned}$$

of course, one can replace $\Sigma_{\tau_m\le T}$ by $\int_0^T$ and simultaneously $\{\tau_m\}$ by $\{t\}$. For brevity, we shall sometimes write $\bar{F}_m(d\xi)$ or $d\bar{F}_m$ for $\bar{F}(d\xi\times\{\tau_m\})$. Adjusting for differences of

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[1] The usual definitions – see Jacod (1979) p.91 – are $\bar{X} = \int I\{\mathcal{J}\}\, dX$, $\bar{\bar{X}} = X - \bar{X}$; but then the Lévy measure $\bar{\bar{F}}$ of $\bar{\bar{X}}$ will be insufficiently smooth at points of $\mathcal{J}$ to satisfy the conditions of Theorem 6. Thus we are in effect assuming that $\bar{\bar{X}}$ (as usually defined) can be altered on $\mathcal{J}$ so as to satisfy these conditions and still be a PII–semimartingale independent of $\bar{X}$.

notation and the replacement of integrals by sums, the properties of $\bar{\mu}$ and $\bar{F}$ and the integrals which they define are mainly analogous to those explained in Section 4, up to the point where portfolios were introduced. We shall therefore only comment on a few points. An additional property of $\bar{F}$ is that the sum $\Sigma_{\tau_m \le T} \int_\Xi |\xi| d\bar{F}_m$ converges absolutely for each T. Thus the second and third sums on the right of (1) are of finite variation and comprise $\bar{V}^{\lambda d}$, while the first sum is taken as the martingale part $\bar{M}^{\lambda d}$. If each [0,T] contains only a finite subset of $\mathcal{J}$, the right–hand side of (1) reduces to $\Sigma_{\tau_m \le T} \int_\Xi \xi^\lambda \bar{\mu}(d\xi \times \{\tau_m\})$, and then compensated integrals are not needed. If $\bar{F}$ is factorised in the form $\bar{f}(d\xi,t)d\bar{G}(t)$ as in (4.3), the function $\bar{G}$ with $\bar{G}(0) = 0$ is corlol, non–decreasing and finite on $\mathcal{T}$, with jumps at points of $\mathcal{J}$ and constant otherwise.

The definitions of integrals of the form $\int_0^T \int_\Xi W \cdot d\bar{\mu}$ and $\int_0^T \int_\Xi W \cdot d\bar{F}$ with $W \in \mathfrak{P}$ are analogous to those in Section 4. An integral $\int_0^T \int_\Xi W \cdot d(\bar{\mu} - \bar{F})$ – cf. (4.8) and (4.10) – now assumes the form

$$\Sigma_{\tau_m \le T} \int_\Xi W_m(\xi) \cdot (\bar{\mu} - \bar{F})(d\xi \times \{\tau_m\}), \qquad \text{...(7.2)}$$

where $W_m(\xi) = W(\omega, \xi, \tau_m)$. The expression corresponding to (4.9) which must be locally integrable if (2) is to be well defined is

$$\Sigma_{\tau_m \le T} \int_\Xi \left[|W_m|^2 \cdot I_{|W_m| \le 1} + |W_m| \cdot I_{|W_m| > 1} \right] d\bar{F}, \qquad \text{...(7.3)}$$

and then (2) represents a compensated–jump local martingale $\mathcal{L}$ with $\mathcal{L}(0) = 0$ such that $\Delta \mathcal{L}_t = 0$ for $t \notin \mathcal{J}$ and

$$\Delta \mathcal{L}(\omega, \tau_m) = \int_\Xi W_m(\xi) \cdot (\bar{\mu} - \bar{F})(d\xi \times \{\tau_m\}); \qquad \text{...(7.4)}$$

this should be compared with (4.11), noting the appearance of $\bar{F}$. The analogue of (4.13) remains valid, as does the assertion that a PII–local martingale $\mathcal{L}$ (possibly with fixed discontinuities) satisfying $E|\mathcal{L}_T| < \infty$ for $T \in \mathcal{T}$ is a true martingale.

Now let $\bar{\Pi}$ be the set of all sequences $(\bar{\pi}_m)$ with $\bar{\pi}_m \in \mathcal{S}$ for each m. According to the definition in Section 2, an element $\pi \in \Pi^{+s}$ has the form

$$\pi_t = \bar{\bar{\pi}}_t I_{\{t \notin \mathcal{J}\}} + \Sigma_m \bar{\pi}_m I_{\{t = \tau_m\}} \qquad \text{...(7.5)}$$

where $\bar{\bar{\pi}}$ is a *collor* element of Π^{+s} and $\bar{\pi} \in \bar{\Pi}$.

We denote by $\bar{\bar{x}}^{\pi}$ the process defined by (2.5–6) with X, M^d, V^d replaced by $\bar{\bar{X}}$, $\bar{\bar{M}}^d$, $\bar{\bar{V}}^d$ and π by $\bar{\bar{\pi}}$, and write $\bar{\bar{\Psi}}(\pi,T) = \bar{\bar{\Psi}}(\bar{\bar{\pi}},T)$, $\bar{\bar{\psi}}(\pi,t) = \bar{\bar{\psi}}(\bar{\bar{\pi}},t)$ for the functions calculated like $\Psi(\pi,T)$, $\psi(\pi,t)$ in the proof of Theorem 6 with the same replacements and $\bar{\bar{\mu}}$, $\bar{\bar{F}}$ instead of μ, F. In the same way, $\bar{x}^{\pi}$ will be the process defined by (2.5–6) with $\bar{X}$, $\bar{M}^d$, $\bar{V}^d$ and $\bar{\pi}$. The calculation of $\bar{x}^{\pi}$ and its expectation (if defined) is analogous to (4.14–16), except that a term $\Sigma_{\tau_m \le T} \int_{|\xi|\le 1} \Sigma_{\lambda} \bar{\pi}_m^{\lambda} \xi^{\lambda} \cdot d\bar{F}_m$ must be added to $\bar{x}_T^{\pi\mu o}$, $\bar{x}_T^{\pi F O}$, $\bar{x}_T^{\pi}$ and $E\bar{x}_T^{\pi}$; then one obtains simply

$$E\bar{x}_T^{\pi} = \Sigma_{\tau_m \le T} \int_{\Xi} \ell n(\Sigma_{\lambda} \bar{\pi}_m^{\lambda} e^{\xi^{\lambda}}) \bar{F}(d\xi \times \{\tau_m\}). \qquad \text{...(7.6)}$$

We retain the notation $\Psi(\pi,T)$ for the function (1.1) calculated from X and define

$$\bar{\Psi}(\pi,T) = \bar{\Psi}(\bar{\pi},T) = (1-b)^{-1} \ell n \, E e^{(1-b)\bar{x}^{\pi}(T)} \qquad \text{...(7.7)}$$

for $t \in \mathscr{T}$ and $\pi \in \Pi^{+s}$, or equivalently $\bar{\pi} \in \bar{\Pi}$. At present, $(1-b)\Psi = \infty$ is permitted, but $\bar{\bar{\Psi}}$ is finite by the assumptions of Theorem 6.

Now note that, with probability one, $\bar{\bar{X}}_T$ is unaltered for all $T \in \mathscr{T}$ if in (4.1) the points of $\mathscr{J}$ are omitted from the time integrals, so that $\bar{\bar{x}}^{\pi}$, $\bar{\bar{\Psi}}(\pi,T)$ are unaltered also. Taking into account the independence of $\bar{X}$ and $\bar{\bar{X}}$, it is clear that

$$\Psi(\pi,T) = \bar{\Psi}(\bar{\pi},T) + \bar{\bar{\Psi}}(\bar{\bar{\pi}},T) \qquad \text{...(7.8)}$$

for $T \in \mathscr{T}$ and $\pi \in \Pi^{+s}$. Consequently an element π^* maximises $\Psi(\pi,T)$ among all $\pi \in \Pi^{+s}$ for each $t \in \mathscr{T}$ iff both

(i) $\bar{\bar{\pi}}^*$ maximises $\bar{\bar{\Psi}}(\bar{\bar{\pi}},T)$ among all *collor* $\pi \in \Pi^{+s}$ for each $T \in \mathscr{T}$ – or equivalently, under the assumptions of Theorem 6, $\bar{\bar{\pi}}_t^*$ maximises $\bar{\bar{\psi}}(\check{\pi},t)$ on $\mathscr{S}$ for each $t \in \mathscr{T} \setminus \mathscr{J}$, and

(ii) $\bar{\pi}^*$ maximises $\bar{\Psi}(\bar{\pi},T)$ among all $\bar{\pi} \in \bar{\Pi}$ for each $T \in \mathscr{T}$, or equivalently for each $T \in \mathscr{J}$.

Thus, under present definitions, optimal portfolio choice at fixed times of discontinuity of X may be considered separately from choice at other times. We now confine attention to problem (ii).

We require a formula for $\bar{\Psi}(\bar{\pi},T)$. For each $\check{\pi} \in \mathscr{S}$, define

$$(1-b)\psi_m(\check{\pi}) = \ell n \, E\left[\Sigma_{\lambda} \check{\pi}^{\lambda} e^{\Delta x^{\lambda}}\right]^{(1-b)} = \ell n \int_{\Xi} \left[\Sigma_{\lambda} \check{\pi}^{\lambda} e^{\xi^{\lambda}}\right]^{(1-b)} \bar{P}_m(d\xi), \qquad \text{...(7.9)}$$

where $\bar{P}_m$ is the law of the vector $\Delta\bar{X}_m$. The definition of a fixed discontinuity means that $\bar{P}_m$ is not degenerate at the origin, so that $(1-b)\psi_m > -\infty$ always. If $\mathscr{J}$ has at most a finite number of points in each finite interval of $\mathscr{T}$, it is not hard to see that

$$\bar{\Psi}(\bar{\pi},T) = \Sigma_{\tau_m \leq T}\, \psi_m(\bar{\pi}_m). \qquad ...(7.10)$$

In the general case, it still seems fairly obvious that (10) is valid if $\bar{\Psi}(\bar{\pi},\cdot)$ is finite on $\mathscr{T}$, or equivalently if the series on the right is absolutely convergent for each T, but it is of interest to outline an argument analogous to that used in the proof of Theorem 5.

Let $\bar{\eta} = \exp\{(1-b)\bar{x}\}$, $\bar{x} = \bar{x}^{\bar{\pi}}$, $\Delta\bar{x}_m = \bar{x}(\tau_m) - \bar{x}(\tau_m-)$. We form $\mathscr{L}\bar{\eta}$ as in (4.17) with M^d, x, π replaced by $\bar{M}^d$, $\bar{x}$, $\bar{\pi}$, noting that now we have simply

$$(\mathscr{L}\bar{\eta})_T = \int_0^T (1/\bar{\eta}_{t-})d\bar{\eta}_t = \Sigma_{\tau_m \leq T}\, [\bar{\eta}(\tau_m)/\bar{\eta}(\tau_m-) - 1] = \Sigma_{\tau_m \leq T}\, [e^{(1-b)\Delta\bar{x}_m} - 1].$$

Taking expectations, replacing $\bar{\mu}$ by $\bar{F}$ where appropriate and simplifying we have

$$E(\mathscr{L}\bar{\eta})_T = \Sigma_{\tau_m \leq T} \int_{\Xi} \left[(\Sigma_\lambda \bar{\pi}_m^\lambda e^{\xi^\lambda})^{(1-b)} - 1\right] \bar{F}_m(d\xi),$$

and it follows from the properties of $\bar{F}$ that the sum on the right either converges absolutely or diverges to $+\infty$, the equality being valid in either case. Further, since the integrand vanishes at $\xi = \{0\}$, we may replace $\bar{F}_m$ by $\bar{P}_m$ and write the equation as

$$E(\mathscr{L}\bar{\eta})_T = \Sigma_{\tau_m \leq T}\, E\left[e^{(1-b)\Delta\bar{x}_m} - 1\right] = \Sigma_{\tau_m \leq T} \left[e^{(1-b)\psi_m(\bar{\pi}_m)} - 1\right]. \qquad ...(7.11)$$

Now, if $E\bar{\eta}$ is finite on $\mathscr{T}$, the change–of–variables formula shows that $(\mathscr{L}E\bar{\eta})_T$ is given by either of the sums in (11) and that these sums converge absolutely. Thus $\mathscr{L}E\bar{\eta} = E\mathscr{L}\bar{\eta}$ whenever one side or the other is defined and finite on $\mathscr{T}$, which may be compared with (4.25–6); on calculating $E\bar{\eta} = \mathscr{E}\mathscr{L}(E\bar{\eta}) = \mathscr{E}E(\mathscr{L}\bar{\eta})$ and adjusting the constants we obtain (10). Conversely, suppose that the sum on the right of (10) converges absolutely for each T, denote its value by $(1-b)^{-1}\ell n\, \zeta_T$, and note that ζ_T defines a (deterministic) semimartingale with $\zeta_0 = 1$. Calculating $(\mathscr{L}\zeta)_T$ we again obtain the sum in the middle term of (11), which is absolutely convergent and equal to $E(\mathscr{L}\bar{\eta})_T$, hence also to $(\mathscr{L}E\bar{\eta})_T$; but then $E\bar{\eta}_T = \zeta_T < \infty$, which yields (10).||

Thus the series on the right of (10) converges absolutely iff the left side is defined and finite. We fix T and note some conditions which are sufficient to ensure that this convergence occurs for *all* $\bar{\pi} \in \bar{\Pi}$, and further that $\bar{\Psi}(\bar{\pi},T)$ is uniformly bounded. Using

(4.34) if $b < 1$, (4.37) if $b > 1$, it is seen that $\psi_m(\check{\pi})$ is finite for all $\check{\pi} \in \mathscr{S}$ if

$$\Sigma_\lambda \int_\Xi e^{(1-b)\xi^\lambda} \bar{P}_m(d\xi) = \Sigma_\lambda E e^{(1-b)\Delta x^\lambda(\tau_m)} < \infty. \qquad ...(7.12)$$

Thus $(1-b)\Sigma_{\tau_m \le T} \psi_m(\bar{\pi}_m)$ is uniformly bounded *above* for $\bar{\pi} \in \Pi$ if

$$\Pi_{\tau_m \le T} \Sigma_\lambda \int_\Xi e^{(1-b)\xi^\lambda} \bar{P}_m(d\xi) < \infty, \qquad ...(7.13)$$

and uniformly bounded *below* if

$$\Pi_{\tau_m \le T} \int_\Xi e^{(1-b)\min_\lambda \xi^\lambda} \bar{P}_m(d\xi) > 0 \quad \text{in case } b < 1,$$

$$\Pi_{\tau_m \le T} \int_\Xi e^{(1-b)\max_\lambda \xi^\lambda} \bar{P}_m(d\xi) > 0 \quad \text{in case } b > 1. \qquad ...(7.14)$$

We now assume – without necessarily adopting (13–14) – that $\bar{\Psi}(\bar{\pi},T)$ is defined and finite for all $\bar{\pi} \in \bar{\Pi}$ and $T \in \mathscr{T}$ and that (12) is satisfied for each m. As in Section 4, it follows from (12) by dominated convergence that ψ_m is continuous on $\mathscr{S}$. Thus a maximum exists at some point $\bar{\pi}^*_m$, and it further follows from (12) that necessary conditions may be obtained by differentiating under the integral sign in (9). These conditions are also sufficient, and the maximising point is unique, because ψ_m is strictly concave on $\mathscr{S}$. Thus (as might have been expected) the portfolio problem can be solved for each τ_m separately; we omit further details.

We shall not attempt an exhaustive discussion of non–existence, but clearly such cases arise for $b < 1$ $(b > 1)$ if, for some m, $(1-b)\psi_m(\check{\pi}) = \infty$ for some (every) $\check{\pi}$. If some interval $[0,T]$ contains an infinite subset of $\mathscr{J}$, there are the additional possibilities that the terms of the series (10) are all finite, but that their sum diverges to $+\infty$ $(-\infty)$ for some (all) π.

It is of interest to ask whether, if an optimal element of Π^{+s} exists, it may in some cases be possible to choose a *collor* optimal element even in the presence of fixed discontinuities. Consider first cases with $\bar{\bar{X}} \equiv 0$. If $\mathscr{J}$ consists of a single time $\tau > 0$ and π^* is optimal sure, then any π with $\pi(\tau) = \pi^*(\tau)$ is also optimal, and a collor version is obviously defined by setting $\pi(t) = \pi^*(\tau)$ for $t \le \tau$. Again, if $\mathscr{J} = (\tau_m)$ has no point of accumulation in $\mathscr{T}$, the times τ_m can be enumerated in an ascending sequence with intervals of positive length between successive times, and if π^* is optimal sure we obtain a collor version by setting $\pi(t) = \pi^*(\tau_m)$ on $(\tau_{m-1}, \tau_m]$. The situation is quite different if $\mathscr{J}$ has a finite point of accumulation. Suppose for instance that $\mathscr{J}$ consists of an ascending sequence (τ_m) together with its finite limit $\tau_\diamond$. Let π^* be optimal sure and suppose that for each m the vector $\pi^*(\tau_m) = \pi^*_m$ is the unique element of $\mathscr{S}$ maximising ψ_m, similarly for $\pi^*(\tau_\diamond) = \pi^*_\diamond$ and $\psi_\diamond$. In the absence of a suitable relationship between the laws of $\Delta\bar{X}_m$ and $\Delta\bar{X}_\diamond$ there need be no convergence of π^*_m to $\pi^*_\diamond$, so that in general there will be no collor function π satisfying $\pi(\tau_m) = \pi^*_m$ and $\pi(\tau_\diamond) = \pi^*_\diamond$. (The difficulty disappears if, for example, the $\bar{F}_m$ are absolutely continuous with respect to $\bar{F}_\diamond$ with derivatives $\theta_m(\xi) = d\bar{F}_m/d\bar{F}_\diamond$ tending to 1 uniformly in $\xi \neq 0$ as $m \to \infty$, and if for each λ the variable $e^{(1-b)\xi\lambda}$ is $\bar{F}_\diamond$–integrable. The argument is similar to that in Section 4, the inequalities (4.34) or (4.37) being used to justify the passage to the limit under the integral sign, and the uniqueness of solutions to show that $\pi^*_\diamond = \lim \pi^*_m$.) The case where $\bar{\bar{X}}$ is a continuous PII and $\mathscr{J}$ consists of a single time $\tau > 0$ has been illustrated in Foldes (1991a) S.7. If π^* is optimal with (say) π^*_t unique for each $t \in \mathscr{T}$, there is in general no reason why π^*_t should converge to π^*_τ as $t \uparrow \tau$, so π^* need not be collor at τ. In this case there seem in general to be no grounds for imposing conditions on the characteristics of X which would ensure convergence. Note that the conclusion that no collor optimum exists does not stand if there is an interval of the form $[\tau-\delta, \tau)$ during which $\bar{\bar{X}}$ is constant, or if there is an interval $(\tau-\delta, \tau]$ during which no portfolio revision is permitted – roughly speaking, if 'the market is closed pending an announcement'. The case where $\bar{\bar{X}}$ is a PII with movable discontinuities and $\mathscr{J} = \{\tau\}$ is similar.

8. LOGARITHMIC UTILITY

In case $b = 1$, an optimal sure portfolio plan is an element π^* of Π^s which maximises $\Psi^{\ell}(\pi,T)$, or equivalently $Ex^{\pi}(T)$, for each T among all $\pi\epsilon\Pi^s$. We shall review this case briefly, noting only the main alterations to the preceding theory. We assume that X has no fixed discontinuities.

To obtain an explicit formula for $Ex^{\pi}(T)$ with π sure, consider the right–hand side of (2.5) line by line. The first line is a martingale with zero expectation. The second line is deterministic, so $Ex^{\pi c}$ is just the expression in that line, which is always defined and finite. If X has (moving) discontinuities, there are additional terms $x^{\pi FO} + x^{\pi F1}$ given by (4.16) with (4.15); thus, with obvious notation,

$$\Psi^{\ell}(\pi,T) \;=\; \Psi^{\ell c} + \Psi^{\ell o} + \Psi^{\ell 1} \;=\; Ex_T^{\pi c} + x_T^{\pi FO} + x_T^{\pi F1} \qquad \text{...(8.1)}$$

if $x_T^{\pi F1} = Ex_T^{\pi\mu 1}$ is defined, and we set $\Psi^{\ell}(\pi,T) = \Psi^{\ell 1}(\pi,T) = -\infty$ otherwise. The calculation of a formula for $\ell n\, Ee^{x^{\lambda} - x^*}$ – see (1.14) – is performed in the same way as for similar expressions with $b \neq 1$.

Suppose first that X is continuous and reconsider the proof of Theorem 1 in Section 3. It is found that $\Psi^{\ell c}(\pi,T)$ is given by (3.4) with $b = 1$, and clearly $\ell n\, Ee^{x^{\pi} - x^*}$ is given by (3.5) with $b = 1$; the proof then goes through, with $\Psi^{\ell c}$ in place of Ψ^c, without substantial change. The proof for general X with $\Pi = \Pi^{+s}$ also proceeds along the same lines as for $b \neq 1$, but since in this case the theorem has been proved in Foldes (1991a) by a simpler method and without restrictive assumptions (except, for brevity, that Ex_t^{π} is always finite) we shall not go into details. Returning to the case of continuous X, the rest of Section 3 applies with only routine changes. In particular, the programming conditions for an optimum remain valid if we set $b = 1$, and Theorems 3 and 4 stand.

Passing to Section 4, we need to replace the bounds for the integrands in J^{FO} and J^{F1} by analogous bounds – uniform with respect to π – for the integrands in $x^{\pi FO}$ and $x^{\pi F1}$. For $x^{\pi FO}$ we refer to the second line of (4.15), replacing μ by F. Using (4.38), it is seen that the integrand is non–negative. On the other hand, the integrand is bounded above by $[\max_\lambda \xi^\lambda - \min_\lambda \xi^\lambda] I_{\{|\xi|\leq 1\}}$, or again, using $\ell n\, z < z-1$, by $\Sigma_\lambda [e^{\xi^\lambda} - \xi^\lambda - 1] I_{\{|\xi|\leq 1\}}$. Turning to $x^{\pi F1}$ and referring to the third line of (4.15), we obtain from (4.38) a lower bound for the integrand of $\Sigma_\lambda [\pi^\lambda \xi^\lambda] I_{\{|\xi|>1\}}$, which in turn is bounded below by $-\Sigma_\lambda |\xi^\lambda| I_{\{|\xi|>1\}}$. An upper bound is obtained from

$$\ell n\, \Sigma \pi^\lambda e^{\xi^\lambda} \leq \ell n\, \Sigma e^{|\xi^\lambda|} \leq \ell n\, e^{\Sigma |\xi^\lambda|} = \Sigma |\xi^\lambda|.$$

On the other hand, if π is the one–security portfolio λ, we have

$$x_T^{\lambda F1} = \int_0^T \int_{|\xi|>1} \xi^\lambda F(d\xi, dt), \qquad \text{...(8.2)}$$

and if this is defined and finite the integral with ξ^λ replaced by $|\xi^\lambda|$ must also be finite. Combining these remarks we have – cf. Proposition 1 (iii) –

PROPOSITION 4. For given T, $\Psi^\ell(\pi,T)$ is finite for every $\pi \in \Pi^{+s}$ iff $\Psi^\ell(\lambda,T)$ is finite for each λ.

The rest of Section 4 requires only routine changes. Introducing assumptions (4.49) and (4.50–51) and writing, for $\breve{\pi} \in \mathscr{S}$ and $\xi \in \mathfrak{R}^\Lambda$,

$$h^\ell(\breve{\pi},\xi) = \ell n(\Sigma_\lambda \breve{\pi}^\lambda e^{\xi^\lambda}) - I_{\{|\xi|\leq 1\}} \Sigma \breve{\pi}^\lambda \xi^\lambda, \qquad \text{...(8.3)}$$

we define $h^{\ell o}$, $h^{\ell 1}$, $\psi^{\ell d}$, $\psi^{\ell o}$, $\psi^{\ell 1}$ by analogy with the definitions of h^o, h^1 etc. – see (4.28) and (4.53). The bounds on the integrands in $x^{\pi Fo}$ and $x^{\pi F1}$ mentioned above are bounds on $h^{\ell o}(\pi_t,\xi)$ and $h^{\ell 1}(\pi_t,\xi)$ which are uniform with respect to π for $t \leq T$. Each upper or lower bound for $h^{\ell o}$ and $h^{\ell 1}$ which is $F(d\xi,dt)$–integrable on $\mathfrak{R}^\Lambda \times [0,T]$ defines a corresponding upper or lower bound for $\psi^{\ell o}(\breve{\pi},t)$ and $\psi^{\ell 1}(\breve{\pi},t)$ on $\mathscr{S}$ for each $t \leq T$; and if F–integrable upper and lower bounds are in force in both cases for each T then $\psi^{\ell o}$ and $\psi^{\ell 1}$, hence also $\psi^\ell = \psi^{\ell o} + \psi^{\ell 1}$, are continuous on $\mathscr{S} \times \mathscr{T}$. In fact, it can be shown as in Section 4 that $\psi^{\ell o}$ is continuous in any case. The concavity properties of the functions are as in Section 4, and it follows that Proposition 2 holds for $b = 1$ provided that (4.35) is replaced by the condition that the integrals (8.2) are defined and finite for all λ and T, and that ψ, Ψ, J^{Fo}, J^{F1} etc. are replaced by ψ^ℓ, Ψ^ℓ, x^{Fo}, x^{F1} etc.

Turning to Section 5 (and skipping the proof of Theorem 1), we replace the formula for ψ in (5.7) by the corresponding expression for ψ^ℓ and obtain programming conditions as before. There are minor differences in the calculation, but the conditions are the same as (5.9), provided that one sets $b = 1$ in (5.8). Theorem 6 then stands with ψ replaced by ψ^ℓ and the assumptions of Theorem 4 and Proposition 2 revised as above.

9. CONCLUSION: CASES OF NON–EXISTENCE OF AN OPTIMUM

We conclude with an informal survey of cases where either no optimum exists in the PII/CRRA model or where existence has not been proved. Here we take into account both the results of the present paper concerning *sure portfolio plans* and the results of Foldes (1991a) concerning *portfolio–cum–saving plans* which were summarised in Section 1. Until further notice we assume $b \neq 1$.

Because of Theorems 1 and 2, non–existence of an optimal PS plan is (subject to minor conditions) equivalent to non–existence of an optimal sure PS plan. Further, because of Theorem 4 of Foldes (1991a), cases of non–existence of an optimal sure plan can be classified into (i) cases where there is no optimal sure portfolio plan, i.e. no $\pi^* \epsilon \Pi^s$ which maximises (1.1) for each T, and (ii) cases where such a π^* does exist but there is no c^* which is π^*–optimal. A case of type (ii) is equivalent to a case of non–existence of an optimal c–plan in a CRRA model of certainty with one asset, a problem which has been extensively studied. As stated above – see (1.8–10) – an optimal c^* exists under our assumptions iff $\mathfrak{N}^{\pi^*} < \infty$. We shall not go further into the classification of parameter values for which this criterion may fail, but merely remark that the definition of an optimal π^* includes the requirement that $Ee^{(1-b)x^*(T)} < \infty$ for each T, and then it is always possible to find some discount density q for which a π^*–optimal element c^* exists.

We turn to cases of non–existence of an optimal sure π^* when $\Pi = \Pi^{+s}$, taking as benchmark the situation where all assumptions mentioned in Theorem 6 are in force and relaxing them one at a time. If $b < 1$, the presence in Π^{+s} of *some portfolio plan* π such that $Ee^{(1-b)x^{\pi}(T)} = \infty$ at some T implies non–existence, and it has been shown that this is equivalent to the presence of *some asset* λ such that $Ee^{(1-b)x^{\lambda}(T)} = \infty$ for some T. Symmetric stable processes with infinite variance (which have sometimes been proposed as models for speculative prices) provide examples. If $b > 1$, there obviously cannot be an optimum if $Ee^{(1-b)x^{\pi}(T)} = \infty$ at some T for *every* $\pi \epsilon \Pi^{+s}$, and according to

Proposition 1 above this possibility is excluded if $Ee^{(1-b)x^{\lambda}(T)} < \infty$ for all T for each asset λ; whether it can happen that, at some T, $Ee^{(1-b)x^{\lambda}(T)} = \infty$ for each λ but $Ee^{(1-b)x^{\pi}(T)} < \infty$ for some $\pi \in \Pi^{+s}$, and whether an optimal sure π^* can exist in such a case, remain open questions. Note that problems of this type cannot occur when X is continuous. We now restrict attention to cases with $Ee^{(1-b)x^{\pi}(T)}$ finite for all T and all sure π.

There then remain — the classification is somewhat arbitrary — various possible causes of non–existence due to insufficient smoothness of the characteristics of X with respect to time. If X is continuous, or has a continuous component, cases of non–existence of an optimal element in Π^{os} or Π^{+s} can occur if $\langle M^c \rangle$ and V^c do not possess collor derivatives; (however, if measurable derivatives exist, conditions for the existence of an optimal sure measurable π are readily obtained). There are analogous cases of non–existence in Π^{+s} when X has movable discontinuities and the functions $f(d\xi,.)$ and $g(.)$ are insufficiently smooth — indeed, this is clear from the non–uniqueness of the decomposition (2.1) of X. If X has a set $\mathcal{J}$ of fixed discontinuities with a finite point of accumulation, or if X has fixed discontinuities as well as either a continuous component or a component with movable discontinuities, there is in general no optimal *collor* π; however an optimal sure $\pi \in \Pi^{+s}$ will exist under conditions stated above. Once again, there are cases of non–existence due to infinite expectations. In addition, if X is continuous and $\Pi = \Pi^{os}$, singularity of a covariance matrix $\langle M^c \rangle_T - \langle M^c \rangle_S$ can give rise to non–existence in the form of an infinite sequence of ever–better portfolio plans, diverging in norm and yielding unbounded 'arbitrage' profits.

Finally, if $b = 1$, the problems of the existence of an optimal c^* and of an optimal π^* can be stated quite separately if one takes the functional as given in the form (1.7), and then an optimal c^* exists iff the functions q and $q \cdot |\ell n\, q|$ are Lebesgue integrable on $\mathcal{T}$ — see Foldes (1991a) eq.(6.6). The cases of non–existence of an optimal sure portfolio plan broadly parallel those for $b \neq 1$. In particular, if $\Pi = \Pi^{+s}$, existence fails if (i) for some π, $Ex_T^{\pi} = \infty$ for some T, or (ii) for every π, $Ex_T^{\pi} = -\infty$ (or Ex_T^{π} is undefined) for some T. Cases involving infinite values of Ψ^{ℓ} cannot arise if Ex_T^{λ} is defined and finite for all λ and T. There then remain possibilities of non–existence due to insufficient smoothness of the characteristics of X. If X is continuous and $\Pi = \Pi^{os}$, there can again be non–existence due to singularity of a covariance matrix.

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