Coordination, Learning, and Delay*

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Abstract

This paper studies how the introduction of social learning with costs to delay affects coordination games with incomplete information. We present a tractable noisy dynamic coordination game with social learning and costs to delay. We show that this game has a unique monotone equilibrium. A comparison of the equilibrium of the dynamic game with the equilibria of analogous static coordination games explicates the role of social learning. The analysis is carried out for both endogenous and exogenous order of moves in the dynamic game.

In the limit as noise vanishes, social welfare is strictly ranked in these games, with the highest welfare achieved in the dynamic game with endogenous ordering. We demonstrate that exogenous asynchronicity is not a substitute for endogenous asynchronicity. We also show that under endogenous ordering, as noise vanishes, the efficiency of coordination is maximized at intermediate costs to delay. The robustness of these results is illustrated numerically away from the complete information limit, when closed forms are not available. Our results have implications for the initial public offerings of debt, as well as for the adoption of new technology under incomplete information.

Keywords: Coordination, equilibrium selection, social learning, network externalities, global games.

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1 Introduction

In many applied problems in economics an agent's payoff from taking an action depends on some underlying (unknown) state of the world and increases in the mass of other agents taking similar actions. Many such settings are also inherently dynamic, encompassing several time periods. Thus agents are presented with several occasions to act, and may be able to (noisily) observe the actions of others who make choices before them. In the presence of private information, such observation may help agents improve their knowledge of payoffs. However, there may be a cost to delay in making choices. We begin by providing a leading example of such a problem.

Consider a group of investors deciding whether to invest their resources in a safe domestic venture or in a risky emerging market project. The risky project requires the participation of a critical mass of speculators to succeed. Potential investors have private information (from a combination of publicly available information and personal research) about the underlying exogenous value of the project. Since investors operate in a common market, those who choose to wait before investing can at least noisily observe the actions of others who made their choices earlier. This can provide them with better information. But delay comes at a cost of remaining vested in the project for a shorter period of time, and therefore enjoying less of its benefits.[1] Similar examples include the adoption of new technologies with network externalities[2], currency crises[3], and bank runs[4].

At the heart of the example we have just described lies a coordination problem. An individual wishing to invest in the risky project must be convinced that enough fellow investors will also participate. Thus, we might expect equilibrium underinvestment relative to the social optimum. However, the standard coordination problem is complicated here by dynamics and social learning. Individuals have multiple periods to act. There may be an incentive to wait and collect more information. This may delay, or even discourage, investment. On the other hand, observing investment by some investors may encourage others to join in, and thus lead to greater investment. Finally, the additional information produced by observational learning may help agents make better choices. Given these complications, it is natural

[1] Chari and Kehoe (2000) study this problem. However, they do not take into account strategic complementarities. In their model, the success of the project depends only on an exogenous state variable. Two related models that analyze settings with endogenous timing and private information in the absence of strategic complementarities are Caplin and Leahy (1993) and Chamley and Gale (1994).

[2] This problem is analyzed by Choi (1997). We discuss this paper further in Section 6.5.

[3] This problem is analyzed by Morris and Shin (1998), in a static model, ignoring learning. They extend the equilibrium selection techniques of Carlsson and van Damme (1993) to analyze the problem in a unique-equilibrium setting.

[4] Analyzed by Goldstein and Pauzner (2000), also ignoring learning, and also in the tradition of Carlsson and van Damme (1993).

to ask: How might the presence of dynamics and social learning affect the level of equilibrium investment relative to benchmark cases?

In order to address this question we model the stylized features outlined above using a noisy dynamic coordination game with Bayesian social learning and costs to delay. To establish a benchmark for comparison we also analyze the same coordination problem in the absence of dynamic elements and learning, i.e., using *static* coordination games. To avoid the usual difficulties created by multiple equilibria in static coordination games we rely on the work of Carlsson and van Damme (1993) and Morris and Shin (1998, 2000). These authors demonstrate that in the presence of private information, a unique equilibrium is selected in a large class of static coordination games, commonly referred to as *global games*. The static global games analysis provides a tractable benchmark for our results. Comparing our results to these earlier analyses would address the question raised above. Formally stated: *How does the introduction of social learning with delay costs affect the probability of coordinated risk-taking in noisy supermodular games?*[5] Does learning make a difference? If so, how? In order to complete the comparison, however, we must first overcome a theoretical hurdle.

Dynamic coordination games, like their static counterparts, have multiple equilibria when payoffs are common knowledge.[6] Many such games can have multiple equilibria even in the absence of complete information, e.g. Chamley (2001), Angeletos, Hellwig, and Pavan (2002), and Dasgupta (1999). This makes it difficult to compare these games with their static counterparts, and therefore obscures the role of dynamics and learning.

The theory of equilibrium selection in dynamic supermodular games has received recent scholarly attention. In an important set of papers, Burdzy, Frankel, and Pauzner (2001), Frankel and Pauzner (2000), and Frankel (2000) have established that when payoffs are affected by a stochastic parameter with sufficient stationarity and frequent innovations, a unique equilibrium is selected in a class of dynamic coordination games where agents are offered random opportunities to switch between actions.[7] However, agents in these models cannot use the observed actions of others to Bayes update their beliefs about the state of the world. The evolving state variable is observed publicly, and the current value incorporates all available information about future values. Their results are, therefore, not directly applicable to the class of problems of interest to us.

To facilitate a precise comparison with static benchmarks, we propose a simple model in

[5]Marx and Matthews (2000) study a related question in a complete information public goods model, with dynamics but without learning. They find that dynamics can, under certain conditions, increase contributions to a public project relative to static benchmarks.

[6]For an analysis of dynamic coordination games with complete information, see Gale (1995), who demonstrates multiplicity. In particular, for a continuum player version of Gale's model, which shares features with the model we present below, he demonstrates that there is a *continuum* of possible equilibria.

[7]In recent work, Levin (2000) extends their analysis to study overlapping generations games.

which some of the Carlsson-van Damme/Morris-Shin equilibrium selection arguments can be extended to incorporate social learning.[8] In our model, a mass of agents choose whether to invest in a safe project or in a risky project of uncertain underlying value. The underlying state of the world is indexed by a variable θ, which becomes known at the end of the game, when consumption occurs. Investors have two opportunities to invest before the end of the game, at times t_1 and t_2. The receipts of an agent who invests in the risky project depend on whether the project is successful. When the project succeeds, it generates a high payoff which is continuously compounded between the time of investment and the end of the game. Thus, there is a cost associated with investing later in the game. When the project fails, it pays nothing. The project succeeds at θ if a critical mass of agents participate, and this required mass is inversely related to the value of θ. Agents do not know θ for sure, but receive informative private signals about it. Agents who act in the later period, observe a noisy signal of an aggregate statistic based on the proportion of investors in the earlier period. While this statistic provides them with further information on the underlying state, as we have noted already, there is a cost associated with delayed investment.

We first establish the benchmark static analysis for this model. The static models, which enforce simultaneous moves, are, by definition, devoid of social learning. We define two natural static games, based upon the time (t_1 or t_2) at which they are played. Following these static models we examine dynamic extensions, which allow us to incorporate learning. There are two natural ways to do this. We may allow for asynchronicity while prespecifying exogenously the order in which the different agents must act. Alternatively we may consider a model where agents are allowed to choose both their actions and the time at which they act We consider each of these four models in turn.

1.1 Summary of Results

To explore the implications of our dynamic analysis, we first establish some characterization results. We show that as long as noise is small enough there is a unique monotone equilibrium in each of the dynamic games we consider. This is true regardless of whether the order of actions is specified exogenously (Proposition 3), or chosen endogenously (Proposition 4). In the limit as noise vanishes, we can solve for these equilibria in closed form. We show that these dynamic equilibria are "well behaved": As we vary model parameters to bring the dynamic games "close" to the limiting static games, the dynamic equilibria converge

[8]It is worth emphasizing here that the purpose of this exercise is not to show that incorporating social learning into canonical global games perserves the standard equilibrium selection results. This is not true in general. The purpose is, rather, to set up a tractable model in which transparent closed form comparisons can be made to study the role of dynamics and learning.

smoothly to their static counterparts. These convergence results are shown in Corollaries 1 and 2 and discussed below in Sections 4 and 5.

The comparison of the equilbria of the static and dynamic games of exogenous and endogenous order provides insights into the role of dynamics and learning. Our results address two related but distinct issues. The first pertains to the equilibrium probability of coordinated investment. The second pertains to social welfare. We deal with these in turn.

As noise vanishes, we show that there exists a *strict ranking of the probability of coordinated investment across the different games.* The endogenous order dynamic game maximizes the probability of investment. This is followed by the first period static game, which in turn is followed by the exogenous order dynamic game. The lowest probability of investment is achieved in the second period static game. This is summarized in Corollary 3 and discussed below in Sections 5 and 6.

In a related finding, we show that *exogenous ordering cannot substitute for endogenous ordering.* As noise vanishes, for almost all parameter values, there exists no ex ante exogenous ordering of agents that can replicate the probability of coordinated achieved by the endogenous order dynamic game. This is because the endogenous coordination game utilizes the *revealed preference* of a group of agents to invest early, while the exogenous order game does not. We illustrate that this result is robust to the presence of significant amounts of private information in the games. This is discussed further in Section 6.2.

We demonstrate that as noise vanishes, the *probability of coordinated investment in the endogenous order dynamic game is maximized for intermediate cost to delay.* An intuition for this follows from the observation that the efficiency of coordination depends on the total mass of agents who can be persuaded to invest during the course of the game. The cost of delay has opposite effects on the masses of agents who invest early or late. A large cost to delay persuades more agents to invest early. But at the same time it dissuades agents who did not invest early from doing so later based on their updated information. We call this non-monotone relationship the *coordination effect* of introducing a costly delay option. We illustrate numerically that it is robust to the presence of significant amounts of private information. This is discussed further in Sections 6.3 and 6.4.

We show that the effect of introducing learning with exogenous ordering can be given a particularly clean characterization. The relationship between the equilibria of the exogenous order dynamic game and the two benchmark static games is essentially determined by the exogenous parameter specifying the division of players between the two periods. However, we show that later players in the dynamic game are able to use the additional information obtained by observing their predecessors to make more accurate choices.

We now turn to the question of social welfare. In the limit as noise vanishes, social welfare is a monotone increasing function of the equilibrium probability of coordinated investment. Thus, in the noise-free limit, social welfare is ranked as above: highest in the endogenous order game, followed by the first period static game, followed by the exogenous order game, and finally the second period static game. However, away from the limit, in addition to the coordination effect, the introduction of the costly option to delay has two other effects. When the option to delay is exercised, it leads to better information and higher welfare (the *learning effect*), but since the option is costly, leads to lower payoffs and therefore lower welfare (the *direct payoff effect*). The total welfare effect of introducing a costly delay/learning option into a coordination game results from the interaction of these three effects.

We illustrate that for low levels of private information, the coordination effect dominates the learning effect and social welfare is maximized at intermediate levels of delay costs. However, for high levels of private information, the learning effect dominates the coordination effect, and thus, for sufficiently noisy endogenous order dynamic coordination games, social welfare is maximized at minimal cost of delay. The interaction of the coordination, learning, and direct payoff effects is summarized in Section 6.4.

1.2 Applications

Our model has implications for at least two classes of applied problems. We outline them here. A more detailed discussion is provided in Section 6.5. First, consider a government financing an uncertain project by offering a debt contract, in a setting in which secondary markets for the debt contract may be missing or illiquid. Under these circumstances, our model suggests that it may be beneficial to "stagger" the initial offering to allow investors multiple opportunities to invest, and sort themselves over time. Then, under the results outlined above, the coordination effect will ensure that the project will succeed with higher probability than if the entire debt package was offered simultaneously.

Second, our results provide a fresh perspective on the question of whether it is beneficial or harmful to allow firms who are switching between technologies to have the option to delay or not. The so called "penguin effect"[9] can lead to socially suboptimal delay in this context. Choi (1997) suggests that in settings with incomplete information and network externalities, it may be socially optimal for agents to forfeit their option to wait and learn and to make choices simultaneously. In direct contrast with Choi, we find that introducing a costly option to delay and learn can enable agents to sort themselves efficiently over time, and lead to strict gains in the efficiency of coordination. Thus, the penguin effect, while present in our model,

[9] The tendency for agents in strategic settings to wait to act second, in order to gain more information, avoid intermediate or final miscoordination costs from temporary or permanent "stranding" in a ex post suboptimal technology. See Farrell and Saloner (1986)

enhances rather than diminishes social welfare.

The rest of the paper is organized as follows. In the next section we describe the investment problem. In section 3 we analyze the problem using the static approach of Morris and Shin. Sections 4 and 5 extend the analysis to include dynamic elements. In section 4 the problem is analyzed using a dynamic coordination game with exogenously specified order of actions. Section 5 relaxes the exogeneity of order. Sections 6 and 7 discuss and conclude.

2 The Investment Project

The economy is populated by a continuum of risk neutral agents, indexed by $[0, 1]$, each of whom has one unit of resources to invest. They must choose between investing in a safe project, which gives a gross payoff of 1, and a risky project of uncertain value. Uncertainty is summarized by a state variable θ which is distributed $N(0, 1)$ and is revealed at time T, when consumption occurs. There are two periods in which an agent might be able to invest in the risky project: $t \in \{t_1, t_2\}$. We require that $T > t_2 > t_1$, i.e. at the times when agents have opportunities to invest, the value of the project is unknown.

Proceeds from investing in the risky project depend on whether the project succeeds or not. The success of the project, in turn, depends on the actions of the agents and the realized value of θ. In particular, if p denotes the total mass of agents who invest at the times when opportunities are available, then investment succeeds if $p \geq 1 - \theta$. Payoffs from the risky project can be summarized as follows:

- When the project fails, it pays 0.
- When the project succeeds, it pays an instantaneous rate of return $R > 0$, which is continuously compounded over the length of time that an agent has held the investment. Thus, for an agent who invests at time t_i, returns are $e^{R(T-t_i)}$, conditional on the success of the project.

Our payoffs are motivated by, and very similar to, those of Chari and Kehoe (2000). The major difference is that we incorporate strategic complementarities, i.e., we allow the success or failure of the project to depend not only on the exogenous state, but also on the endogenous number of agents who choose to invest.[10]

We now perform some useful normalizations. We recast the game in terms of payoffs to *switching* from the safe project to the risky one, and divide through by $e^{R(T-t_1)}$. Let

[10] A second, minor, difference between our models is that Chari and Kehoe (2000) allow agents $T_1 \geq 2$ occasions to invest, where $T_1 < T$, while we set $T_1 = 2$. The generalization of our model to include more than two periods presents no conceptual difficulties, but comes at great algebraic cost, given the strategic complementarities. We conjecture that the results will be very similar.

$c = \frac{1}{e^{R(T-t_1)}}$, and let $k = 1 - e^{R(t_1 - t_2)}$. We label the act of switching at $t = t_1$ by I_1, and at $t = t_2$ by I_2. The act of never switching is denoted N. Thus, we may now represent agent's utilities by the following schedule:

$$u(I_1, p, \theta) = \begin{cases} 1 - c & \text{if } p \geq 1 - \theta \\ -c & \text{otherwise} \end{cases} \quad (1)$$

$$u(I_2, p, \theta) = \begin{cases} (1 - k) - c & \text{if } p \geq 1 - \theta \\ -c & \text{otherwise} \end{cases} \quad (2)$$

$$u(N, p, \theta) = 0 \quad (3)$$

Note that $k \in (0, 1 - c)$, because $t_2 \in (t_1, T)$. Thus, k represents a cost to delay, the payoff forfieted by an agent due to her delay in switching.

At the beginning of $t = t_1$ agents observe the state of fundamentals with idiosyncratic noise. In particular, each agent i receives the following signal at the beginning of the game:

$$x_i = \theta + \sigma\epsilon_i \quad (4)$$

where ϵ is distributed Standard Normal in the population and independent of θ.

We now present a sequence of (progressively more complex) games that can be used to study this investment problem. We begin with the benchmark static case analyzed by Morris and Shin, and then extend by introducing dynamic elements.

3 The Benchmark Static Game

To analyze this investment problem within the framework of static global games in the style of Morris and Shin requires that we place an ad hoc restriction on the actions of players: they must all either move at $t = t_1$, or they must all move at $t = t_2$. This defines two natural static global games, which are mutually exclusive.

The first such game is one in which all players act at $t = t_1$ and payoffs are given by (1) and (3). We label this game $\Gamma_{st,1}$. We label the other game, in which players all move at $t = t_2$ and payoffs are given by (2) and (3), by $\Gamma_{st,2}$. We analyze $\Gamma_{st,1}$ and extend our results by symmetry to $\Gamma_{st,2}$.

It is useful to begin with a preliminary definition. Note that in these games, agents' strategies map from their private information into their action spaces.

Definition 1 *An agent i is said to follow a* monotone *strategy if her chosen actions are increasing in her private information, i.e., if her strategy takes the form:*

$$\sigma_i(x_i) = \begin{cases} I & \textit{when } x_i \geq x^* \\ N & \textit{otherwise} \end{cases}$$

We shall call equilibria in monotone strategies *monotone equilibria*. Monotone equilibria can be given a natural economic interpretation: when an agent chooses to invest, she correctly believes (in equilibrium) that all agents who have more optimistic beliefs than her also choose to do so.

If a continuum of players follow monotone strategies, a threshold level emerges naturally in the underlying state variable of the game. Therefore, we look for monotone equilibria which take the form $(x^*_{st,1}, \theta^*_{st,1})$ where agent i invests iff $x_i \geq x^*_{st,1}$ and investment is successful iff $\theta \geq \theta^*_{st,1}$. Now we may state:

Proposition 1 (Morris and Shin) [11] *If $\sigma < \sqrt{2\pi}$, there is a unique monotone equilibrium in $\Gamma_{st,1}$. As $\sigma \to 0$, it is given by the pair:*

$$x^*_{st,1} = c, \quad \theta^*_{st,1} = c$$

Proof: The following are necessary for the equilibrium:

The marginal agent, who receives signal $x^*_{st,1}$ must be indifferent between investing or not, i.e.

$$Pr(\theta \geq \theta^*_{st,1} | x^*_{st,1}) = c$$

Since $\theta | x \sim N(\frac{x}{1+\sigma^2}, \frac{\sigma^2}{1+\sigma^2})$, the indifference condition can be written as:

$$1 - Pr(\theta < \theta^*_{st,1} | x^*_{st,1}) = 1 - \Phi(\frac{\theta^*_{st,1} - \frac{x^*_{st,1}}{1+\sigma^2}}{\frac{\sigma}{\sqrt{1+\sigma^2}}}) = c$$

Thus,

$$x^*_{st,1} = (1+\sigma^2)\theta^*_{st,1} + \sigma\sqrt{1+\sigma^2}\Phi^{-1}(c) \tag{5}$$

The critical mass condition requires that:

$$Pr(x \geq x^*_{st,1} | \theta^*_{st,1}) = 1 - \theta^*_{st,1}$$

Substituting the indiffirence condition into the critical mass condition we get

$$\Phi(\sigma\theta^*_{st,1} + \sqrt{1+\sigma^2}\Phi^{-1}(c)) = \theta^*_{st,1} \tag{6}$$

Consider the function

$$F(\theta^*_{st,1}) = \Phi(\sigma\theta^*_{st,1} + \sqrt{1+\sigma^2}\Phi^{-1}(c)) - \theta^*_{st,1}$$

[11] This result is a special case of Morris and Shin (2000): Proposition 3.1. It can be obtained by setting the precision of the public signal to 1.

Clearly as $\theta^*_{st,1} \to 1$, $F(\cdot) < 0$, and as $\theta^*_{st,1} \to 0$, $F(\cdot) > 0$. Differentiating yields

$$F'(\theta^*_{st,1}) = \sigma\phi(\cdot) - 1$$

If $\sigma < \sqrt{2\pi}$, then $F'(\theta^*_{st,1}) < 0$ for all $\theta^*_{st,1}$, which establishes the first part of the result. Letting $\sigma \to 0$ in (5) establishes the second part. ■

The corresponding result for $\Gamma_{st,2}$ follows immediately:

Proposition 2 (Morris and Shin) *If $\sigma < \sqrt{2\pi}$, there is a unique monotone equilibrium in $\Gamma_{st,2}$. As $\sigma \to 0$ it is given by the pair:*

$$x^*_{st,2} = \frac{c}{1-k}, \quad \theta^*_{st,2} = \frac{c}{1-k}$$

We now extend our analysis to introduce dynamic elements. The simplest way to achieve this is to require that some exogenous proportion of agents have to choose their actions at $t = t_1$, and the rest must do so at $t = t_2$. Even in this simplest of dynamic frameworks, we are able to incorporate Bayesian social learning, as we show below.

4 The Dynamic Game with Exogenous Order of Actions

We now modify the game to last the length of the investment project: $t \in \{t_1, t_2\}$. The continuum of agents is divided up (exogenously) into two (possibly unequal) groups. Agents $i \in [0, \lambda]$ must choose their actions at $t = t_1$. Agents $i \in (\lambda, 1]$ must choose their actions at $t = t_2$. The payoffs to this game are given by (1 - 3).

We can now incorporate Bayes social learning. Agents who act in period 2 are able to observe a statistic based on the proportion of time 1 agents who chose to invest, which we denote by p_1. Hence, they effectively observe a "market share". However, agents observe such a market share statistic with some idiosyncratic noise, which may be small. We shall be particularly interested in the case where the observation becomes essentially public, i.e. in the limit as such idiosyncratic noise vanishes. Thus, agents $(\lambda, 1]$ receive an additional signal:

$$y_i = \Phi^{-1}(p_1) + \tau\eta_i \tag{7}$$

where η is Standard Normal in the population, and independent of ϵ. The specific transformation of p_1 by the inverse standard normal CDF is an algebraic simplification only (to obtain closed forms), and serves no other purpose in the arguments that follow. As is apparent, the standard case of perfect observation of the past (as is common in the literature on herds and cascades, see Bikhchandani, Hirshleifer, and Welch 1992 for example) is obtained in the limit as $\tau \to 0$. We label this game Γ_{ex} and look for Bayes Nash equilibria of this game.

Players $(\lambda, 1]$ observe two noisy signals, x and y. Let $s(x, y)$ denote a sufficient statistic for (x, y). We look for monotone equilibria which take the form $(x^*_{ex}, s^*_{ex}, \theta^*_{ex})$, such that:[12]

1. Players $[0, \lambda]$ invest iff $x_i \geq x^*_{ex}$
2. Players $(\lambda, 1]$ invest iff $s_i \geq s^*_{ex}$
3. Investment is successful iff $\theta \geq \theta^*_{ex}$

Necessary conditions for such an equilibrium are as follows. Conditional upon receiving signal x^*_{ex}, player i for $i \in [0, \lambda]$ must be indifferent between investing and not investing:

$$Pr(\theta \geq \theta^*_{ex} | x^*_{ex}) = c \tag{8}$$

Conditional upon receiving signals that lead to sufficient statistic s^*_{ex}, player i for $i \in (\lambda, 1]$ must be indifferent to investing and not investing:

$$Pr(\theta \geq \theta^*_{ex} | s^*_{ex}) = \frac{c}{1-k} \tag{9}$$

Finally, at state θ^*_{ex} just the correct proportion of agents must choose to invest for investment to be successful:

$$\lambda Pr(x \geq x^*_{ex} | \theta^*_{ex}) + (1-\lambda) Pr(s \geq s^*_{ex} | \theta^*_{ex}) = 1 - \theta^*_{ex} \tag{10}$$

Note that $\theta | x$ is distributed $N(\frac{x}{1+\sigma^2}, \frac{\sigma^2}{1+\sigma^2})$. The mass of people who invest in period 1 at state θ is

$$p_1 = \Phi(\frac{\theta - x^*_{ex}}{\sigma})$$

Substituting into the definition of the second period signal, y, we get:

$$y_i = \frac{\theta - x^*_{ex}}{\sigma} + \tau \eta_i$$

Defining $z_i = \sigma y_i + x^*_{ex}$ we get

$$z_i = \theta + (\sigma\tau)\eta_i$$

and thus $z_i | \theta$ is distributed $N(\theta, \sigma^2\tau^2)$. Then, using Bayes's Rule we know that

$$\theta | x_i, z_i \sim N\left[\frac{\frac{1+\sigma^2}{\sigma^2}\frac{x_i}{1+\sigma^2} + \frac{1}{\sigma^2\tau^2} z_i}{\frac{1+\sigma^2}{\sigma^2} + \frac{1}{\sigma^2\tau^2}}, \frac{1}{\frac{1+\sigma^2}{\sigma^2} + \frac{1}{\sigma^2\tau^2}}\right]$$

[12]When a sufficient statistic exists, as it does in our problem, restricting attention to monotone equilibria where second period agents condition upon their sufficient statistics is without loss of generality.

Substituting for z_i,

$$\theta|x_i, y_i \sim N\left[\frac{x_i + \frac{\sigma}{\tau^2}y_i + \frac{1}{\tau^2}x^*_{ex}}{1+\sigma^2+\frac{1}{\tau^2}}, \frac{\sigma^2}{1+\sigma^2+\frac{1}{\tau^2}}\right]$$

Thus if we define:

$$s_i = \frac{x_i + \frac{\sigma}{\tau^2}y_i + \frac{1}{\tau^2}x^*_{ex}}{1+\sigma^2+\frac{1}{\tau^2}} \tag{11}$$

then

$$\theta|x, y \equiv \theta|s \sim N\left[s, \frac{\sigma^2}{1+\sigma^2+\frac{1}{\tau^2}}\right] \tag{12}$$

Since s_i is a linear function of two conditionally Normal variables x and y, it is easy to see that:

$$s_i|\theta \sim N\left[\frac{1+\tau^2}{1+\tau^2+\sigma^2\tau^2}\theta, \frac{\sigma^2\tau^2(1+\tau^2)}{(1+\tau^2+\sigma^2\tau^2)^2}\right] \tag{13}$$

Now we can rewrite the necessary conditions for the equilibrium as follows.

Equation (8) can be re-written as:

$$x^*_{ex} = (1+\sigma^2)\theta^*_{ex} + \sigma\sqrt{1+\sigma^2}\Phi^{-1}(c) \tag{14}$$

Using (12), equation (9) can be rewritten as:

$$s^*_{ex} = \theta^*_{ex} + \frac{\sigma}{\sqrt{1+\sigma^2+\frac{1}{\tau^2}}}\Phi^{-1}(\frac{c}{1-k}) \tag{15}$$

Finally, substituting from (14) and (15) into equation (10) we get:

$$\lambda(1-\Phi(\sigma\theta^*_{ex}+\sqrt{1+\sigma^2}\Phi^{-1}(c)))+(1-\lambda)(1-\Phi(\frac{\sigma\tau}{\sqrt{1+\tau^2}}\theta^*_{ex}+\frac{\frac{\sigma\tau}{\sqrt{1+\tau^2+\sigma^2\tau^2}}\Phi^{-1}(\frac{c}{1-k})}{\frac{\sigma\tau\sqrt{1+\tau^2}}{1+\tau^2+\sigma^2\tau^2}})) = 1-\theta^*_{ex}$$

Rearranging, we get:

$$\lambda\Phi(\sigma\theta^*_{ex}+\sqrt{1+\sigma^2}\Phi^{-1}(c))+(1-\lambda)\Phi(\frac{\sigma\tau}{\sqrt{1+\tau^2}}\theta^*_{ex}+\frac{\sqrt{1+\tau^2+\sigma^2\tau^2}}{\sqrt{1+\tau^2}}\Phi^{-1}(\frac{c}{1-k})) = \theta^*_{ex} \tag{16}$$

Equations (8), (9), and (16) are the dynamic counterparts of equations (5) and (6) in the proof of Proposition 1. Thus, taking the derivative, and re-utilizing methods used above, we note that if

$$\lambda\sigma + (1-\lambda)\frac{\sigma}{\sqrt{1+\frac{1}{\tau^2}}} < \sqrt{2\pi}$$

then there is a unique solution to this equation. Letting $\sigma \to 0$ enables us to obtain closed forms, and we can state:

Proposition 3 *If* $\sigma < \frac{\sqrt{2\pi}}{\lambda+(1-\lambda)\frac{\tau}{\sqrt{1+\tau^2}}}$ *there is a unique monotone equilibrium in* Γ_{ex}*. In the limit as* $\sigma \to 0$*, it is given by the triple:*

$$x^*_{ex} = \lambda c + (1-\lambda)\frac{c}{1-k} \quad s^*_{ex} = \lambda c + (1-\lambda)\frac{c}{1-k} \quad \theta^*_{ex} = \lambda c + (1-\lambda)\frac{c}{1-k}$$

We note in passing that as we let $\tau \to \infty$, and thus eliminate learning in this game, the condition for uniqueness converges to the usual static condition for uniqueness: $\sigma < \sqrt{2\pi}$.

Now that we have demonstrated the existence and uniqueness of monotone equilibria in Γ_{ex}, we can compare the selected equilibrium to those of $\Gamma_{st,1}$ and $\Gamma_{st,2}$. A clean comparison can be obtained by comparing the selected threshold levels in the fundamentals in the different games. The findings are summarized in:

Corollary 1 *As* $\sigma \to 0$*:*

- $\theta^*_{st,1} < \theta^*_{ex} < \theta^*_{st,2}$
- *As* $\lambda \to 1$*,* $\theta^*_{ex} \to \theta^*_{st,1}$
- *As* $\lambda \to 0$*,* $\theta^*_{ex} \to \theta^*_{st,2}$

Thus, the outcome in the dynamic game with Bayes learning when the order is specified exogenously is not fundamentally different from the outcomes in the individual static games. The differences are driven solely by the parameter determining the exogenous ordering. As all agents are forced to act in the first or second periods the selected equilibrium converges smoothly to the selected equilibria of the corresponding static games.

Intuitively, by making players act according to the exogenous division parameterized by λ, we are effectively forcing them to play two static coordination games, but with different payoffs. The outcome is simply a convex combination of the outcomes in the two static games, weighted by the mass of agents that play each of them.

However, period 2 players in Γ_{ex} have access to more precise information than players in $\Gamma_{st,2}$. Thus, we would expect them to do better on average than players in $\Gamma_{st,2}$. It turns out that they do. The relevant question is: When the project succeeds (fails) what proportion of later players choose to invest (not invest) in Γ_{ex} versus players in $\Gamma_{st,2}$? To answer this, we note that the proportion of players who choose a particular action in an arbitrary game Γ at any level of fundamentals θ is determined by the difference between θ and θ^*_Γ. In Γ_{ex}, the proportion of period 2 agents who choose to invest at state θ is given by $Pr(s \geq s^*_{ex}|\theta)$. Using the definitions and results above, this can be rewritten to be $\Phi\left(\frac{\frac{\theta-\theta^*_{ex}}{\sigma}}{\sqrt{1+\frac{1}{\tau^2}}} - \frac{\sigma\tau}{\sqrt{1+\tau^2}}\theta^*_{ex} - \frac{\sqrt{1+\tau^2+\sigma^2\tau^2}}{\sqrt{1+\tau^2}}\Phi^{-1}(\frac{c}{1-k})\right)$. The proportion of agents who choose to invest in $\Gamma_{st,2}$ at state θ is given by $Pr(x \geq x^*_{st,2}|\theta)$. This can similarly be rewritten as $\Phi\left(\frac{\theta-\theta^*_{st,2}}{\sigma} - \sigma\theta^*_{st,2} - \sqrt{1+\sigma^2}\Phi^{-1}(\frac{c}{1-k})\right)$. Label $\delta_\Gamma = \theta - \theta^*_\Gamma$. It is easy to see that there exists $\bar{\tau} > 0$ such that for all $\tau \leq \bar{\tau}$, if $\delta_{\Gamma_{st,2}} = \delta_{\Gamma_{ex}} > 0$, then

$$\Phi\left(\frac{\delta_{\Gamma_{ex}}}{\frac{\sigma}{\sqrt{1+\frac{1}{\tau^2}}}} - \frac{\sigma\tau}{\sqrt{1+\tau^2}}\theta^*_{ex} - \frac{\sqrt{1+\tau^2+\sigma^2\tau^2}}{\sqrt{1+\tau^2}}\Phi^{-1}(\frac{c}{1-k})\right) >$$

$$\Phi\left(\frac{\delta_{\Gamma_{st,2}}}{\sigma} - \sigma\theta^*_{st,2} - \sqrt{1+\sigma^2}\Phi^{-1}(\frac{c}{1-k})\right)$$

In words, when investment is successful, and learning is accurate enough, a larger proportion of period 2 agents choose to invest (thus, choose the right action) in Γ_{ex} than in $\Gamma_{st,2}$. If $\delta_{\Gamma_{st,2}} = \delta_{\Gamma_{ex}} < 0$, then the inequality is reversed. Thus, when investment fails, a larger proportion of period 2 agents choose not to invest in Γ_{ex} than in $\Gamma_{st,2}$. In other words, on average later agents may be able to improve their welfare in the dynamic game. We shall return to a more detailed discussion of welfare in Section 6.[13]

5 The Dynamic Game with Endogenous Order of Actions

We now further augment the original game to allow agents to endogenize the order of actions. The payoffs of the game are still given by (1-3) and the information structure is summarized as in the previous section by (4) and (7). However, now agents may also choose when to invest, if at all. In particular, in period 1, agents have the choice to invest or not. If they invest, then their choice is final. If they choose not to invest, however, they get another opportunity in period 2 to make the same choice, based on the additional information they receive at that time. As we have noted earlier, the payoffs to the investment project given in (1-3) induce an endogenous cost to delay in investing. Now that they may choose both their actions and the timing of their actions, agents will rationally trade off the possible excess gains to acting early against the option value of waiting and collecting more information in period 2. Note that even in the presence of this tradeoff, there is no information externality

[13] Readers familiar with the literature on global games will have noticed that the uniqueness results proved thus far are restricted to monotone strategy equilibria. For static global games Carlsson and van Damme (1993, later generalized by Frankel, Morris, and Pauzner 2000) prove a stronger result: the unique monotone equilibrium is also the unique strategy profile surviving the iterated deletion of dominated strategies. Existing arguments for this stronger result do not generalize to our dynamic game due to Bayesian learning. The existence of non-monotone equilibria, which are complex objects in this setting, remains an open question. However, even in the potential presence of such equilibria, the results presented here show that dynamics and learning lead to possible outcomes that pareto-dominate anything that can be achieved in analogous static settings (see below, Section 6).

in our model. In our continuum player game, the precision of posterior information about the state does not depend on the proportion of agents who choose to invest early. Thus, the social value of information does not depend on the mass of early investors. There is no free-rider problem in the production of information.[14] We call this game Γ_{en} and look for Bayes Nash equilibria.

As in the game with exogenous ordering, we look for equilibria in which agents choose monotone strategies with thresholds (x^*_{en}, s^*_{en}), such that:

1. Invest at $t = t_1$ iff $x_i \geq x^*_{en}$. Otherwise choose to wait.

2. Conditional on reaching $t = t_2$ with the option to invest, invest iff $s_i \geq s^*_{en}$

In $\Gamma_{st,i}$ and Γ_{ex} it was apparent that when agents followed monotone strategies there were corresponding equilibrium thresholds in the fundamentals above which investment would be successful, and below which it would fail. This characterization is not immediate in the current game (since the decisions to invest or not in the two periods are not independent) and requires closer examination.

When agents follow monotone strategies as outlined above, at any θ, a mass $Pr(x \geq x^*_{en}|\theta) + Pr(x < x^*_{en}, s \geq s^*_{en}|\theta)$ will choose to invest. Thus, investment is successful at θ if and only if:

$$Pr(x \geq x^*_{en}|\theta) + Pr(x < x^*_{en}, s \geq s^*_{en}|\theta) \geq 1 - \theta$$

Is there a critical θ^* above which investment is successful and below which it is not? The answer is in the affirmative, as we show below:

Lemma 1 *Fix any* (x^*, s^*). *Let*

$$G(\theta) = Pr(x \geq x^*|\theta) + Pr(x < x^*, s \geq s^*|\theta) - 1 + \theta$$

There is a unique solution to

$$G(\theta) = 0$$

The proof is in the appendix.

Given Lemma 1, we can now look for monotone equilibria of the form $(x^*_{en}, s^*_{en}, \theta^*_{en})$ where x^*_{en} and s^*_{en} are defined as above, and investment is successful if and only if $\theta \geq \theta^*_{en}$.

Necessary conditions for such equilibria are as follows:

The indifference equation for those players who arrive at period 2 with the option to invest:

$$Pr(\theta \geq \theta^*_{en}|s^*_{en}) = \frac{c}{1-k} \tag{17}$$

[14] Bolton and Harris (1999) study such a free-rider problem in the absence of payoff externalities.

The critical mass condition is:

$$Pr(x \geq x^*_{en}|\theta^*_{en}) + Pr(x < x^*_{en}, s \geq s^*_{en}|\theta^*_{en}) = 1 - \theta^*_{en} \tag{18}$$

We can rewrite equation (17) as

$$s^*_{en} = \theta^*_{en} + \frac{\sigma}{\sqrt{1+\sigma^2+\frac{1}{\tau^2}}}\Phi^{-1}(\frac{c}{1-k}) \tag{19}$$

Substituting this into equation (18) gives us:

$$Pr(x \geq x^*|\theta^*_{en}) + Pr(x < x^*, s \geq \theta^*_{en} + M|\theta^*_{en}) = 1 - \theta^*_{en}$$

where $M = \frac{\sigma}{\sqrt{1+\sigma^2+\frac{1}{\tau^2}}}\Phi^{-1}(\frac{c}{1-k})$. Now we note:

Lemma 2 *Fix any x^*. Let $\hat{\theta}$ be defined by $G(\hat{\theta}, x^*) = 0$ where*

$$G(\theta, x^*) = Pr(x \geq x^*|\theta) + Pr(x < x^*, s \geq \theta + M|\theta) - 1 + \theta$$

If $\sigma < \frac{\sqrt{2\pi}}{1+\frac{\tau}{\sqrt{1+\tau^2}}}$,

1. *For each x^*, there is a unique $\hat{\theta}$.*
2. *$\frac{d\hat{\theta}}{dx^*} \in (0, \frac{1}{1+\sigma^2})$.*

The proof is in the appendix. Now consider the third equation characterizing the monotone equilibrium, the indifference condition of players in period 1. In period 1, agents trade off the expected benefit of investing in period 1 against the expected benefit of retaining the option value to wait. Thus the marginal period 1 investor who receives signal x^*_{en} must satisfy:

$$Pr(\theta \geq \theta^*_{en}|x^*_{en}) - c = Pr(\theta \geq \theta^*_{en}, s \geq s^*_{en}|x^*_{en})[(1-k)-c] + Pr(\theta < \theta^*_{en}, s \geq s^*_{en}|x^*_{en})(-c) \tag{20}$$

Lemma 2 implies that we can write $\theta^*_{en} = \theta^*_{en}(x^*_{en})$ where $0 < \frac{d\theta^*_{en}(x^*_{en})}{dx^*_{en}} < \frac{1}{1+\sigma^2}$. Using this, and substituting from equation (19) into equation (20), we can express the period 1 indifference condition purely in terms of x^*_{en}, as $L(x^*_{en}) = R(x^*_{en})$, where

$$L(x^*_{en}) = Pr(\theta \geq \theta^*_{en}(x^*_{en})|x^*_{en}) - c$$

$$R(x^*_{en}) = (1-k-c)Pr(\theta \geq \theta^*_{en}(x^*_{en}), s \geq \theta^*_{en}(x^*_{en})+M|x^*_{en}) - cPr(\theta < \theta^*_{en}(x^*_{en}), s \geq \theta^*_{en}(x^*_{en})+M|x^*_{en})$$

Given the posterior distribution of θ given x, and Lemma 2, we know that $L(\cdot)$ is monotone increasing in x^*_{en}. Since s is positively but imperfectly correlated with θ conditional on

x, intuitively the first term in $R(\cdot)$ also increases in x^*_{en} but at a slower rate than $L(\cdot)$.[15] In addition, the rate of increase of this term is "dampened" because it is multiplied by $1-k-c<1$. The second term the $R(\cdot)$ has an ambiguous rate of change with x^*_{en}, since it represents the intersection of two events, one of which becomes more likely as x^*_{en} increases, while the other becomes *less* likely under the same circumstances. Heuristically, therefore, the rate of change of the second term of $R(\cdot)$ due to x^*_{en} is small. Thus, based on this informal argument, we would expect that $L(x^*_{en})$ increases *faster* in x^*_{en} than $R(x^*_{en})$, which implies that there is a unique x^*_{en} which solves $L(\cdot)=R(\cdot)$. A more formal argument, given in the appendix, establishes that this is true, and we can state:

Proposition 4 *If* $\sigma < \frac{\sqrt{2\pi}}{1+\frac{\tau}{\sqrt{1+\tau^2}}}$, *there exists a unique monotone equilibrium in* Γ_{en}.

The proof is in the appendix.[16]

While we cannot give closed form to the equilibrium thresholds in general, a clean characterization emerges as we let noise become small. Observe that as we let $\tau \to 0$, equation (18) reduces to:

$$x^*_{en} = \theta^*_{en} + \sigma\Phi^{-1}(\frac{1-k}{c}\theta^*_{en}) \tag{21}$$

At the same time, equation (20) becomes:

$$\Phi\left(\frac{\frac{x^*_{en}}{1+\sigma^2}-\theta^*_{en}}{\frac{\sigma}{\sqrt{1+\sigma^2}}}\right) = \frac{c}{c+k}$$

Combining these two, we get:

$$\Phi\left(\frac{\frac{\theta^*_{en}+\sigma\Phi^{-1}(\frac{1-k}{c}\theta^*_{en})}{1+\sigma^2}-\theta^*_{en}}{\frac{\sigma}{\sqrt{1+\sigma^2}}}\right) = \frac{c}{c+k}$$

Which simplifies to:

$$\Phi\left(\frac{\Phi^{-1}\left(\frac{1-k}{c}\theta^*_{en}\right)}{\sqrt{1+\sigma^2}} - \frac{\sigma\theta^*_{en}}{\sqrt{1+\sigma^2}}\right) = \frac{c}{c+k}$$

Clearly, as $\sigma \to 0$, the unique solution to this is given by

$$\theta^*_{en} = \frac{c^2}{(c+k)(1-k)}$$

Thus we can now summarize:

[15] As we let $\tau \to 0$, $s \to \theta$, $M \to 0$, and thus the first term in $R(\cdot)$ becomes identical to the first term in $L(\cdot)$ while the second term vanishes.

[16] This condition reduces to the familiar condition $\sigma < \sqrt{2\pi}$ as $\tau \to 0$.

Proposition 5 *In the limit as $\tau \to 0$, $\sigma \to 0$, the unique equilibrium thresholds of Γ_{en} can be written as:*

$$x^*_{en} \to \frac{c^2}{(c+k)(1-k)}, \quad s^*_{en} \to \frac{c^2}{(c+k)(1-k)}, \quad \theta^*_{en} \to \frac{c^2}{(c+k)(1-k)}$$

Two important sets of properties about these limiting thresholds are immediate. First, as the cost to delay gets arbitrarily large or small (i.e. as k tends to the boundaries of its feasible range), the thresholds converge smoothly to the unique thresholds of the corresponding static games.[17]

Corollary 2 *Convergence to static games:*

- *As $\tau \to 0$, $k \to 0$, $\sigma \to 0$, $x^*_{en} \to \infty$, $s^*_{en} \to c$, $\theta^*_{en} \to c$.*
- *As $\tau \to 0$, $k \to 1-c$, $\sigma \to 0$, $x^*_{en} \to c$, $s^*_{en} \to \infty$, $\theta^*_{en} \to c$*

Thus, as the cost of delay becomes small, nobody invests in the first period, and the entire mass of agents play the static game (with vanishing noise) in the second period. Similarly, as the cost of delay becomes large, nobody who waits till the second period ever invests, and the entire mass of agents play a static coordination game in the first period.

A more interesting conclusion emerges upon comparison of the thresholds of the endogenous order dynamic game with those of the static games and exogenous order dynamic game as noise vanishes. In particular, a clean and economically important result is apparent when comparing the threshold levels of the fundamentals in the unique monotone equilibria of these games.

Corollary 3 *As noise vanishes, for all $c \in (0, \frac{1}{2}), k \in (0, 1-c), \lambda \in (0,1)$:*

$$\theta^*_{en} < \min[\theta^*_{st,1}, \theta^*_{st,2}] < \theta^*_{ex}$$

Thus, when θ is in $[0,1]$ coordinated investment becomes *more probable* when we let agents choose both how to act and when to act. The endogenous sorting of agents acts as an implicit coordination device which makes it more likely that they shall coordinate efficiently for any given level of the fundamentals. We shall return to discuss this in further detail later in the paper.

Note that θ^*_{en} has a non-monotonic relationship with k. It is minimized at $k = \frac{1-c}{2}$. We shall return to discuss this property further in Section 6.3. Figure 1 plots the limiting thresholds in the different games for $c = 0.3$, $\lambda = 0.5$, over different values of k.

[17] To understand the behavior of x^*_{en} and s^*_{en} as it pertains to Corollary 2, it is easiest to use equations (21) and (17) respectively.

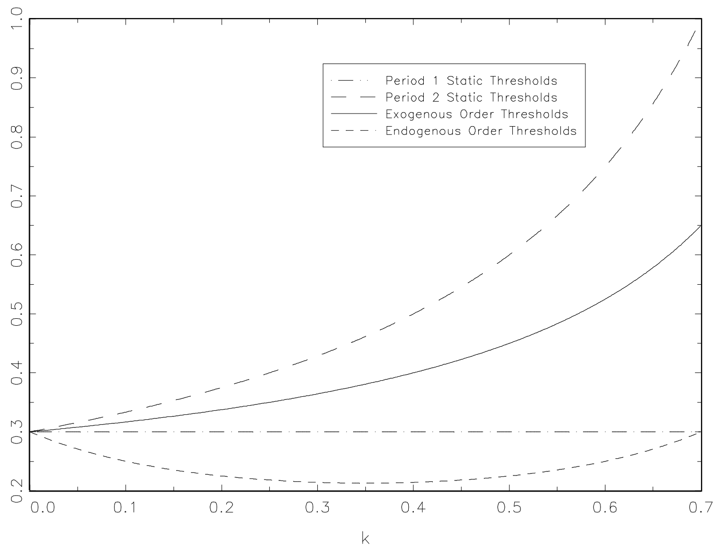


Figure 1: Limiting Thresholds: $c = 0.3, \lambda = 0.5$

Finally, we consider whether there is "excess delay" in this equilibrium. In the unique monotone equilibrium of Γ_{en}, a proportion of agents choose to postpone their investment decision until period t_2. Such waiting can be socially costly, since there is a cost to delay. Loosely adopting Bolton and Harris's (1999) terminology, we may be tempted to ask whether there is "too little experimentation" in our model: whether "too few" agents choose to invest early in equilibrium.[18] While Corollary 3 ensures that there is no decentralized solution that leads to a higher probability of investment than in Γ_{en}, it remains of interest to examine whether a informationally constrained social planner may be able to do better.[19] Is there a rule of behavior that could be implemented by an informationally constrained planner that increases the level of experimentation and also the probability of coordinated investment?

The answer turns out to be in the affirmative, as we show below:

[18] As we have discussed above, there are important differences between the role of experimentation in Bolton and Harris (1999) and in our model. In their model, agents are discrete, and therefore individual experimentation leads to better social information. This creates a free-rider problem in the production of information. This informational externality is missing in our model with a continuum of agents.

[19] For our purposes, an informationally constrained planner is a planner who has no information herself, but may specify the strategies of agents as a function of their own information.

Proposition 6 *There exist monotone decision rules parameterized by the threshold pair* $(\hat{x}, \hat{s})$ *such that* $\forall(\sigma, \tau, c, k)$, *the induced state-variable threshold* $\hat{\theta}(\hat{x}, \hat{s})$ *satisfies:* $\hat{\theta} \leq \theta^*_{en}$.

When (σ, τ) is sufficiently small, this decision rule improves on social welfare relative to Γ_{en}. The result follows immediately upon the construction of a variant of Γ_{en} by eliminating the payoff externality. The proof is presented in the appendix.

6 Discussion

We have presented a sequence of models to study a multi-period investment problem characterized by incomplete information, strategic complementarities, social learning, and costs to delayed decision-making. It is useful to compare the results obtained from these different analyses. We begin by comparing welfare across the different models.

6.1 Welfare

We are particularly interested in welfare comparisons in the limit as social learning becomes public and accurate. It is useful to explicitly write down expressions corresponding to ex ante social welfare in the different games.

In the limit as $\tau \to 0$, we denote ex-ante social welfare in the first-period static coordination game $\Gamma_{st,1}$ by $W_{st,1}(c, \sigma)$. It is given by:

$$Pr(\theta \geq \theta^*_{st,1}, x \geq x^*_{st,1})(1 - c) + Pr(\theta < \theta^*_{st,1}, x \geq x^*_{st,1})(-c)$$

Similarly, for $\Gamma_{st,2}$, welfare $W_{st,2}(c, k, \sigma)$ is given by:

$$Pr(\theta \geq \theta^*_{st,2}, x \geq x^*_{st,2})([1 - k] - c) + Pr(\theta < \theta^*_{st,2}, x \geq x^*_{st,2})(-c)$$

For the exogenous order dynamic game, Γ_{ex}, welfare $W_{ex}(c, k, \lambda, \sigma)$ is defined as:

$$\lambda[Pr(\theta \geq \theta^*_{ex}, x \geq x^*_{ex})(1 - c) + Pr(\theta < \theta^*_{ex}, x \geq x^*_{ex})(-c)] + (1 - \lambda)Pr(\theta > \theta^*_{ex})([1 - k] - c)$$

Finally, for the endogenous order dynamic game, Γ_{en}, ex-ante social welfare $W_{en}(c, k, \sigma)$ is given by:

$$Pr(\theta \geq \theta^*_{en}, x \geq x^*_{en})(1 - c) + Pr(\theta < \theta^*_{en}, x \geq x^*_{en})(-c) + Pr(\theta > \theta^*_{en}, x < x^*_{en})([1 - k] - c)$$

Note that as we let noise vanish in the games, i.e., as $\sigma \to 0$, the product probability terms simplify and we get the following clean welfare ranking:

Remark 1 *As noise vanishes, for all* $c \in (0, \frac{1}{2}), k \in (0, 1 - c), \lambda \in (0, 1)$*:*

$$W_{en}(c, k) > W_{st,1}(c) > W_{ex}(c, k, \lambda) > W_{st,2}(c, k)$$

As $\sigma \to 0$, ex-ante welfare in each game becomes a monotone decreasing function of its unique equilibrium fundamental threshold. The lower the threshold, the higher is ex-ante social welfare. Thus, Remark 1 follows immediately upon inspection of Corollary 3.

It is also interesting to perform a welfare comparison away from the limit, i.e., for strictly positive σ. Figures 2 through 5 demonstrate this comparison for a representative set of parameter values. In each case, we set $c = 0.3$, and vary k over its permissible range. For the exogenous order dynamic game, we set $\lambda = 0.5$. We then plot social welfare for small (Figures 2 and 3) and large (Figures 4 and 5) levels of private information. We omit plotting $W_{st,2}$ as it is of little interest. As is apparent upon inspection of the figures, over wide ranges of parameter values, the welfare rankings summarized in Remark 1 are robust to the presence of private information. In particular, welfare in Γ_{en} is always higher than in $\Gamma_{st,1}$ and Γ_{ex}. For low values of k, and for high levels of noise, however, welfare under Γ_{ex} can occasionally be greater than welfare under $\Gamma_{st,1}$. The intuition for this is straightforward. Since we let $\tau \to 0$, learning becomes complete in period 2. Thus, when the cost of delay is sufficiently small, enforcing a large number of agents to act in period 2 provides them with greater information. Since agents make mistakes for large σ in period 1, welfare can be higher in Γ_{ex} than in $\Gamma_{st,1}$. For further discussion of related issues see Section 6.4.

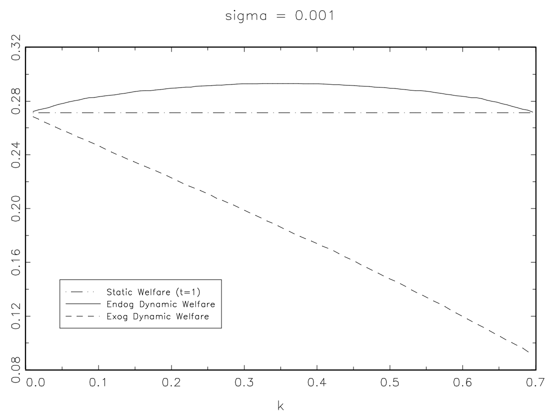
We now consider two properties that emerge from Remark 1 and upon inspection of figures 2 to 5. The first of these is the marked difference between the welfare properties of the games with endogenous and exogenous ordering of agents.

6.2 Exogenous vs. Endogenous Ordering

The dynamic games with exogenous and endogenous order are apparently quite similar. In both games subsets of agents move in each period, and late movers learn from the actions of early movers by paying a delay cost. It may seem, therefore, that there may be some way to parametrize the game with exogenous order of moves to match the equilibrium outcomes of the game with endogenous order. Remarkably, the answer turns out to be no.

Remark 2 *Exogenous asynchronicity is not a substitute for endogenous asynchronicity. As noise vanishes, there exists no* $\lambda \in (0, 1)$ *such that* $W_{ex}(c, k, \lambda) \geq W_{en}(c, k)$.

The intuition behind this apparently surprising conclusion is as follows. The exogenous order game is parametrized by λ: each value of λ corresponds to a specific ex ante ordering of agents. However, any ex ante sorting of agents involves selecting a *homogenous* subsample of agents to make early choices. By definition, only *some* of these agents will choose to invest. The others wont invest in period one, and due to exogenous sorting, lose their investment option forever. However, these same agents who did not invest early in Γ_{ex} *may have invested* ex post in



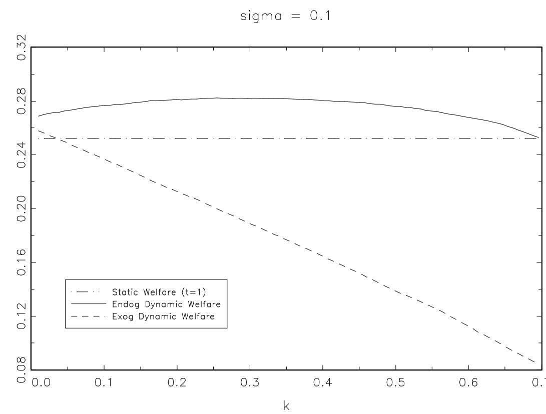
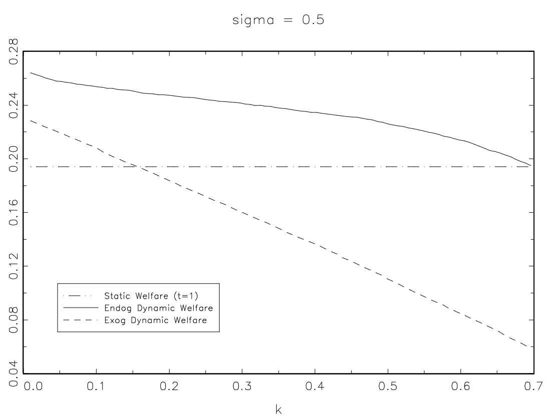


Figure 2: Welfare Comparisons: $\sigma = 0.001$

Figure 3: Welfare Comparisons: $\sigma = 0.1$



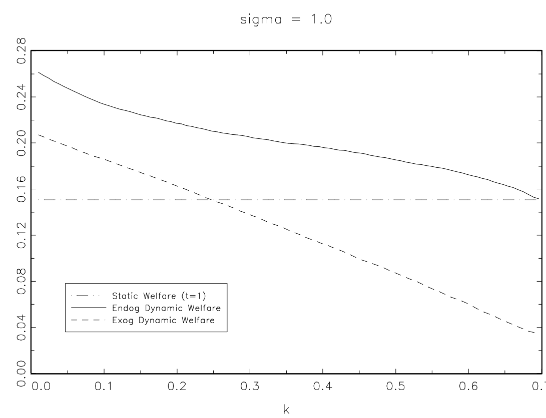


Figure 4: Welfare Comparisons: $\sigma = 0.5$

Figure 5: Welfare Comparisons: $\sigma = 1.0$

Γ_{en}, where they would have have another chance to do so. Thus, for any given mass of early investors, there is always a larger pool of second period investors under endogenous ordering than under exogenous ordering. In other words, endogenous ordering is more efficient, since it exploits the *revealed preference* of a subgroup of agents to make early decisions.

6.3 Efficiency Gains At Intermediate Costs of Delay

From inspection of Figures 2 and 3 it is apparent that when noise is small in Γ_{en}, the welfare of agents is maximized for *intermediate* costs of delay. In the limiting case as noise vanishes, the formal result is implied by Proposition 5:

Remark 3 *As noise vanishes, welfare in the endogenous order game is maximized for intermediate costs to delay. In particular, there exists $k^* \in (0, 1-c)$ such that $k^* = argmax_k W_{en}(c, k)$.*

This conclusion too may seem surprising, but it is simple to explain. When θ is in $[0, 1]$, ex ante social welfare increases in the total proportion of agents who invest in the game (adding up the proportions in periods 1 and 2). A *high* cost of delay makes it unattractive for agents to wait. Thus, increasing the cost of delay persuades more agents to invest early. However, ex post in period two, a high cost of delay makes it *unattractive* for the remainder of agents to invest. Thus, increasing the cost of delay has opposite influences on the mass of agents who choose to invest in periods one and two. As a result, to maximize the total mass of agents who invest, it is natural that an intermediate cost of delay would be optimal.

Let us explore this intuition further. At state θ^*_{en}, the proportion of agents investing is (by definition) equal to $1 - \theta^*_{en}$. Therefore, using the characterization given in Proposition 5 we can heuristically write the limiting critical mass condition as follows:

$$p(\theta^*_{en}) = 1 - \theta^*_{en} = 1 - \frac{c^2}{(c+k)(1-k)}$$

It is useful to decompose $1 - \frac{c^2}{(c+k)(1-k)}$ in the following manner:

$$1 - \frac{c^2}{(c+k)(1-k)} = (1 - \frac{c}{c+k}) + \frac{c}{c+k}(1 - \frac{c}{1-k}) \tag{22}$$

The right hand side of equation (22) consists of two terms. The first term represents the proportion of agents investing early in the limit, at the critical threshold state. The second represents the proportion of agents investing late.

The first term on the RHS of (22) always increases in k: raising the cost of delay encourages more agents to invest early. The second term always decreases in k: raising the cost of delay discourages agents from investing late. An interior extremum arises because the first term increases very fast for small values of k but increases very slowly for larger values of k.

Why is this so? In our model the value of social learning is independent of the proportion of early investors. The benefit from learning is thus *fixed*, and independent of k. Thus, the higher is k, the lower the incentive to wait. In addition, what determines the change in behavior as we vary k, is the *percentage* increase in k. At low values of k, a given increase

Δk leads to a much larger percentage increase in the cost of delay. For a fixed benefit of learning, this persuades a large proportion of agents to give up the delay option and invest early. Hence, starting at (essentially) zero cost of delay (when almost everybody waits), small increases in k lead to large increases in the proportion of early investors. While the proportion of late investors decreases, this decrease is swamped by the increase in early investors. Hence, the probability of coordination increases in k.

This effect disappears at higher values of k. Now, a given Δk implies a smaller percentage increase in the cost of delay. Thus, the decrease in the mass of late investors is not compensated for by the increase in early investors for higher k. Thus, increasing k past a certain point leads to less investment, and decreases the efficiency of coordination.

It is important to note here that the essential ingredients of this argument do not depend on the size of the noise. The formal result is shown only for the case where noise vanishes, as we can obtain closed forms only in this case. However, there is no reason to suspect that the phenomenon of improved coordination at intermediate costs of delay is affected by the size of σ or τ. We shall illustrate this point numerically in Section 6.4.

Another puzzle remains. Careful readers may have noticed that while welfare in the endogenous order game is maximized at intermediate ranges of k for small noise (figures 2 and 3), at higher levels of noise (figures 4 and 5), welfare is maximized for *low* costs of delay. To understand this dichotomy, we must understand precisely how the costly option to wait makes a difference in our dynamic coordination games.

6.4 The Costs and Benefits of the Option to Wait

The introduction of a costly option to wait into a dynamic coordination game has three effects. First, when the option is exercised, it leads to better information, and therefore higher welfare. We call this the *learning effect*. However, since the option is costly, its use leads to lower payoffs, and therefore lowers welfare. Let us call this the *direct payoff effect*. Finally, the option to wait and the resultant endogenous asynchronicity improves coordination, by lowering (for intermediate values of k) the threshold above which investment is successful (and therefore the ex ante probability of successful coordinated risk taking). We call this the *coordination effect*. The total welfare gains for different levels of k in Γ_{en} result from the interaction of these three effects.

Note that the learning effect is independent of the size of k, since the informativeness of observational learning is independent of the measure of agents who choose to invest early (as long as the measure is strictly positive). The direct payoff effect is clearly increasing in the size of k. Thus, for low levels of k, the positive learning effect dominates the negative direct payoff effect. However, the coordination effect has a non-monotonic relationship with

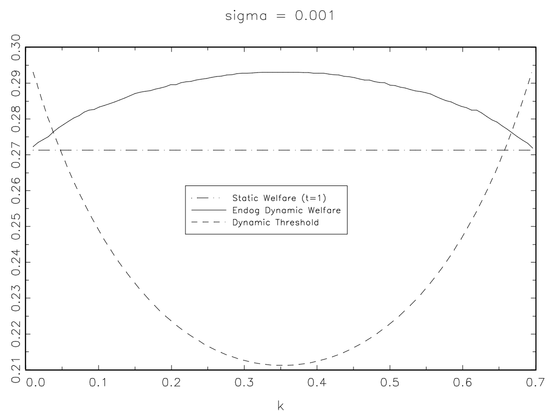


Figure 6: Effects: $\sigma = 0.001$

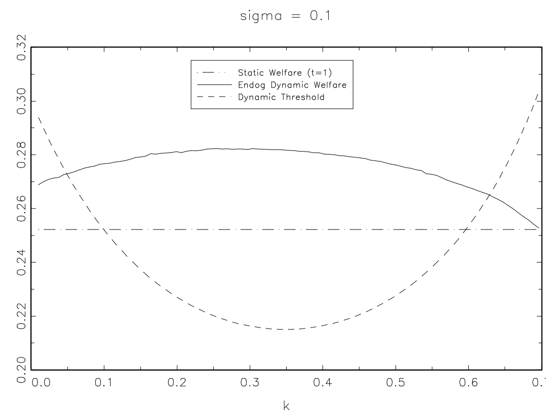


Figure 7: Effects: $\sigma = 0.1$

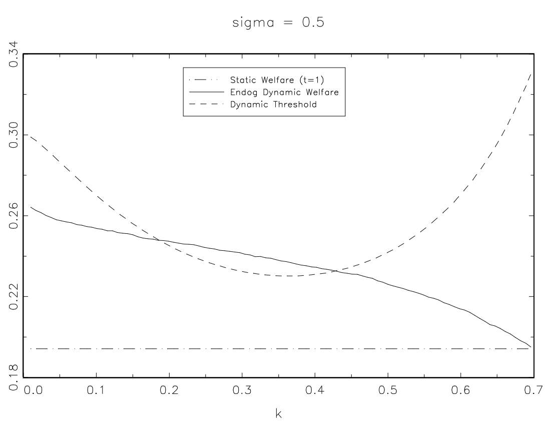


Figure 8: Effects: $\sigma = 0.5$

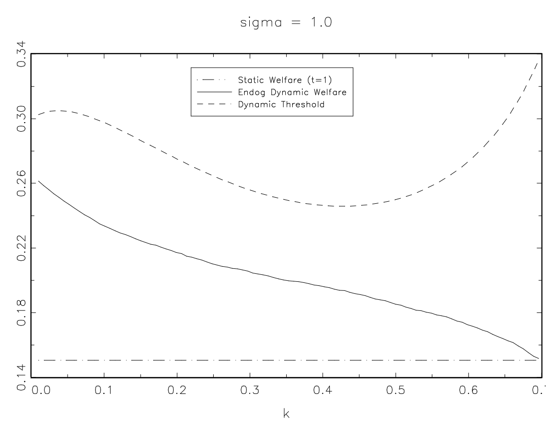


Figure 9: Effects: $\sigma = 1.0$

k. As noise vanishes, this relationship is clearly demonstrated by Figure 1. We plot the effect of k on the coordination threshold (along with welfare plots) in figures 6 through 9. As we have noted above in Section 6.3, and as is apparent upon inspection of figures 6 through 9, the coordination effect is *noise-independent*. We can now explain the shape of the welfare functions in the endogenous order game by appealing to the intuition that we have just built up.

With endogenous ordering, the proportions of early and late investors are sensitive to k. As Corollary 2 indicates, the outcomes of the endogenous order dynamic coordination game converge smoothly to those of the two limiting static games as k tends to the limits of its permissible range. As k gets very small, most investors choose to wait and the game resembles very closely the the second period static coordination game. As k becomes very large, essentially all agents choose to act early or not at all, so that the game resembles the first period static coordination game. For intermediate levels of k agents sort themselves over time.

When social learning becomes public and perfect, agents who choose to wait do not make errors in period 2. With positive noise, agents who decide to act in period 1 may still make mistakes. When noise is small, the chances that agents investing in period one will make mistakes is small. Thus, what matters for social welfare is how well agents are able to coordinate, i.e., the coordination effect dominates the learning effect. Thus, welfare in Γ_{en} tends to track the coordination threshold as a function of k. Welfare follows a bell-shaped curve as a function of k. This is best seen in figure 6. However, when noise gets large, the value of the learning effect becomes much larger. For small values of k, most agents choose to wait, get the benefit of the learning effect, and welfare is high. But for somewhat larger values of k, a significant proportion of agents choose to invest early. These agents tend to make many mistakes, since σ is large. Thus, welfare can be significantly reduced, even though the positive contribution of the coordination effect is maximized at intermediate values of k. Thus, when noise is large, welfare in Γ_{en} can be a monotonic decreasing function of k.

6.5 Applications

The sequence of models outlined in this paper contain the stylized features observed in at least two large classes of applied problems. The first of these is the financing of risky projects where there are increasing returns to scale from participation. The second is the adoption of new technologies in the presence of uncertainty and network externalities. The welfare results presented above have implications for both of these problems. We consider them in turn.

6.5.1 Staggered Debt Offerings

Consider an emerging market government that wants to float a bond to finance a long-term investment project using foreign investment. In addition, suppose that for reasons that we do not model, a secondary market in such bonds is likely to be absent or highly illiquid, with high transaction costs. Our results imply that when uncertainty about the state of the emerging economy is not overly large, it may be better for the government to float the bond in two pieces over time, and to provide information about initial rates of participation. In a nutshell, it may be optimal to "stagger" the initial offering of debt.

Let θ represent the underlying value of the emerging economy. It is natural to assume that if the underlying fundamentals of the economy are realized to be very good ($\theta > 1$) then domestic government can be wealthy enough to unilaterally finance the project, and the project succeeds even without foreign participation. On the other hand, if the economy ends up in a very bad state ($\theta < 0$) the project may fail even if all available foreign investors participated. Under these circumstances, we have demonstrated that in all our models, there is some threshold, θ^*, above which the project succeeds endogenously and below which it fails. Assuming that the emerging market government cares only about the success or failure of the project, its goal must be to make θ^* as low as possible. Then, Corollary 3 implies that it is best to offer investors at least two opportunities to invest, and let them choose endogenously between the two.

6.5.2 The Penguin Effect

Now consider a group of firms choosing whether to switch between a safe current technology and a risky unknown technology characterized by network externalities. In this context, it is unclear whether offering firms the option to delay switching is beneficial. While the option to delay can lead to the provision of more information, it can also lead to "too much" waiting, which can be socially suboptimal. There may be a tendency for players to delay making choices because doing so lets them make more informed choices, avoid interim payoff losses, and avoid being "stranded" in a suboptimal technology by later adopters who do not conform. Farrell and Saloner (1986) term this general phenomenon the "penguin effect", by analogy to penguins who often delay entering the water, hoping that others might do so first to test for the presence of predators. In a complete information model with multiple equilibria, they identify parameter ranges in which the option to wait can be harmful, because it leads to socially suboptimal delay. In a more recent paper, Choi (1997) provides a model of technology adoption under incomplete information, in which the penguin effect reappears. In his model, the use of technology by one user reveals its value to other users. Thus, the fear of being stranded in an ex post inferior technology may lead people to always want to choose second, which can produce socially suboptimal delay. Under certain ranges of parameters, Choi (1997) shows that forfeiting the option to wait and learn may be socially optimal.

Our results provide a different perspective on the penguin effect. We show that when a large number of firms are allowed the option to delay switching to obtain more information at some cost, they will sort themselves over time efficiently. In particular, for intermediate costs of delay, such endogenous sorting can improve efficiency, and lead to strict welfare improvements. This is true no matter how small the level of private information and therefore how small the benefit from waiting and learning. Thus, even though the penguin effect is present in our model for the same reasons as in Choi (1997)[20], its presence leads to *improved*, not diminished social welfare.

7 Concluding Remarks

In this paper we have explored the role of learning and delay in coordination problems under incomplete information. We conclude with some remarks on the generality of these results. We begin with theoretical considerations.

For tractability and closed forms, we have made two main simplifying assumptions in this model. The first is the assumption of Gaussian noise. This assumption, taken together with our choice of market statistic function, $\Phi^{-1}(\cdot)$, allows us to construct a simple one-dimensional sufficient statistic. We conjecture that the results shall not change substantively by relaxing these assumptions. As $\tau \to 0$, it makes no difference what market statistic function we choose, as long as it is monotone increasing. We conjecture that the ordinal properties of our results will hold true for models with noise generated from any one-dimensional exponential family and for any choice of monotone increasing market statistic function.

From the perspective of applications, a natural extension would allow agents to choose the cost of delay, rather than pre-specify it in the model. It would also be desirable to let the cost of delay depend on the actions of agents in the early period. We believe that these modifications hold promise for further interesting results.

[20]Note that as $\tau \to 0$, the use of one technology by a positive measure subset of agents fully reveals the value of the technology to agents who wait, just as in Choi (1997).

8 Appendix

8.1 Proofs

Lemma 1 *Fix any* (x^*, s^*)*. Let*

$$G(\theta) = Pr(x \geq x^*|\theta) + Pr(x < x^*, s \geq s^*|\theta) - 1 + \theta$$

There is a unique solution to

$$G(\theta) = 0$$

Proof: Since $s = \frac{\tau^2 x + \sigma y + x^*}{1+\tau^2+\sigma^2\tau^2}$, writing $x = \theta + \sigma\epsilon$, $y = \frac{\theta - x^*}{\sigma} + \tau\eta$, and substituting, we get $s = \frac{1+\tau^2}{1+\tau^2+\sigma^2\tau^2}\theta + \frac{\sigma\tau}{1+\tau^2+\sigma^2\tau^2}(\tau\epsilon + \eta)$. Then $s \geq s^* \equiv \gamma \geq \frac{1+\tau^2+\sigma^2\tau^2}{\sigma\tau}s^* - \frac{1+\tau^2}{\sigma\tau}\theta$, where $\gamma = \tau\epsilon + \eta$. Thus, we can rewrite:

$$G(\theta) = 1 - \Phi(A(\theta)) + \int_{-\infty}^{A(\theta)} \int_{B(\theta)}^{\infty} f(\epsilon, \gamma) d\gamma d\epsilon - 1 + \theta$$

where $A(\theta) = \frac{x^* - \theta}{\sigma}$ and $B(\theta) = \frac{1+\tau^2+\sigma^2\tau^2}{\sigma\tau}s^* - \frac{1+\tau^2}{\sigma\tau}\theta$. Differentiating under the double integral:

$$G'(\theta) = -A'(\theta)\phi(A(\theta)) + A'(\theta)\int_{B(\theta)}^{\infty} f(A(\theta, \gamma) d\gamma - B'(\theta)\int_{-\infty}^{A(\theta)} f(\epsilon, B(\theta) d\epsilon + 1$$

Writing the joint densities as products of conditionals and marginals:

$$f(\epsilon = A(\theta), \gamma) = \phi(A(\theta)) f(\gamma | \epsilon = A(\theta))$$

$$f(\epsilon, \gamma = B(\theta)) = \hat{\phi}(B(\theta)) f(\epsilon | \gamma = B(\theta))$$

writing $\phi(\cdot)$ to denote the standard normal PDF of ϵ, and $\hat{\phi}(\cdot)$ to denote the (non-standard) Normal PDF for γ. Finally,

$$A'(\theta) = -\frac{1}{\sigma}, B'(\theta) = -\frac{1+\tau^2}{\sigma\tau}$$

Now we can rewrite $G'(\theta)$ as:

$$\frac{1}{\sigma}\phi(A(\theta))\left[1 - \int_{B(\theta)}^{\infty} f(\gamma|\epsilon = A(\theta) d\gamma\right] + \frac{1+\tau^2}{\sigma\tau}\hat{\phi}(B(\theta))\int_{-\infty}^{A(\theta)} f(\epsilon|\gamma = B(\theta) d\epsilon + 1$$

i.e. $G'(\theta) > 0$. Note that $\lim_{\theta \to \infty} G(\theta) = \infty$, and $\lim_{\theta \to -\infty} G(\theta) = -\infty$. Thus there exists a unique solution to $G(\theta) = 0$. ■

Lemma 2 *Fix any* x^* *Let* $\hat{\theta}$ *be defined by* $G(\hat{\theta}, x^*) = 0$ *where*

$$G(\theta, x^*) = Pr(x \geq x^*|\theta) + Pr(x < x^*, s \geq \theta + M|\theta) - 1 + \theta$$

If $\sigma < \frac{\sqrt{2\pi}}{1+\frac{\tau}{\sqrt{1+\tau^2}}}$,

1. *For each* x^*, *there is a unique* $\hat{\theta}$

2. $\frac{d\hat{\theta}}{dx^*} \in (0, \frac{1}{1+\sigma^2})$

Proof: As above, we know that $s = \frac{1+\tau^2}{1+\tau^2+\sigma^2\tau^2}\hat{\theta} + \frac{\sigma\tau}{1+\tau^2+\sigma^2\tau^2}(\tau\epsilon + \eta)$. Since $s^* = \hat{\theta} + M$, $s \geq s^* \equiv \gamma \geq \sigma\tau\hat{\theta} + \frac{1+\tau^2+\sigma^2\tau^2}{\sigma\tau}M$. Let

$$B(\hat{\theta}) = \sigma\tau\hat{\theta} + \frac{1 + \tau^2 + \sigma^2\tau^2}{\sigma\tau}M$$

Note that $B'(\hat{\theta}) = \sigma\tau$, and so, using the proof of Lemma 1,

$$\frac{\partial G(\hat{\theta}, x^*)}{\partial \hat{\theta}} = \frac{1}{\sigma}\phi(A(\hat{\theta}, x^*)) \left[1 - \int_{B(\hat{\theta})}^{\infty} f(\gamma|\epsilon = A(\hat{\theta}, x^*)d\gamma\right] - \sigma\tau\hat{\phi}(B(\hat{\theta})) \int_{-\infty}^{A(\hat{\theta},x^*)} f(\epsilon|\gamma = B(\hat{\theta})d\epsilon + 1$$

where $\hat{\phi}(\cdot)$ denotes the non-standard Normal pdf of γ. Let

$$P_1 = \int_{B(\hat{\theta})}^{\infty} f(\gamma|\epsilon = A(\hat{\theta}, x^*)d\gamma$$

$$P_2 = \int_{-\infty}^{A(\hat{\theta},x^*)} f(\epsilon|\gamma = B(\hat{\theta})d\epsilon$$

Since the variance of γ is $1 + \tau^2$, $\hat{\phi}(\cdot) < \frac{1}{\sqrt{2\pi}\sqrt{1+\tau^2}}$, and $P_2 \leq 1$, clearly if $\sigma < \frac{\sqrt{2\pi}}{\frac{\tau}{\sqrt{1+\tau^2}}}$, $\frac{\partial G(\hat{\theta},x^*)}{\partial \hat{\theta}} > 0$. Similarly,

$$\frac{\partial G(\hat{\theta}, x^*)}{\partial x^*} = -\frac{1}{\sigma}\phi(A(\hat{\theta}, x^*)) \left[1 - P_1\right] < 0$$

By the implicit function theorem

$$\frac{d\hat{\theta}(x^*)}{dx^*} = -\frac{\frac{\partial G(\hat{\theta},x^*)}{\partial x^*}}{\frac{\partial G(\hat{\theta},x^*)}{\partial \hat{\theta}}}$$

Let $Q = -\frac{\partial G(\hat{\theta},x^*)}{\partial x^*}$, where $Q > 0$. Then,

$$\frac{d\hat{\theta}(x^*)}{dx^*} = \frac{Q}{Q - \sigma\tau\hat{\phi}(\cdot)P_2 + 1}$$

It is easy to check, that when $\sigma < \frac{\sqrt{2\pi}}{1+\frac{\tau}{\sqrt{1+\tau^2}}}$

$$\frac{1}{1+\sigma^2} - \frac{d\hat{\theta}(x^*)}{dx^*} > 0$$

Since $\sigma < \frac{\sqrt{2\pi}}{1+\frac{\tau}{\sqrt{1+\tau^2}}}$ implies that $\sigma < \frac{\sqrt{2\pi}}{\frac{\tau}{\sqrt{1+\tau^2}}}$, we are done. ■

Proposition 4 *If* $\sigma < \frac{\sqrt{2\pi}}{1+\frac{\tau}{\sqrt{1+\tau^2}}}$, there exists a unique monotone equilibrium in Γ_{en}.

Proof: Initially, agents trade off the expected benefit of investing in period 1 against the expected benefit of retaining the option value to wait. Thus the marginal period 1 investor who receives signal x^*_{en} must satisfy:

$$Pr(\theta \geq \theta^*_{en}|x^*_{en}) - c = Pr(\theta \geq \theta^*_{en}, s \geq s^*_{en}|x^*_{en})[(1-k)-c] + Pr(\theta < \theta^*_{en}, s \geq s^*_{en}|x^*_{en})(-c) \tag{23}$$

Since $\theta^*_{en} = g(x^*_{en})$, we can rewrite equation (19) as:

$$s^*_{en} = g(x^*_{en}) + M \tag{24}$$

Write x for x^*_{en} and let

$$G(x) = Pr(\theta \geq \theta^*_{en}|x) - c - (1-k-c)Pr(\theta \geq \theta^*_{en}, s \geq s^*_{en}|x) + cPr(\theta < \theta^*_{en}, s \geq s^*_{en}|x)$$

Note that

$$Pr(\theta \geq \theta^*_{en}|x) = 1 - \Phi(\frac{\theta^*_{en} - \frac{x}{1+\sigma^2}}{\frac{\sigma}{\sqrt{1+\sigma^2}}})$$

Let $A(x) = \frac{\theta^*_{en} - \frac{x}{1+\sigma^2}}{\frac{\sigma}{\sqrt{1+\sigma^2}}}$. Given x,

$$s = \frac{\tau^2 x + \theta + \sigma\tau\eta}{1+\tau^2+\sigma^2\tau^2}$$

Rearranging terms, we can write this as

$$s = \frac{x}{1+\sigma^2} + \frac{\sigma}{1+\tau^2+\sigma^2\tau^2}\left[\frac{z}{\sqrt{1+\sigma^2}} + \tau\eta\right]$$

where $z = \frac{\theta - \frac{x}{1+\sigma^2}}{\frac{\sigma}{\sqrt{1+\sigma^2}}}$. is distributed $N(0,1)$ conditional on x. Let $\gamma = \frac{z}{\sqrt{1+\sigma^2}} + \tau\eta$. Then, $s \geq s^*$ is equivalent to

$$\gamma \geq \frac{1+\tau^2+\sigma^2\tau^2}{\sqrt{1+\sigma^2}}A(x) + \tau\sqrt{1+\tau^2+\sigma^2\tau^2}\Phi^{-1}(\frac{c}{1-k})$$

Let

$$B(x) = \frac{1+\tau^2+\sigma^2\tau^2}{\sqrt{1+\sigma^2}}A(x) + \tau\sqrt{1+\tau^2+\sigma^2\tau^2}\Phi^{-1}(\frac{c}{1-k})$$

Now, we may rewrite:

$$G(x) = 1 - \Phi(A(x)) - (1-k-c)Pr(z \geq A(x), \gamma \geq B(x)) + cPr(z < A(x), \gamma \geq B(x))$$

Differentiating under the double integral and rearranging we get:

$$G'(x) = -\phi(A(x))A'(x)\left[1-(1-k)P_1\right] + B'(x)\hat{\phi}(B(x))\left[(1-k)P_2 - c\right]$$

where by $\hat{\phi}(\cdot)$ we denote the non-standard normal density of γ, and P_1 and P_2 are defined as follows:

$$P_1 = \int_{B(x)}^{\infty} f(\gamma|z = A(x))d\gamma$$

$$P_2 = \int_{A(x)}^{\infty} f(z|\gamma = B(x))dz$$

Using standard formulae for computing conditional distributions of Normal random variables (see, for example, Greene 1996), we know that:

$$z|\gamma = B(x) \sim N(A(x) + \frac{\tau\sqrt{1+\sigma^2}}{\sqrt{1+\tau^2+\sigma^2\tau^2}}\Phi^{-1}(\frac{c}{1-k}), \frac{\tau^2(1+\sigma^2)}{1+\tau^2+\sigma^2\tau^2})$$

Thus,

$$P_2 = \int_{A(x)}^{\infty} f(z|\gamma = B(x))dz = \frac{c}{1-k}$$

and therefore

$$G'(x) = -\phi(A(x))A'(x)[1-(1-k)P_1]$$

Under the conditions of the theorem $A'(x) < 0$ and therefore the proof is complete. ■

Lemma 3 *Fix any $\hat{\theta} \in [0,1]$. Let $\hat{x}$ be defined by $H(\hat{x}, \hat{\theta}) = 0$, where*

$$H(\hat{\theta}, x) = Pr(\theta \geq \hat{\theta}|x) - c - Pr(s \geq \hat{\theta} + M, \theta \geq \hat{\theta}|x)(1-k-c) + Pr(s \geq \hat{\theta} + M, \theta < \hat{\theta}|x)(c)$$

Then

$$\frac{d\hat{x}(\hat{\theta})}{d\hat{\theta}} > 0$$

Proof: Using the proof of Proposition 4 above, we can write

$$H(\hat{\theta}, x) = 1 - \Phi(A(x,\hat{\theta})) - (1-k-c)Pr(z \geq A(x,\hat{\theta}), \gamma \geq B(x,\hat{\theta})) + cPr(z < A(x,\hat{\theta}), \gamma \geq B(x,\hat{\theta}))$$

where

$$A(x,\hat{\theta}) = \frac{\hat{\theta} - \frac{x}{1+\sigma^2}}{\frac{\sigma}{\sqrt{1+\sigma^2}}}$$

$$B(x,\hat{\theta}) = \frac{1+\tau^2+\sigma^2\tau^2}{\sqrt{1+\sigma^2}} A(x,\hat{\theta}) + \tau\sqrt{1+\tau^2+\sigma^2\tau^2}\,\Phi^{-1}(\frac{c}{1-k})$$

Note that

$$\frac{\partial A}{\partial x} < 0, \quad \frac{\partial A}{\partial \hat{\theta}} > 0$$

Then, using the same analysis as above, we have:

$$\frac{\partial H}{\partial x} = -\phi(A(x))\frac{\partial A}{\partial x}[1-(1-k)P_1]$$

where $P_1 = \int_{B(x,\hat{\theta})}^{\infty} f(\gamma|z = A(x,\hat{\theta}))d\gamma$, where $z|x \sim N(0,1)$ and $\gamma = \frac{1}{\sqrt{1+\sigma^2}}z + \tau\eta$ as above. Similarly,

$$\frac{\partial H}{\partial \hat{\theta}} = -\phi(A(x))\frac{\partial A}{\partial \hat{\theta}}[1-(1-k)P_1]$$

Thus, $\frac{\partial H}{\partial x}$ and $\frac{\partial H}{\partial \hat{\theta}}$ are of opposite sign and bounded away from zero. Thus, by the implicit function theorem: $\frac{d\hat{x}(\hat{\theta})}{d\hat{\theta}} > 0$. ■

Proposition 6 *There exist monotone decision rules parameterized by the threshold pair* $(\hat{x}, \hat{s})$ *such that* $\forall(\sigma,\tau,c,k)$*, the induced state- variable threshold* $\hat{\theta}(\hat{x},\hat{s})$ *satisfies:* $\hat{\theta} \leq \theta^*_{en}$.

Proof: We construct the game $\hat{\Gamma}_{en}$ in which payoffs are still given by 1 through 3, but the condition for the success of the project $p \geq 1-\theta$ is replaced by the condition $\theta \geq 0$. This eliminates the payoff externality.

In this game, agents who arrive at t_2 with the option to invest will choose to invest if $s \geq \hat{s}$ where

$$Pr(\theta \geq 0|\hat{s}) = \frac{c}{1-k}$$

i.e.,

$$\hat{s} = \frac{\sigma\tau}{\sqrt{1+\tau^2+\sigma^2\tau^2}}\Phi^{-1}(\frac{c}{1-k})$$

Agents at t_1 will choose to invest if $x \geq \hat{x}$ where:

$$Pr(\theta \geq 0|\hat{x}) - c = Pr(\theta \geq 0, s \geq \hat{s}|\hat{x})(1-k-c) + Pr(\theta < 0, s \geq \hat{s}|\hat{x})(-c)$$

Using methods similar to those given above, we can rewrite this to be:

$$0 = H(\hat{x},\hat{\theta}) = 1-\Phi(A(\hat{x},\hat{\theta}))-(1-k-c)Pr(z \geq A(\hat{x},\hat{\theta}), \gamma \geq B(\hat{x},\hat{\theta}))+cPr(z < A(\hat{x},\hat{\theta}), \gamma \geq B(\hat{x},\hat{\theta}))$$

where z, γ, $A(\cdot)$, and $B(\cdot)$ are defined as in the proof of Proposition 4 and $\hat{\theta} = 0$. Now, appealing to Lemma 3, and because $\theta^*_{en} \geq 0 = \hat{\theta}$, we conclude that $\hat{x} \leq x^*_{en}$. In addition, it is clear that $\hat{s} \leq s^*_{en}$.

Now, returning to Γ_{en}, let a social planner force agents to play according to $(\hat{x},\hat{s})$. Let $\hat{\theta}$ be the level of θ above which $p(\theta) \geq 1-\theta$ under $(\hat{x},\hat{s})$. Since $\hat{x} \leq x^*_{en}$ and $\hat{s} \leq s^*_{en}$, it follows that $\hat{\theta} \leq \theta^*_{en}$. ■

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