Financing Constraints, Irreversibility, and Investment Dynamics*

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Abstract

We develop a structural model of an industry with many entrepreneurial firms in order to investigate the cyclical behaviour of aggregate fixed investment, variable capital investment and output. In particular, we consider an environment in which the entrepreneur cannot borrow unless the debt is secured by collateral and cannot sell fixed capital without liquidating her whole business. We show that, when these entrepreneurs experience persistent idiosyncratic and aggregate shocks, the interplay between financing constraints and irreversibility of fixed capital is essential to explain several common observations. It helps to explain why inventory investment is very volatile and procyclical, especially during recessions, and why the output and inventories of small firms are more volatile and more cyclical than that of large firms. The model is also consistent with the observations that inventory investment leads the business cycle, and that both fixed and inventory investment are sensitive to the net worth of firms, even when marginal productivity of capital is taken into account.

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1 Introduction

This paper studies the effects of financing constraints on the cyclical fluctuations of investment in fixed capital, variable capital and output. We develop a structural dynamic model of an entrepreneurial firm who is subject to both borrowing constraints and irreversibility of fixed capital. We use the model to simulate industry dynamics with heterogeneous entrepreneurs who experience both idiosyncratic and aggregate uncertainty, and we illustrate how the irreversibility of fixed capital amplifies the effect of financing constraints on variable capital investment. Such amplification effect is both quantitatively and qualitatively important. In particular it helps to explain why aggregate inventory investment is very volatile and procyclical, and why the drop in inventories accounts for a large part of the decline in business spending in recessions.

This paper is motivated by a large body of theoretical literature which has shown that asymmetric information and contract incompleteness may prevent firms to access external finance and make them unable to fund profitable investment opportunities[1]. Are these imperfections relevant for the investment decisions of firms? To what extent they affect aggregate investment and production dynamics over the business cycle?

Most of the existing empirical literature, following the seminal paper by Fazzari, Hubbard and Petersen (1988)[2], addresses only the first question. It shows that investment is significantly correlated to cash flow, especially for firms likely to face capital markets imperfections. but the relevance of this finding has been seriously questioned. Kaplan and Zingales (1997 and 2000) find that the investment-cash flow correlation is stronger for firms which are financially very wealthy and, according to their selection criteria, surely not financially constrained[3]. Gomes (2001) and Ericson and Whited (2001) show that measurement errors are the most likely cause of the positive correlation between investment and cash flow.

The approach we follow in this paper is instead to develop a structural model of firm behaviour under financing and irreversibility constraints, and to use the model to address both questions regarding the effect of financing constraints at firm level and the consequences for aggregate investment and output dynamics. The model has three distinctive features. First, output is produced by an entrepreneur who operates a concave risky technology using two complementary factors of production, fixed and variable capital. Both factors take one period to produce output. Fixed capital cannot be disinvested unless the whole business is sold. Second, the entrepreneur's only source of external finance is debt secured by collateral asset. Therefore her borrowing capacity depends on the value of her assets. Third, the entrepreneur is risk averse and discounts future consumption at a rate higher than the market interest rate. This implies that she never accumulates enough financial wealth to eliminate the chances to face borrowing constraints.

It is well known that future expected financing constraints may affect current consumption decisions (see Zeldes, 1989, and Carroll, 2001). In this paper we show that,

[1] Stiglitz and Weiss (1981), Besanko and Thakor (1986), Milde and Riley (1988), Hart and Moore (1998), Albuquerque and Hopenhayn (2000).

[2] See Hubbard (1998) for a review of this literature.

[3] Similar evidence is produced by Cleary (1999), who studies a larger sample of 1317 US firms.

for an entrepreneurial firm, future expected financing constraints also significantly affect investment decisions: anticipating a risk of binding financing constraints in the future, the entrepreneur reduces the investment spending in the risky technology, and keeps some financial assets (or spare borrowing capacity) as precautionary saving motive.

More importantly we show that the effect of current and future expected financing constraints on variable capital are amplified when fixed capital is irreversible. Consider for example an entrepreneur who faces a persistent negative productivity shock. She would like to reduce the amount of wealth invested in the risky technology, which is likely to have a low return for a while. Since fixed capital is irreversible, she cannot reduce it, and as a consequence she expects a lower return on her assets due to the inefficiently high level of fixed capital. If the productivity will keep being low for a while, her financial wealth will decrease rapidly, until she may become financially constrained. Anticipating this possibility, she tries to reduce the exposure to the risky technology and to increase the share of her wealth invested in risk free assets, but she can do this only by reducing her variable capital investment, before she becomes actually constrained to do so. Hence following a negative productivity shock variable capital drops more than it would have done if fixed capital were reversible. This overshooting of variable capital is very large when, after a particularly large or prolonged negative shock, both constraints are contemporaneously binding: in this case not only fixed capital is inefficiently high, but due to a binding financing constraint variable capital is constrained to be inefficiently low. Therefore not only the volatility of variable capital increases, but expected returns decrease dramatically, due to the unbalanced use of factors of production, and this increases the probability that the entrepreneur will face financing constraints also in the future.

When the bad period ends and productivity start to rise, the entrepreneur is very cautious in investing in fixed capital, given the high cost of the irreversibility constraint in case of a new negative productivity shock in the future. Thus she mainly invests in variable capital, which overshoots upwards. This discussion illustrates that the combination of irreversibility and financing constraints greatly amplifies the volatility of variable capital relative to fixed capital, and hence also relative to output. Such amplification effect is more severe for smaller firms, which are more likely to suffer from both irreversibility and financing constraints[4].

The other important implication of the model is that the fluctuations in investment, especially in variable capital, are partly driven by net worth fluctuations, even when expected productivity is properly accounted for[5].

In the paper we quantify the implications of these effects for firm dynamics by simulating an artificial economy with many heterogeneous entrepreneurs. The cross-sectional distribution of net worth and fixed capital among entrepreneurs is determined by both idiosyncratic and aggregate uncertainty, and it affects the way aggregate output and in-

[4]In fact a large multi-establishment firm could avoid the irreversibility of fixed capital simply by shutting down some of the plants, and it is also less likely to face the informational and contractual problem that prevent a smaller firm to access financial markets.

[5]In reality investment, productivity and net worth are highly correlated in the business cycles, even in the absence of financing imperfections. Hence we mean that financing imperfections in our model imply that investment is very sensitive to net worth even conditional on the productivity shock.

vestment react to aggregate shocks.

Despite the model is relatively stylised, we are able to use it to explain the observed dynamics of aggregate investment and output, and of small firms as opposed to large firms. In fact because of the time lag for the installed fixed and variable capital to produce output, the stock of variable capital at the end of one period has a natural interpretation as input inventories, such as raw materials and work in progress. Likewise the change in the stock of variable capital can be interpreted as inventory investment.

Moreover, even though the entrepreneurs in the simulated economy are ex ante all identical, in equilibrium the heterogeneity of entrepreneurs with respect to the net worth implies that entrepreneurs with smaller businesses are on average more financially constrained. This is because they became small after experiencing low productive periods recently, and tend to have smaller net worth relative to output.

We calibrate the model using the long run averages of a four digit US industrial sector, with annual data from 1962 to 1996. We also directly estimate from the same sector the moments of the distribution of the productivity shock, and use them to simulate, for many periods, several artificial economies, with and without financing imperfections and irreversibility of fixed capital.

The comparison of the simulated data with the empirical data show that the interactions between financing and irreversibility constraints are able to explain why the US manufacturing data show that aggregate inventory investment is much more volatile and procyclical than fixed investment, and that during downturns it accounts for a large part of the drop in business spending. This is an important result because the excessive volatility of inventories is a stylised fact that has recently received considerable attention, because of its importance in business cycle dynamics, and because the drop in inventories accounts for a large part of the GDP decline in recessions[6] (Ramey and West, 1999). The choice of explaining these stylised facts focusing on input rather than output inventories is supported by Ramey (1989), who shows that input inventories are larger and much more volatile than finished goods inventories during recent US recessions.

Moreover the interaction between the financing and irreversibility constraints implies that inventory investment, at firm level, is more sensitive than fixed investment to the availability of internal finance, even conditional on expected productivity. This result is consistent with a study which shows that "*inventory investment for small firms absorb from 15% to 40% of cash flow fluctuations*" (Carpenter, Fazzari and Petersen, 1998). The model is also able to match the observed fact that inventory investment leads the fluctuations in output (Stock and Watson, 1998), and that small firms are more procyclical than large ones in inventories and output. This result is consistent with Kashap, Lamont and Stein (1994), Gertler and Gilchrist (1994) and Oliner and Rudebusch (1996), who compare the behaviour of small versus large manufacturing firms after Romer dates, that represent episodes of tight monetary policy that led to a recession[7].

[6] Stock and Watson (1998) report that "*Changes in business inventories, which constitute but a small fraction of total GDP, account for one-fourth of the cyclical movements in GDP*". In addition of being procyclical at business cycle frequencies, Hornstein (1998) shows that fluctuations of GDP at frequency shorter than business cycle are almost entirely driven by changes in inventories.

[7] Moreover Bernanke Gertler and Gilchrist (1996) show that one third of aggregate fluctuations in the

As a robustness check of the ability of the model to explain investment dynamics, we also simulate the artificial economy with the same sequence of aggregate shocks estimated from the annual data, from 1962 to 1986, of the chosen US manufacturing sector . We find that the presence of both financing and irreversibility constraints improves the ability of the model to predict empirical data, in particular the large drop in variable capital and inventories during the recession in the beginning of the '80s.

Finally, in the last section of the paper we briefly illustrate some empirical evidence in favour of the assumption of financing constraints at firm level. We describe the results obtained by Caggese (2003), who analyses a data-set of small and medium sized Italian manufacturing firms, which includes both balance sheet data and qualitative response data about entrepreneurs' financing problems. Caggese (2003) shows that the premium of productivity of variable capital over the cost of capital, which our model predicts to be the best indicator of the intensity of financing constraints, is strongly positively correlated with the likelihood that the entrepreneurs state problems in obtaining external funds to finance new investment projects.

Three recent papers adopt a similar approach to our paper, and analyse an economy with heterogenous entrepreneurs where financing constraints are binding for a fraction of them in equilibrium: Cooley and Quadrini (2001), Cooley Quadrini and Marimon (2002) and Gomes (2001). Cooley and Quadrini (2001) show that financing imperfections in a model of industry dynamics help to explain a stilised fact regarding growth dynamics of firms which was not explained by models based only on technological shocks. Cooley, Quadrini and Marimon (2002) focus on financing imperfections in the context of long term contracts between firms and banks. They show that imperfect enforceability makes the diffusion of new technologies sluggish and amplifies their impact on aggregate output. Gomes (2001) builds a model with heterogeneous firms and financing constraints that replicates some stylised facts about industry dynamics and shows that cash flow is not significant in reduced form investment regressions when average Q is properly measured. Moreover our paper is also related to the real option theory literature, and in particular to Bertola and Caballero (1994).

Our paper is substantially different from all those above, because we focus on the interactions between financing and irreversibility constraints, and on business cycle dynamics rather than growth dynamics of firms. Moreover we analyse a multifactor technology and use the model to explain the empirical evidence about both fixed and inventory investment. Thus our paper provides an added value with respect to the existing literature on several important aspects: we model theoretically and quantify with calibrations the effects of precautionary saving on the risky investment of an entrepreneurial firm. We model theoretically an amplification effect between irreversibility and financing constraints and show that such effect is essential to explain several stylised facts regarding business cycle dynamics of aggregate investment. In this respect we complement and extend the findings of Bertola and Caballero (1994), , who only focus on the effects of irreversibility at firm level on the behaviour of aggregate fixed investment.

The paper is organised as follows: section 2 illustrates the theoretical model; section 3

US manufacturing sector can be accounted for by the difference between small and large firms.

describes the solution method and simulation results; section 4 presents the conclusions.

2 The model

We consider an economy populated by many infinitely lived entrepreneurs, some active and some retired, and many competitive banks. We assume that each active entrepreneur (henceforth E) chooses consumption and investment in order to maximise the expected value of her lifetime utility function. All entrepreneurs have same preferences and have access to the same technology. Utility from consumption is measured by a concave function $U(x_t)$, where x_t is consumption at time t.

$$U'(.) > 0;\ U''(.) < 0;\ U'(0) = \infty;\ U'(\infty) = 0 \tag{1}$$

$E's$ subjective *discount rate β is such that:*

$$\beta R < 1$$

where $R = 1 + r$, and r is the lending/borrowing interest rate. This implies that E is impatient[8], and that she chooses her optimal capital structure balancing her desire to borrow in order to anticipate consumption with her desire to save in order to avoid future borrowing constraints.

Regarding the technology, we assume that E can invest in the risky technology of the business she owns and manages. k_t and l_t are respectively the stock of fixed and variable capital, installed at or before time $t-1$, which will generate output at time t. Variable capital represents variable inputs such as raw materials and work in progress, while fixed capital represents fixed inputs such as plant and equipment. The time to build assumption implies that borrowing constraints are relevant when E needs additional funds to exploit new investment opportunities. It also allows us to interpret l_t as end of period $t-1$ inventories of variable capital. Output y_t is produced according to a Cobb-Douglas production function:

$$y_t = e^{\theta_t} k_t^{\alpha} \left(l_t + l^E\right)^{\kappa} \text{ with } \alpha > 0; \kappa > 0; \alpha + \kappa < 1 \tag{2}$$

l^E is a small fixed amount of variable capital supplied[9] each period by E. θ_t is the productivity shock. All prices are assumed constant and normalised to 1. Regarding the factors of production, variable capital is non-durable, while fixed capital is durable:

$$1 = \delta_l > \delta_k \tag{3}$$

δ_l and δ_k are respectively the depreciation factors of variable and fixed capital[10]. Moreover fixed capital is reversible, while fixed capital is irreversible, and can only be disinvested if E sells her whole business. In this case she cannot start a new business and must retire. The retired E simply manages her wealth until she dies.

[8] Kiyotaki and Moore (2002) show that such feature arise endogenously in a general equilibrium model with heterogenous agents and financing imperfections.

[9] It can be interpreted as E's effortless labour supply.

[10] Full depreciation of variable capital is not necessary, but it simplifies the exposition.

Irreversibility of fixed capital is justified by the fact that in many industries plant and equipment do not have a secondary market because they cannot be easily converted to other productions. Yet we allow fixed capital to be used as collateral by assuming that such conversion is easier if the whole of the assets is sold. The assumption that fixed capital is irreversible conditional on the continuation of the activity is consistent with the empirical evidence on a very large sample of US manufacturing plants analysed by Caballero, Engel and Haltiwanger (1995). Therefore conditional on continuation E is subject to the following constraints:

$$k_{t+1} \geq (1-\delta_k)\, k_t \tag{4}$$

$$l_{t+1} \geq 0 \tag{5}$$

The productivity shock θ_t is assumed to follow a stationary autoregressive stochastic process:

$$\begin{aligned} \theta_t &= \overline{\theta} + \rho\theta_{t-1} + \zeta_t; \text{ with } 0 \leq \rho < 1 \\ \zeta_t &\sim iid\left(0, \sigma_\zeta^2\right) \end{aligned} \tag{6}$$

We introduce in the problem financial markets imperfections assuming that equity finance and risky debt are not available. At time t E can borrow from (and lend to) the banks one period debt, with face value b_{t+1}, at the market riskless rate r. A positive (negative) b_{t+1} indicates that E is a net borrower (lender). Banks only lend secured debt, and the only collateral they accept is the next period residual value of physical capital. Therefore at time t the amount of borrowing is limited by the following constraint[11]:

$$b_{t+1} \leq \tau_k k_{t+1} \tag{7}$$

τ_k is the share of fixed capital value that can be used as collateral[12]:

$$\tau_k \leq 1-\delta_k \tag{8}$$

From (3) and (8) it follows that $\tau_l = 0$. The timing of the model is represented in figure 1: E inherits from time $t-1$ the stocks of fixed and variable capital k_t and l_t, and the stock of debt b_t. Then θ_t is realised, y_t is produced and b_t repaid. Residual wealth w_t is:

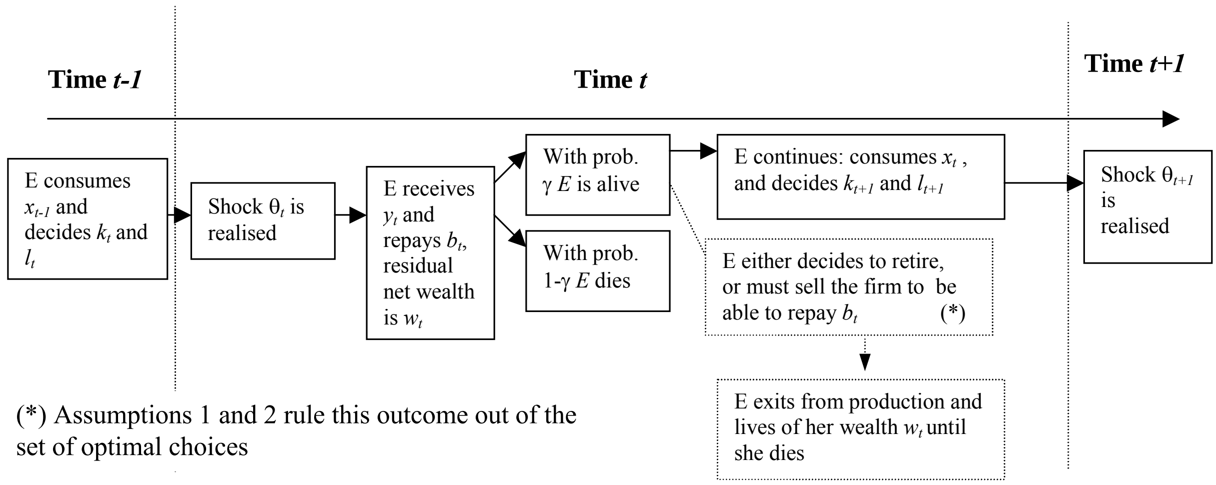
$$w_t = y_t + (1-\delta_k)k_t - b_t \tag{9}$$

After producing E dies with probability 1-γ. If she survives, she either retires or continues activity. Retirement can happen for two different reasons: i) E must retire after having

[11] The rationale for assumptions *f1-f3* is that E can hide the revenues from the production. Being unable to observe such revenues the lenders can only claim, as repayment of the debt, the value of E's physical assets (Hart and Moore, 1998). Therefore E can only lend or borrow one period secured debt at the market interest rate r offered by the banks. We also implicitly assume that in any default and renegotiation of the debt with the bank E has all the bargaining power. Otherwise the bank could use the threat of liquidation of fixed capital to enforce the repayment of uncollateralised debt.

[12] $\tau_k < 1-\delta_k$ can be motivated by assuming that E can 'steal' a $1-\tau_k$ fraction of the residual value of capital.

Figure 1: The timing of the model



sold the business to repay b_t; ii) E chooses to retire because the liquidation value of the assets is greater than the continuation value of the business.

The interaction between financing and irreversibility constraints implies that forced and voluntary retirement can happen in equilibrium. Intuitively because after a negative shock E may be forced to sell the fixed assets to repay the debt. But even if she manages to repay the debt without being forced to sell the assets and retire, she may be left with no funds to invest in variable capital, and hence unable to generate output. If the negative shock is persistent, then the expected return from the risky technology may be lower than the expected return from selling the assets and retire. While this is an interesting intuition to explore in future research, it goes beyond the scope of this paper. Here we simply notice that the presence of endogenous retirement implies that the discount factor of the problem becomes endogenous as well, and this makes the dynamic maximisation problem extremely difficult to solve, even numerically. Therefore in order to simplify the analysis we restrict the set of parameters in order to rule out forced and voluntary retirement from the set of possible outcomes. More specifically, we assume the following:

Assumption 1: $l^E \geq l^E_{\min}(\Theta)$

Assumption 2: $w_0, \overline{\theta}, \rho$ and σ^2_ζ are such that: i) the net present value of the expected utility from consumption is always higher conditional on continuing activity than on retiring; ii) constraint (5) is never binding with equality.

Θ is the vector of parameters: $\Theta' = \left\{\overline{\theta}, \rho, \sigma^2_\zeta, \beta, R, \delta_k, \tau_k, l^E\right\}$. Assumptions 1 and 2 imply that E does not retire voluntarily, and is never forced to retire, and that it is never optimal to sell part of l^E rather than using it in her own production (proof in Appendix A). $l^E_{\min}$ is defined in equation (40) of Appendix A. It represents the minimum amount of l^E that always allows E to generate enough revenues to repay the debt without liquidating k_t. Therefore she is never forced to retire. Assumption *2* ensures that an active E never voluntary retires. It is important to note that assumptions *1* and *2* do not affect the qualitative results of the model. In fact they rule out the extreme outcomes that would increase the expected cost of irreversibility and financing constraints, and hence would strengthen rather than weaken model's results.

The continuing E borrows new one period debt with face value b_{t+1}, receiving the discounted value b_{t+1}/R. The net worth w_t plus the new borrowing b_{t+1}/R are allocated between consumption and investment. Therefore the budget constraint faced by E is the following:

$$x_t + l_{t+1} + k_{t+1} = w_t + b_{t+1}/R \tag{10}$$

We denote the expected lifetime utility at time t of an active E, after θ_t is realised, by $V_t(w_t, \theta_t, k_t \mid \Theta, w_0, k_0, \theta_0)$, where w_t, θ_t and k_t are the three state variables of the problem.

$$V_0(w_0, \theta_0, k_0) = \underset{\begin{pmatrix} k_{t+1} = k(w_t, \theta_t, k_t) \\ l_{t+1} = l(w_t, \theta_t, k_t) \\ b_{t+1} = b(w_t, \theta_t, k_t) \end{pmatrix}_{t=0,1,\ldots\infty}}{MAX} E_0 \left\{ \sum_{t=0}^{\infty} \beta^t U(x_t) \right\} \tag{11}$$

The problem is defined by (11) subject to (4) (5), (7) and (10). These constraints define a compact and convex feasibility set for l_{t+1}, k_{t+1}, b_{t+1} and x_t, and the law of motion of w_{t+1} conditional on w_t, k_t and θ_t is continuous. Therefore, given the assumptions on θ_t and the concavity of the production function, a solution to the problem exists and is unique[13].

Let μ_t and λ_t be the Lagrangian multipliers associated to the constraints (4) and (7). Taking the first order conditions of (11) with respect to b_{t+1}, l_{t+1} and k_{t+1} it is possible to show that the solution of the problem is given by the optimal sequence of $\{k_{t+1}, l_{t+1}, x_t, \lambda_t, \mu_t \mid k_t, w_t, \theta_t, \Theta\}_{t=0}^{\infty}$ that satisfies (4), (12), (13), (14) and (15), plus the standard complementary slackness conditions on λ_t and μ_t:

$$U'(x_t) = \beta R E_t [U'(x_{t+1})] + R\lambda_t \tag{12}$$

$$U'(x_t) = \beta E_t [U'(x_{t+1})(MPK_{t+1} + 1 - \delta)] + \mu_t + \lambda_t \tau_k - \beta(1 - \delta_k) E_t \left[\mu_{t+1}\right] \tag{13}$$

$$U'(x_t) = \beta E_t [U'(x_{t+1}) MPL_{t+1}] \tag{14}$$

$$D_k k_{t+1} + l_{t+1} + x_t \leq w_t \tag{15}$$

$D_k = 1 - \frac{\tau_k}{R}$ is the downpayment required to purchase one additional unit of fixed capital. Equation (15) combines together the budget constraint (10) and the collateral constraint (7) and implies that the downpayment necessary to buy k_{t+1}, l_{t+1} and x_t must be lower than E's net worth[14]. The term $(1 - \delta)\gamma\beta E_t \left[\mu_{t+1}\right]$ in equation (13) is the cost of future expected irreversibility constraints. λ_t and μ_t are positive when respectively the financing and the irreversibility constraint is binding, and are equal to zero otherwise. Since it is not possible to obtain an analytical solution to this problem, in the next section we will solve it using a numerical method, while in the remainder part of this section we

[13] See Stokey and Lucas (1989), Chapter 9.2.

[14] The optimal choices of k

will describe the main qualitative features of the model. We first analyse the solution without financing problems, then we analyse the solution without irreversibility problems, and finally we explain how the two problems interact together. In order to illustrate the intuition of the model, let's use (12) to rewrite (13) and (14) in the following way:

$$E_t\left(MPK_{t+1}\right) = UK + \frac{1}{E_t\left[U'\left(x_{t+1}\right)\right]}\left\{\frac{1}{\gamma\beta}\left[\left(R-\tau_k\right)\lambda_t - \mu_t\right] - cov^{xk}_{t+1} + \left(1-\delta_k\right)E_t\left(\mu_{t+1}\right)\right\} \tag{16}$$

$$E_t\left(MPL_{t+1}\right) = UL + \frac{1}{E_t\left[U'\left(x_{t+1}\right)\right]}\left(\frac{1}{\gamma\beta}R\lambda_t - cov^{xl}_{t+1}\right) \tag{17}$$

The first term on the right and side of (16) and (17) is the user cost of capital:

$$UK = R - (1-\delta)\,;UL = R \tag{18}$$

while the remaining terms on the right hand side represent current and future expected costs of financing and irreversibility constraints. cov^{xk} and cov^{xl} are the following covariances:

$$cov^{xk}_{t+1} = cov\left[U'\left(x_{t+1}\right), MPK_{t+1}\right] \tag{19}$$

$$cov^{xl}_{t+1} = cov\left[U'\left(x_{t+1}\right), MPL_{t+1}\right] \tag{20}$$

2.1 Solution with the irreversibility problem only.

In this subsection we rule out current and future expected financing constraints by assuming that the entrepreneur can borrow upfront future expected earnings. Moreover we also rule out risky debt, by assuming that the utility function is linear in consumption, and that E discounts the future at the market interest rate:

$$U\left(x_t\right) = x_t;\;\; R\beta = 1$$

Hence the following transversality condition is also necessary:

$$\lim_{t\to\infty}\beta^t b_t = 0$$

These changes imply that we are considering a standard profit maximising problem where E is never financially constrained and consumption and financing decisions are irrelevant for investment choices. Therefore $\lambda_t = cov^{xk}_{t+1} = cov^{xl}_{t+1} = 0$ for any t, and the first order conditions (16) and (17) simplify to:

$$E_t\left(MPK_{t+1}\right) = UK + \left(1-\delta_k\right)E_t\left(\mu_{t+1}\right) - R\mu_t \tag{21}$$

$$E_t\left(MPL_{t+1}\right) = UL \tag{22}$$

Equations (21) and (22) and constraint (4) determine μ_t, k_{t+1} and l_{t+1}. They describe the solution to a version of a well known irreversible investment problem (e.g. see Bertola and Caballero, 1994). The main difference with the irreversible investment literature is that we allow for a reversible factor of production to be used in conjunction with the irreversible one. The intuitive consequence is that l_{t+1}, the reversible factor, is more volatile than k_{t+1}, the irreversible one, both after a positive and a negative shock[15]. This does not necessarily imply that variable capital is volatile also in absolute terms. In fact the more the two factors of production are complementary, the more the irreversibility of fixed capital also reduces variable capital volatility.

2.2 Solution with financing constraints only

In this section we rule out current and future expected irreversibility constraints by assuming that both variable capital and fixed capital are reversible. Hence the irreversibility constraint (4) no longer applies, and $\mu_t = E_t\left(\mu_{t+1}\right) = 0$, for any t. Let's first consider the case without financing problems as well. In this case equations (16) and (17) reduce to:

$$E_t\left(MPK_{t+1}\right) = UK \tag{23}$$

$$E_t\left(MPL_{t+1}\right) = UL \tag{24}$$

Equation (12) evaluated at $\lambda_t = 0$ and equations (23) and (24) determine x_{t+1}, k_{t+1} and l_{t+1}, the optimal levels of consumption and investment when E faces no constraints.

Let's now consider the case with financing constraints. Equation (15) implies that there exists a minimum level of net worth $\underline{w}_t = \underline{w}(\theta_t, k_t)$ such that E is not financially constrained at time t, and (15) is not binding. Therefore there are two possibilities: If $w_t \in (w_t^{\min}, \underline{w}_t)$ then E is financially constrained . Equation (15) is binding with equality and together with equations (12), (16) and (17) evaluated at $\mu_t = E_t\left(\mu_{t+1}\right) = 0$ determine λ_t, x_t, k_{t+1} and l_{t+1}. In this situation E borrows up to the limit without being able to exhaust all profitable investment opportunities. Intuitively (15) binding with equality means that one additional unit of wealth allows E to increase either k_{t+1} by $1/D_k$ units, or l_{t+1} and x_t by 1 unit. $\lambda_t > 0$ represents the shadow cost of not being able to increase investment because of the lack of additional borrowing. Therefore financing constraints affect both investment and consumption decisions. More precisely, by substituting recursively in equation (12) we obtain:

$$U'\left(x_t\right) = R \sum_{j=0}^{\infty} \left(\gamma\beta\right)^j E_t\left(\lambda_{t+j}\right) \tag{25}$$

[15] This is evident from the comparison between (21) and (22). After a negative productivity shock at time t that reduces the marginal productivity of capital, E reduces l_{t+1}, to ensure that (22) is satisfied. When (4) is binding, k_{t+1} cannot be reduced, and as a consequence marginal productivity of capital is lower than the user cost UK. This is compensated by a positive μ_t on the right hand side of (22). Instead after a positive productivity shock E wants to invest more in both factors. Therefore (4) is not binding and $\mu_t = 0$. Instead $E_t\left(\mu_{t+1}\right) > 0$ because, applying the same reasoning made before, (4) can be binding at time $t+1$ conditional on a future negative shock. The positive $E_t\left(\mu_{t+1}\right)$ represents the cost associated to future expected irreversibility. Such cost increases the required marginal productivity of fixed capital $E_t\left(MPK_{t+1}\right)$, thereby reducing k_{t+1}. Therefore k_{t+1} increases less than l_{t+1} after a positive shock.

Equation (25) shows that expected marginal utility from consumption is increasing in the shadow cost of current and future expected borrowing constraints. If $w_t \in [\underline{w}_t, \infty)$ then E is unconstrained today, but could face borrowing constraints in the future. Since $\lambda_t = \mu_t = E_t \left[\mu_{t+1} \right] = 0$ then (16) and (17) become:

$$E_t \left(MPK_{t+1} \right) = UK - \frac{cov_{t+1}^{xk}}{E_t \left[U' \left(x_{t+1} \right) \right]} \tag{26}$$

$$E_t \left(MPL_{t+1} \right) = UL - \frac{cov_{t+1}^{xl}}{E_t \left[U' \left(x_{t+1} \right) \right]} \tag{27}$$

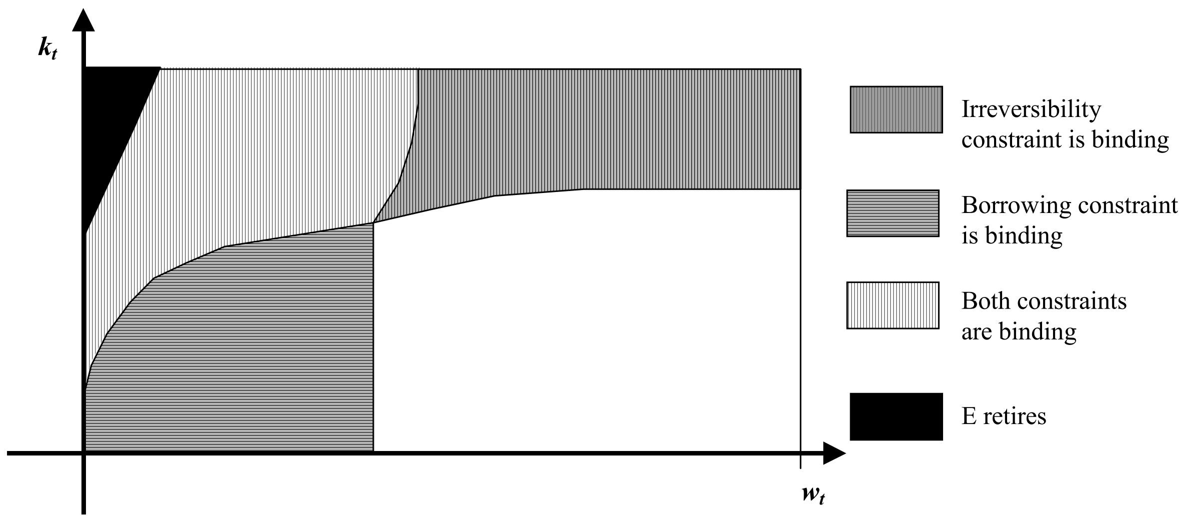
Equations (12), (26) and (27) determine the optimal consumption x_t and investment k_{t+1} and l_{t+1}. The covariances cov_{t+1}^{xk} and cov_{t+1}^{xl} between marginal productivities and marginal utility of consumption are negative, and become large in absolute value when w_t is small. This is because the closer is w_t to $\underline{w}_t$, the more likely is that the borrowing constraint will be binding in the future conditional on a negative productivity shock. Hence such shock at the same time reduces marginal productivity of capital and increases marginal utility of consumption (see equation 25). From (26) and (27) it is easy to see that the more negative are cov_{xk} and cov_{xl}, the higher is the optimal marginal productivity of capital and the lower are the optimal investment levels k_{t+1} and l_{t+1}. Hence future expected financing constraints reduce optimal investment choices of a risk averse E. It is well known in consumption literature that financing constraints may reduce current consumption because of a precautionary saving motive (see Zeldes, 1989 and Carroll, 2001). This is to our knowledge the first paper to show the same effect for investment choices. In the next section we will show that, for realistic parameter choices, this precautionary saving effect on investment, is quantitatively relevant: between two identical entrepreneurs who do not currently face borrowing constraints, the richer one invests in the risky technology up to 18% more than the poorer one, the difference being entirely due to future expected financing problems.

It is finally worth mentioning that financing constraints do not alter the optimal mix between fixed and variable capital, when next period residual fixed capital is fully collateralisable (a formal proof is in appendix B). The intuition is that the advantage of fixed capital over variable capital for an unconstrained E (lower user cost) is identical to the advantage for a constrained E (collateral value). This result means that current and future expected financing problems affect the trade off between safe and risky investment but not necessarily the mix between assets in the risky investment. This result is reversed when financing and irreversibility constraints coexist, as it is shown in the simulations in section 3.

2.3 Solution with the financing and the irreversibility constraints.

We consider now the solution of the problem with both constraints. Figure 2 summarises the different types of optimal policy functions $k_{t+1}(w_t, k_t \mid \theta_t)$ and $l_{t+1}(w_t, k_t \mid \theta_t)$ in the $\{k_t, w_t\}$ space, conditional on a certain productivity shock. The black area on the top left corresponds to a situation in which E is either forced to sell the firm to repay the debt, or finds convenient to retire. This area is ruled out by assumptions *1* and *2.* Instead of

Figure 2: Policy functions for a given productivity shock



describing in detail such solutions, we focus only on the most interesting feature: the fact that irreversibility and financing constraints interact and amplify each other. When both constraints are binding, μ_t is determined by equation (16). By substituting recursively we obtain:

$$\mu_t = \sum_{j=0}^{\infty} (1-\delta)^j \left[\Gamma_{t+1+j} + \frac{1}{\beta} (R - \tau_k) E_t (\lambda_{t+j}) - cov_{t+j}^{xk} \right] \tag{28}$$

Equation (28) shows that μ_t, the shadow cost of the irreversibility of fixed capital increases in the present and expected costs of financing constraints. The term Γ_{t+1+j} on the right hand side represents the expected cost of overinvestment in terms of expected marginal utility:

$$\Gamma_{t+1+j} = E_t [UK - MPK_{t+1+j}] E_t [U' (x_{t+1+j})] \tag{29}$$

The term $\frac{1}{\gamma\beta} (R - \tau_k) E_t (\lambda_{t+j}) - cov_{t+j}^{xk}$ represents the direct[16] additional cost of current and future expected financing constraints. This implies that financing constraints amplify the cost of irreversibility, and increase the cautious investment effect on fixed capital. More importantly, the reverse is also true: the irreversibility constraint increases the chances of facing financing constraints now, because fixed capital cannot be liquidated, and also in the future, because the unbalanced use of factors leads to lower wealth accumulation. As a result the cost of future expected financing constraints is higher when fixed capital is irreversible. A quantification of this amplification effect is computed in the next section in figure 7.

[16] An indirect cost of future expected financing constraints is also in $E_t [U' (x_{t+1+j})]$, which also increases in λ_{t+j+1}.

Figure 3: Summary statistics

	Output	Fixed Inputs	Variable Inputs
Empirical Data*	4695	2319	3320
Simulated Data**	4722	2259	3310

* Average (1962-1995) values for the average firm in the Industrial Machinery Sector, at constant 1987 prices; ** average values for an entrepreneur in the simulated economy.
Empirical variables are measured as follows: output is the value of industry shipments; fixed inputs are plant and equipment; variable inputs are total cost of materials and labour plus changes in inventories.

3 Numerical Solution and simulation

3.1 Model's solution

We solve the dynamic nonlinear system of equations defined by (4), (12), (13), (14) and (15) using a numerical method (see appendix C). Adding the subscript i to indicate the *i-th* entrepreneur, the solution consists in the optimal policy functions $k_{i,t+1}(w_{i,t}, \theta_{i,t}, k_{i,t})$ and $l_{i,t+1}(w_{i,t}, \theta_{i,t}, k_{i,t})$, the associated Lagrange multipliers $\lambda_{i,t}(w_{i,t}, \theta_{i,t}, k_{i,t})$ and $\mu_{i,t}(w_{i,t}, \theta_{i,t}, k_{i,t})$ and the value function $V_{i,t}(w_{i,t}, \theta_{i,t}, k_{i,t})$.

Uncertainty regarding output is modeled in the following way:

$$y_{i,t} = e^{\theta_{i,t}} k_{i,t}^{\alpha} \left(l_{i,t} + l^{E} \right)^{\kappa} \tag{30}$$

Where:

$$\theta_{i,t} = \theta_{i,t}^{f} \varepsilon_t$$

$\theta_{i,t}^{f}$ is the idiosyncratic productivity shock, and ε_t is the economy-wide demand shock common to all entrepreneurs. Both are first order autoregressive stochastic processes with autocorrelation coefficients ρ_θ and ρ_ε and variances σ_θ^2 and σ_ε^2 respectively. In practice $\theta_{i,t}^{f}$ and ε_t are modeled as a two states and a 6 states symmetric Markow process respectively. We calibrate the model to match the aggregate yearly data on output and capital stock for the "Industrial Machinery Sector" in the US from 1962 to 1995(source: NBER-CES manufacturing industry database). Hence in this section the ability of the model to explain the cyclical fluctuations of aggregate output and investment will be measured by comparing the simulated statistics from the artificial economy with the corresponding empirical data for the "Industrial Machinery Sector". This sector has been chosen as a generic representative one, but the theoretical model can be applied to any other sector where productive units use a combination of reversible and irreversible factors of production[17] and can be subject to borrowing constraints. The main advantage to use a single four digits sector rather than the whole manufacturing industry is that we have less aggregation problems in the empirical data.

Therefore we believe that the results obtained in this section, regarding the ability of the model to explain firm dynamics, are robust: the chosen sector is populated by many

[17] It is not necessary for the result that the collateral is provided by the irreversible factor. Moreover the results are quantitatively significant also when the depreciation rate of the irreversible factor is high, as is the case, for example, of technology spending.

Figure 4: Summary of parameters

$\alpha = 0.09$	$\rho_\theta = 0.45$
$\kappa = 0.86$	$\sigma_\theta = 0.11$
$\delta = 0.15$	$\beta = 1/(1+r)$
$\overline{\theta}^f = 0.643$	$\gamma = 0.99$
$r = 0.03$	$\tau = 0.7(1\text{-}\delta)$
$\rho_\varepsilon = 0.91$	l^E
$\sigma_\varepsilon = 0.024$	

firms, 4963 according to the 1997 census of the manufacturing industry, the majority of which are small firms: 63% of all the establishments have less than 20 employees, and more than 95% of the firms have only one establishment. Therefore it is reasonable to assume that a large share of these firms may be affected by borrowing and irreversibility[18] constraints, like the entrepreneurial businesses in the simulated artificial economy.

The chosen parameter values are reported in figure (4).The technological parameters α, κ, δ and $\overline{\theta}^f$ (the unconditional expectation of $\theta^f_{i,t}$), given the chosen annual real interest rate $r = 3\%$, imply that the steady state level of output, fixed capital and variable capital for an entrepreneur in the simulated economy match the same statistics for the average firm in the US industrial machinery sector, as is shown in figure 3. The parameters of the aggregate shock ρ_ε and σ_ε are directly estimated using the information, included in the NBER dataset, about the total factor productivity growth. The parameters of the firm specific shock ρ_θ and σ_θ are chosen conservatively with respect to the parameters adopted in similar studies that analyse industry dynamics, like Cooley and Quadrini (2001) and Gomes (2001)[19].

Among the remaining parameters the value of l^E implies that the variable inputs provided by the entrepreneur matter for around 10% of the average value of external inputs. The value of β implies that $\beta(1 + r) = 0.99 < 1$, so that agents are impatient in their consumption decisions. Finally, the value of τ implies that 70% of the residual fixed capital is collateralisable.

The next three figures illustrate the policy functions for chosen values of $\theta^f_{i,t}, \varepsilon_t$ and $k_{i,t}$. In figure 5 we plot $k_{i,t+1}(w_{i,t})$ and $l_{i,t+1}(w_{i,t})$, which are the investment decisions as a function of financial wealth, for a productive firm ($\theta^f_{i,t}$ is high) in a recessive economy (ε_t is the lowest among the six states of the world), for $k_{i,t} = 0$ (irreversibility is not binding). For very small wealth levels the financing constraint is currently binding, ad the policy functions of fixed and variable capital are very steep. The kink in these two functions represents the wealth level for which the borrowing constraint is no longer

[18] A large firm cound avoid irreversibility constraints simply by shutting down some of its establishments.

[19] The comparison with these two studies is relevant given that both concern the same topic of our work and both calibrate industry dynamics using yearly data. Gomes 2001 chooses the values of $\rho_f = 0.62$ and $\delta_f = 0.15$, while Cooley and Quadrini 2001 choose higher standard deviation ($\delta_f = 0.28$) but no persistency ($\rho_f = 0$). The results we obtan in this section are however robust to a wide range of such parameters.

Figure 5: Policy functions: high idiosyncratic shock and lowest aggregate shock.

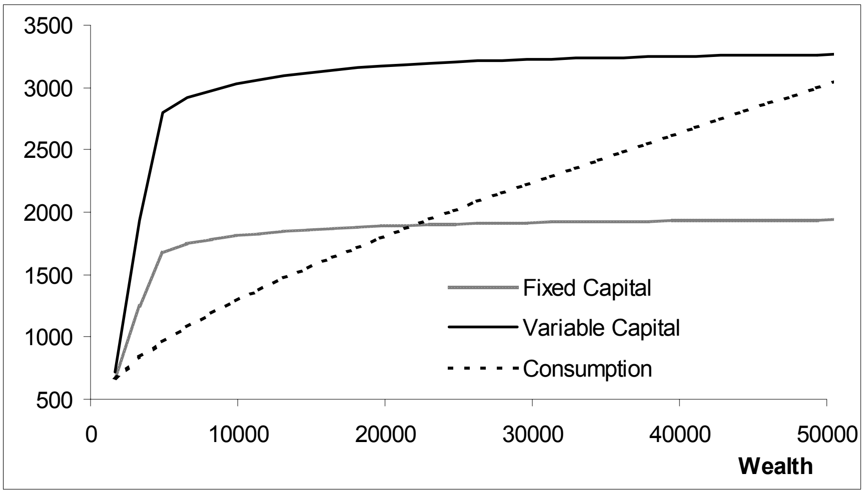
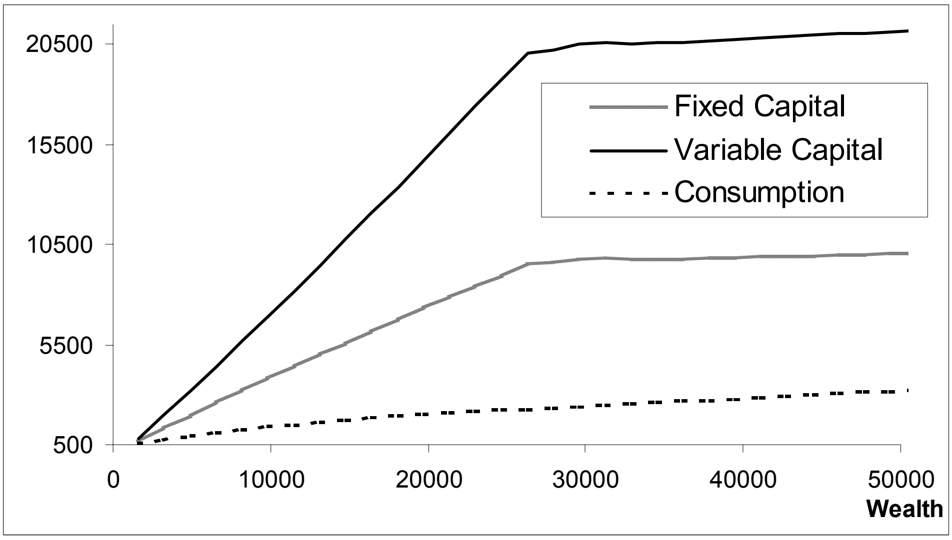


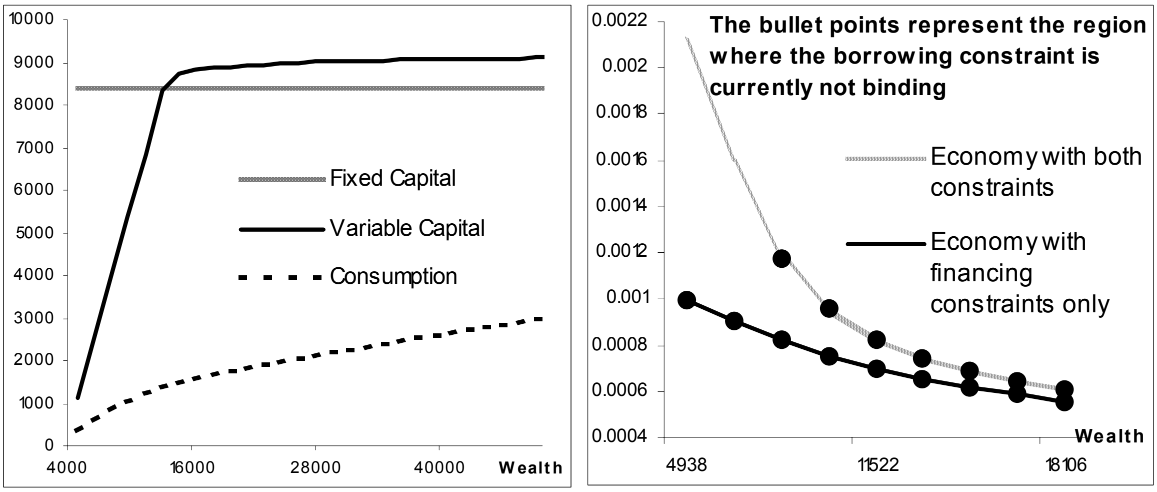
Figure 6: Policy functions: high idiosyncratic shock and highest aggregate shock.



binding. After the kink optimal capital is still increasing in wealth, because the increase in wealth decreases future expected financing problems. This "precautionary saving effect" is quantitatively important. Consider two entrepreneurs identical in everything except that their financial wealth, and both currently not financially constrained. Future expected financing problems imply that the richer entrepreneur invests in the risky technology up to 18% more than the poorer entrepreneur.

Figure 6 represents the policy functions for a productive firm ($\theta_{i,t}^{f}$ is high) in a booming economy (ε_t is the highest), with $k_{i,t} = 0$. On the one hand the region with a binding borrowing constraint is larger, given the high investment needs of E. On the other hand when the borrowing constraint is not binding E has few future expected financing problems. As a result the richer unconstrained entrepreneur invests only 11% more in the risky technology with respect to the poorer one.

Figure 7: **On the left:** policy functions for high idiosyncratic shock, lowest aggregate shock and binding irreversibility constraint on fixed capital. **On the right:** present discounted value of the cost of future expected borrowing constraints



The left hand side of figure 7 represents an entrepreneur in the same situation as in figure 5 but with $k_{i,t} = 9875$. This entrepreneur has too much fixed capital given the current state of the economy, and hence $k_{i,t+1}$ is a flat line equal to $(1-\delta)\, k_{i,t}$, indicating that the irreversibility constraint is binding. The consequence is that the region in which the borrowing constraint is binding is now larger than in figure 5. Moreover when the borrowing constraint is not binding both k_{t+1} and l_{t+1} are inefficiently high. Therefore if there is a sequence of negative aggregate shocks, financial wealth decreases more rapidly towards the binding constraints region, and variable capital declines faster as well, moving leftward along the policy function schedule. The consequence is that variable capital is more volatile than it is the case when fixed capital is reversible.

In order to quantify the amplification effect of irreversibility on borrowing constraints, it is useful to remember that, from equation (12):

$$\sum_{j=1}^{\infty} \beta^j E_t\left(\lambda_{t+j}\right) = \frac{U'\left(x_{i,t}\right) - R\lambda_{i,t}}{R} \tag{31}$$

Equation (31) shows that the discounted sum of the shadow cost of future expected financing constraints, on the left hand side, can be computed using the information about marginal utility of consumption and shadow cost of currently binding constraints at time t . The right hand side of figure 7 illustrates the value $\sum_{j=1}^{\infty} \beta^j E_t\left(\lambda_{t+j}\right)$ as a function of financial wealth, for an entrepreneur who suffers the worst possible shock[20]. We compare two different economies, with and without irreversibility of fixed capital. The figure shows that the cost of future expected financing problems is much higher in the economy with irreversibility of fixed capital. More importantly, this is true also when the borrowing

[20]the aggregate shock goes from the highest state to the lowest state and the firm specific shock goes from high to low. This entrepreneur has the same level of fixed capital than in the left hand side of figure 7.

constraint is not currently binding. This means that the irreversibility of fixed capital amplifies not only current but also future expected borrowing constraints.

3.2 Dynamics of aggregate output and investment

We now use the model's solution to simulate the investment and production path of many heterogeneous entrepreneurs. The aim is to show how the combination of irreversibility and financing constraints generates a behaviour of aggregate investment consistent with the empirical evidence.

In the simulated economy, the behaviour of aggregate investment and production depends on the heterogeneity of the entrepreneurs in terms of the state variables. All entrepreneurs are identical ex ante, but each of them is subject to different realisation of the idiosyncratic productivity shock $\theta^f_{i,t}$, which is uncorrelated across entrepreneurs and serially correlated for each entrepreneur. Therefore at time t entrepreneurs have different values of $w_{i,t}$ and $k_{i,t}$, depending on $\left\{\theta^f_{i,j}\right\}_{j=0}^{t}$. The distribution of $\{w_{i,t}, k_{i,t}\}$ at time t depends on the parameters set Θ and on the aggregate shock ε_t.

The statistics illustrated in the remainder of this section are generated by the simulation of an economy of 5000 firms, for 2000 periods. In each period the number of new entrepreneurs is constant, since they are infinitely lived. Nevertheless the model could be easily modified in order to allow finite lives and entry-exit dynamics. In this case if we allowed new entrants to be less wealthy than the average surviving entrepreneur we would increase the impact of borrowing constraints on investment dynamics. This would increase quantitatively the importance of some of the findings in this section, but would not modify them qualitatively. Therefore entry-exit dynamics are not necessary to generate the main results of the paper[21].

We simulate four different economies corresponding to the four versions of the model described in the previous section: without any constraint, with one of the two constraints only, and with both constraints. Figure 8 shows some statistics for the simulated economies compared with the empirical data of the industrial machinery sector used to calibrate the model. We choose to represent variable capital with the total cost of materials excluded energy. This is the empirical variable most similar to the definition of variable capital we employed in our theoretical model[22]. We do not report any simulated data about consumption in the economies without borrowing constraints, since in these economies we impose, by construction, a separation between investment and consumption decisions.

The last column of figure 8 shows that in the economy without constraints, not surprisingly, aggregate capital and output have an implausibly large volatility. Such volatility is smaller in the economy with the irreversibility constraint only, but it is still very far from the empirical data[23].

[21] Simulation results for artificial economies with entry-exit dynamics are available upon request.

[22] If we had the information about inventories in the different stages of fabrication we could have computed a more precise variable, which would have included only materials used in the period.

[23] This result is consistent with Bertola and Caballero (1994) who note that, when irreversibility of capital is the only friction to firm level investment, a model of industry dynamics can replicate observed investment volatility only with implausibly large level of the idiosyncratic shocks.

Figure 8: Aggregate volatilities: comparison between empirical data and simulated data

Variable	Empirical data	Simulated data			
		Both Constraints	Only financing constraint	Only irreversibility constraint	No constraints
Standard deviation of variable capital (l_t)	7.647[1]	11.836	13.482	22.862	35.366
Standard deviation of fixed capital (k_t)	1.982[2]	7.035	12.570	14.489	35.366
Standard deviation of of consumption (x_t)	n.a.	1.492	1.270	n.a.	n.a.
Standard deviation of output (y_t)	6.926[3]	10.682	12.413	21.018	33.489
St. dev. of var. capital relative to output	**1.104**	**1.105**	**1.081**	**1.091**	**1.059**
Correlation between $y_t - y_{t-1}$ and $l_{t+1} - l_t$	**0.229**[4] **0.367**[5]	**0.265**	**0.157**	**0.101**	**-0.056**

1. Cost of materials, excluded energy; 2. Real capital stock (plant + equipment); 3. Value of industry shipments; 4. $l_{t+1} - l_t$ measured as change in cost of materials; 5. $l_{t+1} - l_t$ measured as inventory investment.

On the contrary the volatilities of the economy simulated with both constraints are much closer to the empirical volatilities. Fixed capital is less volatile in comparison with the other simulated economies, while variable capital is relatively more volatile. To understand this finding it is useful to remember that the irreversibility constraint induces a cautious behaviour in fixed capital investment: the entrepreneur is less willing to increase the size of her fixed assets because a future negative shock could make them inefficiently large. Conditional on such negative shock too much fixed capital lowers profits and cash flow, and this is particularly damaging when the entrepreneur also faces borrowing constraints. As a consequence the entrepreneur is even more cautious in investing in fixed capital when she faces both irreversibility and borrowing constraints.

The last two rows of figure 8 show that the economy with both constraints is the only one among the four simulated economies to be able to match the empirical data across two important dimensions: i) the volatility of variable capital relative to output; ii) the correlation between lagged changes in sales and inventory investment.

Variable capital is relatively more volatile in the model with both constraints than in the other simulated models, and in line with the empirical data. This is because in such

Figure 9: Distributional features

	Simulated data			
Variable	Both Constraints	Only financing constraint	Only irreversibility constraint	No constraints
Avg % of firms with a binding borrowing constraint	36.3	58.2	0	0
Avg % of firms with a binding irreversibility constraint	37.4	0	60.7	0
Avg % of firms with both constraints binding	10.6	0	0	0
Avg. ratio of debt over phisical assets	0.1115	0.1745	n.a.	n.a.
Avg. ratio of consumption over output	0.2602	0.2839	n.a.	n.a

and G7 countries (Ramey and West, 1999). The correlation is negative in the economy without constraints because the productivity shock is mean reverting and hence it affects current output more than variable capital investment, which is a forward looking variable. A positive correlation is present in the model with irreversibility only, because a positive aggregate shock reduces the number of firms with a binding irreversibility constraint, and hence increases the sensitivity of investment to future shocks. In the model with financing constraints the correlation is also positive because a positive aggregate shock increases financial wealth and therefore increases investment by reducing current and future expected financing constraints. The sum of the two effects implies that in the economy with both constraints the correlation coefficient is higher and closer to the value observed in reality.

Figure 9 illustrates the main distributional features of the simulated economies. The economy with both constraints has an average ratio of constrained firms of 46.9%, of which 10.6% also have the irreversibility constraint binding. This value averages a very volatile ratio, which in some periods falls below 10%, when there are on average low investment needs, and in other periods rises up to 80%.

Figure 10 quantifies the effect of financing constraints on investment. We compute the percentage change in optimal capital if wealth increases by 1%, other variables kept constant. The calculation is reported only for productive entrepreneurs, defined as those with an high idiosyncratic shock in a given period. Therefore the figure shows the elasticity of investment to changes in financial wealth. For example an elasticity equal to 1.4 means that if the wealth of all productive entrepreneurs increases by 1%, keeping constant expected productivity, their stock of capital increases by 1.4%. Such elasticity is equal to zero in the economies without financing imperfections, where investment decisions are not affected by capital structure, and therefore these economies are omitted. Figure 10 shows that aggregate investment is positively affected by changes in financial wealth in the economies with financing constraints. Three results are particularly worth mentioning: i) in the economy with both constraints variable capital is more sensitive to financial

Figure 10: Elasticity of investment to financial wealth for a productive firm**, conditional on the productivity shock and the level of fixed capital.

	variable investment		fixed investment	
	Both Constr.	Borr. const. only	Both Constr.	Borr. const. only
All entrepreneurs				
No constraint is currently binding	0.10	0.09	0.10	0.09
Borrowing constr. is currently binding	1.11	1.16	1.05	1.26
Both constraints are currently binding	1.62		0.95	
Smaller entrepreneurs				
No constraint is currently binding	0.11	n.a.*	0.13	n.a.*
Borrowing constr. is currently binding	1.23	1.31	1.19	1.40
Both constraints are currently binding	1.75		0.95	

*No firm is in this category; ** A firm with a positive idiosyncratic productivity shock.

Figure 11: Volatility and procyclicality of aggregate investment

		All entrepreneurs				Small entrepreneurs			
Volatilities relative to aggregate investment	Empirical data	Both constr.	Only irrev. constraint	Only borr. constraint	No constr.	Both constr.	Only irrev. constraint	Only borr. constraint	No constr.
Fixed investment	2.497	3.491	3.684	5.080	3.061	3.859	2.276	3.506	3.007
Inventory investment	14.779[2] 16.916[3]	12.859	6.931	10.149	4.040	10.650	3.585	4.902	4.024
Drop of inventories in recessions[1]	**86%[4]**	**82%**	**79%**	**65.8%**	**64.1%**	**83.2%**	**75.2%**	**65%**	**67.2%**

1. Absolute drop in inventories as a percentage of the absolute drop of aggregate capital (fixed+inventories) in recessions; 2. measured as change in cost of materials; 3. measured as inventory investment. 4. The empirical series are detrended, in order to make them comparable with the simulated series. Inventories are measured as total inventories.

wealth than fixed capital[26]. This result is consistent with empirical evidence at firm level (Carpenter Fazzari and Petersen, 1998). ii) Entrepreneurs who are not currently financially constrained also show a positive elasticity around 10%. This is due to the fact that the increase in wealth reduces future expected financing problems. This effect is small but not negligible, and it increases up to 20% at the beginning of expansion periods, when entrepreneurs engage in precautionary saving given the uncertainty about how long the expansion will be. iii) Entrepreneurs with both constraints are those with the highest elasticity of variable capital investment with respect to wealth. This is especially true for smaller entrepreneurs. These very high values of the elasticity explain why variable capital is relatively more volatile in the economy with both constraints than in the other simulated economies.

[26] This result is obtained despite the distortion caused by the presence of the minimum level of variable capital l^E supplied by the entrepreneur, which biases downwards the elasticity of variable investment to financial wealth. This is why in the economy with only borrowing constraints fixed capital is more sensitive to financial wealth than variable capital.

Figure 11 summarises the aggregate consequences of financing and irreversibility constraints on investment volatility. As illustrated in the introduction, empirical evidence on US data provided by Ramey (1989) and by Blinder and Maccini (1991), and on G7 countries provided by Ramey and West (1999) show that inventories, especially in raw materials and work in progress, are very volatile and procyclical. This is especially true during recessions, when they account for a large fraction of the drop in business spending. Moreover Bernanke Gertler and Gilchrist (1996) show that the drop in inventories during recessions is more pronounced for smaller firms.

Figure 11 shows that the model with both constraints is consistent with all these findings. The first two rows report the volatility of fixed and inventory investment relative to the volatility of aggregate investment. Since in the model l_{t+1} can be interpreted as the end of the period t stock of input inventories, for the simulated data we measure inventory investment as the change in the stock of variable capital. For the empirical data we follow the same strategy used in figure 8, and propose two alternative measures of investment in input inventories: i) the change in the cost of materials, which coincides with the theoretical variable and is also closely correlated to the unobservable investment in materials inventories; ii) the investment in total inventories. The figure shows that inventory investment is significantly more volatile in the economy with both constraints than in any of the other simulated economies, and much closer to the empirical data.

Our simulations also provide a direct estimation of the relative importance of the drop in inventories during recessions. We define as recessions the episodes of several periods of decline in output (in real terms) which start when output is consistently above trend (greater than 0.75 of the standard deviation of the trend deviations of output) and end when output is below trend. We show not only that the aggregate drop in inventories during recessions is larger for the economy with both constraints, but also that the difference is driven by a larger drop in the inventories of smaller firms, consistently with empirical evidence. Necessary element for this result is the amplification effect between irreversibility and financing constraints. During recessions some entrepreneurs suffer from both constraints, and hence they are forced to cut dramatically investment in variable capital. Small entrepreneurs are on average less rich, and suffer more of this problem.

The comparison with empirical data shows that once again the economy with both constraints is the best performing one. However it must be noted that the value of 86% computed for the empirical data is very volatile, being the average between 82% (for the '69-'71 period) and 90% (for the '79-'83 period).

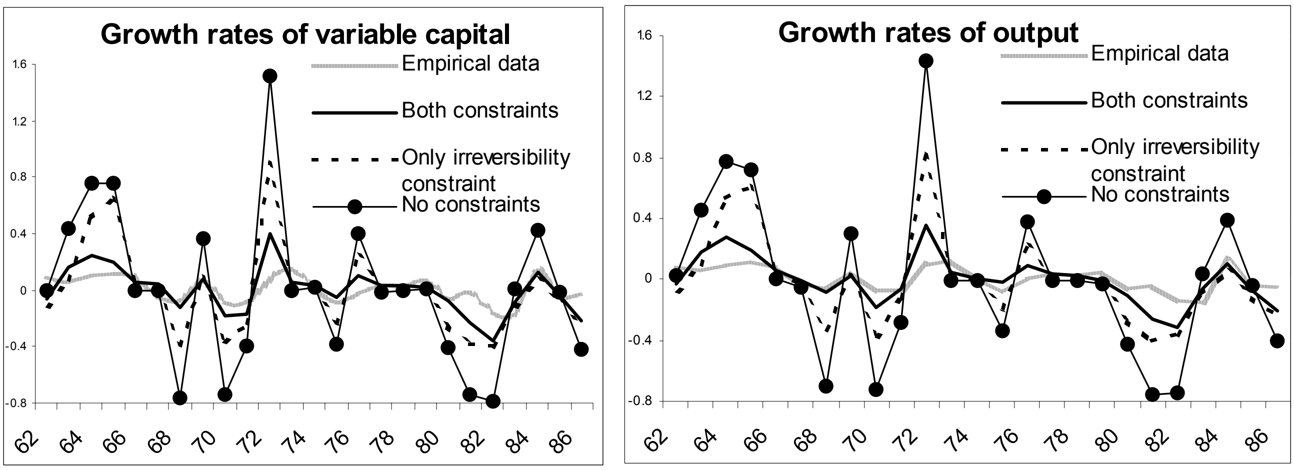
Figure 12 reports some information about the correlation between output and investment. In the economies without irreversibility constraint (first four rows) output and investment are contemporaneously correlated. This is natural since output directly depends on the inputs through the production function. The correlation is relatively low because inputs are decided one period in advance, based on the expected productivity shock $E_{t-1} \; \theta^f_{i,t}\varepsilon_t$, while output depends on the realised shock $\theta^f_{i,t}\varepsilon_t$.

The economies with the irreversibility constraint (last four rows) show that variable investment leads the fluctuations in output. This is because many entrepreneurs cannot immediately adjust fixed capital following an aggregate shock, and hence also the

Figure 12: Cross correlations between investment and output

		Correlation($input_t$, $output_{t+k}$)				
		-2	-1	0	1	2
Economy without constraints	Fixed investment	-0.01	0.02	0.48	0.40	0.34
	Variable investment	0.22	0.23	0.25	0.21	0.18
Economy with borr. Constr. only	Fixed investment	0.22	0.29	0.59	0.54	0.50
	Variable investment	-0.18	-0.13	0.20	0.19	0.17
Economy with Irreversibility only	Fixed investment	0.23	0.38	0.67	0.63	0.57
	Variable investment	-0.22	-0.17	0.20	0.20	0.19
Economy with both constraints	Fixed investment	0.45	0.58	0.78	0.74	0.70
	Variable investment	-0.18	-0.11	0.18	0.20	0.19

Figure 13: Growth rates of variable capital and output, comparison between empirical data and simulated economies



adjustment in output is a bit lagged with respect to the adjustment in variable capital. This feature, which is more evident in the model with both constraints, is consistent with empirical evidence of US business cycles (Stock and Watson, 1998).

3.2.1 Comparison with empirical aggregate data.

In figures 13-16 we directly compare the aggregate time series of the US Industrial Machinery Sector, between 1962 and 1986, with the time series computed using the simulated economies. We used the data about total factor productivity to estimate the aggregate productivity shock ε_t for the period 1962-1986. We then simulated the artificial economies feeding into the simulation the shock estimated above. The aim is to compare the dynamics of output and inventories predicted by the model with the ones observed in reality. If financing and irreversibility constraints are important to explain investment volatility, especially during recessions, then we expect the economy with both constraints to be more

Figure 14: Fit between empirical and simulated growth rates of aggregate capital and output

R^2 of an OLS regression of the empirical growth rate on a constant and the corresponding simulated growth rate of variable capital

	Simulated economies			
Growth rates of:	Both Constraints	Only borr. constraint	Only irrev. constraint	No constraints
Fixed capital	0.220	0.030	0.130	0.001
Variable capital	0.475	0.439	0.379	0.387
Output	0.526	0.502	0.444	0.463

consistent with empirical data, especially in recessionary periods. We focus on the period from '62 to '86 which includes large swings in investment and output, as opposed to a less volatile period from '87 to '96.

Figure 13 compares the growth rate of output and variable capital for the empirical data and for three simulated economy: without constraints, with irreversibility only, and with both constraints[27]. The economy with both constraints is closer to the empirical data. The other two economies are much more volatile, especially the one without constraints. This result confirms the statistics showed in figure (8).

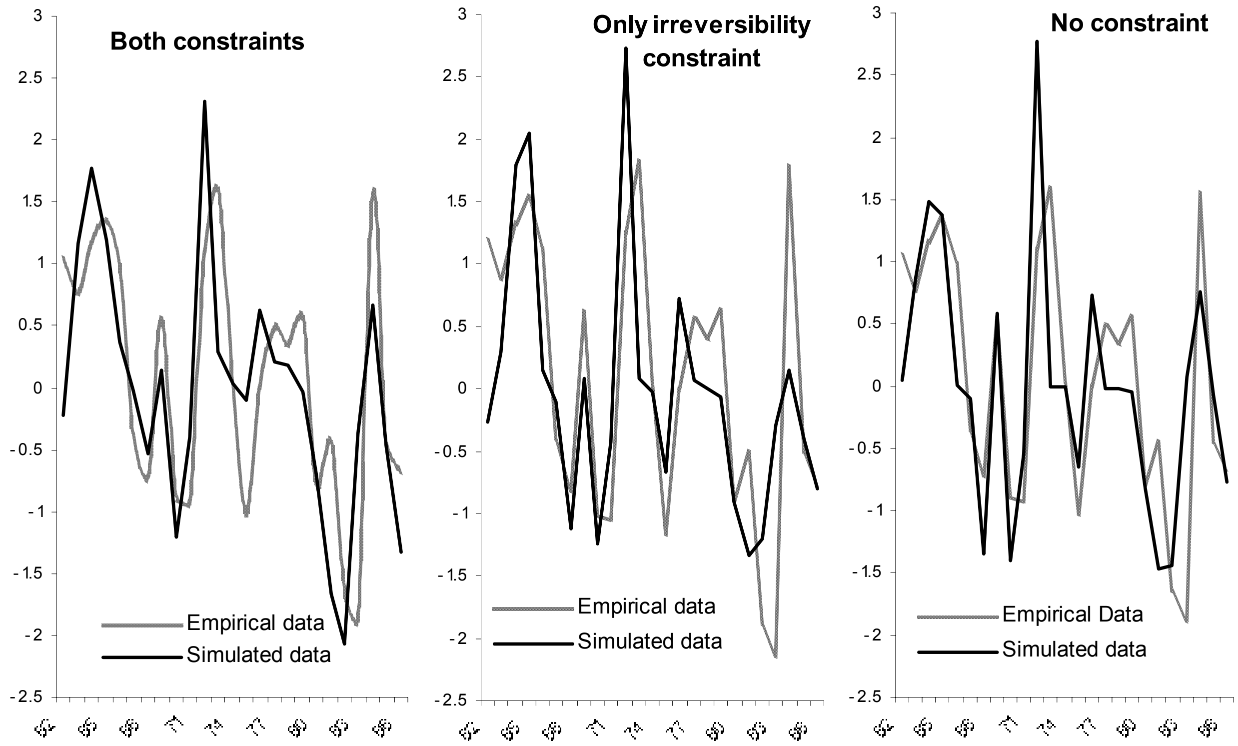
Figure 14 quantifies the fit between empirical and simulated data: it reports the R^2 of OLS regressions with the observed growth rate of capital and output as independent variable and the corresponding simulated growth rate and a constant as independent variables. The economy with both constraints has the best fit with the empirical growth rates. The gain in R^2 is bigger for capital than for output, and this result supports the validity of the main intuition of the model: the interactions between the two constraints are useful in explaining the behaviour of firms in the business cycle, especially regarding investment dynamics. The result is particularly relevant regarding variable capital. In fact although the biggest gain in R^2 regards fixed capital, this could be due to the fact that the model with both constraints, by inducing a more cautious behaviour fixed capital investment, captures the effect of adjustment costs not included in the model.

Moreover if the model correctly represents empirical data, such ability to replicate empirical growth rates should be more evident in years of large productivity shocks, when the two constraints are more likely to be binding. This intuition is confirmed by figure 15[28], which compares empirical and simulated growth rates of variable capital. The comparison is facilitated by the fact that the growth rates are rescaled using their standard deviations in the '62-'86 period. The model with both constraints is better in replicating empirical data when growth rates are large, both during expansion and contraction phases.

[27] The economy with borrowing constraints only is omitted in order to make these graphics more readable

[28] The graphic about output growth rates shows similar results, and is therefore omitted.

Figure 15: Growth rates of variable capital (rescaled)



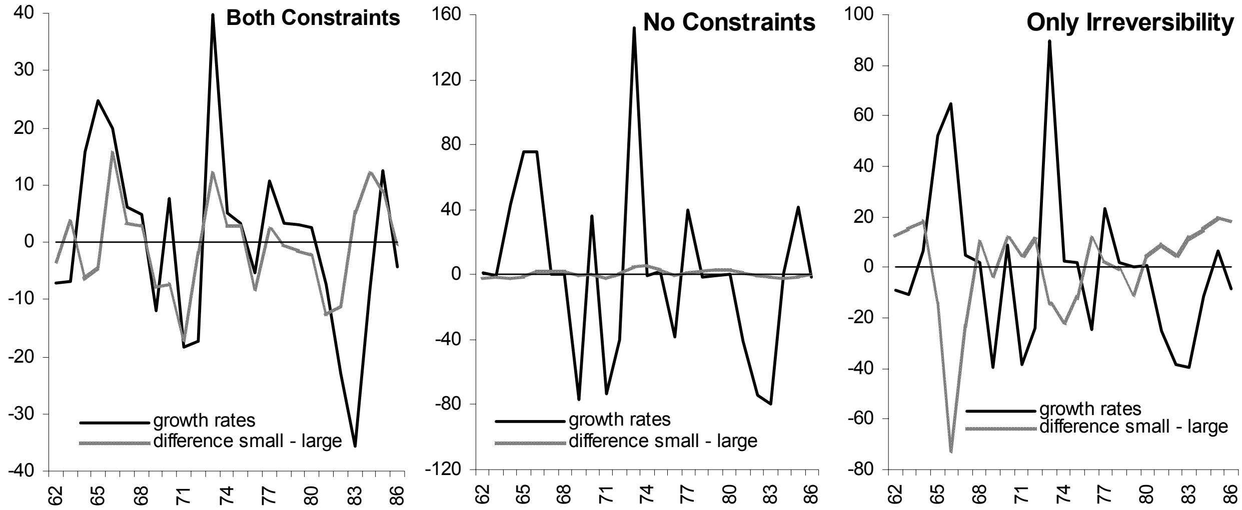
This is especially true for the recession of the beginning of the '80s, which is especially underestimated by the model with irreversibility only, and for the expansion in the mid '70s, which is especially overestimated by the model without constraints.

Unfortunately we do not have data available about small firms for the industrial sector we used as a benchmark, and hence we cannot directly compare the ability of the model to predict the behaviour of small as opposed to large firms in the period. Nonetheless our simulation results can be compared to the empirical literature on the US manufacturing sector. In particular to Bernanke, Gertler and Gilchrist (1996) who show that small firms are more volatile and procyclical than large ones both in inventories and output. Figure 16 is consistent with these findings, showing that in the economy with both constraints small firms are more procyclical than large ones, both in downturns and upturns. The same result is not obtained in the other two economies. The economy with irreversibility only has the opposite result, while the economy without constraints has no difference between small and large firms.

3.3 Empirical microeconomic evidence

In the above simulations we showed that the interaction between irreversibility and financing constraints explains a number of stylised facts about the dynamics of aggregate investment. An essential condition for the results derived above is the presence of financing constraints on firm investment decisions. In this section we report some direct empirical evidence of such constraints. The idea is simple: because variable capital in-

Figure 16: Simulated growth rate of aggregate variable capital and difference in the cumulative growth rates of small and large firms



vestment is reversible, the "premium" of expected marginal productivity over user cost of variable capital reflects the tightness of current and future expected financing constraints. In Caggese 2003 this premium is estimated using a unique dataset of 561 Italian manufacturing firms with both balance sheet and qualitative data. The sample contains the following information: i) 11 years of balance sheet data available from 1982 to 1992. This panel is a subset of the dataset produced by Centrale dei Bilanci, which is the largest and most reliable source of data about Italian firms. ii) Qualitative information from the First Mediocredito Centrale Survey on Small and Medium Italian Manufacturing Firms. Among the information in the survey we have the statements from the entrepreneurs about the financing problems they faced in the 1989-1991 period. Entrepreneurs were asked whether they had any of the following problems regarding investment financing: Q1) lack of collateral; Q2) lack of medium-long term financing; Q3) too high cost of banking.

The premium in expected productivity of variable capital, which constitutes a financing constraints indicator, is computed in Caggese (2003) as the difference between expected marginal productivity and user cost of variable capital. Using equation (17):

$$E_t\left(\Psi_{i,t+1}\right) = E_t\left(MPL_{i,t+1}\right) - UL_{i,t} \tag{32}$$

Where:

$$E_t\left(\Psi_{i,t+1}\right) = \frac{1}{E_t\left[U'\left(x_{t+1}\right)\right]}\left(\frac{1}{\gamma\beta}R\lambda_t - cov_{t+1}^{xl}\right) \tag{33}$$

$E_t\left(\Psi_{i,t+1}\right)$ as defined in (33) is an indicator of the intensity of current and future expected financing constraints. It is monotonously increasing in the shadow cost of being unable to borrow to finance new investment. Equation (32) implies that such indicator can be estimated as the difference between expected marginal productivity and user cost of capital. In Caggese (2003) the empirical counterparts of $UL_{i,t}$ and $E_t\left(MPL_{i,t+1}\right)$ are estimated using the panel of balance sheet data for Italian manufacturing firms, and an empirical counterpart of $E_t(\Psi_{i,t+1})$, called ${}_t\hat{\Psi}^w_{i,t+1}$, is obtained. The validity of ${}_t\hat{\Psi}^w_{i,t+1}$ as an indicator of financing constraints is tested using the direct information about financing problems available in the Mediocredito Centrale survey. Caggese (2003) considers the entrepreneurs that stated problems in accessing external finance in the 1989-91 period (questions Q1, Q2 and Q3). Such problems are directly related to the ${}_t\hat{\Psi}^w_{i,t+1}$ variable. The bigger ${}_t\hat{\Psi}^w_{i,t+1}$, the higher the shadow value of additional funding for the *i-th* entrepreneur and the higher the probability that she answers positively to one of the questions regarding financing constraints. Among the 561 firms considered, 21.6% of their entrepreneurs indicate one of the three problems in accessing bank credit during the 1989-1991 period. Using this information 4 dichotomous variables, $ration_i^j$ with $j = \{1, 2, 3, 4\}$, are constructed. They have value 0 if the $i - th$ entrepreneur does not state any financing problem, 1 if she answers positively to questions Q1, Q2 and Q3 respectively ($j = 1, 2$ and 3) or states any of the three problems ($j = 4$). Figure 17 from Caggese (2003) shows the result of a regression of $ration_i^j$ on $\overline{\Psi}_i^w$, which is the average value of ${}_t\hat{\Psi}^w_{i,t+1}$ in the period covered by the Mediocredito Centrale survey:

$$\mathbf{ration}_i^j = \alpha_0 + \alpha_1 \overline{\Psi}_i^w + \alpha_2 \dim_i \tag{34}$$

$\overline{\Psi}_i^w = \left(\sum_{t=1989}^{1992} {}_t\hat{\Psi}^w_{i,t+1}\right)$. The time interval used to compute $\overline{\Psi}_i^w$ includes 1989, 1990 and 1991, the period which the questions refer to, and 1992, the year in which the questionnaire has been compiled. $\dim_i$ is the size of the ith firm in number of employees, included to control for size effects. The first column of figure 17 is relative to the whole sample and to $ration_i^4$ as dependent variable. The coefficient relative to $\overline{\Psi}_i^w$ is positive and significant. The second and third columns repeat the same regression for larger (more than 300 employees, 19% of the sample) and smaller (less than 300 employees, 81% of the sample) firms.

In order to interpret this result, we note that this estimation is done under the assumption that the productivity shock is stationary plus the condition that $\alpha + \kappa < 1$. Hence firms are assumed to be in different steady state sizes, according to their fixed effects A_i. Each firm evolves around such steady state according to the realisations[29] of its idiosyncratic shock. Therefore the result illustrated in figure 17 is consistent with the assumption that the higher the average size of firms, the less likely they are to face the informational or contractual problems which causes the financing constraint (7). This assumption is realistic as large Italian firms usually have strong links with financial intermediaries and do not face tight borrowing constraints like the one represented by equation (7).

The strong correlation between $ration_i^j$ and $\overline{\Psi}_i^w$ for small and medium firms below 300 employees is confirmed by the probit regression results in figure 17. The last four columns show that the $\overline{\Psi}_i^w$ coefficient is positive and strongly significant, especially for the specification ($j = 4$) that pools together the three different questions. This result shows that ${}_t\hat{\Psi}^w_{i,t+1}$ is a valid indicator of the intensity of financing constraints, supporting the validity of our theoretical model and our empirical approach, and rejecting the view

[29] This stationarity assumption is reasonable in this context, given that the time series is 11 years only.

Figure 17: Relation between stated financing problems and the financing constraints indicator

Probit regression: $ration_i^j = \alpha_0 + \alpha_1 \overline{\Psi}_i^W + \alpha_2 \dim_i$

	Whole sample	Larger firms[1]	Smaller firms[2]			
Dependent variable	*Ration*[4] (all problems)	*Ration*[4] (all problems)	*Ration*[4] (all problems)	*Ration*[1] (Low collateral)	*Ration*[2] (Lack of bank credit)	*Ration*[3] (High cost of debt)
α_0	-0.64***	-0.69	-0.78***	-2.05***	-1.48***	-1.04***
	(0.09)	(0.42)	(0.16)	(0.31)	(0.20)	(0.17)
α_1	0.24**	-0.16	0.30***	0.36*	0.28**	0.29**
	(0.11)	(0.36)	(0.12)	(0.20)	(0.14)	(0.13)
α_2	-0.0006*	-0.0007	0.0005	0.001	0.002*	0.0005
	(0.0003)	(0.0007)	(0.001)	(0.002)	(0.001)	(0.001)
Obs with ration=0	341	70	271	341	310	296
Obs with ration=1	92	11	81	11	42	56
(% of total)	(21.2%)	(13.6%)	(23%)	(3.1%)	(11.9)	(15.9%)
Total obs	433	81	352	352	352	352

Standard error in parenthesis; 1: More than 300 employees; 2: Less than 300 employees; * Significant at 90% confidence level; ** significant at 95% confidence level; *** significant at 99% confidence level; $ration_i$ = 1 if the entrepreneur stated financing constraints, and 0 otherwise; dim_i = dimension in number of employees; Ψ_i^W = premium in the expected productivity of variable capital;

of efficient financial markets[30].

4 Conclusions

In this paper we illustrated a structural model of an entrepreneurial firm subject to both borrowing constraints and irreversibility of fixed capital. The solution of the optimal investment problem shows that not only expected productivity but also financing problems affect investment and saving decisions of the entrepreneurs. In particular the proportion of wealth allocated to risky projects rather than safe assets is negatively affected by future expected financing constraints. This precautionary saving effect is not negligible: consider two entrepreneurs identical in everything except than in their endowment of financial wealth. Our simulations show that the richer one may invest up to 18% more in the risky technology than the poorer one, only because she expects less future expected financing constraints.

More importantly we showed that irreversibility and financing problems are complementary: the irreversibility problem amplifies the effect of financing constraints on variable capital both during upturns and downturns. By simulating an artificial economy with many heterogeneous entrepreneurs, we showed that this amplification effect, which has not been studied yet in the literature, is essential in explaining why aggregate inventory investment is very volatile and procyclical, and why the drop in inventories accounts for a large part of the decline in business spending in recessions. Our theory accounts also for

[30] This result is robust to possible biases induced by measurement errors, for at least three reasons: i) the qualitative and quantitative information come from different sources (see Caggese 2002 for a description of the samples). This reduces the probability that those entrepreneurs that declare financing constraints also manipulate their balance sheets data to show that their investment is inefficiently low; ii) we condition for firms size, thus ruling out the possibility that $\overline{\Psi}_i^w$ is on average higher for small firms, which are also more likely to state financing constraints; iii) the result is not driven by sectorial differences: in Caggese 2002 it is shown that financing constraints are equally distributed in the different industrial sectors.

the observed fact that most of the volatility in aggregate inventories and output is created by small firms, that inventory investment leads the business cycle, and that both fixed and inventory investment are sensitive to the net worth, even when marginal productivity of capital is taken into account.

Although all the simulations in the paper refer to one specific US four digits manufacturing sector, similar results could be obtained for any sector where productive units satisfy the following assumptions: both financing and irreversibility constraints are binding for a non negligible share of firms in equilibrium; firms produce output using a combination of reversible and an irreversible inputs. It is not necessary for the results that the irreversible factor of production is also the source of collateral. Nor it is necessary that variable capital fully depreciates. Moreover our parametrization implies that on average around 45% of the entrepreneurs are financially constrained every period, but the results would hold also for other parametrizations which would generate a lower fraction of financially constrained entrepreneurs. For example, we assume a logaritmic utility function with a unitary coefficient of risk aversion. Using an higher value of the coefficient of risk aversion would imply that entrepreneurs engage in more precautionary saving. This would lower the share of constrained entrepreneurs in equilibrium but would not change the result, because the precautionary saving effect has the same implication of currently binding constraint for the quantitative results derived in the paper.

Finally, the model restricts its attention to entrepreneurial firms. We think that this is not a necessary restriction, as analogous models could be developed to analise similar issues on publicly owned companies. We rather believe that this is one of the strong points of this paper. Hamilton (2000) notes that, as of 1997, business owners constituted approximately 13% of non agricultural employees in the Unites States. This large class of businesses is therefore responsible of both a large share of production and of wealth accumulation in the economy. Hubbard and Gentry (2000) and Heaton and Lucas (2000) study the effect of entrepreneurial risk on aggregate saving and on the portfolio choices of the private sector, while this is the first study to link the issue of entrepreneurial risk to that of investment and output fluctuations of firms.

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Appendix A: proof of proposition 1

We assume that θ_t is a symmetric two state stochastic Markov process:

$$\theta_t \in \{\theta_L, \theta_H\} \;\; with \; \theta_H > \theta_L; \qquad pr(\theta_{t+1} = \theta_t) = \epsilon > 0.5; \qquad pr(\theta_{t+1} \neq \theta_t) = 1 - \epsilon \tag{35}$$

Hence the first order autocorrelation coefficient is $\rho = 2\epsilon - 1 > 0$, and we have that:

$$E_t\left(\theta_{t+1} \mid \theta_t = \theta_H\right) > E_t\left(\theta_{t+1} \mid \theta_t = \theta_L\right) \tag{36}$$

Proof of proposition 1.

We define D_t as the dichotomous variable that represents the retirement choice of E conditional on not dying in period t:

$$D_t = \begin{cases} 1 \text{ if E continues activity} \\ 0 \text{ if E retires} \end{cases}$$

the choice of D_t is subject to the following constraint:

$$\begin{array}{ll} D_t = 0 & if\ y_t + \frac{\tau_k}{R}\left(1-\delta_k\right)k_t < b_t \\ D_t \in \{0,1\} & otherwise \end{array} \tag{37}$$

D_t is constrained to be 0 if the inequality (37) is satisfied[31]. We first prove that assumption *a1* implies that E is never forced to retire. We substitute (7) in (10) obtaining the following:

$$x_t + l_{t+1} + k_{t+1} \leq w_t + \frac{\tau_k}{R} k_{t+1} \tag{38}$$

the left hand side of (38) is constrained downwards by constraints (4) and (5), and by assumption *p1*, which implies that consumption is non-negative. Now let's substitute w_t in (38) using (9) and k_{t+1} and l_{t+1} from constraints (4) and (5) holding with equality. At the limit, for x_t which goes to zero (38) becomes:

$$y_t + \frac{\tau_k}{R}\left(1-\delta_k\right)k_t \geq b_t \tag{39}$$

condition (39) is symmetric to the disequality in (37) and is necessary to ensure that E is not forced to violate the irreversibility constraint to repay the debt. We determine $l^E_{\min}$, the constant level of variable capital supplied by E, as the level of l^E such that (39) is always satisfied for all the possible levels of state variables w_t, k_t and θ_t. The left hand side of (39) is monotonously decreasing in θ_t and l_t, Such that:

$$x_t = w_t - \frac{w_{t+1}}{R} \tag{43}$$

It is then clear that the exit decision is the following:

$$D_t = \begin{cases} 1 \text{ if } \underset{x_t, l_{t+1}, k_{t+1}}{MAX} \ln x_t + \beta E_t (V_{t+1}) > V_t' (w_t) \\ 0 \text{ otherwise} \end{cases} \tag{44}$$

Assumption *a2* simply states that the parameters of the model are chosen such that continuation ($D_t = 1$) is always the optimal choice of a consumption maximising entrepreneur. There is no analytical proof of this result, because there is no analytical solution of the problem. We rather ensure that the numerical solution of the model satisfy this condition for all possible values of the state variables. ■

Appendix B

In section 2.3 we argued that financing constraints do not alter the optimal mix between fixed and variable capital when fixed capital is fully collateralisable ($\tau_k = 1 - \delta_k$). To see this, let's divide (16) by (17), and evaluate such ratio for $\mu_t = E_t \left(\mu_{t+1} \right) = 0.$:

$$\frac{E_t (MPK_{t+1})}{E_t (MPL_{t+1})} = \frac{E_t [U'(x_{t+1})] UK + \frac{1}{\gamma\beta} \left\{ [(R - \tau_k) \lambda_t] - cov_{xk} \right\}}{E_t [U'(x_{t+1})] UL + \frac{1}{\gamma\beta} \left(R\lambda_t - cov_{xl} \right)} \tag{45}$$

Lets consider the two extreme cases: if there are no current and future expected financing constraints then (45) becomes:

$$\frac{E_t (MPK_{t+1})}{E_t (MPL_{t+1})} = \frac{UK}{UL} \tag{46}$$

On the other extreme, if current financing constraints are very strong and λ_t becomes very large, (45) converges to:

$$\lim_{\lambda_t \to \infty} \frac{E_t (MPK_{t+1})}{E_t (MPL_{t+1})} = D_k \tag{47}$$

But when $\tau_k = 1 - \delta_k$ then $D_k = \frac{R-1+\delta_k}{R} = \frac{UK}{UL}$, and the optimal mix between fixed and variable capital is the same.

Appendix C: numerical solution

In order to obtain a numerical solution of the dynamic nonlinear system of equations defined by (4), (12), (13), (14) and (15), we discretise the state space as follows: $k_{i,t}$ and $w_{i,t}$ are discretised in 70 elements each, while θ_t is discretised in 12 elements, which corresponds to the six states of the aggregate shock and the two states of the idiosyncratic shock. First, we formulate an initial guess of the forward variables $E_t \left(\phi_{i,t+1} \right)$, $E_t \left(\phi_{i,t+1} \theta_{i,t+1} \right)$ and $E_t \left(\mu_{i,t+1} \right)$. Second, we solve the static optimisation problem conditional on this guess, for each discrete value of the state variables $w_{i,t}, k_{i,t}$ and $\theta_{i,t}$. This static optimisation problem is nonlinear as well. Conditional on the value of $\theta_{i,t}$ the solution falls in four possible categories, depending on the values of the couple $\{w_{i,t}, k_{i,t}\}$. These categories correspond to the four areas in figure 2. Third, we update the guess of $E_t \left(\phi_{i,t+1} \right)$, $E_t \left(\phi_{i,t+1} \theta_{i,t+1} \right)$ and $E_t \left(\mu_{i,t+1} \right)$. We repeat these steps until the value function converges.