A Structural Model of Corporate Bond Pricing with Co-ordination Failure

By

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A Structural Model of Corporate Bond Pricing with Co-ordination Failure*

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Abstract

It has been suggested (Morris, Shin 2001) that co-ordination failure between bondholders could produce an effect that would explain the systematic mispricing of corporate debt produced by the Merton (1974) framework. In essence, fear of premature foreclosure by other debtors can lead to pre-emptive action, lowering the value of debt. This paper presents a continuous-time bond pricing model integrating this effect, and shows that co-ordination failure can indeed cause bonds to be traded at a discount.

1 Introduction

Some recent research (Morris and Shin 2000) (Morris and Shin 2001) suggests that co-ordination failure among bond holders can have an effect on the price of debt. The problem of co-ordination failure is akin to the problem faced by depositors of a bank which is vulnerable to a run. Even if it is not efficient to attempt to foreclose, e. g. when the issuer of the bond is likely to have sufficient funds at maturity, and premature foreclosure produces a high liquidation cost, fear that other bondholders
to pre-emptive action, and cause the firm to be liquidated or enter restructuring.

The prevention of co-ordination failures is one of the key aims of all bankruptcy codes (cf. e.g. Baird and Jackson 1990, Jackson 1986), and whether or not

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co-ordination failures arise in practice depends on the effectiveness of the legal provisions in bringing structure to formal and informal renegotiation. Empirically, actual deviations from principles laid down by bankruptcy codes such as e. g. the absolute priority of the claims of debtors have been well documented (e. g. Franks and Torous 1994). As such a deviation, co-ordination failure has only relatively recently become the focus of attention. For example, there now is evidence that when creditors do not manage to establish co-ordination arrangements such as e. g. creditor pools, this much decreases the likelihood of successful reorganisation (Brunner and Krahnen 2001).

Merton (1974) pioneered an approach to pricing defaultable debt sometimes referred to as the structural, or firm-value based, or contingent claims approach. Models within this framework explain prices as a function of the process driving the asset value of a company: Bonds are treated as a "bull spread" on the asset value of a firm, and bankruptcy occurs when the face value of debt exceeds the value of assets at some given date.

Empirically, there is evidence that the simple Merton model seems to over-predict prices of bonds especially for firms that have a low asset value volatility, i.e. stable earnings(cf. e.g. Jones, Mason, and Rosenfeld 1984, Anderson and Sundaresan 2000, Eom, Helwege, and Huang 2001). Often, it seems that the Merton model relies rather too heavily on a high volatility to generate high spreads.

Many extensions of the basic Merton model have been suggested, relating to e. g. sub-ordination arrangements, indenture provisions (Black and Cox 1976), coupon bearing bonds (Geske 1977), stochastic interest rates (Shimko, Tejima, and van Deventer 1993, Longstaff and Schwartz 1995) or an optimally chosen capital structure (e. g. Leland 1994). Recently, strategic issues have been examined. However, co-ordination failure has not been the focus of much attention in the bond pricing literature. Typically, research has focussed on games between creditors and debtors (Anderson and Sundaresan 1996, Mella-Barral and Perraudin 1997), rather than on games between creditors, such as co-ordination failure.

The aim here will be to derive a structural model of bond prices, with the simplest possible assumptions, integrating co-ordination failure. The model presented below is intended as a demonstration of how co-ordination failure could be integrated into a continuous time bond pricing model, and should easily be extendable to more complicated cases, such as the ones referred to above. It will be shown that models of this kind can indeed produce a discount especially for bonds of firms that have a low asset value volatility. The result is different from that of Morris and Shin (2001), but some of the mechanics driving it are essentially the same. Morris and Shin produce a model where the critical asset value or trigger point at which the firm goes bankrupt is a decreasing function of the asset value. When people ignore this, they underprice debt. In the case that will be presented here, the critical asset value or trigger point will not be a function of the asset value, so this effect does not arise. However, costs associated with early reorganisation and a non-optimal point at which reorganisation occurs can produce a discount. A preliminary empirical calculation indicates that the model presented here can possibly explain some of the empirical difficulties of the Merton model.

2 The model

2.1 Co-ordination failure in a bond context

In order to produce co-ordination failure, agents have to be able to act in a way that imposes a (negative) externality on other bondholders. This action will be related to forcing reorganisation, which will only be possible if the company is legally insolvent.

Typically, bankruptcy codes stipulate that to qualify for bankruptcy proceedings, debtors have to be delinquent in servicing their debt (a 'cessation of payments' standard). The cessation of payments could be modelled explicitly, by e.g. postulating that the firm produces a cash-flow that is proportional to the asset value, and has to make per period payments to creditors that are a fraction of the par value of debt. In this case, if the asset value falls below a certain level, the generated cash-flow will be insufficient to make the payments. If the firm is not able to raise money externally, it would have to default on at least a part of the payments, and would be legally insolvent. The level of the asset value at which this would occur would be proportional to the par value of debt. For simplicity, the 'cessation of payments' is not modelled explicitly here - we assume that the firm is deemed insolvent when the asset value reaches a fraction of the par value of debt - call this the legal insolvency level. Alternatively, one could postulate a net worth covenant stipulating that the firm is insolvent as soon as the asset value falls below the specified fraction of the par value of debt.

Now suppose that if the firm is insolvent as specified above, it can be forced into reorganisation by a sufficiently large fraction of the bondholders. Reorganisation in this case would typically involve an exchange of bonds for other securities or cash.

Suppose the bondholders that force reorganisation are able to extract money from the firm at the expense of those bondholders who do not participate in forcing reorganisation. Even in mature markets it seems that bankruptcy provisions are insufficient to prevent this from happening: Take, for instance, the case of the US. The Trust Indenture Act of 1939 seems to stipulate that holders of the same class of bonds should be treated on equal terms in a restructuring. However, there are examples where this was not the case. In the case of Eastern Airlines, for instance, the court held that a side payment to bondholders voting in favour of an exchange (and no side payment to bondholders who did nothing or voted against the exchange) was not a violation of existing laws (cf. e.g. Roe 1987). Arguably, the amounts involved here were relatively small (a payment of USD 35 for every USD 1000 of face value), but one can imagine that in jurisdictions with a less developed bankruptcy code (e. g. emerging markets), these kind of events are even more likely to be observed.

Suppose that forcing reorganisation to gain at the expense of others is costly, e. g. because bondholders need to pay their lawyers to threaten the firm, or because the firm is forced into a panic sale of some of its assets. Also, assume that the cost imposed on the bondholders who do not attempt to force reorganisation is stark: their bonds become worthless. This sets the cost imposed on non-participating bondholders at a very high level, but simplifies the model and serves to emphasise the results.

Finally, assume that in order to successfully force reorganisation, those bondholders attempting to force reorganisation have to represent a large enough fraction of all bondholders. This mirrors provisions in some bankruptcy codes, or could represent the weight of the parties in informal negotiations. Below, we will assume that this fraction has to be bigger than the current asset value divided by the insolvency level. This assumption implies that it is easier to succeed in forcing reorganisation the lower the current asset value of the firm (i.e. the more distressed the firm is). Also, it makes it impossible to force reorganisation if the firm is not insolvent.

Then co-ordination failure could arise as follows: Even if it is not efficient to force reorganisation because of direct and indirect costs associated with it, fear that other bondholders may attempt to force reorganisation may lead to pre-emptive action. The question to ask in this context is at what critical level of the asset value bondholders will rush to force reorganisation. We will call this level the 'trigger point' or 'reorganisation boundary', because it is the level at which bondholders will act.

2.2 The setup

As in the Merton (1974) model, the bond can be priced as a function of the asset value process. We will first set up a discrete time game, where in every period agents have to decide whether to attempt to force reorganisation or not. We will then be able to derive a trigger point for the asset value, for which the firm will just be reorganised. The asset value changes between periods, such that when we take the continuous time limit, the asset value process will turn out to be a geometric Brownian motion. This will then allow us to price the bond as a combination of barrier options on the asset value using standard procedures, where the barrier is given by the trigger point.

2.2.1 Payoffs

Formally, denote the face value of debt by D. Then if the asset value of the firm falls below the legal insolvency level AD, the holders of the bond can attempt to reorganise. Actual reorganisation will take place immediately before t only when the fraction of bond holders who attempt to force reorganisation l is larger than or equal to $\frac{V_t}{AD}$, where V_t is the asset value of the firm at time t. So if the asset value exceeds AD, bond holders cannot force reorganisation, and as V_t decreases it becomes easier to force reorganisation. Attempting to force reorganisation produces an immediate cost $KV_{t-\Delta}$ (proportional to the value of the assets). If the firm is actually reorganised, an agent that has participated in reorganising receives her share of the asset value $V_{t-\Delta}$, whereas agents that have not participated receive . If the firm is not reorganised, both types of agents still hold the bond, which will be worth B_t.

The table below illustrates the instantaneous payoffs that agents need to take into account when making the decision to participate in attempting to extract money or not. Note that both types of agents will still be holding the bond in the next period if the firm is not reorganised.

	firm reorganised	firm not reorganised
attempt to reorganise.	$- K \;\; V_{t-\Delta}$	$-KV_{t-\Delta}$
do not attempt to reorganise		

2.2.2 Information content of prices

A necessary ingredient for co-ordination failure to arise is uncertainty about the actions of other agents. Without common knowledge of the fundamentals (the asset value in our case) of an issuer, agents will not be completely sure of how other agents will act. Suppose there is private as well as public information, then provided that private information is sufficiently precise in relation to public information, i. e. there is sufficient uncertainty about the actions of others, this will create co-ordination failure.

In a comment on Morris and Shin's (2000) paper, Atkeson (2000) doubts that the co-ordination failure idea is applicable to pricing debt. He argues that if agents can see prices, there will be no co-ordination failure, because all information will be revealed in the prices - there is no role for private information, and hence uncertainty about the information of other agents. This is not necessarily the case, depending on the timing assumptions.

Suppose that agents have to make a decision as to whether or not to force reorganisation after they have received a signal, but before trading occurs and the information contained in the private signals is integrated into prices. Suppose that after they have made a decision on reorganising, there is a tatonnement process that integrates all information into prices - they submit their orders according to a price set by the Walrasian auctioneer, the auctioneer revises the

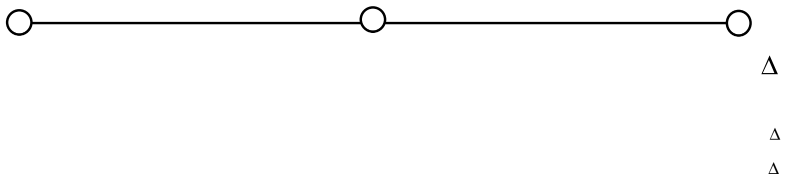


Figure 1: Timing assumptions

price and invites new orders etc. Then there might still be co-ordination failure, basically because private information is not already public at the time the reorganisation decision needs to be made.

2.2.3 Timing

Time increments are of size Δ. At time t, identical agents know the asset value of this period, V_t. We will later let the number of agents tend to infinity, and will subsequently index them by the unit interval. Relative changes in the asset value are normally distributed. The bonds trade at a price B_t which incorporates the information V_t. Let q denote a time increment that is smaller than Δ ($< q < \Delta$). At $t \quad q$, agents receive a signal X_i about the increase in the asset value - subscript i indexes the different agents, we omit the time subscript to simplify notation. They form a posterior given their information. Given their posterior, they make a decision as to whether or not to attempt to force reorganisation.

After it has been determined whether the firm will be reorganised or not in this period, we proceed to the next period: agents submit their orders to a Walrasian auctioneer, who then determines the price $B_{t+\Delta}$ incorporating all the information in the private signals about $V_{t+\Delta}$. Suppose that this reveals $V_{t+\Delta}$. We see that as a consequence of these timing assumptions, only public information will be incorporated into prices, and that there will be no asymmetry between agents. This is important as it allows pricing by standard martingale techniques.

At the maturity of the bond, the holders receive the minimum of the face value or the asset value of the firm. There is no cost to reorganisation (the firm will be wound up in any case).

2.2.4 Information

The relative increase in the asset value is normally distributed around a drift.

$$V_{t+\Delta} - V_t \quad \mu_V V_t \Delta \quad V_t \eta_t, \;\; \eta_t \sim NID\left(\;, \frac{}{\alpha}\right)$$

Very shortly afterwards, agents receive a signal X_i (subscript i indexes the different agents) about this increase with a distribution conditional on the asset value V_t given by

$$X_i \quad V_{t+\Delta} \quad V_t \varepsilon_i, \;\; \varepsilon_i \sim NID\left(\;, \frac{}{\beta}\right),$$

with $Cov \; \eta_t, \varepsilon_i$, i. e. the noise is orthogonal to the innovations in the fundamental.

From the signal X_i and the public information V_t, agents form a posterior about the value of the firm in period $t \quad \Delta$, $V_{t+\Delta}$ which is also normally distributed.

2.3 The solution

2.3.1 Basic procedure

We follow the same procedure as Morris and Shin (2001) to solve the model. Suppose that agents follow a switching strategy around a certain posterior belief. Given the posterior belief around which agents switch, we can work out how many of them will attempt to force reorganisation, given the asset value in the next period (posterior beliefs will be centered around this asset value in the next period). We can therefore work out what the critical next-period asset value is for which the firm will be reorganised, given the belief in this period around which agents switch. This is the trigger point.

2.3.2 The discrete time trigger point

In the appendix, section 5.1, the following solution is derived (equation 5):

$$V_t^* \quad AD\Phi\left\{\frac{\alpha}{\sqrt{\beta}}\left(\frac{V_t^*}{V_{t-\Delta}} - \quad - \mu_V \Delta\right) \quad \frac{\sqrt{\alpha \quad \beta}}{\sqrt{\beta}}\Phi^{-1}\{ \quad - K\}\right\} \qquad (1)$$

The trigger point V_t^* is unique if $AD\frac{1}{\sqrt{2\pi}}\frac{\alpha}{\sqrt{\beta}}\frac{1}{V_t} <$ (condition 1). (cf appendix, section 5.1.8, proposition 2).

2.3.3 Continuous time limit

Now take the continuous time limit. If we want the asset value process to tend to a geometric Brownian motion, we need

$$_{\Delta \to dt} \alpha \qquad \frac{}{\sigma_V^2 dt},$$

i.e. public information about the innovation in the asset value to be proportional to time. So the variance of the innovation is $O \; \Delta$, or the precision is $O \; \frac{1}{\Delta}$.

Now a sufficient condition for the uniqueness of the equilibrium described in equation (1) in continuous time, *regardless of the asset value, the parameter A and the face value of debt*, is that

$$\beta \quad o \; \frac{}{\Delta^2} \; ,$$

because this ensures that condition (1) (s.a.) is always satisfied. This is just to say that we need the quality of private information to be sufficiently high in relation to the quality of public information in order for agents to be sufficiently uncertain about the actions of others to obtain co-ordination failure. As $\Delta \to dt$, $\Delta^2 \to$, and hence β grows at a faster rate than α. Consequently, $\frac{\alpha}{\sqrt{\beta}}$ tends to zero, so condition (1) will be satisfied for any finite V_t. Also, $\frac{\sqrt{\alpha+\beta}}{\sqrt{\beta}} \to$. The resulting trigger point equation then reduces to

$$V^* \; t \qquad AD \qquad - K$$

or, since the solution is constant,

$$V^* \qquad AD \qquad - K \; . \tag{2}$$

The solution to this equation is always unique, of course. If we let the intermediate time period tend to period immediately following it, reorganisation is forced at t whenever $V \; t$ hits $AD \quad - K$, i. e. when the asset value is a *fraction* $\quad - K$ of the legal insolvency level, since $\; < K < \;$. So here reorganisation actually occurs at a later point than legal insolvency. The reorganisation boundary or trigger point is a decreasing function of the cost of forcing reorganisation - agents are reluctant to force reorganisation if it is costly to do so.

Due to the special assumptions about the costs of forced reorganisation, this function turns out to be quite simple here - it is constant. Of course, one could allow for more general types of costs, but these do not in general produce a closed form solution.

2.3.4 Interpretation of the continuous time limit

The way in which limits are taken actually produces a situation where agents receive the same information in the limit. However, strategic uncertainty remains in the sense that for the marginal agent, the fraction of bondholders that forces reorganisation is still a random variable in the limit, which actually turns out to be uniformly distributed. This type of result has been discussed at length elsewhere (Morris and Shin 2002). A formal proof is in the appendix, section 5.2.

Suppose that we instead start with the assumption that there is no private information, and hence no co-ordination failure. All agents now have the same information, and will therefore either all attempt to force reorganisation, or all just hold the bond in equilibrium if they follow switching strategies. Suppose there is a strategy that specifies a path for the switching point. At every node of the game, it does not pay to deviate from the strategy when it specifies attempting to force reorganisation (because in this case deviating *always* implies loosing $V\ t$ - holding the bond when everyone else forces reorganisation is not a good idea), and it does not pay to deviate from a strategy when it specifies holding the bond (deviating *always* implies incurring $-KV\ t$ - attempting to force reorganisation alone will not be successful). It follows that all paths of a trigger point below AD can be supported. Taking the limit of our discrete time game has allowed us to eliminate all equilibria but one, even though in the continuous-time co-ordination failure case in the limit, agents also all have the same information.

2.4 Pricing

Prices are determined *ex ante* to receiving the signal. This implies that the price incorporates the *ex ante* probabilities of incurring the cost and benefits of forcing reorganisation and of actual reorganisation. Since agents are ex ante identical, the price will e. g. incorporate the ex ante probability of receiving a signal that implies te iogih2(y)6ifofoa6isioti-0-lh2(y)6h2(y)6h1.4(ne)TJ/TT8stes2.10710TD0Tc()Tj/TT21T59

1. $\Pi_1 \; T \quad D - \quad D - V_T, \quad$, if $s > T$

2. $\Pi_2 \; s - \Delta \quad q$ or $\Pi_2 \; s - \Delta \quad q \quad - K \;\; V_{s-\Delta}$, depending on whether the agent participated in forcing reorganisation or not, if $s \leq T$.

   Note that (1) and (2) are exclusive.

3. $\Pi_2 \; t - \Delta \quad q \quad - K V_{t-\Delta}$, which is incurred every time an attempt at forcing reorganisation is made but reorganisation does not actually take place, for all $t < s$.

2.4.2 Continuous time payoffs

In continuous time, the pricing will be simpler. Examine the second and third component. Conditional on the asset value in the next period, the probability that an agent receives a signal which prompts her to force reorganisation is $\Phi\left\{\frac{1}{V_t}\sqrt{\beta}\; X_t^* - V_{t+\Delta}\;\right\}$. We see that as β tends to infinity, this probability tends either to or to . What this means is that because all agents essentially receive the same information (as the signal becomes infinitely precise), the agents will either all attempt to force reorganisation, or will all refrain from doing so. So for any agent, the ex ante probability of forcing reorganisation when the other agents do not do so tends to zero. Also, the probability of not attempting to force reorganisation if all other agents are doing so tends to zero. In continuous time, this makes the pricing of the third component trivial - the probability of attempting to force reorganisation when others do not do so tends to zero, hence this probability will also tend to zero under the equivalent martingale measure, and hence this component has a price of zero. The pricing of the second component is also simplified, because here the probability of not receiving any money in the case of early reorganisation tends to zero. Note that due to the assumption that $q \to \Delta$, the payoff to the second component will occur at s in continuous time. We therefore have two mutually exclusive payoffs:

1. $\Pi_1 \; T \quad D - \quad D - V_T, \quad$, if $s > T$
2. $\Pi_2 \; s \quad - K \;\; V_{s-dt}$, if $s \leq T$.

(In the following, we use replaceV_{s-dt} with V_s, arguably this will not produce a different price.)

2.4.3 Equivalent Martingale Measure

With these payoffs, the model looks very similar to the standard Black and Cox (1976) case. The difference here is that the absorbing boundary is given by our trigger point (in their case it is the covenant), and that the payoff upon hitting this boundary is not equal to the asset value, but to a fraction $- K$ of the asset value. We know that as long as one 'derivative' on the underlying (the asset value) is traded, e. g. equity, then the asset value is implicitly tradeable, and hence in our simple case its drift under the equivalent martingale measure defined by the money market account numeraire will be equal to the risk-free interest rate.

2.4.4 Bond price

The bond price will be equal to the discounted expected value of the payoffs under the equivalent martingale measure (Q).

$$B \; V_t, t, T \qquad E_t^Q \left[e^{-r(T-t)} \Pi_1 I \;\; s > T \qquad e^{-r(T-s)} \Pi_2 I \;\; s \leq T \;\; \right]$$

(Here, r denotes the constant risk free interest rate and t denotes the present.)

At this stage pricing is straightforward. It was first explored by Black and Cox (1976). For a good recent treatment, cf. e. g. Ericsson and Reneby (1998).

The bond can be viewed as a portfolio of barrier options. We can view Π_1 as a combination of a long position in a down-and-out call with strike price (which is of course just equivalent to a down-and-out position in the underlying asset value), and a short position in a down-and-out call in with a strike price of D. This captures the fact that a bond can be viewed as a bull spread on the asset value. We can view Π_2 as a long position in $\quad - K \;\; V^*$ units of a dollar-in-boundary claim (a claim that pays one dollar in the case the boundary is hit before maturity). The price is the sum of prices of these positions:

$$\begin{aligned} B \; V \;\; t \;\; , t, T, V^* \qquad & F_{C,DO} \; V \;\; t \;\; , \;\; , t, T \;\; - F_{C,DO} \; V \;\; t \;\; , D, t, T \\ & \quad - K \;\; V^* F_{DIB} \; V \;\; t \;\; , t, T \end{aligned}$$

where $F_{C,DO} \; V \;\; t \;\; , Z, t, T$ denotes the price of a down-and-out call with strike price Z on the underlying V at t with maturity T. Similarly, $F_{DIB} \; V \;\; t \;\; , t, T$ denotes the price of a dollar-in-boundary claim. The interested reader is referred to the appendix, section 5.3 for details of how the components are priced.

2.4.5 Equity price

Similarly, we can interpret equity as a down-and-out call option on the asset value. Hence the price of equity is given by

$$E \; V \;\; t \;\; , t, T, V^* \qquad F_{C,DO} \; V \;\; t \;\; , D, t, T \;\; .$$

The sum of the value of equity and the value of debt equals the market value of the firm:

$$M \; V \;\; t \;\; , t, T, V^* \qquad F_{C,DO} \; V \;\; t \;\; , \;\; , t, T \qquad\quad - K \;\; V^* F_{DIB} \; V \;\; t \;\; , t, T$$

This is in essence the sum of the price of the down-and-out asset value of the firm, plus the price of the down-and-in claim representing the recovery value in the case of default. Also, note that the market value is not equal to the asset value of the firm.

2.4.6 Extension to coupon-paying debt

An extension to coupon-paying debt is simple, using methods described by Ericsson and Reneby (1998). If we assume that coupon i is paid in case the boundary is not reached prior to its maturity t_i, and not paid if the boundary is reached and reorganisation takes place at or before maturity, then we can think of the coupon essentially as a down-and-out binary cash call with the strike price and barrier equal to the trigger point. If the coupon rate is c of face value D, then the value of debt will be increased by

$$B_c \quad cD \sum_i F_{BCC,DO} \ V \ t \ , t, t_i, V^*$$

The value of equity will be decreased by

$$E_c \quad - \quad - \varkappa \ cD \sum_i F_{BCC,DO} \ V \ t \ , t, t_i, V^* \ \ ,$$

where $F_{BCC,DO}$ denotes the pricing function of a down-and-out binary cash call, and $\varkappa$ is the tax rate - this captures the value of the tax shield to equity.

3 Evaluation

3.1 Comparison

3.1.1 Black-Cox and Merton

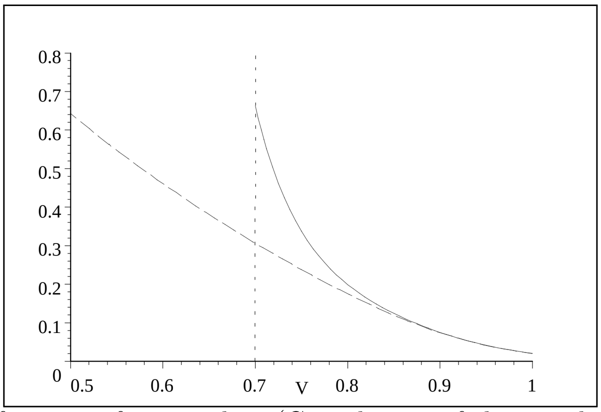
In the Black and Cox (1976) case, the amount recovered if reorganisation is forced is simply $V\ t$. It is trivial to show that this is always more than $B\ t$ (intuitively, this is the case since $B\ t$ represents a claim to $V\ T$ in some states of the world, and a claim to a value less than $V\ T$ in others, whereas $V\ t$ represents a claim to $V\ T$ in all states of the world). Reorganisation will always be forced if it is possible to do so, and $B\ t$ (representing the value of holding the bond) is smaller than the payoff to reorganisation, which is $V\ t$ in the Black Cox case. So reorganisation is always forced if the asset value hits the covenant, basically because there are no bankruptcy costs.

Note that in the co-ordination failure model, if the cost of attempting to force reorganisation (K) is set zero, the amount recovered in the case of early reorganisation is $V\ t$, and the trigger point tends to AD, which is simply the value of the legal insolvency level - if a net worth covenant exists, this will be the legal insolvency level. So the Black and Cox (1976) model is a special case of our model. If the cost of attempting to force reorganisation are zero, then there are sometimes benefits associated with doing so, but never any costs, so it becomes a dominant strategy. Hence people will force reorganisation as soon as they can - leading to the Black Cox default point. Given that there are no costs to forcing early reorganisation, we also have the Black Cox recovery value, and hence the Black Cox price. Note that the Black Cox price will always be higher, as there are no costs to reorganisation and reorganisation occurs when there is still more of the asset value left (the Black Cox trigger point is higher).

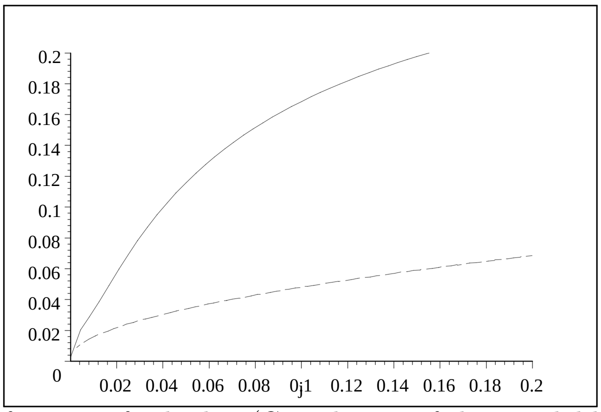
If we set K , it never pays to force reorganisation. The bondholders never do, so there is no reorganisation before maturity. The pricing formula reduces to the Merton (1974) formula.

The effect producing the price that differs from Merton price is twofold: Firstly, the possibility of early default - i.e. receiving money before the maturity of the bond - increases the value of the bond, as in the Black-Cox case. But secondly, since there is a cost of reorganisation K, this decreases the value of the bond, ceteris paribus. Because these two effects conflict, the co-ordination failure model does not produce a discount vis-a-vis the Merton case in all situations.

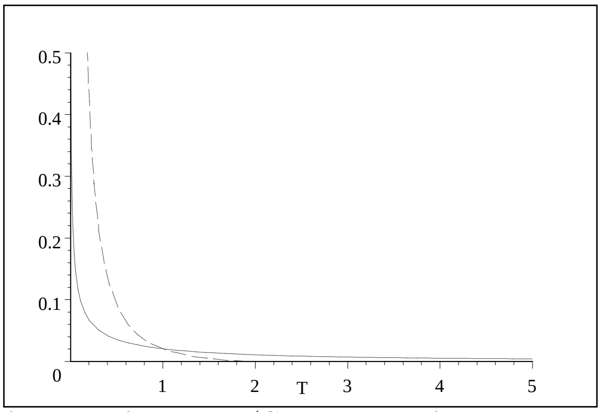
We can plot the differences in predicted spreads as a function of the asset value, the asset value volatility and maturity. All plots assume an asset value V , an asset value of volatility of σ_V . , a time to maturity of $T-t$, an interest rate of r . , a parameter A and a parameter K . unless otherwise specified. This implies a default boundary at of the face value of debt. We see that for these parameters spreads react more abruptly to the asset value, they increase stronger in asset value volatility, and they exhibit more abrupt behaviour with respect to maturity.



Spreads as a function of asset value. (Co-ordination failure - solid line, Merton - dashed line)



Spreads as a function of volatility (Co-ordination failure - solid line, Merton - dashed line)



Spreads as a function of maturity (Co-ordination failure - solid line, Merton - dashed line)

3.1.2 Other cases

Morris-Shin Morris and Shin (2001) refer to a version of equation (5), and argue that if one assumes the trigger point to be fixed, one would underprice debt, as the trigger point is actually a decreasing function of the asset value. So as the asset value decreases, the trigger point moves up. Ignoring this effect would cause overpricing. The effect mentioned by Morris and Shin (2001) does not cause the difference in predicted price here, because the continuous time limit of the trigger point (equation 2) is not a function of the asset value - it is constant.

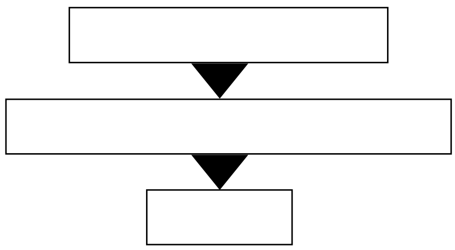
Optimal early reorganisation Suppose that agents co-ordinate to force early reorganisation when it is optimal to do so. Then early reorganisation will occur when $B\ t\ \leq\quad -K\ \ V\ t$, i. e. when it is better to liquidate than to hold the bond. In this situation which is more general than the case examined by Black and Cox it is not necessarily the case that early reorganisation will occur as soon as the asset value hits the covenant. In fact, the bond now resembles an American-type security, where the holders of the bond have the (American) option to force reorganisation. It is unlikely that a closed form solution for the price of such a bond can be found in general.

However, it is possible to argue that this price, reflecting the *optimum* point at which to force reorganisation, will always be equal to or bigger than the price that is produced by co-ordination failure, where reorganisation is forced at a point that will not in general be optimal. Hence co-ordination failure would reduce the price vis-a-vis this kind of bond.

3.2 Testable implications

The co-ordination failure model is observationally equivalent to any geometric Brownian motion - based model that produces the same trigger point and payoffs. The key testable implication of the model is that there is a relationship between the fraction that is recovered, and the asset value at which the company goes bankrupt, as specified by the trigger point equation (2) and the recovery fraction $-K$.

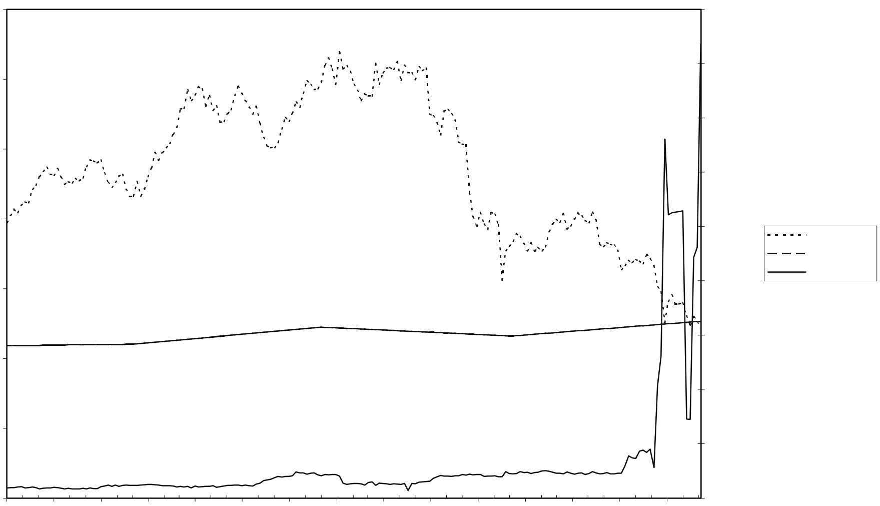
More generally, note that the present model is a special case of a very generic bond pricing approach (Ericsson and Reneby 1998) that views a bond as a portfolio of simple and barrier claims. This also goes for other bond pricing models integrating strategic interaction (Mella-Barral and Perraudin 1997). In essence, incorporating elements of strategic interaction into bond pricing produces arguments as to what the simple and barrier claims should be that make up the bond. Strategic interaction produces different payoffs in different states of the world which determine the price. Of course, any other model that produces the same payoffs will also produce the same price.



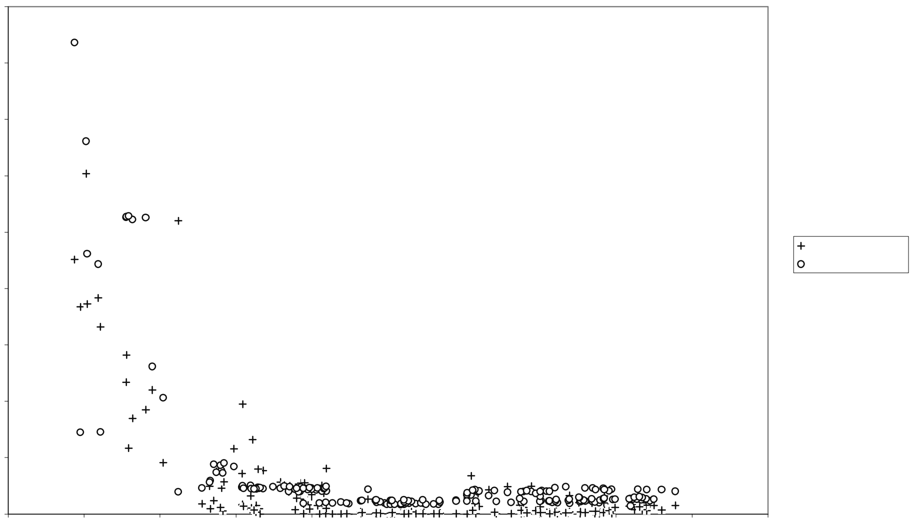
3.3 An indicative empirical example

To see whether a model with the payoffs described here could possibly improve on the predictive performance of the Merton model, the predicted spreads are compared for the case of Xerox Corp., which underwent major restructuring in December 2000 (there are several inaccuracies implicit in the method used, it is intended merely as a back-of-the-envelope calculation and mirrors what a practitioner might do).

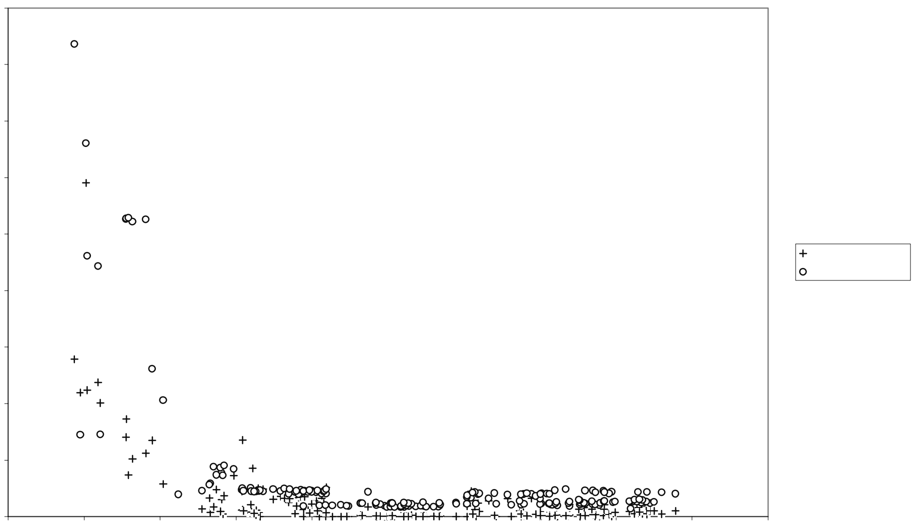
For weekly data from the end of 1996 to the end of 2000, obtained from Datastream, Bloomberg and Multex, the above formula for the market value of equity and the volatility of equity (implied volatilities from call options) is numerically inverted to obtain the asset value and asset value volatility. Figures for total liabilities excluding equity are linearly interpolated to obtain a time series for liabilities. Yields to maturity from one bond picked at random are calculated. A similar (in terms of maturity and coupon) US Treasury Note is picked, and its yield also calculated. We can obtain a (coupon) spread as the difference between the two yields. Assuming a recovery value of about 50% and a parameter of A , i.e. the firm is insolvent as soon as liabilities exceed assets, one obtains from the trigger point equation a cost of reorganisation K of about 30%. This roughly matches the average recovery fraction for a senior unsecured bond as reported e. g. by Standard & Poor's for all types of credit events. Using these values, one can calculate the predicted spread for the co-ordination failure bond and the Merton bond (ignoring coupons), and can plot the actual spreads and the predicted spreads as a function of the asset value of the firm. Note that Xerox Corp. was downgraded by Standard & Poor's from A to BBB in April of 1999.



The fall of Xerox Corp.



Co-ordination failure model



Merton model

We see that for the dataset at hand, the Merton model seems to underpredict spreads close to default, whereas the co-ordination failure model seems to produce a better fit. Both models underpredict spreads further away from default.

The error seems to be small, relatively constant and positive for large asset values. A possible guess as to why this might be is some liquidity advantage of the underlying benchmark (the T-Note), or a factor related to the calculation of the spread.

We can calculate the prediction error as the difference in spreads, relative to the actual spread. The average prediction error is negative for both models (i.e. the models tend to underpredict spreads on average), and are both in the range of about -60%.

In terms of squared prediction error, the co-ordination failure model outperforms the Merton model in 67.36% of the time periods. For the absolute prediction error, the co-ordination failure model also outperforms the Merton model in 67.36% of the time periods. The co-ordination failure model itself is outperformed in 96.89% of the time period by a simple Random Walk model predicting that the spread today will be equal to the spread tomorrow, both in terms of squared and absolute error. These results seem to be in line with other research (Eom, Helwege, and Huang 2001), which generally suggests that including a cost to bankruptcy improves fit, but that structural models are generally worse at predicting spreads than the simple Random Walk model. Comparing the correlation between predicted and actual spreads is also interesting: Correlation coefficients for the Random Walk model (i.e. the first-order autocorrelation coefficient) is 0.8809, the coefficient for the co-ordination failure model is 0.8625 and for the Merton model it is 0.4890. It appears that the co-ordination failure model is able to pick up quite a large part of at least the non-linear relationship between variables. An element of linear mis-specification seems to remain. As indicated above, this might be due to for example a liquidity advantage of the benchmark.

To see how sensitive the results are to the specification of the parameter K, the model was also run using K . , which would correspond to a recovery fraction (of par value) of about 64%. In this case, the co-ordination failure model outperforms the Merton model in 67.88% of the time periods both in terms of squared and absolute error, and the correlation between the co-ordination failure predicted spreads and actual spreads rises to 0.8628. The performance of the co-ordination failure model vis-a-vis the random walk model is unchanged. Taking a parameter of K . , which would correspond to a recovery fraction of 36%, the co-ordination failure model outperforms the Merton model of 62.18% of the time in terms of both absolute and squared prediction errors, and the correlation coefficient between predicted and actual spreads is 0.8615. Again, the performance vis-a-vis the random walk model is not changed. It appears that the results are not very sensitive to the parameter K.

4 Concluding Remarks

When examining cases of reorganisation, it is possible to observe situations in which holders of the same bond manage to receive different amounts of cash or different securities in a reorganisation. The extent to which this is possible differs across jurisdictions, but even in mature markets with elaborate bondholder protection legislation like the US, these kind of situations do seem to arise. Consequently, co-ordination failure is likely to be an issue for holders of bonds.

This paper attempts to derive a continuous-time structural bond pricing model incorporating co-ordination failure by postulating private information between periods in which trading occurs. In the limit, the private information vanishes, so standard martingale pricing methods are applicable, but strategic uncertainty and the equilibrium produced by it remain.

Theoretically, the model suggests that there should be a difference between the legal insolvency level and the point at which firms are actually reorganised, and that the point at which firms are reorganised depends negatively on the cost associated with reorganisation. Furthermore, the price produced here is different from the Merton price. A preliminary empirical back-of-the-envelope calculation seems to suggest that a model with payoffs as described by the co-ordination failure model appears to produce a better fit to actual data.

5 Appendix

5.1 Solution of the discrete time model

5.1.1 Basic procedure

We follow the same procedure as Morris and Shin (2001) to solve the model. Suppose that agents follow a switching strategy around a certain posterior belief. Given the posterior belief around which agents switch, we can work out how many of them will attempt to force reorganisation, given the asset value in the next period (posterior beliefs will be centered around this asset value in the next period). We can therefore work out what the critical next-period asset value is for which the firm will be reorganised, given the belief in this period around which agents switch.

Also, we can use the fact that agents will switch if they believe that they will obtain a higher utility from doing so. Once we have defined utilities, this allows us to derive the critical posterior belief, given a critical next period asset value for which the firm is reorganised.

So we have two equations in two unknowns, which can then be solved for the critical asset value for which the firm is reorganised - the trigger point.

5.1.2 Information

For convenience, the assumptions about information are restated here. The relative increase in the asset value is normally distributed around a drift.

$$V_{t+\Delta} - V_t \quad \mu_V V_t \Delta \quad V_t \eta_t,\ \ \eta_t \sim NID\left(\ ,\frac{\ }{\alpha}\right)$$

Very shortly afterwards, agents receive a signal X_i (subscript i indexes the different agents) about this increase with a distribution conditional on the asset value V_t given by

$$X_i \quad V_{t+\Delta} \quad V_t \varepsilon_i,\ \ \varepsilon_i \sim NID\left(\ ,\frac{\ }{\beta}\right),$$

with $Cov\ \eta_t, \varepsilon_i$, i. e. the noise is orthogonal to the innovations in the fundamental.

5.1.3 Posteriors

From the signal X_i and the public information V_t, agents form a posterior about the value of the firm in period $t \quad \Delta$, $V_{t+\Delta}$ which is normal with mean and variance given by

$$\rho_i \quad E\ V_{t+\Delta}|X_i \qquad \frac{\alpha \qquad \mu_V \Delta\ V_t \quad \beta X_i}{\alpha \quad \beta}$$

and

$$Var\ V_{t+\Delta}|X_i \qquad \frac{V_t\ ^2}{\alpha \quad \beta}.$$

5.1.4 Critical value of $V_{t+\Delta}$ for which there is forced reorganisation

Given the posterior belief around which agents switch, we work out how many of them will attempt to force reorganisation, given the asset value in the next period (posterior beliefs will be centered around this asset value in the next period). We then work out what the critical next-period asset value is for which the firm will be reorganised, given the belief in this period around which agents switch.

Suppose agents follow a switching strategy around ρ^*, i. e. agents attempt to force reorganisation when their posterior is below ρ^*. Then an agent will not force reorganisation if and only if the private signal is bigger than

$$X^* \quad \frac{\alpha \quad \beta}{\beta}\rho^* - \frac{\alpha}{\beta} \qquad \mu_V \Delta\ V_t.$$

Conditional on state $V_{t+\Delta}$, the distribution of X_i is normal with mean $V_{t+\Delta}$ and precision $\frac{\beta}{V_t^2}$. So the ex ante probability for any agent of attempting to force reorganisation is equal to

$$\Phi \left\{ \frac{}{V_t} \sqrt{\beta}\ X^* - V_{t+\Delta} \right\}.$$

As the number of agents tends to infinity, the fraction of agents that forces reorganisation will be equal to this ex ante probability for any individual agent by the law of large numbers.

Since the firm is reorganised if the fraction that forces reorganisation is $l \geq \frac{V_{t+\Delta}}{AD}$, the critical value of $V_{t+\Delta}$ (denoted by $V^*_{t+\Delta}$) for which the firm is reorganised at t is given by $V^*_{t+\Delta} \quad AD\Phi \left\{ \frac{1}{V_t} \sqrt{\beta}\ X^* - V^*_{t+\Delta} \right\}$ or

$$V^*_{t+\Delta} \quad AD\Phi \left\{ \frac{}{V_t} \left(\frac{\alpha}{\sqrt{\beta}}\ \rho^* - \quad \mu_V \Delta\ V_t \quad \sqrt{\beta} \left(\rho^* - V^*_{t+\Delta} \right) \right) \right\}. \tag{3}$$

5.1.5 Utility, budget constraint

The utility function is additively separable across time. Noting that agents believe that their actions as individuals will not affect the bond price (since they are atomistic), consumption in any period t in which trading occurs (this does not apply to the interim period) conditional on the firm not having been reorganised prior to this period, is equal to:

$$c_t \quad e_t \quad \xi_{t-\Delta} B_t - \xi_t B_t$$

where e_t denotes all non-bond-related items and ξ_t denotes the amount invested in the bond from period t to period $t \quad \Delta$. Note that as a consequence of the timing assumptions, agents are identical in the periods in which trading occurs and prices will adjust such that they will hold the same amounts ξ of the bond.

Consumption in the intermediate period $t \quad q$ is as described in the payoff matrix in the main text. Note that retrading is not allowed at $t \quad q$, and hence the positions ξ will still be the same as at time t. Denote consumption in the case the agent decides to force reorganisation (f) and the firm is actually forced into reorganisation (R) by

$$c_{t+q}\ fR \qquad - K\ \ V_t \xi_t,$$

consumption in the intermediate time period in the case the agent decides to force reorganisation (f) but the firm is not reorganised (C) by

$$c_{t+q}\ fC \quad -K V_t \xi_t$$

and consumption in the intermediate time period in the case the agent decides not to force reorganisation (n) (in that case it does not matter whether the firm is reorganised or not) by

$$c_{t+q} \quad nR \qquad c_{t+q} \quad nC \qquad .$$

5.1.6 Critical value of ρ

We now use the fact that agents will switch if they believe that they will obtain a higher utility from doing so. We then derive the critical posterior belief, given a critical next period asset value for which the firm is reorganised.

Now the marginal agent (one that is indifferent between forcing reorganisation or not) has a posterior over the asset value which has its mean just at the switching point (i.e. ρ for this agent is equal to ρ^*). For her the expected utility of just holding the bond should just equal the expected utility of forcing reorganisation. This *defines* the switching point. Using u to denote the instantaneous utility, δ to denote the discount factor from time $t \quad \Delta$ to the intermediate time $t \quad q$ and F to denote the posterior cumulative distribution (given the belief) over the asset value $V_{t+\Delta}$ we can write:

$$\int_{-\infty}^{V^*_{t+\Delta}} u \quad c_{t+q} \quad fR \quad dF \qquad \int_{V^*_{t+\Delta}}^{\infty} u \quad c_{t+q} \quad fC \quad dF \qquad \int_{-\infty}^{\infty} \delta u \quad c_{t+\Delta} \quad dF$$

$$\int_{-\infty}^{V^*_{t+\Delta}} u \quad c_{t+q} \quad nR \quad dF \qquad \int_{V^*_{t+\Delta}}^{\infty} u \quad c_{t+q} \quad nC \quad dF \qquad \int_{-\infty}^{\infty} \delta u \quad c_{t+\Delta} \quad dF$$

Note that the utility at $t \quad q$ does not depend on $V_{t+\Delta}$. We can therefore write:

$$u \quad c_{t+q} \quad fR \qquad \left(V_{t+\Delta} < V^*_{t+\Delta}\right) \quad u \quad c_{t+q} \quad fC \qquad \left(V_{t+\Delta} > V^*_{t+\Delta}\right)$$

$$u \quad c_{t+q} \quad nR \qquad \left(V_{t+\Delta} < V^*_{t+\Delta}\right) \quad u \quad c_{t+q} \quad nC \qquad \left(V_{t+\Delta} > V^*_{t+\Delta}\right)$$

We can write this probability as:

$$\left(V_{t+\Delta} > V^*_{t+\Delta}\right) \qquad \Phi\left\{\frac{\sqrt{\alpha \quad \beta}}{V_t} \quad \rho^* - V^*_{t+\Delta} \quad \right\}$$
$$(\qquad - \qquad \left(V_{t+\Delta} < V^*_{t+\Delta}\right)) \, .$$

We insert this and rearrange to obtain

$$\rho^* - V^*_{t+\Delta} \quad \frac{V_t}{\sqrt{\alpha \quad \beta}} \Phi^{-1} \left\{ \frac{u \ c \ fR \quad - u \ c_t \ nR}{u \ c_t \ fR \quad - u \ c_t \ fC} \right\}$$

Now take limits as the number of agents goes to infinity. This will imply that the amount any individual agent holds goes to zero, $\xi_t \rightarrow$. Note that we have a fraction of functions of ξ_t, and can apply l'Hopital's rule. In the limit, $c_{t+q} \ fR \quad c_{t+q} \ fC \quad c_{t+q} \ nL \quad c_{t+q} \ nC \quad c_{t+q}$, so

$$\lim_{\xi_t \rightarrow 0} \frac{u \ c_{t+q} \ fR \quad - u \ c_{t+q} \ nR}{u \ c_{t+q} \ fR \quad - u \ c_{t+q} \ fC} \quad \frac{u' \ c_{t+q} \quad - K \ V_t}{u' \ c_{t+q} \quad - K \ V_t \quad u' \ c_{t+q} \ KV_t}$$
$$- K.$$

So the limit of our equation is

$$\rho^* - V^*_{t+\Delta} \quad \frac{V_t}{\sqrt{\alpha \quad \beta}} \Phi^{-1} \{ \quad - K \} . \tag{4}$$

We need $< K <$ for this function to be well defined. Note that $- K$ is the fraction of the asset value recovered in the case of early reorganisation, if the agent has attempted to force reorganisation.

This equation together with (3) pins down the critical value of beliefs and the asset value.

5.1.7 Equilibrium forced reorganisation

Combining equations (4) and (3) we can solve for the failure point at which the asset value in the next period causes failure in this period:

$$V^*_{t+\Delta} \quad AD\Phi \left\{ \frac{\alpha}{\sqrt{\beta}} \left(\frac{V^*_{t+\Delta}}{V_t} - \quad - \mu_V \Delta \right) \quad \frac{\sqrt{\alpha \quad \beta}}{\sqrt{\beta}} \Phi^{-1} \{ \quad - K \} \right\} \tag{5}$$

Reorganisation at time $t \quad q$ will occur when V hits V^* at $t \quad \Delta$.

5.1.8 Uniqueness

To simplify notation, define

$$Z \quad \frac{\alpha}{\sqrt{\beta}} \left(\frac{V^*_{t+\Delta}}{V_t} - \quad - \mu_V \Delta \right) \quad \frac{\sqrt{\alpha \quad \beta}}{\sqrt{\beta}} \Phi^{-1} \{ \quad - K \}$$

and

Condition 1 $AD \frac{1}{\sqrt{2\pi}} \frac{\alpha}{\sqrt{\beta}} \frac{1}{V} <$

Proposition 2 *The trigger point $V^*_{t+\Delta}$ is unique if condition (1) is satisfied.*

Proof. This is a version of the proof in Morris and Shin (2001). A sufficient condition for a unique solution is that the slope of

$$AD\Phi\{Z\}$$

is less than one everywhere. This slope is equal to

$$AD\varphi\{Z\}\frac{\alpha}{\sqrt{\beta}\,V_t}.$$

It reaches a maximum where the argument of the normal density is , the maximum there will be $\frac{1}{\sqrt{2\pi}}$. Hence a sufficient condition for a unique solution is that

$$AD\frac{}{\sqrt{\ \pi}}\frac{\alpha}{\sqrt{\beta}\,V_t} < \ .$$

■

5.2 Uncertainty in the limit

It can be shown that the marginal or pivotal agent views the fraction of bondholders that attempt to force reorganisation as a random variable that is uniformly distributed in the continuous-time limit, and hence that strategic uncertainty remains. Note that these kind of results have been discussed at length elsewhere (Morris and Shin 2002).

Proposition 3 *The distribution of l given the belief ρ^* of the marginal agent is uniform in the limit*

Proof. The proportion of people who receive a lower signal X^* is

$$l \quad \Phi\left(\frac{\sqrt{\beta}}{V_t}\ X^* - V_{t+\Delta}\ \right).$$

The question to ask is: What is the probability that a fraction less than z of the other bondholders receive a signal higher than that of the marginal agent, conditional on the marginal agent's belief, or what is $\quad - l\ < z \mid \rho^*$?

Now the event

$$- l < z$$

is equivalent to

$$- \Phi\left(\frac{\sqrt{\beta}}{V_t}\ X^* - V_{t+\Delta}\ \right) < z$$

or (rearranging)

$$V_{t+\Delta} < X^* \quad \frac{V_t}{\sqrt{\beta}}\Phi^{-1} \quad - z\ .$$

So the probability we are looking for is $\left(V_{t+\Delta} < X^* \quad \frac{V_t}{\sqrt{\beta}}\Phi^{-1} \quad - z \mid \rho^*\right)$. The posterior of the marginal agent over $V_{t+\Delta}$ has mean ρ^* and variance $\frac{V_t^2}{\alpha+\beta}$, hence this probability is

$$- l \quad < z \mid \rho^* \qquad \Phi\left(\frac{\sqrt{\alpha \quad \beta}}{V_t}\left(X^* \quad \frac{V_t}{\sqrt{\beta}}\Phi^{-1} \quad - z \, - \rho^*\right)\right).$$

Now as we take limits, $\rho^* \rightarrow X^*$, since private information becomes infinitely

$$f^M \; Y \; t \; , s \; \equiv \frac{Y \; t}{\sqrt{\pi \; s - t \; ^3}} e^{-\frac{1}{2}\frac{1}{(s-t)} \; Y(t)+\mu_Y^M(s-t) \; ^2}. \tag{7}$$

The point mass for the first passage time is given by

$$^M \; s \leq T \quad \int_t^T f^M \; Y \; t \; , s \; ds \quad \Phi\left(\frac{-Y \; t \; - \mu_Y^M \; T - t}{\sqrt{T-t}}\right) \tag{8}$$
$$e^{-2\mu_Y^M Y(t)} \Phi\left(\frac{-Y \; t \quad \mu_Y^M \; T - t}{\sqrt{T-t}}\right)$$

where Φ is the normal cumulative density. The density of $Y \; T$, given that Y has not hit before maturity, is given by

$$f^M \; Y \; T \; \; Y \; t \; , t, T \quad \frac{}{\sqrt{\pi \; T - t}} e^{-\frac{1}{2}\left(\frac{Y(T)-Y(t)-\mu_Y^M(T-t)}{\sqrt{T-t}}\right)} \tag{9}$$
$$- \; e^{-2\mu_Y^M Y(t)} \frac{}{\sqrt{\pi \; T - t}} e^{-\frac{1}{2}\left(\frac{Y(T)+Y(t)-\mu_Y^M(T-t)}{\sqrt{T-t}}\right)}$$

Using the definition of $Y \; t$, we can change variables and work out the corresponding results under Q, where $\mu_Y^Q \quad \frac{r-\frac{1}{2}\sigma_V^2}{\sigma}$, or the measure G, where $\mu_Y^G \quad -\frac{r-\frac{1}{2}\sigma_V^2}{\sigma}$.

Use r to denote $r - \frac{1}{2}\sigma_V^2$, x to denote $\frac{V^*}{V(t)}$ and θ to denote $\frac{r+\frac{1}{2}\sigma_V^2}{\sigma_V^2}$, then:

$$^G \; s \leq T \quad \Phi \; d_5 \quad x^{-2\theta}\Phi \; d_6 \tag{10}$$

where

$$d_5 \quad \frac{x \quad \left(r \quad \frac{1}{2}\sigma^2\right) \; T - t}{\sigma_V\sqrt{T-t}} \tag{11}$$

and

$$d_6 \quad \frac{x - \left(r \quad \frac{1}{2}\sigma^2\right) \; T - t}{\sigma_V\sqrt{T-t}}. \tag{12}$$

Also,

$$f^Q \quad V \; T \quad V \; t \; t, T \tag{13}$$
$$\frac{}{\sqrt{\pi \; T - t}} e^{-\frac{1}{2}\left(\frac{\ln V(T) - \ln V(t) - \tilde{r}(T-t)}{\sigma_V\sqrt{T-t}}\right)}$$
$$-\left(\frac{V^*}{V \; t}\right)^{\frac{2\tilde{r}}{\sigma_V^2}} \frac{}{\sqrt{\pi \; T - t}} e^{-\frac{1}{2}\left(\frac{\ln V(T) - \ln \; V(t)x^2 \; - \tilde{r}(T-t)}{\sigma_V\sqrt{T-t}}\right)}$$

5.3.3 Prices of constituent claims

Dollar-in-boundary claim The price of a claim that pays when the boundary is hit (dollar-in-boundary claim) is

$$F_{DIB} \; V \; t \; , t \qquad E^{Q}\left[e^{-r(T-s)} I \;\; s \leq T \;\right] \qquad \int_{t}^{T} e^{-r(s-t)} f^{Q} \;\; Y \;\; t \;\; , s \;\; ds$$

which (completing the squares and integrating) yields (see e.g. Ericsson and Reneby 1998, in the appendix)

$$F_{DIB} \;\; V \;\; t \;\; , t \qquad e^{- \;\; \mu_Y^Q - \mu_Y^G \;\; Y(t) \;\; G} \qquad s \leq T \;\; . \tag{14}$$

Noting that

$$\frac{\left(\mu_Y^Q \qquad \sqrt{\left(\mu_Y^Q\right)^2 \qquad r}\right)}{\sigma_V} \qquad \frac{r}{\sigma_V^2}$$

we can write this as

$$F_{DIB} \;\; V \;\; t \;\; , t \qquad x^{\frac{2r}{\sigma_V^2}} \left(\Phi \;\; d_5 \qquad x^{-2\theta} \Phi \;\; d_6 \;\; \right)$$

or

$$F_{DIB} \;\; V \;\; t \;\; , t \qquad \frac{V \;\; t}{V^*} \left\{x^{2\theta} \Phi \;\; d_5 \qquad \Phi \;\; d_6 \;\; \right\} \tag{15}$$

Down-and-out claim Define the payoff $\Pi_{Tr} \;\; V \;\; T$ as the payoff $\Pi \;\; V \;\; T$ truncated at V^*:

Then we can derive the price of the down-and-out claim in terms of the prices of the truncated down-and-out claims with starting values for the process of $V \;\; t$ and $V \;\; t \;\; x^2$, using the distribution given in (9):

$$F_{\Pi,DO} \;\; V \;\; t \;\; , t \qquad F_{\Pi_{Tr}} \;\; V \;\; t \;\; , t \;\; - x^{\frac{2\tilde{r}}{\sigma_V^2}} F_{\Pi_{Tr}} \left(V \;\; t \;\; x^2, t\right) \tag{16}$$

For a good exposition, see e. g. Björk (1998).

Down-and-out call For example, consider a down-and-out call. For pricing a down-and-out call with strike price Z, we need to know what the price of a truncated call is. For a call whose price is truncated at V^*, with a starting value of the process equal to S, the price will be the simple Black-Scholes price if $V^* \leq Z$, i. e. if the truncated payoff is just equal to the normal call payoff:

$$F_C \;\; S, Z, t \qquad S\Phi \;\; d_1 \;\; - e^{-r(T-t)} Z \Phi \;\; d_2 \tag{17}$$

where

$$d_2 \; S \qquad \frac{\left(\frac{S}{Z}\right) \quad r \;\; T-t}{\sigma_V \sqrt{T-t}} \tag{18}$$

and

$$d_1 \; S \qquad d_2 \quad \sigma_V \sqrt{T-t}. \tag{19}$$

If $V^* > Z$, the price will be different. Denote the price of the truncated call as $F_{C,Tr}$. It is given by

$$F_{C_{Tr}} \;\; S, Z, t \qquad S\Phi \;\; d_3 \;\; - e^{-r(T-t)} Z \Phi \;\; d_4 \tag{20}$$

where now

$$d_4 \; S \qquad \frac{\left(\frac{S}{V^*}\right) \quad r \;\; T-t}{\sigma_V \sqrt{T-t}} \tag{21}$$

and

$$d_3 \; S \qquad d_4 \quad \sigma_V \sqrt{T-t} \tag{22}$$

We can now insert these pricing functions into the equation for a down-and-out-price to obtain the pricing functions $F_{C,DO}$.

Down-and-out call with strike price By the above formulas, this is given by

$$F_{C,DO} \;\; V \;\; t \;\; , \;\; , t \qquad V \;\; t \;\; \left[\Phi\left\{d_3 \;\; V \;\; t \;\;\right\} - x^{2\theta}\left(\Phi\left\{d_3\left(V \;\; t \;\; x^2\right)\right\}\right)\right] \tag{23}$$

Down-and-out call with strike price D, price of equity Suppose we have a call with a strike price of D, which is bigger than V^*, so that the payoff function is not truncated, then its price is

$$\begin{aligned} F_{C,DO} \;\; V \;\; t \;\; , D, t \qquad & V \;\; t \;\; \left(\Phi\left\{d_1 \;\; V \;\; t \;\;\right\} - x^{2\theta}\Phi\left\{d_1\left(V \;\; t \;\; x^2\right)\right\}\right) \\ & e^{-r(T-t)} D \left(x^{\frac{2\tilde{r}}{\sigma_V^2}} \Phi\left\{d_2\left(V \;\; t \;\; x^2\right)\right\} - \Phi\left\{d_2 \;\; V \;\; t \;\;\right\}\right) \end{aligned} \tag{24}$$

This is also the value of equity (E).

Note that the derivative of the value of equity with respect to the asset value is

$$\frac{\partial E}{\partial V} \quad \Phi\left\{d_1 \; V \;\right\} \quad \frac{r}{\sigma_V^2}\left(\frac{x}{V^{*\;\;2}}\right)\left(Vx^2\Phi\left\{d_1\left(Vx^2\right)\right\} - e^{-r(T-t)}D\Phi\left\{d_2\left(Vx^2\right)\right\}\right) \tag{25}$$

So using Ito's lemma, we can now find the volatility of equity as

$$\sigma_E \quad \frac{V}{E}\frac{\partial E}{\partial V}\sigma_V \tag{26}$$

This is a necessary ingredient for using the market value and volatility of equity to obtain the implied asset value and asset value volatility.

Bond price Recall that the dollar-in-boundary claim is

$$F_{DIB} \; V \; t \;, t \quad \frac{V \; t}{V^*}\left[x^{2\theta}\Phi\left\{d_5\right\} \quad \Phi\left\{d_6\right\}\right]. \tag{27}$$

We will need $\quad - K \; V^*$ of these - this is what we get for holding the bond in case of early default.

Putting it all together, we can calculate the price of the bond as

$$\begin{aligned} B \; V \; t \;, t, T, V^* \quad & F_{C,DO} \; V \; t \;, \;\;, t, T \\ & - F_{C,DO} \; V \; t \;, D, t, T \\ & \quad - K \quad V^* \; F_{DIB} \; V \; t \;, t \end{aligned} \tag{28}$$

$$\begin{aligned} B \; V \; t \;, t, T, V^* \quad & V \; t \; \left[\Phi\left\{d_3 \; V \; t \;\right\} - x^{2\theta}\left(\Phi\left\{d_3\left(V \; t \; x^2\right)\right\}\right)\right] \\ & - V \; t \; \left(\Phi\left\{d_1 \; V \; t \;\right\} - x^{2\theta}\Phi\left\{d_1\left(V \; t \; x^2\right)\right\}\right) \\ & - e^{-r(T-t)}D\left(x^{\frac{2\tilde{r}}{\sigma_V^2}}\Phi\left\{d_2\left(V \; t \; x^2\right)\right\} - \Phi\left\{d_2 \; V \; t \;\;\right\}\right) \\ & \quad - K \; V^*\frac{V \; t}{V^*}\left[x^{2\theta}\Phi\left\{d_5\right\} \quad \Phi\left\{d_6\right\}\right] \end{aligned} \tag{29}$$

Now note that $-d_3 \; V \; t \quad d_6$, and $d_5 \quad d_3\left(V \; t \; x^2\right)$ (see list below).
So we can write

$$B \; V \; t \;, t, T, V^* \quad e^{-r(T-t)}D\left[\Phi\left\{d_2 \; V \; t \;\;\right\} - x^{\frac{2\tilde{r}}{\sigma_V^2}}\Phi\left\{d_2\left(V \; t \; x^2\right)\right\}\right] \tag{30}$$

$$\begin{aligned} & V \; t \; \left[\Phi\left\{-d_1 \; V \; t \;\right\} \quad x^{2\theta}\Phi\left\{d_1\left(V \; t \; x^2\right)\right\}\right] \\ & - V \; t \; \left[K\left(x^{2\theta}\Phi\left\{d_5\right\} \quad \Phi\left\{d_6\right\}\right)\right] \end{aligned} \tag{31}$$

where

$$x \quad \frac{V^*}{V \ t}$$

$$\theta \quad \frac{r \quad \frac{1}{2}\sigma_V^2}{\sigma_V^2}$$

$$r \quad r - -\sigma^2$$

$$d_1 \ V \ t \quad \frac{\left(\frac{V(t)}{D}\right) \quad \left(r \quad \frac{1}{2}\sigma_V^2\right) \ T-t}{\sigma_V\sqrt{T-t}}$$

$$d_2 \ V \ t \quad \frac{\left(\frac{V(t)}{D}\right) \quad \left(r - \frac{1}{2}\sigma_V^2\right) \ T-t}{\sigma_V\sqrt{T-t}}$$

$$d_3 \ V \ t \quad \frac{- \quad x \quad \left(r \quad \frac{1}{2}\sigma_V^2\right) \ T-t}{\sigma_V\sqrt{T-t}}$$

$$d_1 \ V \ t \ x^2 \quad \frac{\left(\frac{V(t)}{D}\right) \qquad x \quad \left(r \quad \frac{1}{2}\sigma_V^2\right) \ T-t}{\sigma_V\sqrt{T-t}}$$

$$d_2 \ V \ t \ x^2 \quad \frac{\left(\frac{V(t)}{D}\right) \qquad x \quad \left(r - \frac{1}{2}\sigma_V^2\right) \ T-t}{\sigma_V\sqrt{T-t}}$$

$$d_3 \ V \ t \ x^2 \quad \frac{x \quad \left(r \quad \frac{1}{2}\sigma_V^2\right) \ T-t}{\sigma_V\sqrt{T-t}}$$

$$d_5 \quad \frac{x \quad \left(r \quad \frac{1}{2}\sigma_V^2\right) \ T-t}{\sigma_V\sqrt{T-t}} \qquad d_3 \ V \ t \ x^2$$

$$d_6 \quad \frac{x - \left(r \quad \frac{1}{2}\sigma_V^2\right) \ T-t}{\sigma_V\sqrt{T-t}} \qquad -d_3 \ V \ t$$

as before.

5.3.4 Relation to Merton price

Note that the bond price is equal to the Merton price plus the difference between the price of a normal call option with strike D and a down-and-out call option with strike D (this will always be positive), minus $KV \ t$ units of a dollar-in-boundary claim. We can interpret the call options as Merton-type equity and co-ordination failure equity respectively. If we set K , we obtain the Merton price: because there is no benefit to early reorganisation, it will never occur, and we are back at the Merton model, where there is no early default.

$$B \ V \ t \ ,t,T,V^* \qquad B_{Merton} \ V \ t \ ,t,T \tag{32}$$

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