Conditions for optimality in the infinite-horizon portfolio-cum-saving problem with semimartingale investments

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Abstract:

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ABSTRACT

A model of optimal accumulation of capital and portfolio choice over an infinite horizon in continuous time is formulated in which the vector process representing returns to investments is a general semimartingale. Methods of stochastic calculus and calculus of variations are used to obtain necessary and sufficient conditions for optimality involving martingale properties of the 'shadow price' processes associated with alternative portfolio-cum-saving plans. The relationship between such conditions and 'portfolio equations' is investigated. The results are applied to special cases where the returns process has stationary independent increments and the utility function has the 'discounted relative risk aversion' form.

KEY WORDS: Investment, portfolio, martingales, semimartingale calculus, optimisation, economic theory.

1. INTRODUCTION

This paper is concerned with the necessary and sufficient conditions for optimal saving and portfolio choice over an infinite horizon in continuous time by an investor who seeks to maximise the integral of expected utility. Both utility and the return to investment are subject to risk, the sources of risk and the investor's information structure being specified in a very general form; in particular, the vector process representing asset prices – or more generally returns – is only assumed to be a semimartingale (which in general need not be square integrable, or continuous, or even special). Consumption and capital are constrained to be non-negative. Assets are divisible and can be traded at market prices without transaction costs, short sales being permitted in one version of the model if the returns process is continuous but forbidden in the general case.

Conditions for optimality can be cast in various forms. We shall consider primarily a set of conditions which extend those for optimal saving with a single asset given in [F1], where an optimal plan was characterised by (a) a finite-value condition for the welfare functional, (b) a local martingale condition for the 'shadow price' process, defined as the product of the returns process and the marginal utility process evaluated along the optimal plan, and (c) a transversality condition at infinity. When several assets are available, these conditions still apply if the shadow price process is taken to be that determined by the returns to the optimal portfolio plan, but they are supplemented by martingale properties of the shadow prices determined by returns to individual assets. Briefly, the shadow price process of an asset is in all cases a supermartingale, and it is a local martingale if the asset is always held in the optimal portfolio in (strictly) positive amounts or if the market returns process is continuous and short sales are permitted. We also investigate the relationship between martingale conditions, which characterise an optimum in terms of intertemporal comparisons of utility, and 'portfolio equations' which lay down relations to be satisfied by the optimal portfolio vector at each time and state. Abstract portfolio equations are derived, and applied to the special case where the market returns process is a process with stationary independent increments (PSII) and utility has the discounted constant relative risk aversion (CRRA) form.

The contribution of the present work lies partly in the form of the results and the methods by which they are derived, which are motivated by certain ideas about the underlying economic problem, partly in the conditions under which the results are proved. It may be useful to set the stage with some comments on these points.

One of the guiding ideas of economics is the equi-marginal principle, which was stated by W.S. Jevons as follows:

> "....when the person remains satisfied with the distribution he has made, it follows that no alteration would yield him more pleasure; which amounts to saying that an increment of commodity would yield exactly as much utility in one use as in another." [J2] pp.115–116.

The martingale property of shadow prices associated with an optimum – which roughly speaking requires the equalisation across time, in each investment actually undertaken, of the conditional expectation of the marginal utility of consumption adjusted for return – may evidently be regarded as an application of this principle. This suggests that the property in question should owe almost everything to optimality as such, and almost nothing to special technical assumptions. Now, if justice is to be done to this reasoning, the property must be derived in a sufficiently general setting, and I would argue that the kind of model which it is most useful to consider for this purpose is one in which time is treated as continuous and open-ended – an infinite horizon as such is inessential – and which admits a wide class of information structures and returns processes, allowing in particular for discontinuous change. This unfortunately requires a good deal of technique! Nevertheless it seems worthwhile to attempt a mathematical treatment which captures the above ideas as directly as possible. The methods adopted here – comprising mainly martingale theory (including semimartingale calculus) together with a little calculus of variations – appear well suited to the purpose; they are probably as elementary as the problems will allow and seem to yield the best results within the prescribed framework. However the arguments are partly tailor-made and do not extend readily to wider stochastic control problems.

A few comments about the assumptions regarding time, prices and information are in order. An important reason for treating time as continuous is that discrete-time analysis places arbitrary constraints on the timing of both exogenous events and decisions, and it is difficult to know to what extent these constraints are responsible for the results obtained. For example, a variational argument which in a discrete-time model of optimal saving yields a true martingale property for the shadow price process leads in general only to a local martingale property in an analogous continuous-time model, as a comparison between [F2] and [F1] shows. Treating time as open-ended is important because arbitrary terminal valuations affect planning during the whole preceding period. In particular, processes constructed by taking 'backward' conditional expectations of integrable terminal variables will automatically be uniformly integrable martingales, a procedure which may prejudge essential aspects of the investigation. Obviously it is not suggested that closed-horizon models are useless, but such an assumption does appear to be less appropriate in a problem involving optimal accumulation or the pricing of 'fundamental' assets than (say) in the valuation of options with a fixed expiry date. Concerning the modelling of price movements, suffice it to say that, whatever the theoretical (or even empirical) attractions of Brownian motion or symmetric stable processes, the fact remains that prices in practice are always piecewise constant functions of time, and it is nice to have a model which allows for this possibility. The continuing controversy about the choice of statistical models for speculative prices emphasises the need for a flexible approach. It is also desirable to allow for various patterns in the arrival of information: some data come almost in a continuous stream, others as discrete announcements, which may occur at predictable times (e.g. budget speeches) or at totally inaccessible times (e.g. disaster reports); some information is private, some public and accompanied by price changes. It is not usually appropriate to assume a priori that the investor's filtration is precisely that generated by the history of market price movements; to do so is indeed to prejudge the whole question of market efficiency.

These remarks may seem obvious, but taken as a whole they define a programme which has not, to my knowledge, been carried out. It is difficult to give a brief review of the state of the subject because of the diversity of problems, models, assumptions, techniques and the form of results to be found in the literature on continuous-time portfolio problems. Besides, this literature shades on the one hand into work on stochastic growth models and on the other into work on equilibrium in asset markets and the pricing of contingent claims – not to mention parallel work in a discrete-time setting and on related problems in stochastic control. A few classificatory remarks may however be helpful.

Portfolio problems may be "free-standing" or may be imbedded in models of market equilibrium; we do not exclude the latter group from consideration but focus attention on the aspect of individual optimisation. In both cases, it is necessary to distinguish between two types of martingale property which are of economic significance: those of shadow prices associated with an investor's optimal plan, considered with reference to the 'given' probability measure which embodies his opinions, and those of (suitably discounted) market prices or returns under a probability measure equivalent to the given one. The latter type of property is important because market equilibrium (and for that matter individual optimality) requires that there be no possibility of sure profit from arbitrage, and this is known to be equivalent, in a suitable finite-horizon model, to the existence of a 'martingale measure' for prices, i.e. a measure equivalent to the given one such that the vector of prices, divided (say) by the price of a riskless asset, becomes a martingale – see for example [HP]. The relationship between the two types of property can be derived from the fact that the ratio, at each (ω,t), between any two shadow prices is the same as the ratio between the corresponding asset prices. In a closed finite-horizon version of our model, the condition that the vector of shadow prices associated with an optimum is a martingale implies the existence of a martingale measure for asset prices; it is enough to take as the Radon–Nikodym derivative of the measure transformation the shadow price of any one asset, e.g. a riskless one. However this argument will not work in the case of an infinite horizon, because (apart from very special cases) the shadow prices will not be uniformly integrable martingales – see Section 7 below for further discussion.

Papers on continuous-time portfolio theory may conveniently be classified according to the techniques used; we mention only some representative contributions. The first and largest group uses methods of dynamic programming applied to a model of consumption and portfolio choice driven by Brownian motion, mostly in a finite (closed) horizon setting; usually utility is written as the product of a 'felicity' function depending only on current consumption and an exponential discount factor, special attention being paid to the CRRA case, cf. (6.1–2) below. Well-known members of this group include [M1], which also considers a model with a Poisson process, and [CIR], which imbeds the portfolio problem in a diffusion model of market equilibrium. A rigorous development of the former model in the case where the price process is just a Brownian motion appears in [KLSS], which considers the case of an infinite horizon among other possibilities. The model in [CIR] is extended to the case of an infinite horizon, albeit under rather restrictive assumptions, in [N], which also explains some of the difficulties facing such an extension. The main emphasis in these papers is on solving the Bellman equation, though [CIR] also gives pricing formulae for assets which are equivalent to martingale properties of shadow prices as defined here. From our present standpoint, the main limitations of this group of models are the classes of driving processes considered and the additional difficulties encountered in the infinite-horizon case.

A second, very recent, group of papers considers finite-horizon models driven by Brownian motion rather similar to those of the first group, but using martingale methods – see [KLS], [CH1,2], also [K] for further references and a discussion relating the saving-cum-portfolio problem to option pricing and market equilibrium. The use made in these papers of the Girsanov transformation and the representation of martingales as Brownian integrals involves in an essential way the assumptions that the horizon is finite (closed) and that the filtration is that defined by the Brownian fields. The emphasis is on a constructive approach, leading to sufficient conditions for an optimum and equations which can be solved to obtain it; the existence of a martingale measure for prices plays an important part in the argument, but martingale properties of shadow prices do not.

Separate mention should be made of the papers [A1,2], which consider a portfolio model with a log-price process which is the sum of an absolutely continuous term, a Brownian integral and a marked point process taking a finite number of values. Effective use is made of the exponential formula for semimartingales to manipulate the various returns formulae (a method which will be used relentlessly in the present paper) but in other respects the methods are mainly Markovian, in particular dynamic programming is used. There is an interesting discussion of the problem of ruin. The model is limited to a finite horizon and utility depends only on terminal wealth.

A final group of techniques is based on application of a theory of duality. In [P] a combination of the duality theory of [R] with martingale methods is used to derive necessary and sufficient conditions for portfolio optimality in a finite-horizon model where utility depends only on terminal wealth and the price proces is a general semimartingale. Mention should also be made of the related paper [BP] which, although it considers only a one-security model, is of interest here for its interpretation of dual variables as shadow prices of information. Another paper in this group is [BM], which applies the duality theory of [B1]; although mainly concerned with long-run dynamics, this paper gives a sufficiency theorem applicable to the infinite-horizon portfolio problem where the driving proces is the sum of an absolutely continuous process and a Brownian integral. The use of a stochastic duality theory such as [B1,2] or [B3] is indeed an obvious alternative to the present direct approach to the study of shadow prices. This method would have the advantage of drawing on a well-developed and versatile theory. On the other hand, the theory requires extensive technical preliminaries, in the course of which contact with the basic ideas is easily lost; also, for convenient application it is desirable to make assumptions which are not particularly appropriate to the economic problem (e.g. L_2 bounds for martingales and control sets, possibly compact control sets, finite horizon) and which our approach, although in other respects of much more limited generality, does not require.

* * *

The rest of the paper is arranged as follows. Section 2 sets out various definitions, details of the model and some preliminary results. Section 3 deals with martingale conditions for optimality in various forms and under various assumptions. The main results appear in Theorem 1, followed by a number of Propositions which are needed for its proof but which are also of some independent interest. The argument up to this point depends largely on the change–of–variables formula for semimartingales, in particular the exponential formula, and on the multiplicative and additive decompositions of supermartingales. The necessity part of Theorem 1 depends on Theorem 2, which asserts that all shadow prices defined by an optimal plan are supermartingales; this is proved separately in Section 4 by methods which combine basic properties of martingales with elements of the classical calculus of variations. The relationship between martingale properties of shadow prices and portfolio equations is derived in an abstract, general form in Section 5. It appears that portfolio equations can be regarded as defining the compensators (dual predictable projections) of certain processes derived from a logarithmic form of the shadow prices, but the economic interpretation of this relationship is still rather obscure. Section 6 derives the explicit form of the portfolio equations in case the market returns process is a PSII (satisfying certain conditions) and utility has the discounted CRRA form, showing that the abstract equations are indeed a generalisation of known results. Finally Section 7 takes up the connection, in the present model, between martingale properties of shadow prices and martingale measures for asset prices.

2. THE MODEL

The model considered here is essentially an extension of that in [F1], and changes in notation and assumptions have been kept to a minimum to facilitate cross-reference.

Let $\underline{I} = [0,\infty)$, equipped with its Borel sets and Lebesgue measure, be the time domain, and let $(\Omega,\underline{\underline{A}},P)$ be a complete probability space with a filtration $\underline{A} = (\underline{A}_t;\ t\in\underline{I})$ satisfying the usual conditions, where $\underline{\underline{A}} = \underline{A}_\infty$ while $\underline{A}_0 = \underline{A}_{0-}$ is generated by the P-null sets. $\underline{A}$ represents the investor's <u>information structure</u> and P his <u>beliefs</u>. In the product space $\Omega \times \underline{I}$ we define in the usual way the σ-algebras of progressive, optional and predictable sets, as well as the corresponding classes of processes. The following conventions apply to processes unless we state or imply otherwise. They will be defined for all $\omega\in\Omega$ and $t\in\underline{I}$. Properties involving measurability or integrability will refer to $(\underline{A},P)$. Processes will be assumed, or may easily be shown to be, at least progressively measurable, and processes of a given class which differ only on null progressive sets will be identified. Scalar processes will take finite, or occasionally extended, real values, and vector processes will be just finite families of scalar processes. For a scalar process ξ, $\xi > 0$ means $\xi(\omega,t) > 0$ for all (ω,t) and $\xi \geqslant 0$ means $\xi(\omega,t) \geqslant 0$ for all (ω,t), modulo null sets, while similar notation for a vector process means that the condition applies to each component. All martingales (true, sub, super, local) will by definition be corlol (continuous on the right with limits on the left). The qualifier 'local' will be omitted as being part of the definition in the case of (local) semimartingales and processes (locally) of finite variation; but we shall distinguish explicitly between local martingales and (true) martingales, and similarly for supermartingales and processes of integrable variation. Processes will not jump at $t = 0$, i.e. we set $\xi_{0-} = \xi_0$, so that for stochastic integrals we have $\int_{[0,T]} = \int_{(0,T]}$, which we usually write as $\int_0^T$; the formula for integration by parts is adjusted accordingly. As regards notation, we shall chop and change between $\xi(t)$ and ξ_t as convenience dictates, and sometimes even omit the time variable. Note also that the terms positive, negative, increasing, decreasing have their strict meaning throughout, but $\uparrow$, $\downarrow$ mean non-decreasing, non-increasing.

Let a finite number of assets (also called securities) indexed by $\lambda = 1,\ldots,\Lambda$ be available at all times. For each λ there is given a semimartingale x^λ with $x^\lambda(\omega,0) = 0$ called the log-returns or compound interest process for λ; note that by definition the values $x^\lambda(\omega,t)$ and $x^\lambda(\omega,t-)$ are finite on $\underline{T}$ for each ω. The formula $z^\lambda = \exp\{x^\lambda\}$ defines a positive semimartingale with $z^\lambda(0) = 1$, called the returns or price process for λ; $z^\lambda(\omega,t)$ represents the value at t in state ω of one unit of capital invested at zero time in asset λ (with instantaneous reinvestment of dividends etc. in the same asset), the return being measured in suitable natural units such as corn or money adjusted for changes in the price level of consumption goods. We write X, Z for the corresponding vectors, called the market log-returns and returns processes; thus $X = (x^1,\ldots,x^\Lambda)$ denotes a vector process, X(t) or X_t a vector random variable etc. Decompositions of x^λ are written variously as

$$x^\lambda = M^\lambda + V^\lambda = M^{\lambda c} + M^{\lambda d} + V^{\lambda c} + V^{\lambda d} \qquad \ldots(2.1)$$

where M^λ is a local martingale, V^λ a process (locally) of finite variation, $M^{\lambda c} + M^{\lambda d}$ is a decomposition of M^λ into continuous and compensated jump local martingales, and $V^{\lambda c} + V^{\lambda d}$ is a decomposition of V^λ into continuous and discontinuous processes of finite variation; all these processes vanish at $t = 0$, and in general only $M^{\lambda c}$ is uniquely defined.

A portfolio plan π will be a finite vector process with components π^λ, $\lambda = 1,\ldots,\Lambda$, which is defined for $t > 0$, adapted and left continuous with finite right limits (hence predictable) and which satisfies

$$\sum_\lambda \pi^\lambda(\omega,t) = 1 \qquad \ldots(2.2)$$

for all (ω,t). The left continuity expresses the fact that any jump $\Delta x(t)$ accrues to the portfolio $\pi(t) = \pi(t-)$ held immediately before t, not to the portfolio $\pi(t+)$ chosen at t, except that at t = 0 we arbitrarily set $\pi(0) = \pi(0+)$. We denote by Π^o the set of all such π, and by Π^+ the subset satisfying $\pi \geqslant 0$, or explicitly

$$0 \leqslant \pi^\lambda(\omega,t) \leqslant 1 \qquad \ldots(2.3)$$

for all (ω,t) and each λ; the restriction $\pi \geqslant 0$ means that short sales are forbidden. The set of all π which are admissible in a particular problem is denoted by Π, and (for reasons which will be clarified below) we assume that Π is Π^o or Π^+ if X is continuous but $\Pi = \Pi^+$ if X has jumps; sometimes we write simply Π when it is not necessary to specify which case is considered.

Before proceeding, it is useful to recall some facts about the Doléans integral equation

$$z(T) = 1 + \int_o^T z(t-)d\zeta(t) \qquad \ldots(2.4)$$

defined in [Dd]. Given a semimartingale ζ such that $\zeta(0) = 0$; there exists one and only one semimartingale z satisfying this equation for all t, a.s. It is given by

$$z_T = \exp(\zeta_t - \tfrac{1}{2} < \zeta^c >_T) \prod_{t \leqslant T} (1 + \Delta\zeta_t)e^{-\Delta\zeta(t)} \qquad \ldots(2.5)$$

where the product term converges absolutely for all T, a.s., and defines a process of finite variation, see [M2]IV.25; here we have written $< \zeta^c >$ instead of $< \zeta^c, \zeta^c >$ to denote the angle brackets process of the continuous martingale part of ζ. The process z is called the (martingale) exponential of ζ – we shall say mart–exp for short and write $z = \&(\zeta)$. If ζ is a local martingale, so is z. Conversely, if z is a given semimartingale with $z(0) = 1$ such that $z(t) > 0$ and $z(t-) > 0$ always, then the process $1/z(t-)$ is locally bounded and one can define the stochastic integral

$$\zeta(T) = \int_o^T \frac{dz(t)}{z(t-)} \qquad \ldots(2.6)$$

which is the unique semimartingale satisfying (4), see [Dd]p.186, also [J1]Ch.VI, esp. Exs. 6.1–6.2; we shall call ζ the mart–log of z and write $\zeta = \&^{-1}(z) = \pounds(z)$. From

(6) it is clear that, if z is a local martingale, then so is ζ.

Now, given a portfolio plan $\pi \epsilon \Pi$, the portfolio returns process z^π is defined as the unique semimartingale satisfying the equation

$$z^\pi(T) = 1 + \int_o^T z^\pi(t-) \sum_\lambda \pi^\lambda(t-) \frac{dz^\lambda(t)}{z^\lambda(t-)} \qquad \text{....(2.7)}$$

for all $T \epsilon \underline{I}$, a.s. An intuitive interpretation is given after eq.(10) below. To justify the definition formally, consider first the semimartingales ζ^λ, ζ^π defined by

$$\zeta^\lambda(T) = \int_o^T \frac{dz^\lambda(t)}{z^\lambda(t-)}, \qquad \zeta^\pi(T) = \int_o^T \sum_\lambda \pi^\lambda(t) d\zeta^\lambda(t), \qquad \text{....(2.8)}$$

noting that ζ^λ is well defined as a stochastic integral and equal to $\pounds(z^\lambda)$ because the assumptions about x^λ imply that $z^\lambda(t)$ and $z^\lambda(t-)$ are positive; then the integral ζ^π is also well defined because $\pi^\lambda(t)$ is left continuous with right limits, hence locally bounded – see [DM]VIII.8 – so that $\pi^\lambda(t)/z^\lambda(t-)$ has the same properties. Now (7) just says $z^\pi = \&(\zeta^\pi)$.

The definition of Π adopted above is intended to ensure that only portfolio plans π are admitted for which $z^\pi(t)$ is always positive; this may be regarded as a necessary condition for the investor's solvency. Referring to (5), it is seen that $z^\pi(t) > 0$ for all t iff $1 + \Delta\zeta^\pi(t) > 0$ for all t, and then $z^\pi(t-) > 0$ also. By the definitions of π and z^λ we have

$$z^\pi_t/z^\pi_{t-} = 1 + \Delta\zeta^\pi_t = 1 + \sum_\lambda \pi^\lambda_t \Delta z^\lambda_t/z^\lambda_{t-} = \sum_\lambda \pi^\lambda_t z^\lambda_t/z^\lambda_{t-}$$

$$= \sum_\lambda \pi^\lambda_t e^{\Delta x^\lambda(t)}, \qquad \text{....(2.9)}$$

so that it would actually be enough to choose as the admissible set the subset Π^1 of Π^o for which the expression (9) is positive for all t, a.s. This condition is obviously satisfied if X is continuous, and then $\Pi^1 = \Pi^o$. At the other extreme, if each x^λ can have arbitrarily large upward jumps at any time - more precisely, if a.s. $\sup \Delta x_t^{\lambda} = \infty$ for each t and λ - then $\Pi^1 = \Pi^+$ and the solvency condition rules out short sales. Many intermediate cases are possible, but in general it is awkward to work with Π^1, and it must be borne in mind that in practice short sales may be forbidden even if they could be undertaken without any risk of involvency; hence the definition of admissible portfolio plans which we have adopted. Incidentally, it may seem puzzling that the requirement of solvency apparently imposes no restrictions on short sales when X is continuous; the clue is that a restriction is built into the definition of π by virtue of the assumption that $\pi(t) = \pi(t-)$ is always finite. (To make the point informally, let q^λ denote the number of λ-shares held, write $z = \sum q^\lambda z^\lambda$, $\pi^\lambda = q^\lambda z^\lambda/z$, and consider what happens to π^λ if $z(t) \to 0$ as $t \uparrow T$; cf. [A2]p.215.)

Having ensured that $z^\pi > 0$, $z_-^\pi > 0$, we can define a <u>portfolio log-returns</u> or <u>compound interest</u> process x^π by $z^\pi = \exp\{x^\pi\}$. Also, $z^\pi = \&(\zeta^\pi)$ now implies $\zeta^\pi = \pounds(z^\pi)$, i.e. (7) can be written as $\zeta^\pi = \int dz^\pi/z_-^\pi$, so that (7-8) can be combined in a more symmetrical form as

$$\zeta^\pi(T) = \int_o^T \frac{dz^\pi(t)}{z^\pi(t-)} = \int_o^T \sum_\lambda \pi^\lambda(t)\, \frac{dz^\lambda(t)}{z^\lambda(t-)} = \int_o^T \sum_\lambda \pi^\lambda(t)\, d\zeta^\lambda(t). \qquad \ldots (2.10)$$

This says essentially that the instantaneous rate of proportional increase in portfolio value is the portfolio-weighted average of the rates for the individual assets. Note further that, if the 'single-asset portfolio λ' is defined by

$$\pi^\lambda(\omega,t) = 1, \quad \pi^\ell(\omega,t) = 0 \quad \text{for } \ell \neq \lambda, \quad \text{all } (\omega,t), \qquad \ldots (2.11)$$

the processess ζ^π, z^π, x^π may be identified with ζ^λ, z^λ, x^λ; in future we shall replace the superscript π by λ in appropriate cases without special comment.

For later reference we derive formulae expressing x^π explicitly in terms of the x^λ.

Write $z = e^x$ in eqs. (7–10). For a single-asset portfolio, the change-of-variables formula yields

$$\zeta_T^\lambda = \int_o^T \frac{dz^\lambda(t)}{z^\lambda(t-)}$$

$$= x_T^\lambda + \tfrac{1}{2} < x^{\lambda c} >_T + \sum_{t \leq T} [e^{\Delta x^\lambda(t)} - 1 - \Delta x_t^\lambda] \qquad \ldots (2.12)$$

where the sum converges absolutely for all T, a.s. Consequently, using (10),

$$\zeta_T^\pi = \int_o^T \sum_\lambda \pi_t^\lambda \, dx_t^\lambda + \tfrac{1}{2} \int_o^T \sum_\lambda \pi_t^\lambda \, d < x^{\lambda c} >_t$$

$$+ \sum_{t \leq T} \sum_\lambda \pi_t^\lambda [e^{\Delta x^\lambda(t)} - 1 - \Delta x_t^\lambda], \qquad \ldots (2.13)$$

$$< \zeta^{\pi c}, \zeta^{\pi c} >_T = \int_o^T \sum_\lambda \sum_\ell \pi_t^\lambda \pi_t^\ell \, d < x^{\lambda c}, x^{\ell c} >_t . \qquad \ldots (2.14)$$

Now consider (5) with $z = z^\pi$, $\zeta = \zeta^\pi$, substitute for ζ, $< \zeta^c >$, $\Delta\zeta$ from (13), (14), (9), calculate $x^\pi = \ln(z^\pi)$ and rearrange to obtain

$$x_T^\pi = \int_o^T \sum_\lambda \pi_t^\lambda \, dx_t^\lambda + \tfrac{1}{2} \int_o^T \sum_\lambda \pi_t^\lambda \, d < x^{\lambda c} >_t$$

$$- \tfrac{1}{2} \int_o^T \sum_\lambda \sum_\ell \pi_t^\lambda \pi_t^\ell \, d < x^{\lambda c}, x^{\ell c} >_t$$

$$+ \sum_{t \leq T} [\Delta x_t^\pi - \sum_\lambda \pi_t^\lambda \Delta x_t^\lambda] \qquad \ldots (2.15)$$

where

$$\Delta x_t^\pi = \ln [\sum_\lambda \pi_t^\lambda \exp(\Delta x_t^\lambda)], \qquad \ldots (2.16)$$

the sum in the last line of (15) converging absolutely for all T, a.s. Using the decomposition (1), this can be written more explicitly as follows:

$$x^{\pi} = \int \sum_{\lambda} \pi^{\lambda} \, dM^{\lambda c} + \tfrac{1}{2} \int \sum_{\lambda} \pi^{\lambda} \, d < M^{\lambda c} > - \tfrac{1}{2} \int \sum_{\lambda} \sum_{\ell} \pi^{\lambda} \pi^{\ell} \, d < M^{\lambda c}, M^{\ell c} >$$

$$+ \int \sum_{\lambda} \pi^{\lambda} \, dV^{\lambda c}$$

$$+ \int \sum_{\lambda} \pi^{\lambda} \, dM^{\lambda d} + \sum_{t \leq I} [\Delta x^{\pi} - \sum_{\lambda} \pi^{\lambda} \Delta M^{\lambda}] . \qquad \ldots (2.17)$$

In each of the formulae (13), (15), (17), the terms in the last line vanish when X is continuous.

Suppose now that the investor has an <u>initial capital</u> $K_o > 0$ and no outside income. Given a portfolio plan π with returns process $z = z^{\pi} > 0$, we shall say that a (progressive) process $\bar{c}$ is a π–feasible <u>consumption plan in natural units</u>, or simply $\bar{c}$–plan, if it is non–negative and a.s. locally integrable (i.e. Lebesgue integrable on finite intervals) and if the equation

$$\bar{k}(I) - K_o = \int_o^I \bar{k}(t-) \frac{dz(t)}{z(t-)} - \int_o^I \bar{c}(t) \, dt \qquad \ldots (2.18)$$

is solved by one and only one semimartingale $\bar{k}$ and this solution is a.s. non–negative on $\underline{I}$; then $\bar{k}$ is called the <u>capital plan in natural units</u> corresponding to $\bar{c}$. It follows from [J1]p.193 that a semimartingale solution of (18) exists and is unique. In fact, it is given explicitly by

$$\bar{k}(I) = K_o \, z(I) - z(I) \int_o^I [\bar{c}(t)/z(t)] dt . \qquad \ldots (2.19)$$

To check this, apply the formula for integration by parts [DM]VIII(19.2) to obtain, using (19),

$$z_T \int_0^T (\bar{c}_t/z_t)dt = \int_0^T \left\{ \int_0^{t-} (\bar{c}_s/z_s)ds \right\} dz_t + \int_0^T z_t(\bar{c}_t/z_t)dt$$

$$= \int_0^T \left\{ K_o - \bar{k}_{t-}/z_{t-} \right\} dz_t + \int_0^T \bar{c}_t \; dt$$

$$= K_o(z_T - 1) - \int_0^T (\bar{k}_{t-}/z_{t-})dz_t + \int_0^T \bar{c}_t \; dt; \qquad \ldots (2.20)$$

it remains to substitute into (19) to obtain (18). Note that (19) shows that $\bar{k}/z$ is absolutely continuous and non-increasing.

To make the economic meaning of (18) quite clear, suppose that $\bar{k}(T) > 0$ for all T, a.s., so that by (19) we also have $\bar{k}(T-) > 0$; then the equation can be rewritten as

$$\int_0^T \frac{d\bar{k}(t)}{\bar{k}(t-)} = \int_0^T \frac{dz(t)}{z(t-)} - \int_0^T \frac{\bar{c}(t)}{\bar{k}(t-)} \, dt, \qquad \bar{k}(0) = K_o, \qquad \ldots (2.21)$$

expressing the fact that the rate of growth of capital equals the rate of growth of portfolio value minus the rate of consumption out of capital. A little more generally, (18) can always be replaced by (19) on a random interval $[0,\bar{\nu})$, where $\bar{\nu}$ is the first arrival time of $\bar{k}_t/z_t$ at zero (with $\bar{\nu} = \infty$ if this level is never reached). Thus a $\bar{c}$-plan can also be defined as a non-negative, locally integrable process such that (21) has a unique semimartingale solution $\bar{k}$ which is positive on some random interval $[0,\bar{\nu})$ and zero on $[\bar{\nu},\infty)$.

The set of all $\bar{c}$-plans which are π-feasible is denoted $\bar{\underline{C}}^{\pi}$ and the set of all $\bar{c}$-plans which are feasible for some $\pi \epsilon \Pi$ is $\bar{\underline{C}}(\Pi)$. A (feasible) <u>portfolio-cum-consumption plan</u> - or simply a <u>plan</u> - in natural units is a pair $(\bar{c},\pi)$ such that $\bar{c} \,\epsilon\, \bar{\underline{C}}^{\pi}$ and $\pi \epsilon \Pi$.

We assume that the investor's aim is to maximise a <u>welfare functional</u> of the form

$$\bar{\varphi}(\bar{c}) = E \int_0^{\infty} \bar{u} \, [\bar{c}(\omega,t); \, \omega,t] \, dt. \qquad \ldots(2.22)$$

The utility function $\bar{u} = \bar{u}(C; \omega,t)$ is defined for $0 \leq C \leq \infty$, $\omega \in \Omega$, $t \in \underline{I}$, and takes values in $[-\infty,\infty]$. Considered as a function of all its variables, it is $\underline{\underline{B}}_{[0,\infty]} \times \underline{\underline{H}}$ measurable (where $\underline{\underline{B}}$ denotes the Borel sets and $\underline{\underline{H}}$ the progressive sets). For fixed (ω,t), $\bar{u}$ is continuous, concave and increasing in C. The marginal utility function $\bar{u}'(C; \omega,t)$, $\bar{u}' = \partial\bar{u}/\partial C$, is defined on the same domain as $\bar{u}$ and takes values in $[0,\infty]$, and for each (ω,t) is continuous and non-increasing in C with $0 < \bar{u}'(C) < \infty$ for $0 < C < \infty$ and $\bar{u}'(0) = \infty$; thus $\bar{u}'$ also is $\underline{\underline{B}}_{[0,\infty]} \times \underline{\underline{H}}$ measurable. The continuity of $\bar{u}$, $\bar{u}'$ at $C = 0$ and $C = \infty$ is, of course, one-sided.

It follows easily from a standard Measurability Lemma for processes, [L]p.503, that for $\bar{c} \in \underline{\bar{C}}(\Pi)$ the utility and marginal utility plans defined by $\bar{u}[\bar{c}(.,.); .,.]$ and $\bar{u}'[\bar{c}(.,.); .,.]$ are $\underline{\underline{H}}$-measurable. The domain of the functional $\bar{\varphi}$ is taken to be $\underline{\bar{C}}(\Pi)$; it is always assumed (or inferred from other assumptions) that for each $\bar{c}$ in this set the positive part of the double integral in (22) is finite, and further that the supremum φ^* of the functional is finite. The portfolio-cum-saving problem is to maximise $\bar{\varphi}$ on $\underline{\bar{C}}(\Pi)$, if possible. A plan $(\bar{c}^*,\pi^*)$ is called optimal if $\bar{\varphi}(\bar{c}^*) = \varphi^*$ (and φ^* is finite).

Let us for the moment fix π, write $x = x^{\pi}$, $z = z^{\pi}$, and consider the problem of optimal saving for that π, i.e. the problem of maximising $\bar{\varphi}$ on $\underline{\bar{C}}^{\pi}$. Given an element $\bar{c}$ and corresponding $\bar{k}$, we introduce new processes c, k by the definition

$$c(\omega,t) = \bar{c}(\omega,t)/z(\omega,t), \qquad k(\omega,t) = \bar{k}(\omega,t)/z(\omega,t). \qquad \ldots(2.23)$$

The solution (19) of (18) then reduces to

$$k(T) = K_o - \int_0^T c(t) \, dt \qquad \ldots(2.24)$$

so that k is absolutely continuous (even if $\bar{k}$ jumps) and its sample derivative has a progressive version defined by $\dot{k}(t) = -c(t)$. Further, the requirement that $\bar{k}(T) \geq 0$ on $\underline{I}$ a.s. is clearly equivalent to

$$\int_0^\infty c(\omega,t)\ dt \leq K_o, \quad \text{a.s.} \qquad \ldots(2.25)$$

We call c and k the <u>consumption and capital plans in π-standardised units</u> – or simply the c and k plans – corresponding to $\bar{c}$ and $\bar{k}$. It is clear that a c-plan can be defined directly as a process $c = c(\omega,t) \geq 0$ which satisfies (25); this definition does not involve k, which can be defined by (24) if desired. We denote by $\underline{C}$ the set of all c-plans; an advantage of working with $\underline{C}$ as the feasible set is that it does not depend on the choice of π. Given any π, each $c \in \underline{C}$ defines an element $\bar{c} = cz^\pi \in \bar{\underline{C}}^\pi$ and every $\bar{c} \in \bar{\underline{C}}(\Pi)$ can be obtained in this way from some c and π. Thus a <u>plan</u> can be specified either as a pair $(\bar{c},\pi)$ or as pair (c,π); in the latter case, the set of all plans is simply $\underline{C} \times \Pi$.

Still keeping π fixed, we may next define a <u>π-standardised utility function</u> u^π by setting

$$u^\pi(C;\ \omega,t) = \bar{u}[Cz^\pi(\omega,t);\ \omega,t] \qquad \ldots(2.26)$$

for all C, ω, t for which $\bar{u}$ is defined, and then a functional

$$\varphi^\pi(c) = E\int_0^\infty u^\pi[c(\omega,t);\ \omega,t]\ dt \qquad \ldots(2.27)$$

whose domain is $\underline{C}$. The analytical properties of u^π, $u'^\pi = \partial u^\pi/\partial C$, φ^π, $u^\pi(c)$, $u'^\pi(c)$ are the same as those of $\bar{u}$, $\bar{u}'$ etc. stated above and need not be repeated. Note that

$$u'^\pi(C,t) = \partial u^\pi(C,t)/\partial C = \bar{u}'(Cz_t^\pi,t)z_t^\pi,$$

$$\therefore\ u'^\pi(c_t,t) = \bar{u}'(\bar{c}_t,t)z_t^\pi \quad \text{if} \quad c_t = \bar{c}_t/z_t^\pi. \qquad \ldots(2.28)$$

The problem of maximising $\bar{\varphi}$ on $\bar{\underline{C}}^{\pi}$ is clearly equivalent to that of maximising φ^{π} on $\underline{C}$, which in turn is essentially the same as the problem of optimal saving with a single asset studied in [F1]. We say that c^* is $\underline{\pi\text{-optimal}}$, or equivalently that $\bar{c}^* = c^* z^{\pi}$ is π-optimal, if φ^{π} attains its supremum on $\underline{C}$ at c^* and $\varphi^{\pi}(c^*)$ is finite. The necessary and sufficient conditions for optimality established in [F1] apply. Before recalling them, we state some further definitions.

Let $(\bar{c}^*,\pi^*)$, or equivalently (c^*,π^*), be a distinguished plan with $\bar{c}^* > 0$ everywhere. When dealing with this 'star' plan we write x^*, z^*, $\bar{\underline{C}}^*$, u^*, u'^* etc. in place of x^{π}, z^{π}, $\bar{\underline{C}}^{\pi}$, u^{π}, u'^{π} etc. For brevity we also write

$$v_t = v(\omega,t) = \bar{u}'[\bar{c}^*(\omega,t);\ \omega,t]. \qquad \text{....(2.29)}$$

Now, for arbitrary $\pi\epsilon\Pi$, we define a process y^{π}, called the $\underline{\text{shadow price process}}$ associated with π, by

$$y^{\pi}(\omega,t) = v(\omega,t)z^{\pi}(\omega,t). \qquad \text{....(2.30)}$$

In particular, $y^{\lambda} = vz^{\lambda}$ defines the shadow price process for asset λ. If $\pi = \pi^*$, we write

$$y^*(t) = v(t)z^*(t) = v(t)\cdot\exp\{x^*(t)\}. \qquad \text{....(2.30a)}$$

Thus, for arbitrary π,

$$y^{\pi}(t) = y^*(t)z^{\pi}(t)/z^*(t) = y^*(t)\cdot\exp\{x^{\pi}(t) - x^*(t)\}. \qquad \text{....(2.31)}$$

Note that in these definitions the marginal utility process v is always evaluated along the star plan. Note also that $0 < \bar{c}^* < \infty$ implies $\infty > v > 0$, and then $0 < z^{\pi}$, $z^* < \infty$ implies $0 < y^{\pi}$, $y^* < \infty$. Usually we write $y^*(0)$ as y_0; we have

$y^{\pi}(0) = y_o$ for every π and $v(0) = y_o$.

The following conditions for an element $c^* \in \underline{C}$ to be π^*-optimal restate [F1]I.5-6 in an alternative form:

Proposition 1: Conditions for π^*-Optimality. Let $(c^*,\pi^*) \in \underline{C} \times \Pi$ be given and let $\bar{c}^* = c^* z^*$. Suppose that

(i) $\bar{c}^* > 0$,

(ii) there is a number $\alpha_o \in (0,1]$ such that

$$\bar{\varphi}(\bar{c}^* - \alpha\bar{c}^*) > -\infty \quad \text{for} \quad 0 < \alpha < \alpha_o. \qquad \ldots(2.32)$$

Then c^* (or equivalently $\bar{c}^*$) is π^*-optimal iff

(a) the process $y^* = (y_t^*;\ t\in\underline{I})$ defined by (2.30a) is an $\underline{A}$-local martingale, ...(2.33)

and

$$\text{(b)} \quad E\int_o^\infty y^*(t)\ c^*(t)dt = K_o\ y^*(0). \qquad \ldots(2.34)$$

If moreover $k^*(I) = K_o - \int_o^I c^*(t)dt$ is a.s. bounded away from zero for each $I\in\underline{I}$, then (a) may be replaced by

(a') y^* is a (true) $\underline{A}$-martingale. ....(2.35)

Remarks. (i) The conditions are given in the above form for brevity, but they can be further refined, in particular by specifying the times which reduce y^*; some additional details are mentioned in Section 4 below.

(ii) The fact that $\bar{c}^* > 0$ if $\bar{c}^*$ is π^*-optimal is actually a consequence of the assumption $\bar{u}'(0) = \infty$. On the other hand, conditions (a) and (b) together with

$\bar{\varphi}(\bar{c}^*) > -\infty$ are sufficient for optimality without assuming $\bar{u}'(0) = \infty$, $\bar{c}^* > 0$, or (32). Qualifications of this kind also apply to Theorem 1 below but are omitted for brevity.

(iii) In the models usually considered, (32) holds for all plans, not just optima.

(iv) Examples show that typically y^* is not uniformly integrable even when (35) holds – cf. [F1]S.1(E) and below, S.6.

3. MARTINGALE PROPERTIES OF SHADOW PRICES

In this Section, (c^*,π^*) – or equivalently $(\tilde{c}^*,\pi^*)$ with $\tilde{c}^* = c^*z^*$, $z^* = \exp\{x^*\}$ – refers to the distinguished 'star' plan introduced above, and v, y^*, y^π are defined with reference to this plan as in (2.29–31). Any optimality properties assumed for the star plan are stated explicitly. It is always assumed that $(c^*,\pi^*) \in \underline{C} \times \Pi$, but the definition of Π varies. Martingale properties refer to $\underline{A}$. The following is our fundamental result for the saving–cum–portfolio problem:

<u>Theorem 1: Martingale Conditions for Optimality</u>

Let Π be Π^o or Π^+ if X is continuous, Π^+ if X has jumps, let (c^*,π^*) be a given plan satisfying $c^* > 0$ and (2.32), and consider the following conditions:

(i) y^λ is a supermartingale for $\lambda = 1,\ldots,\Lambda$;

(ii) y^λ is a local martingale for $\lambda = 1,\ldots,\Lambda$;

(iii) y^π is a local martingale for all $\pi \in \Pi$;

(iv) y^π is a supermartingale for all $\pi \in \Pi$.

<u>Sufficiency</u>: (c^*,π^*) is optimal if c^* is π^*-optimal (i.e. the conditions of Proposition 1 are satisfied) and one of the following holds:

S.(a) Condition (iv);

S.(b) $\pi^* > 0$ and (i) or (ii) or (iii);

S.(c) $\Pi = \Pi^o$ (X continuous) and (ii) or (iii);

S.(d) $\Pi = \Pi^+$, $\pi^* \geq 0$ and (i).

<u>Necessity</u>: If (c^*,π^*) is optimal, the conditions of Proposition 1 are satisfied and

N.(a) Condition (iv) holds;

N.(b) If $\pi^* > 0$, then (i), (ii) and (iii) hold;

N.(c) If $\Pi = \Pi^o$ (X continuous), then (ii) and (iii) hold;

N.(d) If $\Pi = \Pi^+$, $\pi^* \geqslant 0$, then (i) holds in the following, more precise form: each y^λ is a supermartingale with a canonical multiplicative decomposition $y^\lambda = L^\lambda D^\lambda$ – see Remark (i) – satisfying

$$\int_0^I \pi^{*\lambda}(t) \frac{dD^\lambda(t)}{D^\lambda(t-)} = 0 \qquad I \in \underline{\underline{I}}, \text{ a.s.} \qquad \ldots (3.1)$$

Remarks. (i) In Theorem 1, 'supermartingale' may be replaced by 'local supermartingale'. A local supermartingale η satisfying $\eta(0) = 0$ has a unique additive decomposition $\eta = \mu + \partial$ with $\mu(0) = \partial(0) = 0$, where μ is a local martingale and ∂ is non-increasing and predictable (hence also non-positive and locally of integrable variation); conversely a process having such a decomposition is a local supermartingale, [J1]2.13–19, [RW]VI.32. A (strictly) positive local supermartingale has a unique multiplicative decomposition $y = LD$, $L(0) = y(0)$, $D(0) = 1$, where L is a positive local martingale and D is positive, non-increasing and predictable (hence also locally of integrable variation), [J1]pp. 199–201, [M2]p. 317. These decompositions are sometimes called 'canonical', but since they are the only ones considered here we shall omit the adjective.

(ii) The significance of (3.1) is that it provides a 'Kuhn–Tucker' condition for a constrained solution, as may be seen by writing informally

$$\pi^{*\lambda} \geqslant 0, \quad dD^\lambda/D^\lambda_- \leqslant 0, \quad \pi^{*\lambda} dD^\lambda/D^\lambda_- = 0. \qquad \ldots (3.2)$$

Note that, if $\pi^{*\lambda}(t) > 0$ on $\underline{I}$ a.s. for some λ, then condition N.(d) implies that y^λ is a local martingale. The case of an optimum with $\pi^* > 0$, i.e. $0 < \pi^{*\lambda}(t) < 1$ on $\underline{I}$ a.s. for each λ, is of special interest, both because the martingale conditions are simpler and more interesting and because this case arises when all investors hold identical portfolios.

The proof of Theorem 1 will be broken up into several Propositions, some of which are of independent interest. They will yield first the sufficiency assertions, and then – when it has been shown in Section 4 that optimality implies that every y^π is a supermartingale – the necessity assertions also. The following Proposition is useful as a verification theorem for proposed solutions:

Proposition 2. If c^* is π^*-optimal and y^π is a local martingale for every $\pi \in \Pi$, then (c^*, π^*) is optimal.

Proof. Let (c, π) be another plan with $\bar{c} = cz^\pi$ and $k = K_0 - \int c\,dt$ - see (2.23–24) - assume to avoid trivialities that $\bar{\varphi}(\bar{c})$ is finite, and let (χ_n) be a sequence of stopping times which reduce y^* and y^π. The following calculations yield the result:

$\bar{\varphi}(\bar{c}) - \bar{\varphi}(\bar{c}^*)$

$= E\int_0^\infty [\bar{u}(\bar{c}_t) - \bar{u}(\bar{c}_t^*)]\,dt$ — by definition

$\leq E\int_0^\infty (\bar{c}_t - \bar{c}_t^*)\,\bar{u}'(\bar{c}_t^*)\,dt$ — by concavity; (the preceding line provides an integrable lower bound for the integrand)

$= E\int_0^\infty (c_t y_t^\pi - c_t^* y_t^*)\,dt$ — changing to standardised units and using the definitions of y^* and y^π, see (2.29-30)

$= -K_0 y_0 + E\int_0^\infty c_t\, y_t^\pi\,dt$ — by (2.34), since c^* is π^*-optimal; note $y_0 = y^*(0)$

$= -K_0 y_0 + \lim_n E\int_0^{\chi_n} c_t y_t^\pi\,dt$ — by definition of χ_n

$= -K_0 y_0 + \lim_n E\{y^\pi(\chi_n)\int_0^{\chi_n} c_t\,dt\}$ — integration of a martingale w.r.t. a non-decreasing process [M2]VII.16

$= -K_0 y_0 + \lim_n E\{y^\pi(\chi_n)[K_0 - k(\chi_n)]\}$ by (2.24)

$\leqslant -K_0 y_0 + K_0 y_0 = 0$ since $Ey^\pi(\chi_n) = y^\pi(0)$ by local martingale property, $y^\pi(0) = y_o$ and $k \geqslant 0$. ||

...(3.3)

Assertions S.(b)(iii) and S.(c)(iii) of Theorem 1 obviously follow.

Proposition 3. If c^* is π^*-optimal and y^π is a local supermartingale for every $\pi \epsilon \Pi$, then (c^*, y^*) is optimal.

Proof. Let $y^\pi = L^\pi D^\pi$ with $L^\pi(0) = y^\pi(0) = y_o$ be the multiplicative decomposition. Since $0 < D^\pi \leqslant 1$ we have $y_t^\pi \leqslant L_t^\pi$, and on applying this inequality to the fifth line of (3) and assuming that the times χ_n reduce the local martingale L^π the proof of Proposition 2 stands with L^π in place of y^π in lines six to nine of (3). ||

This Proposition completes the proof of S.(a) of Theorem 1 (taking into account the fact that a supermartingale is a local supermartingale).

Before going on to the next proposition, we note some relations between shadow prices and their mart-logs. Suppose for the moment that (c^*, π^*) is any plan such that, for some $\pi^o \epsilon \Pi$, the process $y^o = y^{\pi^o}$ is a positive local supermartingale; (this will always be the case in the sequel, in particular when c^* is π^*-optimal, since then one can set $\pi^o = \pi^*$, $y^o = y^*$). The assumption about y^o implies that the process
$y_-^o = (y^o(t-);\ t \epsilon \underline{I})$ is also positive, [DM]VI.17, [J1]6.20. Let $x = x^o$, $z = z^o$ correspond to π^o. For any other $\pi \epsilon \Pi$, it follows from (2.31) that

$$y^\pi = vz^\pi = y^o z^\pi / z^o = y^o \exp\{x^\pi - x^o\},$$

and since the semimartingales x^π, x^o are right continuous with (finite) left limits it follows that y^π is a positive semimartingale with $y^\pi_- > 0$. Note further that $v = y^*/z^* = y^o/z^o$ is also a semimartingale with $v_- > 0$, and so is corlol (although we have not assumed this property for $\bar{u}'$ or $\bar{c}$). These facts are used repeatedly below without special comment.

Under the conditions stated in the preceding paragraph, we may define the 'mart-log' $\eta^\pi = \pounds(y^\pi)$ – see (2.6) – for every $\pi \epsilon \Pi$, in particular $\eta^\lambda = \pounds(y^\lambda)$ for each $\lambda = 1,\ldots,\Lambda$; explicitly,

$$\eta^\pi(I) = \int_o^I \frac{dy^\pi(t)}{y^\pi(t-)}, \qquad \eta^\lambda(I) = \int_o^I \frac{dy^\lambda(t)}{y^\lambda(t-)}. \qquad \ldots(3.4)$$

Now replace y^π by vz^π, write out $v_t z^\pi_t$ by means of the formula for integration by parts [DM]VIII.18, use the fact that v_- and z^π_- are positive, and introduce the process $\zeta^\pi = \pounds(z^\pi)$ – see (2.6–8); this yields, in abridged notation,

$$\eta^\pi_I = \int_o^I \frac{dv}{v_-} + \int_o^I \frac{dz^\pi}{z^\pi_-} + \left[\int \frac{dv}{v_-} , \int \frac{dz^\pi}{z^\pi_-} \right]_I$$

$$= \int_o^I \frac{dv}{v_-} + \zeta^\pi_I + \left[\int \frac{dv}{v_-} , \zeta^\pi \right]_I \qquad \ldots(3.5)$$

On applying the same procedure to η^λ, with z^λ, ζ^λ in place of z^π, ζ^π, and noting that $\zeta^\pi = \int \Sigma \pi^\lambda d\zeta^\lambda$ by (2.8) and that $\Sigma \pi^\lambda = 1$, it follows readily that

$$\eta^\pi_I = \int_o^I \sum_\lambda \pi^\lambda_t \, d\eta^\lambda_t, \quad \text{i.e.} \quad \int \frac{dy^\pi}{y^\pi_-} = \int \sum_\lambda \pi^\lambda \frac{dy^\lambda}{y^\lambda_-}. \qquad \ldots(3.6)$$

Because of this linear relation, it is often more convenient to work with the mart-logs of shadow prices than with the shadow prices themselves.

Suppose now that π is such that y^π is a positive local supermartingale, write $y^\pi = y$ for short and $y = LD$ for the multiplicative decomposition. Using integration by parts, and bearing in mind that D is predictable and of finite variation, we have

$$y_I = L_I D_I = y_o + \int_o^I D_t dL_t + \int_o^I L_{t-} dD_t \,, \qquad \ldots(3.7)$$

see [DM]VIII.19; (the appearance of y_o is due to our convention that processes do not jump at zero). Using (4) we have, in abridged notation

$$\eta_I = \int_o^I \frac{D}{D_-} \frac{dL}{L_-} + \int_o^I \frac{dD}{D_-} = \mu_I + \partial_I \qquad \ldots(3.8)$$

where $\eta = \eta^\pi$, and $\mu = \mu^\pi$, $\partial = \partial^\pi$ are for the moment just names for the processes defined by the integrals. Note that $L_- > 0$ because L is a positive local martingale, also $D_- > 0$ because D is positive non-increasing, moreover $1/L_-$ and $1/D_-$ are left continuous with right limits, hence locally bounded, so that the integrals are well defined. Moreover μ is a local martingale because L is one, ∂ is non-positive, non-increasing, predictable, and $\mu(0) = \partial(0) = 0$, so that η is a local supermartingale and (8) gives its additive decomposition, justifying the notation $\mu + \partial$. In particular, if y is a local martingale, so is η (because then $D \equiv 1$, $\partial \equiv 0$). Note also that (4), (8) and $y = y_- + \Delta y > 0$, $y_- > 0$ imply

$$\Delta\mu_t + \Delta\partial_t = \Delta\eta_t = \Delta y_t / y_{t-} > -1, \qquad \ldots(3.9)$$

hence also $\Delta\mu_t > -1$ since $\Delta\partial(t) < 0$.

Suppose conversely that we know that $y^\pi = y$ is a positive <u>semi</u>martingale with y_- also positive and that $\eta = \eta^\pi$, as defined by (4), is a local supermartingale with additive decomposition $\eta = \mu + \partial$ (not necessarily of the form given by (8)). The relations (9) remain valid, so that $1 + \Delta\mu > 0$, and on writing out the formula $y = \&(\eta)$ explicitly from (2.5) and rearranging one obtains

$$y = \&(\mu)_I \exp(\partial_I^c) \cdot \prod_{t \le I} \{(1+\Delta\mu_t+\Delta\partial_t)/(1+\Delta\mu_t)\} \qquad \ldots(3.10)$$

where ∂^c denotes the continuous part of ∂. Now $\&(\mu)$ is a local martingale because μ is one. Further, ∂^c is non-positive and non-increasing (because η is a local supermartingale), so that $\exp(\partial^c)$ is in (0,1] and non-increasing. Each term in the product of jumps is also in (0,1] because $\Delta\partial \leqslant 0$ and (9) implies $1 + \Delta\mu > 0$, while $1 + \Delta\mu + \Delta\partial > 0$ because $y > 0$, $y_- > 0$; therefore the product of jumps is in (0,1] and non-increasing. It follows that y is a local supermartingale with decomposition $y = LD$ given by (10) with $L = \&(\mu)$; and if η is a local martingale, then so is y.

Proposition 4. If y^λ is a local martingale for each $\lambda = 1,\ldots, \Lambda$, then y^π is a local martingale for every $\pi\in\Pi$.

Proof. If each y^λ is a local martingale, so is each η^λ, therefore by (6) so is η^π, therefore so is y^π.||

Assertions S.b(ii) and S.(c)(ii) of Theorem 1 follow from Propositions 4 and 2.

Proposition 5. If y^λ is a local supermartingale for each $\lambda = 1,\ldots,\Lambda$, then y^π is a local supermartingale for every $\pi \geqslant 0$.

Proof. Formulae (7-8) apply with y^λ, L^λ, D^λ, η^λ, μ^λ, ∂^λ in place of y, L, D etc. Using (6) we have, in abridged notation, the first equality in

$$\eta^\pi = \int \sum_\lambda \pi^\lambda \frac{D^\lambda}{D_-^\lambda} \frac{dL^\lambda}{L_-^\lambda} + \int \Sigma\, \pi^\lambda \frac{dD^\lambda}{D_-^\lambda} = \mu^\pi + \partial^\pi, \qquad \ldots(3.11)$$

(noting that $\pi^\lambda(t) = \pi^\lambda(t-)$ is left continuous with right limits, hence locally bounded, so that the integrals are well defined). Clearly the first integral is a local martingale and the second is predictable and of finite variation; moreover the second integral is non-positive and non-increasing because $0 \leqslant \pi^\lambda \leqslant 1$. Thus η^π is a local supermartingale with additive decomposition given by the sum of the two integrals, and it is in conformity with our notation for such decompositions to call the integrals μ^π and ∂^π respectively. It follows that y^π is also a local supermartingale.||

Assertion S.(d) of Theorem 1 follows from Propositions 5 and 3. Note that Propositions 4–5 do not depend on optimality as such.

In certain cases, Prop. 5 can be strengthened as follows.

Proposition 6. If y^λ is a local supermartingale for each λ and y^* is a local martingale, then if $\pi^* \geq 0$ we have eq. (3.1); if moreover $\pi^* > 0$, then y^λ is a local martingale for each λ (and consequently y^π is a local martingale for every $\pi \in \Pi$).

Proof. If each y^λ is a local supermartingale we may set $\pi = \pi^*$, $\eta^\pi = \eta^*$ in (11), and then if y^* is a local martingale we have $\partial^* = \sum_\lambda \int \pi^{*\lambda} dD^\lambda / D^\lambda_- = 0$; since this is a sum of non-positive integrals it follows that each integral vanishes and we have (1). If moreover $\pi^{*\lambda}(t) > 0$ always, then since D^λ is non-increasing and positive with $D^\lambda(0) = 1$ it follows from (1) that $D^\lambda(t) = 1$ on $\underline{T}$, a.s.; but then $y^\lambda = L^\lambda$. The final assertion in brackets is due to Prop. 4. ||

Since π^*-optimality of c^* implies that y^* is a local martingale (Proposition 1), assertion S.(b)(i) of Theorem 1 follows from Propositions 6 and 2. Thus all sufficiency assertions are now proved. The following proposition could also be used to prove S.(c).

Proposition 7. If $\Pi = \Pi^o$, if y^π is a local supermartingale for every $\pi \in \Pi^o$ and y^* is a local martingale, then y^π is a local martingale for every $\pi \in \Pi^o$.

Proof. Since y^λ is a local supermartingale for each $\lambda = 1, \ldots, \Lambda$, we have the first equality in (11) for arbitrary π, as in the proof of Proposition 5, and once again the first integral is a local martingale, the second predictable and of finite variation. This time, however, we know in advance that η^π is a local supermartingale, and on writing its decomposition as in (8) it is clear that both equalities hold in (11), with the first integral equal to μ^π, the second to ∂^π. Explicitly, we have for the finite variation term

$$\partial_I^\pi = \int_o^I \sum_\lambda \pi_t^\lambda d\partial_t^\lambda \qquad \text{i.e.} \qquad \int_o^I \frac{dD^\pi}{D_-^\pi} = \int_o^I \sum_\lambda \pi^\lambda \frac{dD^\lambda}{D_-^\lambda} \qquad \ldots (3.12)$$

for every $\pi \epsilon \Pi^o$, with $\partial^\pi(0) = \partial^\lambda(0) = 0$. Of course, $\partial^* \equiv 0$. Now, if all ∂^λ are non–increasing and π can be chosen arbitrarily, subject only to left continuity with right limits and $\Sigma \pi^\lambda = 1$, then ∂^π cannot be non–increasing for all $\pi \epsilon \Pi^o$ unless the ∂^λ are constant on $\underline{I}$ a.s., hence vanish. But then $\eta^\pi = \mu^\pi$ and the result follows.

To be explicit, suppose that ∂^1 does not vanish. Let $\tau > 0$ be a predictable time such that $\partial_\tau^1 < 0$ with positive probability. Choose a number $p < 0$ and let π' be a portfolio policy defined by

$$\pi'^1 = p1 + (1-p)\pi^{*1}$$

$$\pi'^\lambda = (1-p)\pi^{*\lambda} \text{ for } \lambda \neq 1.$$

Then

$$\partial_\tau^{\pi'} = p\partial_\tau^1 + (1-p)\partial_\tau^* = p\partial_\tau^1$$

which takes positive values with positive probability, contrary to $\partial^{\pi'} \leq 0$. //

Remark (iii). The preceding proposition is formulated only for $\Pi = \Pi^o$, and this case is considered here only when X is continuous. However the argument as such does not use continuity, and it could be refined to yield local martingale conditions in cases where $\Pi = \Pi^1$ and short sales are permitted, provided that suitable bounds are imposed on the upward jumps of X and possibly also on negative values of π. This variant will not be pursued here.

Suppose now that (c^*, π^*) is optimal. It will be shown below (Theorem 2) that then y^π is a supermartingale for every $\pi \epsilon \Pi$. For the moment we assume this result and verify the necessity assertions of Theorem 1. Obviously c^* is π^*–optimal, and by Proposition 1 y^* is a local martingale. Now N.(a) and N.(b)(i) follow immediately from Theorem 2, and then N.(d) results from Proposition 6. Assertions N.(b)(ii) and (iii) also follow from Proposition 6, and N.(c)(ii) and (iii) from Proposition 7. This concludes the proof of Theorem 1.

4. THE SUPERMARTINGALE PROPERTY

The purpose of this Section is to show that, if a plan $(\bar{c}^*,\pi^*)$ is optimal, then for every portfolio plan $\pi\in\Pi$ the shadow price process y^π defined by (2.30–31) is a supermartingale; here Π can be Π^0 or Π^+ (among other possibilities). This result will complete the proof of the necessity part of Theorem 1. The argument is presented separately because the methods used are rather different from those in other Sections. The procedure here is an extension of the classical calculus of variations approach used in [F1], and we shall omit details of some steps which are essentially the same. The reader who is prepared to accept the result stated in Theorem 2 can skip this Section without loss of continuity. We begin with some definitions and preliminary results concerning directional derivatives and time changes.

Let $(\bar{c}^*,\pi^*)$ be an optimal plan satisfying $\bar{c}^* > 0$ and (2.32), and define associated processes x^*, $z^* = \exp(x^*)$, $\bar{k}^*$, $c^* = \bar{c}^*/z^*$, $k^* = k_o - \int c^* dt$, $\bar{u}(\bar{c}^*)$, $\bar{u}'(\bar{c}^*) = v$ and $y^* = vz^*$ as in Section 2. Let $(\bar{c},\pi)$ be an arbitrary plan with associated processes x^π, $z^\pi = \exp(x^\pi)$, $\bar{k}^\pi$, $c^\pi = \bar{c}/z^\pi$, k^π and $y^\pi = vz^\pi = y^* z^\pi/z^*$; when no confusion is possible we sometimes drop the superscript π. We write
$\delta\bar{c} = \bar{c} - \bar{c}^*$, $\delta c = c - c^*$ etc.
and note that by (2.23) and (2.29–30) we have

$$\delta\bar{c} = c^\pi z^\pi - c^* z^* = \delta c . z^\pi + c^*(z^\pi - z^*), \qquad \text{.....(4.1)}$$

$$\delta\bar{c} . v = \delta c . y^\pi + c^*(y^\pi - y^*). \qquad \text{.....(4.2)}$$

For $0 < \alpha < 1$, we define a new consumption plan $\bar{c}^\alpha$ (in natural units) by

$$\bar{c}^\alpha = \alpha\bar{c} + (1-\alpha)\bar{c}^* = \bar{c}^* + \alpha\delta\bar{c}. \qquad \text{.....(4.3)}$$

To show that $\bar{c}^\alpha$ is feasible, it is necessary to exhibit a portfolio policy π^α which finances $\bar{c}^{\alpha}$, i.e. which is such that $\bar{c}^{\alpha}\epsilon\underline{\bar{C}}^{\pi^{\alpha}}$. Intuitively, it is clear that π^{α} may be constructed as follows: divide the initial capital K_0 into two funds, in proportions α, $1-\alpha$, invest the first according to π and use it to finance $\alpha\bar{c}$, the second according to π^* and use it to finance $(1-\alpha)\bar{c}^*$. It may be checked that a suitable π^{α} is given (in abridged notation) by

$$\pi^{\alpha\lambda}\bar{k}^{\alpha}_{-} = \alpha\pi^{\lambda}\bar{k}_{-} + (1-\alpha)\pi^{*\lambda}\bar{k}^{*}_{-} \qquad \text{....(4.4)}$$

where $\bar{k}_{-} = \bar{k}(t-)$ etc. and

$$\bar{k}^{\alpha} = \alpha\bar{k} + (1-\alpha)\bar{k}^{*}, \qquad \text{....(4.5)}$$

and that $\bar{k}^{\alpha}$ is feasible and is the capital plan corresponding to $\bar{c}^{\alpha}$; moreover, if π, π^* are in Π^+, so is π^{α}. (There could in general be more than one portfolio policy which finances $\bar{c}^{\alpha}$, but we shall always consider this one).

Given these convexity properties, one can define the directional derivative $D\bar{\varphi} = D\bar{\varphi}(\bar{c}^*,\pi^*;\ \delta\bar{c},\delta\pi)$ of $\bar{\varphi}$ at $(\bar{c}^*,\pi^*)$ in the 'direction' $(\delta\bar{c},\delta\pi)$ by

$$D\bar{\varphi} = \lim_{\alpha\downarrow 0} (1/\alpha)[\bar{\varphi}(\bar{c}^{\alpha}) - \bar{\varphi}(\bar{c}^{*})]$$

$$= \lim_{\alpha\downarrow 0} (1/\alpha)E\int_0^{\infty} [\bar{u}(\bar{c}^{\alpha}) - \bar{u}(\bar{c}^{*})]\ dt; \qquad \text{....(4.6)}$$

the limit exists because of the concavity of $\bar{u}$, and the optimality of $(\bar{c}^*,\pi^*)$ implies that $D\bar{\varphi} \leqslant 0$ for all feasible variations $(\delta\bar{c},\delta\pi)$. Differentiating under the integral sign, writing $v = \bar{u}'(\bar{c}^*)$ and using (2) yields

$$D\bar{\varphi} = E\int \delta\bar{c}.v\ dt = E\int [(y^{\pi}-y^{*})c^{*} + \delta c.y^{\pi}]\ dt. \qquad \text{....(4.7)}$$

The differentiation can be justified, as in [F1], under assumption (2.32). In particular, on setting $\bar{c} = 0$, $\delta\bar{c} = -\bar{c}^*$, $\delta\pi = 0$ we obtain from (7) and (2.32)

$$D^* \stackrel{\text{def}}{=\!=} -D\bar{\varphi}(\bar{c}^*, \pi^*; -\bar{c}^*, 0) = E \int \bar{c}^* v \, dt$$

$$= E \int y^* c^* dt < \infty \qquad \ldots (4.8)$$

and for an arbitrary feasible variation $(\delta\bar{c}, \delta\pi)$ it follows that

$$0 \geq D\bar{\varphi}(\bar{c}^*, \pi^*; \delta\bar{c}, \delta\pi) \geq -D^* > -\infty. \qquad \ldots (4.9)$$

Now define a process $G^* = (G^*_t;\ t\epsilon\underline{T})$ by

$$G^*(I) = \int_0^I c^*(t)\, dt = K_o - k^*(I) \qquad \ldots (4.10)$$

cf. (2.24). Obviously G^* is increasing, absolutely continuous and takes values in $[0,K_o)$. For each $i\epsilon[0,K_o)$, we define a stopping time $\tau_i = \tau(\omega,i)$, called the <u>depletion time</u> at the level i, as the finite solution of

$$G^*[\omega,\ \tau(\omega,i)] = i \qquad \ldots (4.11)$$

if this exists and $\tau(\omega,i) = \infty$ if $G^*(\omega,\infty) < i$. We denote by $\hat{\underline{A}}_i$ the σ-algebra of events prior to (i.e. not later than) τ_i. The family $\tau = (\tau_i;\ 0 \leq i < K_o)$ defines a time change, which corresponds to measuring time by the depletion of 'standardised' capital along the star plan. The family $\hat{\underline{A}} = (\hat{\underline{A}}_i)$ satisfies the usual conditions, and we can define the σ-algebra of $\hat{\underline{A}}$-progressive sets and the corresponding concepts of optional and predictable sets, martingales, stopping times etc. in the usual way. If $\xi = (\xi_t;\ t\epsilon\underline{T})$ is an $\underline{A}$-progressive process (or $\underline{A}$-process for short), its transform under τ is the process $\hat{\xi} = (\hat{\xi}_i;\ 0 \leq i < K_o)$ defined for each ω by

$$\hat{\xi}_i = \xi(\tau_i) I\{i : \tau_i < \infty\} + \hat{\xi}_{\#} I\{i : \tau_i = \infty\} \qquad \ldots (4.12)$$

where the 'variable at infinity' $\hat{\xi}_{\#}$ remains to be defined in each case. Note that $\{i:\tau_i=\infty\} = \{i:G^*_\infty \leq i<K_0\}$, so that $\hat{\xi}_{\#}(\omega)$ is the constant value assigned to $\hat{\xi}(\omega,i)$ when $i \geq G^*(\omega,\infty)$; if $G^*(\omega,\infty) = K_0$ this variable plays no part in what follows and can be defined arbitrarily. In the case of the transforms $\hat{y}^*$ and $\hat{y}^\pi$ we set $\hat{y}^*_{\#} = \hat{y}^\pi_{\#} = 0$ on $\{\omega:G^*(\omega,\infty)<K_0\}$. The time change inverse to τ is by definition G^*, and it may be checked that each G^*_t is an $\hat{\underline{A}}$-stopping time with $\hat{\underline{A}}_{G^*(t)} = \underline{A}_t$. If $\eta = (\eta_i)$ is an $\hat{\underline{A}}$-process we define $\hat{\eta} = \eta(G^*)$, so that $\hat{\hat{\xi}} = \xi$ if ξ is an $\underline{A}$-process. Setting $\hat{G}^*_{\#}$ equal to G^*_∞ (which is well defined as a left limit) we have

$$\hat{G}^*_i = i\wedge G^*_\infty, \quad d\hat{G}^*_i/di = I\{0\leq i<G^*_\infty\}, \qquad \ldots(4.13)$$

provided that the right-hand derivative is taken at $i = G^*_\infty$; this formula is used to transform integrals according to the rule

$$\int_{\tau(i)}^{\tau(j)} \xi(t)c^*(t)\,dt = \int_{\tau(i)}^{\tau(j)} \xi(t)dG^*(t) = \int_i^j \hat{\xi}(\theta)d\hat{G}^*(\theta)$$

$$= \int_i^{j\wedge G^*(\infty)} \hat{\xi}(\theta)d\theta. \qquad \ldots(4.14)$$

The transformed shadow price process $\hat{y}^* = (\hat{y}^*_i;\ 0\leq i<K_0)$ is of special interest. Since c^* is π^*-optimal, it follows from Theorem 5 of [F1] that for the stopping times (χ_n) appearing in Proposition 1 once can take any sequence $(\tau_{i(n)})$ of depletion times such that $i(n)\uparrow K_0$, hence $\tau_{i(n)}\uparrow\infty$, a.s. More precisely, the Theorem asserts that $\hat{y}^*$ is a (true, right continuous) $\hat{\underline{A}}$-martingale satisfying $\hat{y}^*_i > 0$ for $i < G^*_\infty$ (i.e. for $\tau_i<\infty$) and $\hat{y}^*_i = 0$ for $i\geq G^*_\infty$; also, of course, $E\hat{y}^*_i = y_0$ for each i. On applying the time change to the integral in (8) and using these properties we have

$$D^* = E\int_0^\infty y^*_t\, c^*_t\, dt = E\int_0^{G^*(\infty)} \hat{y}^*_\theta\, d\theta = E\int_0^{K_0} \hat{y}^*\, d\theta$$

$$= K_0 y_0, \qquad \ldots(4.15)$$

which agrees with (2.34). Note that, since $y^\pi = y^* z^\pi / z^*$, it follows from the properties of y, $\hat{y}^*$, z^π and z^* that y^π is positive everywhere, $\hat{y}^\pi$ is positive for $i < G^*_\infty$, both processes are right continuous everywhere and $\hat{y}^\pi(0) = y^\pi(0) = y_0$.

We are now ready to prove

Theorem 2: Supermartingale Property: If (c^*, π^*) is optimal in $\underline{C}$xΠ, then for every $\pi \epsilon \Pi$ the process y^π defined by (2.30–31) is an $\underline{A}$–supermartingale.

Proof. (i) We consider a fixed π throughout. It is more convenient to show that $\hat{y}^\pi = (\hat{y}^\pi_i;\ 0 \leq i < K_o)$ is an $\hat{\underline{A}}$-supermartingale; the result for y^π then follows by optional stopping, using $y^\pi(t) = \hat{y}^\pi(G^*_t)$, cf. [F1], proof of T.6.

(ii) As a preliminary step, we show that the average values of $E\hat{y}^\pi(i)$ on intervals of the form [0,h), where $0 < h < K_o$, are uniformly bounded. For this purpose, choose h and define a plan (c,π) by setting, for each ω,

$c_t = (K_o/h)c^*_t I\{0 \leq t < \tau_h\}$.

Since

$k(\tau_h) = K_o - (K_o/h)G^*(\tau_h)$ and $G^*(\tau_h) \leq h$

– see (10) and (13) – it is clear that c is feasible. On substituting into the last term in (7) and rearranging we obtain

$$D\bar{\varphi} = E\{\int_0^{\tau(h)} (K_o/h) y^\pi_t\ c^*_t\ dt - \int_0^\infty y^*_t\ c^*_t\ dt\}.$$

On transforming the integrals to depletion time we have

$$D\bar{\varphi} = (K_o/h)\ E\int_0^h \hat{y}^\pi_\theta\ d\theta - E\int_0^{K_o} \hat{y}^*_\theta\ d\theta,$$

or, taking account of (9) and (15)

$$(1/h)\ E \int_0^h \hat{y}_\theta^\pi \, d\theta \leq D^*/K_o = y_o \qquad \ldots (4.16)$$

(iii) To construct a variation which shows that $\hat{y}$ is a supermartingale, we proceed as follows. First choose $i \geq 0$ and $A \epsilon \hat{\underline{A}}_i$, then I, h, H such that

$$i < i+h \leq I < I+H < K_o, \qquad \ldots (4.17)$$

then ε with $0 < \varepsilon < h$. To avoid trivialities, we assume that $\tau(\omega,i) < \infty$ for $\omega \epsilon A$. The idea is to reduce consumption a little below $\bar{c}^*$ during $[\tau_i, \tau_{i+h})$, invest the resulting saving in π (while leaving $\bar{k}^*$ invested in π^*), then increase consumption after τ_I until a random time ρ is reached when $\bar{k}$ returns to the star path $\bar{k}^*$ (arranging matters if possible so that $\rho = I + H$, at least in the limit as $\varepsilon \downarrow 0$) and thereafter to revert to the star plan; the fact that welfare does not increase, i.e. $D\bar{\varphi} \leq 0$, should produce a supermartingale inequality for $\hat{y}^\pi$.

For brevity, we sometimes write

$$z_i^\pi = z^\pi(\tau_i), \quad z_{i,t}^\pi = z_t^\pi / z_i^\pi$$

with similar notation for z^*. Let

$$B = B(\omega,\varepsilon) = \{t: (\varepsilon/h)(z_{i,t}^\pi / z_{i,t}^*) \geq 1 \text{ and } \tau_i \leq t < \tau_{i+h}\}.$$

Now define a plan $(\bar{c}, \pi^o)$ as follows. For $\omega \notin A$ or $t < \tau_i$, set $\bar{c} = \bar{c}^*$ and $\pi^o = \pi^*$. For $\omega \epsilon A$ and $t \geq \tau_i$, write $\delta\bar{c} = \bar{c} - \bar{c}^*$ and define

$$\delta\bar{c}_t / \bar{c}_t^* = \begin{cases} -(\varepsilon/h)(1 - I_B)\ z_{i,t}^\pi / z_{i,t}^* & \tau_i \leq t < \tau_{i+h} \\ (\varepsilon/H)(z_{i,t}^\pi / z_{i,t}^*) & \tau_I \leq t < \rho \end{cases} \qquad \ldots (4.18)$$

and $\delta\bar{c}_t = 0$ otherwise, where $\rho = \rho(\omega,\varepsilon)$ is a stopping time to be defined. Still for $\omega \epsilon A$ and $t \geq \tau_i$, we take for π^o the portfolio policy which corresponds to investing $\bar{k}^*$

in π^* and any variation $\delta\bar{k}$ defined by (18) in π, and denote by $\rho = \rho(\omega,\varepsilon)$ the smallest solution after $\tau(\omega,i)$ of $\bar{k}(\omega,t) = \bar{k}^*(\omega,t)$ if this exists and $\rho = \infty$ otherwise. For $t \geq \rho$ we again set $\pi^o = \pi^*$. Since $\bar{k}^*$ remains invested in π^* and $\delta\bar{c} < 0$ during $[\tau_i,\tau_I)$ we have $\delta\bar{k} > 0$ during this interval, hence $\rho > \tau_I$ as assumed in the definition (18). For $\omega \notin A$ we set $\rho = \tau_i$, so that ρ is well defined as a stopping time. Clearly $\bar{c}$ is progressive and $\bar{c} > 0$ because of the definition of the set B; and since $\delta\bar{k} > 0$ on $[\tau_i,\rho)$ and $\delta\bar{k} = 0$ for $t > \rho$ it follows that $\bar{k} = \bar{k}^* + \delta\bar{k} > 0$, so that $(\bar{c},\pi^o)$ is feasible. Note that $\bar{c}$ depends on ε but π^o does not.

It would be possible to write out formally the portfolio composition $\pi_t^{o\lambda}$ in terms of $\pi_t^{*\lambda}$ and π_t^λ as in (4-5), but this would only obscure the argument. It is clear enough that π^o is well defined as a portfolio policy (and is in Π^+ if π^* and π are in this set), and we can calculate ρ by considering only the equation of accumulation of the additional fund $\delta\bar{k}(T)$ which is invested in π. Referring to (2.18), it is seen that this equation may be written, for $\omega\epsilon A$ and $\tau_i \leq T < \rho$, as

$$\delta\bar{k}(I) = \int_{\tau(i)}^{T} \delta\bar{k}(t-)[z^\pi(\tau_i)/z^\pi(t-)]\ dz^\pi(t) - \int_{\tau(i)}^{I} \delta\bar{c}(t)\ dt. \qquad \ldots(4.19)$$

As in the transformation of (2.18) into (2.19), this can be rewritten as

$$\delta\bar{k}(I) = -\int_{\tau(i)}^{I} \delta\bar{c}(t)[z^\pi(I)/z^\pi(t)]\ dt \qquad \ldots(4.20)$$

where of course $\delta\bar{k}(\tau_i) = 0$, and ρ is the first value of $T > \tau_i$ for which $\delta\bar{k}(T) = 0$ if such a value exists. Suppose for the moment that this is the case for some $\varepsilon = \varepsilon_0 > 0$; then obviously τ_{i+h} and τ_I are finite. On substituting from (18), writing $\delta\bar{k}(\rho) = 0$ and simplifying one obtains

$$(1/h)\int_{\tau(i)}^{\tau(i+h)} (1-I_B)[\bar{c}_t^*/z_t^*]\ dt = (1/H)\int_{\tau(i)}^{\rho} [\bar{c}_t^*/z_t^*]\ dt. \qquad \ldots(4.21)$$

When $\varepsilon\downarrow 0$, $1-I_B\uparrow 1$ on $[\tau_i,\tau_{i+h})$, and taking into account that $\overline{c^*}/z^* = c^* = dG^*/dt$ it is seen that the left side $\uparrow$ to $(1/h)[G^*(\tau_{i+h}) - G^*(\tau_i)]$, which equals 1 if $\tau_{i+h} < \infty$. On the other hand, the right side is $(1/H)[G^*(\rho) - G^*(\nu_I)]$, which would exceed 1 if we had $\rho > \tau_{I+H}$; therefore $\rho \leq \tau_{I+H}$, and if $\tau_{I+H} < \infty$ then clearly $\rho\uparrow\tau_{I+H}$ as $\varepsilon\downarrow 0$ to maintain equality in (21).

In fact, it can be checked that $\rho(\varepsilon)\uparrow\tau_{I+H}$ as $\varepsilon\downarrow 0$ even if $\tau_{I+H} = \infty$. Then, if $\tau_{I+H} < \infty$, the limit on the left of (21) is 1 and the right side is < 1 for every upper limit of the integral, so that $\lim\rho(\varepsilon) = \infty$. Alternatively, if $\tau_{i+h} = \infty$, then the right side of (21) vanishes identically while the left is positive since $\tau_i < \infty$ by assumption, so that $\rho(\varepsilon) = \infty$ for every ε. Of course, in these cases the equality in (21) is to be replaced by $\geq$.

Now substitute from (18) into (7), replace $v\overline{c^*}$ by y^*c^*, note that $D\overline{\varphi} \leq 0$, and obtain - after cancelling the term z_i^π/z_i^* -

$$0 \geq \int_A dP\ \{\ -(\varepsilon/h)\int_{\tau(i)}^{\tau(i+h)} (1-I_B)\ y_t^*[z_t^\pi/z_t^*]c_t^*\ dt$$

$$+\ (\varepsilon/H)\int_{\tau(I)}^{\rho} y_t^*[z_t^\pi/z_t^*]c_t^*\ dt\}. \qquad \ldots(4.22)$$

On cancelling ε, then letting $\varepsilon\downarrow 0$, we have $1-I_B\uparrow 1$ on $[\tau_i,\tau_{i+h})$ and $\rho(\varepsilon)\uparrow\tau_{I+H}$, so that by monotone convergence we may omit $1-I_B$ and replace ρ by τ_{I+h} without disturbing the inequality. On replacing y^*z^π/z^* by y^π, transforming the integrals to depletion time and rearranging we obtain

$$\int_A dP\ \{(1/h)\int_i^{i+h} \hat{y}^\pi(\theta)d\theta\} \geq \int_A dP\ \{(1/H)\int_I^{I+H} \hat{y}^\pi(\theta)d\theta\}. \qquad \ldots(4.23)$$

Note that, since for fixed i and A these inequalities hold for arbitrary h, I and I+H satisfying (17), they continue to hold if I, I+H are replaced by i, i+h' with $0<h<h'<K_o-i$.

(iv) It remains to replace the inequalities (23), which relate to time averages of $\hat{y}^\pi$, by inequalities for the process itself. For $0 \leq i < i+h < K_0$, define new variables

$$\hat{Y}_i^h = (1/h) \int_i^{i+h} \hat{y}^\pi(\theta)\, d\theta$$

and note that

$$\hat{Y}_i^h \to \hat{y}_i^\pi \text{ a.s. when } h \downarrow 0 \qquad \ldots (4.24)$$

because of the right continuity of $\hat{y}^\pi$. For fixed $h = H$, the inequalities (23) with $A = \Omega$ show that $E\,\hat{Y}_i^h \uparrow$ as $i \downarrow$, and by part (ii) above we have $E\,\hat{Y}_i^h \leq y_0$, so that the $\hat{Y}_i^h$ are all integrable. For fixed i, the process $(\hat{Y}_i^h;\ 0<h<K_0-i)$ is adapted to the filtration $(\hat{\underline{A}}_{i+h};\ 0<h<K_0-i)$, it is right continuous because $\hat{y}^\pi$ has this property, it is integrable, and so by the sentence following (23) it is a supermartingale. On letting $h \downarrow 0$, it follows that the $\hat{Y}_i^h$ converge a.s. and in L^1 to some $\hat{\underline{A}}_i$-measurable variable, see [DM]VI.7, and by (24) this variable is $\hat{y}_i^\pi$. But then we may pass to the limit under the outer integral in the inequalities (23) as $h = H \downarrow 0$, yielding the supermartingale inequality for $\hat{y}^\pi$, i.e.

$$\int_A \hat{y}_i^\pi\, dP \geq \int_A \hat{y}_I^\pi\, dP, \quad A \in \hat{\underline{A}}_i, \quad i < I, \qquad \ldots (4.25)$$

as well as the integrability condition $E\,\hat{y}_i^\pi \leq y_0$. ||

5. PORTFOLIO EQUATIONS

This Section investigates the abstract relationship between martingale conditions, which characterise an optimum in terms of intertemporal comparisons of utility, and 'portfolio equations' which lay down relations to be satisfied by the optimal portfolio vector at each time and state. An example of portfolio equations is given in Section 6.

The present discussion follows on from Section 3. Initially it is assumed, unless otherwise stated, that $\Pi = \Pi^+$, that X can have jumps, and that (c^*,π^*) is a distinguished plan satisfying $\pi^* > 0$ (the case $\Pi = \Pi^o$, X continuous, π^* unrestricted being similar though simpler); the case $\Pi = \Pi^+$, $\pi^* \geq 0$ is considered separately afterwards. We assume for the time being that (c^*,π^*) is optimal and consider <u>necessary</u> conditions.

Let π be another portfolio plan and write $\delta\pi = \pi - \pi^*$, $\delta\pi^\lambda = \pi^\lambda - \pi^{*\lambda}$ etc. The processes y^π, y^* are local martingales by Propositions 1 and 6 (or by Prop. 7 if $\Pi = \Pi^o$), therefore the corresponding processes η^π, η^* defined as in (3.4) are local martingales also. Using (3.5) we have

$$\eta^\pi_I - \eta^*_I = \zeta^\pi_I - \zeta^*_I + [\int \frac{dv}{v_-}, \zeta^\pi - \zeta^*]_I \qquad \ldots (5.1)$$

and $\zeta^\pi - \zeta^*$ is given by (2.13) with $\delta\pi$ in place of π, and then x^λ can be written out explicitly as in (2.1). The square bracket term in (1) can be split up as

$$[\int \frac{dv}{v_-}, \zeta^\pi - \zeta^*]_I = \langle (\int \frac{dv}{v_-})^c, \zeta^{\pi c} - \zeta^{*c}\rangle_I$$

$$+ \sum_{t\leq I} \frac{\Delta v(t)}{v(t-)} \cdot (\Delta\zeta^\pi_t - \Delta\zeta^*_t) \qquad \ldots (5.2)$$

and $\Delta\zeta^\pi - \Delta\zeta^*$ may be obtained from (2.9); explicitly, the angle bracket term in (2) is

$$\int_0^I \sum_\lambda \delta\pi^\lambda_t \, d \langle (\int \frac{dv}{v_-})^c, M^{\lambda c}\rangle_t$$

$$= \int_0^I \sum_\lambda \delta\pi^\lambda_t \, d \langle (\ln v)^c, M^{\lambda c}\rangle_t \qquad \ldots (5.3)$$

while the sum of jumps is

$$\sum_{t\leq I} \frac{\Delta v(t)}{v(t-)} \sum_{\lambda} \delta\pi^{\lambda}(t)\ e^{\Delta x^{\lambda}(t)} . \qquad \ldots(5.4)$$

Writing for brevity

$$N^{\lambda} = V^{\lambda c} + \tfrac{1}{2} < M^{\lambda c},\ M^{\lambda c} > + < (\ln v)^{c},\ M^{\lambda c} > , \qquad \ldots(5.5)$$

substituting into the right-hand side of (1) and rearranging we get, in abridged notation,

$$\int_0^I \sum_{\lambda} \delta\pi^{\lambda}\ dM^{\lambda} + \int_0^I \sum_{\lambda} \delta\pi^{\lambda}\ dN^{\lambda} + \int_0^I \sum_{\lambda} \delta\pi^{\lambda}\ dV^{\lambda d}$$

$$+ \sum_{t\leq I} \sum_{\lambda} \delta\pi^{\lambda}\ [e^{\Delta x^{\lambda}} - 1 - \Delta x^{\lambda} + (\Delta v/v_{-})e^{\Delta x^{\lambda}}] . \qquad \ldots(5.6)$$

This simplifies further, using first $\Delta x^{\lambda} = \Delta M^{\lambda} + \Delta V^{\lambda d}$ and the fact that $V^{\lambda d}$ is just the sum of its jumps, secondly that $y^{*} = ve^{x*}$, $y^{\lambda} = ve^{x^{\lambda}}$, hence

$$1 + \Delta v/v_{-} = v/v_{-} = (y^{*}/y^{*}_{-})e^{-\Delta x^{*}} = (y^{\lambda}/y^{\lambda}_{-})e^{-\Delta x^{\lambda}},$$

and finally $\sum\delta\pi^{\lambda} = 0$. On taking the first term in (6) to the left-hand side of (1) and taking into account the definitions of η^{π}, η^{*} we have

$$\eta^{\pi}_{I} - \eta^{*}_{I} - \int_0^I \sum_{\lambda} \delta\pi^{\lambda}_{t}\ dM^{\lambda}$$

$$= \int_0^I \sum_{\lambda} \delta\pi^{\lambda}_{t} \left[\frac{dy^{\lambda}(t)}{y^{\lambda}(t-)} - dM^{\lambda}_{t} \right]$$

$$= \int_0^I \sum_{\lambda} \delta\pi^{\lambda}_{t}\ dN^{\lambda}_{t} + \sum_{t\leq I} \sum_{\lambda} \delta\pi^{\lambda}_{t} \left[\frac{y^{\lambda}(t)}{y^{\lambda}(t-)} - \Delta M^{\lambda}_{t} \right], \qquad \ldots(5.7)$$

or simply

$$B^{\delta\pi}_{I} = N^{\delta\pi}_{I} + S^{\delta\pi}_{I} \qquad \ldots(5.7a)$$

where $B^{\delta\pi}$ is a new name for the common value of the three lines in (7) and $N^{\delta\pi}$, $S^{\delta\pi}$ are abbreviations for the two terms in the last line. Note that the equality of the three lines in (7) does not depend on optimality or martingale properties of the y^λ as such – except that, as mentioned in the discussion preceding (3.4) and (3.5), the fact that y^* is a positive local martingale ensures that y^λ, y^λ_-, v, v_- are all positive so that the processes η^π, η^* and the other formulae involved in the calculations leading to (7) are well defined. In particular, if we consider the single asset portfolio defined by (2.11) and write

$$N_I^* = \int_0^I \sum_\lambda \pi_t^{*\lambda}\, dN_t^\lambda \,, \qquad M_I^* = \int_0^I \sum_\lambda \pi_t^{*\lambda}\, dM_t^\lambda, \qquad \ldots(5.8)$$

we get, on replacing π by λ in η^π, $B^{\delta\pi}$, $N^{\delta\pi}$, $S^{\delta\pi}$, the formulae

$$\begin{aligned} B_I^{\delta\lambda} &= \eta_I^\lambda - \eta_I^* - M_I^\lambda + M_I^* \\ &= N_I^\lambda - N_I^* + S_I^{\delta\lambda}, \end{aligned} \qquad \ldots(5.9)$$

$$S_I^{\delta\lambda} = \sum_{t<I} \left[\frac{y^\lambda(t)}{y^\lambda(t-)} - \frac{y^*(t)}{y^*(t-)} - \Delta M_t^\lambda + \Delta M_t^* \right]. \qquad \ldots(5.10)$$

Now, the first line in (7) is a local martingale vanishing at zero time, therefore so is the last. Further, it follows from the method of calculation and standard results concerning the change-of-variables formula that the sum of jumps $S_I^{\delta\pi}$ converges absolutely for all $I \in \underline{I}$ a.s., [M2]III.3,8. Since the integral $N^{\delta\pi}$ is a continuous process of finite variation, it follows that $B^{\delta\pi}$ is (locally) of finite variation, and being also a local martingale vanishing at zero it is locally of integrable variation, [DM]VI.83. But $N^{\delta\pi}$ is locally of integrable variation, therefore so is $S^{\delta\pi}$. It follows that the sum-of-jumps process $S^{\delta\pi} = (S_I^{\delta\pi}; I\in\underline{I})$ admits a compensator (dual predictable projection) $\tilde{S}^{\delta\pi}$, and since $S^{\delta\pi} + N^{\delta\pi}$ is a local martingale vanishing at zero we have $\tilde{S}^{\delta\pi} = -N^{\delta\pi}$, so that the compensator is continuous. In other words, $B^{\delta\pi}$ is a compensated jump local martingale, and satisfies

$$0 = \tilde{B}_I^{\delta\pi} = N_I^{\delta\pi} + \tilde{S}_I^{\delta\pi} \quad \text{on } \underline{I}, \text{ a.s., for every } \pi = \pi^* + \delta\pi \in \Pi. \qquad \text{....(5.11)}$$

In particular, for the single-asset portfolios we have

$$0 = \tilde{B}_I^{\delta\lambda} = N_I^{\lambda} - N_I^{*} + \tilde{S}_I^{\delta\lambda} \quad \text{on } \underline{I}, \text{ a.s., for } \lambda = 1,\ldots,\Lambda. \qquad \text{....(5.12)}$$

For brevity we shall consider mainly the system (12), rather than (11), from now on. This system may be regarded as a set of portfolio equations to be satisfied by the optimal portfolio process π^* in case $\Pi = \Pi^+$, $\pi^* > 0$ (or $\Pi = \Pi^o$, X continuous); of course, the equations do not in general determine π^* independently of $\bar{c}^*$, since both the continuous and jump terms in (2) involve v, which depends on $\bar{c}^*$, which in turn depends on π^* by way of the conditions stated in Proposition 1.

The equations (12) have been presented so far as necessary conditions for optimality. Conversely, if (c^*,π^*) is such that c^* is π^*-optimal, so that in particular y^* is a local martingale, the processes η^λ, $S^{\delta\lambda}$ are well defined and the equations (9) are valid, with $S^{\delta\lambda}$ of finite variation. If moreover each $S^{\delta\lambda}$ is locally of integrable variation so that $\tilde{S}^{\delta\lambda}$ is well defined, and if the equations (12) hold, then each $B^{\delta\lambda}$ has a zero compensator and so is a local martingale, therefore so is each $\eta^\lambda - \eta^*$, and since η^* is a local martingale (because y^* is one) the same is true of η^λ, hence of y^λ; but then (c^*,π^*) is optimal by Theorem 1. The case $\Pi = \Pi^o$, X continuous does not require separate detailed consideration. The discussion so far is summed up in part (a) of the following

Theorem 3: Portfolio Equations. Let Π be Π^o or Π^+ if X is continuous, Π^+ if X has jumps. A plan (c^*,π^*) is optimal iff c^* is π^*-optimal and a further condition holds, as follows:

(a) In case either $\pi^* > 0$, or $\Pi = \Pi^o$ (X continuous), the condition is that, for each $\lambda = 1,\ldots,\Lambda$, the process $B^{\delta\lambda}$ defined by (5.9) is locally of integrable variation and its compensator $\tilde{B}^{\delta\lambda}$ vanishes, i.e. the equation (5.12) is satisfied. If X is continuous (with either $\pi^* > 0$ or $\Pi = \Pi^o$), the condition simplifies to the requirement that the processes N^λ, N^* defined by (5.5), (5.8) satisfy

$$N_I^\lambda = N_I^* \quad \text{on } \underline{I}, \text{ a.s., for } \lambda = 1, \ldots, \Lambda. \qquad \ldots (5.13)$$

(b) In case $\Pi = \Pi^+$, $\pi^* \geq 0$, the condition is that, for each λ, $B^{\delta\lambda}$ is locally of integrable variation and $\tilde{B}^{\delta\lambda}$ is non-positive, non-increasing and satisfies (5.14) below. If X is continuous, the condition is simplified by writing $B^{\delta\lambda} = \tilde{B}^{\delta\lambda} = N^\lambda - N^*$ (and then the locally integrable variation is automatic).

It remains to complete the proof of (b). Starting with necessity, a review of the previous argument shows that the calculations up to (10) are still valid but η^π, η^λ are no longer necessarily local martingales. However, η^* is still a local martingale (because y^* is one by Proposition 1). Also, by Theorem 2, each y^λ is a positive supermartingale, so that each η^λ can be decomposed as in (3.8) and

$$\eta^\lambda - \int dD^\lambda/D_-^\lambda = \eta^\lambda - \partial^\lambda$$

is a local martingale. A slightly modified version of the argument preceding (11–12) then shows that $\tilde{B}^{\delta\lambda}$ need no longer vanish as in (12), but instead is equal to ∂^λ and so is non-positive and non-increasing. It then follows from (3.1) or Proposition 6 that

$$\int_0^I \pi^{*\lambda}(t)\, d\tilde{B}^{\delta\lambda}(t) = 0 \quad \text{on } \underline{I}, \text{ a.s.,} \quad \lambda = 1, \ldots, \Lambda. \qquad \ldots (5.14)$$

Turning to sufficiency, the validity of the definitions and calculations up to (10) is established as in the case $\pi^* > 0$, and $\tilde{S}^{\delta\lambda}$, $\tilde{B}^{\delta\lambda}$ are well defined because $S^{\delta\lambda}$ is assumed to be locally of integrable variation. Now $B^{\delta\lambda} - \tilde{B}^{\delta\lambda}$ is by definition a local martingale and $\tilde{B}^{\delta\lambda}$ is non-positive and non-increasing by assumption, implying that $B^{\delta\lambda}$ is a local supermartingale, hence by (9) that η^λ is also a local supermartingale, so that the same is true of y^λ, and then optimality follows from Propositions 3 and 5. ||

Remarks (i) In general, the processes $y^\lambda = vz^\lambda$, $y^* = vz^*$ may jump even if X does not, whether because $\underline{A}$ has a time of discontinuity (sudden arrival of news) or because the function $\bar{u}'$ is discontinuous with respect to t. However – bearing in mind that $\sum\delta\pi^\lambda = 0$, that

$y^\lambda/y^\lambda_- = (v/v_-)(z^\lambda/z^\lambda_-) = (v/v_-)\exp(\Delta x^\lambda)$,

and that $\Delta x^\lambda = \Delta M^\lambda + \Delta V^\lambda$ – the processes $B^{\delta\pi}$, $B^{\delta\lambda}$ defined in (7), (9) can jump only if at least one of the components of X jumps.

(ii) If X has at most a finite number of jumps in a finite time interval, formulae (7–10) may be simplified by absorbing the jumps of M into V^d (and any remaining part of M^d into V^c). If it is known that v also has finite number of jumps in a finite interval, one can form separately the sums

$$S^\lambda_I = \sum_{t\leq I} \frac{y^\lambda(t)}{y^\lambda(t-)}\,, \quad S^*_I = \sum_{t\leq I} \frac{y^*(t)}{y^*(t-)} \qquad \text{....(5.15)}$$

If these processes are locally integrable, one can define their compensators, and then (12) becomes

$$N^\lambda_I + \tilde{S}^\lambda_I = N^*_I + \tilde{S}^*_I \qquad \text{on } \underline{I},\ \text{a.s.},\ \lambda = 1,\dots,\Lambda. \qquad \text{....(5.16)}$$

(iii) In portfolio theory it is often assumed that there is a 'riskless' asset, say the one with index Λ. If this is taken to mean that the process x^Λ is deterministic, then the assumption that X is a semimartingale imposes some limitations on the functional form of $x^\Lambda(t)$, see [J1]2.77. In general, x^Λ can still have jump discontinuities; but if one assumes that X has jumps only at totally inaccessible times, then x^Λ can have no jumps at all. Taking this last case as an example, and adopting also the assumptions leading to (15–16) above, it follows from $M^{\Lambda c} = 0$ and (5) that $N^\Lambda = V^{\Lambda c}$, and since x^Λ does not jump we have $S^\Lambda = 0$. Subtraction from (16) then yields $N^\lambda + \tilde{S}^\lambda = V^{\Lambda c}$ for $\lambda = 1,\dots,\Lambda-1$, with $V^{\Lambda c}$ deterministic.

6. AN EXAMPLE

The example to be considered is that where utility has the 'discounted CRRA' form and the semimartingale X is a PSII with respect to $\underline{A}$ satisfying an integrability condition which ensures that an optimum exists. The solution of this model is well known in the case where X is a Λ-dimensional Brownian motion with drift, see [M1], [KLSS], but even in that case it is of interest to derive the results by new methods and so to convince the reader that the rather abstract objects which we have called 'portfolio equations' deserve the name.

In the 'discounted CRRA' model it is assumed that the welfare functional has one of the forms

$$\bar{\varphi}(\bar{c}) = (1-b)^{-1} E \int_0^\infty \bar{c}(t)^{1-b} e^{-\rho t} dt \qquad 0 < b \neq 1 \qquad \ldots (6.1)$$

$$\bar{\varphi}(\bar{c}) = E \int_0^\infty \{\ln \bar{c}(t)\} e^{-\rho t} dt \qquad \ldots (6.2)$$

where ρ is a constant (not necessarily positive). For brevity we shall consider only (1) explicitly; (in fact, correct expressions for the optimal plan under (2) can be obtained by setting b = 1 in the results derived under (1), but the condition for existence is rather different). Regarding X, we assume that it is a PSII as well as a semimartingale with decomposition (2.1), where M^c and M^d are now true martingales. It may be assumed w.l.o.g. that

$$V_t^{\lambda c} = tm^\lambda \ , \quad M_t^{\lambda c} = \sigma^\lambda W_t^\lambda, \quad < M^{\lambda c}, M^{\ell c} >_t = t\sigma^{\lambda\ell} \qquad \ldots (6.3)$$

where $(W^1, \ldots, W^\Lambda)$ is a vector of standard Wiener processes relative to $\underline{A}$ and m^λ, $\sigma^{\lambda\ell}$ are constants, $\sigma^\lambda = \surd\sigma^{\lambda\lambda} > 0$, and the (covariance) matrix $[\sigma^{\lambda\ell}]$ is symmetric and non-negative definite. Further assumptions about $V^{\lambda d}$, $M^{\lambda d}$ will be introduced below. We take $\Pi = \Pi^+$ if X has jumps, $\Pi = \Pi^o$ or Π^+ if X is continuous.

A convenient feature of this model is that, if an optimal plan exists at all, then there is one with a constant ratio of consumption to capital and a constant portfolio composition. We shall not prove this fact as such, but rather use it as a guide in constructing an optimum. Let us say that a consumption plan c or $\bar{c}$ is <u>constant</u> if there is a number $\theta > 0$ such that

$$c_t/k_t = \bar{c}_t/\bar{k}_t = \theta \qquad \text{for all } (\omega,t), \qquad \ldots(6.4)$$

and that a portfolio plan π is <u>constant</u> if there is a numerical vector $\tilde{\pi} = (\tilde{\pi}^1,\ldots,\tilde{\pi}^\Lambda)$ such that

$$\pi_t^\lambda = \tilde{\pi}^\lambda, \qquad \lambda = 1,\ldots,\Lambda, \qquad \text{for all } (\omega,t); \qquad \ldots(6.5)$$

a plan (c,π) or $(\bar{c},\pi)$ will be <u>constant</u> if both its components are constant. In the sequel we shall usually be concerned with properties of a distinguished 'star' plan and shall leave off the stars when no ambiguity arises; also, if $\pi = \pi^*$ is constant, we shall write π^λ rather than $\pi^{*\lambda}$ or $\tilde{\pi}^{*\lambda}$.

If π is constant, it is clear that the corresponding process $x = x^\pi$ is a PSII, and the problem of optimal saving with π given is equivalent to the problem of optimal saving with a single asset whose log-returns process is a PSII. Let us briefly recall the solution of this problem in a form suitable for present purposes – see [F1]S.1 for details. Let $K_0 = 1$ and consider what conditions are necessary for a constant $\theta > 0$ to define a π-optimal constant consumption plan $\bar{c}$ such that $y^* = \bar{u}'(\bar{c})e^x$ is a true martingale. Since $\dot{k} = -c$ we calculate successively

$$k_t = e^{-\theta t}, \quad c_t = \theta e^{-\theta t}, \quad \bar{c}_t = \theta e^{x(t)-\theta t},$$

$$v_t = (\bar{c}_t)^{-b} e^{-\rho t} = \theta^{-b} e^{-bx(t)+(b\theta-\rho)t},$$

$$y_t^* = v_t e^{x(t)} = \theta^{-b} e^{(1-b)x(t)+(b\theta-\rho)t}, \qquad \text{.....(6.6)}$$

and using he fact that x is a PSII it is readily shown that y^* is a martingale iff $Ey^*(t)$ is finite and constant on $\underline{\mathrm{I}}$. This in turn is equivalent to the conditions that

$$E\ e^{(1-b)x(1)} < \infty \qquad \text{.....(6.7)}$$

and that

$$\theta = n = n(\pi) > 0 \qquad \text{.....(6.8)}$$

where n is defined by

$$e^{-bn} = E\ e^{(1-b)x(1)-\rho}. \qquad \text{.....(6.9)}$$

A calculation then shows that

$$(1-b)\bar{\varphi}(\bar{c}) = [n(\pi)]^{-b} \qquad \text{.....(6.10)}$$

which is finite under (7–8), and then conditions (2.32) and (2.34) of Prop. 1 are also satisfied and the plan is π–optimal. On the other hand, if (7) fails or $n \leq 0$, it can be shown that no π–optimal c–plan can exist. Moreover, a standard convexity argument shows that a π–optimal c–plan is unique, modulo null sets. Thus a π–optimal c (or $\bar{c}$) exists iff (7) holds and $n(\pi) > 0$, and then it is unique and is given explicitly by (6) and (8). Note that (7) holds automatically if x is a continuous PSII.

The condition (7), which is essentially a requirement that a particular value of the bilateral Laplace transform of x(1) exists, is typically not satisfied if x is a PSII whose sample functions can have countably many jumps in a finite interval. In particular, it is not satisfied if x is a symmetric stable process with index in the interval (1,2), which is to my knowledge the only processs of this type to have been proposed as an empirical model for speculative prices. It will therefore be assumed in the rest of this Section that the vector process X has a.s. at most a finite number of jumps in (say) a unit interval of time. As noted in Remark (ii) of Section 5, each $M^{\lambda d}$ can then be absorbed into the corresponding V^{λ}, so that the terms in ΔM^{λ}, ΔM^{*} in (5.7–10) can be omitted.

Consider now what conditions are necessary for a constant plan $(c^*,\pi^*) = (c,\pi)$ to be optimal. Of course, c^* must be π^*–optimal, so that (6–9) above must be satisfied. If $\Pi = \Pi^+$, we shall assume for the time being that $\pi^* > 0$ (which in the case of a constant optimum can in principle be achieved simply by omitting from the list $\lambda = 1,\ldots,\Lambda$ those securities which are not held in the portfolio). Both in this case and in case $\Pi = \Pi^o$ with X continuous, the optimum must satisfy the 'portfolio equations' (5.12). To evaluate N^{λ} and N^*, refer to (5.5) and (5.8) for the definitions, then use (6) above together with (2.17) and (3) to obtain

$$<(\ln v)^c,\ M^{\lambda c}>_T = -\ b < \sum_{\ell} \pi^{\ell}\ M^{\ell c},\ M^{\lambda c}>_T = -\ b\ T \sum_{\ell} \sigma^{\lambda\ell}\ \pi^{\ell} \qquad \ldots(6.11)$$

$$N_T^{\lambda} = T[m^{\lambda} + \tfrac{1}{2}\sigma^{\lambda\lambda} - b\sum_{\ell}\sigma^{\lambda\ell}\ \pi^{\ell}], \qquad N_T^{*} = \sum_{\lambda}\pi^{\lambda}\ N_T^{\lambda} \qquad \ldots(6.12)$$

If X has no jumps, the equations (5.12) reduce to $N^{\lambda}(T) = N^*(T)$ for $\lambda = 1,\ldots,\Lambda$, or simply $N^{\lambda}(1) = N^*(1)$ on cancelling T. If Λ is a riskless asset, they reduce further to $N^{\lambda}(1) = m^{\Lambda}$ for $\lambda = 1,\ldots,\Lambda-1$, or explicitly

$$m^{\lambda} + \tfrac{1}{2}\sigma^{\lambda\lambda} - b\sum_{\ell}\sigma^{\lambda\ell}\ \pi^{\ell} = m^{\Lambda}, \qquad \lambda = 1,\ldots,\Lambda-1. \qquad \ldots(6.13)$$

which are the equations for the Brownian case originally derived by Merton [M1].

We now allow for jumps. Writing $x = x^*$ and $y^\lambda = v \cdot \exp\{x^\lambda\}$, we have from (6) that

$$y_t^\lambda / y_{t-}^\lambda = e^{\Delta x^\lambda(t) - b\Delta x^*(t)}, \qquad y_t^* / y_{t-}^* = e^{(1-b)\Delta x^*(t)} \qquad \ldots (6.14)$$

and $\exp\{\Delta x^*\}$ may be evaluated from (2.16) as $\sum \pi^\ell \exp(\Delta x^\ell)$. Substituting into (5.15) yields

$$S_I^\lambda = \sum_{t \leq I} [\sum_\ell \pi_t^\ell e^{\Delta x^\ell(t)}]^{-b} e^{\Delta x^\lambda(t)} ,$$

$$S_I^* = \sum_{t \leq I} [e^{(1-b)\Delta x^*(t)}] = \sum_{t \leq I} [\sum_\ell \pi_t^\ell e^{\Delta x^\ell(t)}]^{1-b} \qquad \ldots (6.15)$$

and the proceses S^λ, S^* are again PSII. Now S^* is an integrable process if $ES^*(1) < \infty$, and then the compensator $\tilde{S}^*$ (being the predictable process which must be deducted from S^* to get a local martingale) is given by $\tilde{S}^*(T) = T \cdot ES^*(1)$. Since S^* is just the positively weighted average of the S^λ, with constant weights, the preceding condition implies that the S^λ are also integrable with $\tilde{S}^\lambda(T) = T \cdot ES^\lambda(1)$. On the other hand, in view of the assumptions made about the continuous part of X, the condition $ES^*(1) < \infty$ is just (7) with $x = x^*$. It now follows from Remark (ii) of Section 5 that the portfolio equations may be obtained by substituting into either (5.12) or (5.16).

To simplify a little further, let T_m, $m = 1,2,\ldots$ be the jump times of the vector X with $0 < T_1 < \ldots < T_m < T_{m+1} \ldots$, the vectors $\Delta X(T_m)$ being i.i.d. Write simply $\Delta X = (\Delta x^1,\ldots,\Delta x^\Lambda)$ for a representative variable, let $\# < \infty$ denote the expected number of jumps in a unit interval and assume that

$$E\ (\sum_\ell \pi^\ell e^{\Delta x^\ell})^{1-b} < \infty \qquad \ldots (6.16)$$

where $\pi = \pi^*$. Then, on writing

$$\tilde{B}_1^{\lambda} = m^{\lambda} + \tfrac{1}{2}\,\sigma^{\lambda\lambda} - b \sum_{\ell} \sigma^{\lambda\ell}\,\pi^{\ell} + \#\, E\,[\,(\sum_{\ell} \pi^{\ell}\, e^{\Delta x^{\ell}})^{-b}\, e^{\Delta x^{\lambda}}\,]$$

$$\tilde{B}_1^{*} = \sum_{\ell} \pi^{\ell}\,\tilde{B}_1^{\ell}\,, \qquad \tilde{B}_1^{\delta\lambda} = \tilde{B}_1^{\lambda} - \tilde{B}_1^{*}, \qquad \ldots (6.17)$$

the portfolio equations (5.12) or (5.16) are

$$\tilde{B}_1^{\delta\lambda} = 0, \qquad \lambda = 1,\ldots,\Lambda. \qquad \ldots (6.18)$$

These of course contain the well-known results for the discrete-time case, see [H]. If λ is a riskless asset – which according to Remark (iii) of Section 5 does not jump – the equations reduce to $\tilde{B}^{\lambda}(1) = m^{\Lambda}$ for $\lambda = 1,\ldots,\Lambda-1$.

Finally, if $\Pi = \Pi^{+}$ and the assumption $\pi^{*} > 0$ is replaced by $\pi^{*} = \pi \geq 0$, it follows from (3.1–2) and (5.14) that (18) is to be replaced by

$$\pi^{\lambda} \geq 0, \qquad \tilde{B}_1^{\delta\lambda} \leq 0, \qquad \pi^{\lambda}\,\tilde{B}_1^{\delta\lambda} = 0, \qquad \lambda = 1,\ldots,\Lambda. \qquad \ldots (6.19)$$

To sum up, we have

<u>Proposition 8</u>. Let $\bar{\varphi}$ have the 'discounted CRRA' form (6.1), let X be a semimartingale – PSII having at most a finite expected number of jumps per unit time, and let Π be Π^{o} or Π^{+} if X is continuous Π^{+} if X has jumps.

The necessary conditions for a constant plan (c^{*},π^{*}) – i.e. a plan with constant consumption ratio and constant portfolio composition – to be optimal are as follows:

(i) that (6.16) holds at $\pi = \pi^{*}$, or equivalently that (6.7) holds at $x = x^{*}$; (this condition is redundant if X is continuous);

(ii) that $n = n(\pi^{*})$ defined as in (6.9) is positive and $c^{*}(t) = ne^{-nt}$ on $\underline{I}$, a.s.;

(iii) (a) in case either $\pi^{*} > 0$, or $\Pi = \Pi^{o}$ (X continuous), that π^{*} satisfies the equations (6.17–18);

(b) in case $\Pi = \Pi^{+}$, $\pi^{*} \geq 0$, that π^{*} satisfies (6.19).

These conditions are also sufficient for optimality.

Remarks. Verification of sufficiency is straightforward and is omitted for brevity. It can be checked that the y^λ are true martingales for those λ with $\pi^*\lambda > 0$. A question not settled by the above proposition is whether there can be any optimum if there is no constant one. On the other hand, an optimal portfolio plan is in general not unique, cf. [S], so that if there is a constant optimum there may also be optima for which the portfolio plan is not constant.

7. MARTINGALE MEASURES FOR ASSET PRICES

This Section considers briefly the connection, in the present model, between martingale properties of shadow prices and the existence of 'martingale measures' for asset prices.

We first recall some concepts and results. Given $(\Omega,\underline{\underline{A}},P)$ and $\underline{A} = (\underline{\underline{A}}_t)$ with $\underline{\underline{A}} = \underline{\underline{A}}_\infty$ and another probability measure Q, we write $Q \sim P$ if Q is equivalent to P, or more generally $Q \sim P(\underline{\underline{B}})$ if the equivalence is asserted only on some sub σ-algebra $\underline{\underline{B}}$. Let P^B, Q^B denote the restrictions of P, Q to $(\Omega,\underline{\underline{B}})$, in particular write P^t, Q^t if $\underline{\underline{B}} = \underline{\underline{A}}_t$. If $Q \sim P$, there is a positive version L of the Radon–Nikodym derivative dQ/dP, and the formula $L_t = E_P^t L$ defines a positive and uniformly integrable $(P,\underline{A})$- martingale; here E_P^t means $E[\,.\,/\underline{A}_t]$ taken with respect to P. Further, L_t is a version of dQ^t/dP^t and is sometimes written $L_t = E_P^t(dQ/dP)$. Conversely, if (L_t) is a positive, uniformly integrable $(P,\underline{A})$- martingale such that $L_o = 1$, hence $L_t = E_P^t L_\infty$, and if Q is the probability defined on $\underline{A}$ by $dQ/dP = L_\infty$, then

$$L_t = E_P^t(dQ/dP) = dQ^t/dP^t, \qquad \ldots(7.1)$$

see [VSW], [BJ].

These results will be applied to the situation where a positive stopping time χ (the 'horizon') is given and $\underline{A}_\infty$ is replaced by $\underline{A}_\chi$, $\underline{A} = (\underline{A}_t)$ by $\underline{A}^\chi = (\underline{A}_{t\wedge\chi})$, P, Q, P^t, Q^t by the restrictions P^χ, Q^χ, $P^{t\wedge\chi}$, $Q^{t\wedge\chi}$, also E^t by $E^{t\wedge\chi}$, L_t by $L_{t\wedge\chi}$, L_∞ by L_χ. Then we say for short that the properties considered in the preceding paragraph hold 'up to χ'; thus '$Q \sim P$ up to χ' means '$Q^\chi \sim P^\chi$' etc.

Returning to our model, let the Λ-dimensional semimartingales X, Z be defined as usual, let an optimal plan be given for which the processes y^λ are all local martingales, and bear in mind the interpretation of $z_t^\lambda = \exp\{x_t^\lambda\}$ as the market price of asset λ at time t and of $y_t^\lambda = v_t z_t$ as the shadow price. Select one of the assets, say Λ, and define the $(\Lambda-1)$-semimartingale $\tilde{Z}$ with components $(\tilde{z}_t^\lambda;\ t\epsilon\underline{I})$, $\tilde{z}_t^\lambda = z_t^\lambda/z_t^\Lambda$, noting that

$$\tilde{z}_t^\lambda = z_t^\lambda/z_t^\Lambda = \exp\{x_t^\lambda - x_t^\Lambda\} = y_t^\lambda/y_t^\Lambda. \qquad \ldots(7.2)$$

The processes $\tilde{z}^\lambda$ represent '<u>Λ-discounted prices</u>' (the usual assumption, which we shall not need, being that Λ is a riskless asset). Given the investor's beliefs P, another probability Q, and a horizon χ, we say that Q is a <u>martingale measure for $\tilde{Z}$ up to χ</u> if $Q^\chi \sim P^\chi$ and for each $\lambda = 1,\ldots,\Lambda-1$ the stopped process $(\tilde{z}_{t\wedge\chi}^\lambda)$ is a uniformly integrable $(Q^\chi,\underline{A}^\chi)$-martingale. These definitions are straightforward extensions of those used in the finite-horizon theory of the pricing of contingent claims, see [HP].

Now let (χ_n) be a sequence of times reducing each of the $(P,\underline{A})$-local martingales (y_t^λ), $\lambda = 1,\ldots,\Lambda$, and fix $\chi = \chi_n$, so that the $(y_{t\wedge\chi}^\lambda)$ are uniformly integrable $(P^\chi,\underline{A}^\chi)$-martingales. Suppose w.l.o.g. that $y^\lambda(0) = y^*(0) = 1$ for each λ, so that in particular y_χ^Λ is a positive, $\underline{A}_\chi$-measurable variable with $Ey_\chi^\Lambda = 1$, and we may define a probability Q by $y_\chi^\Lambda = dQ/dP^\chi$. Then $Q^\chi = Q$ on $\underline{A}_\chi$ and we have, as in (1),

$$y_{t\wedge\chi}^\Lambda = E_P^{t\wedge\chi}\ y_\chi^\Lambda = dQ^{t\wedge\chi}/dP^{t\wedge\chi}. \qquad \ldots(7.3)$$

It can now be shown that Q^χ is a martingale measure for $\tilde{Z}$ up to χ. In fact, for $A\epsilon\underline{A}_{t\wedge\chi}$ we have

$$\int_A \tilde{z}_\chi^\lambda\, dQ^\chi$$

$$= \int_A (y_\chi^\lambda/y_\chi^\Lambda)\,dQ^\chi \qquad \text{by (2)}$$

$= \int_A y^\lambda_\chi \, dP^\chi$ definition of Q^χ

$= \int_A y^\lambda_{t\wedge\chi} \, dP^\chi$ martingale property of $(y^\lambda_{t\wedge\chi})$

$= \int_A y^\lambda_{t\wedge\chi} \, dP^{t\wedge\chi}$ obvious

$= \int_A (y^\lambda_{t\wedge\chi}/y^\Lambda_{t\wedge\chi}) dQ^{t\wedge\chi}$ by (3)

$= \int_A \tilde{z}^\lambda_{t\wedge\chi} \, dQ^{t\wedge\chi}$ by (2)

$= \int_A \tilde{z}^\lambda_{t\wedge\chi} \, dQ^\chi$ obvious.|| ....(7.4)

As an immediate consequence, we have

Proposition 9. Let an optimal plan be given such that the shadow price processes y^λ, $\lambda = 1,\ldots,\Lambda$, are $(\underline{A},P)$-local martingales reduced by a sequence (χ_n) of positive stopping times. For each n, let $\underline{A}_n = \underline{A}_{\chi_n}$, $\underline{A}^n = (\underline{A}_{t\wedge\chi_n};\ t\in\underline{I})$, $P^n = P/\underline{A}_n$, and let Q^n be the probability measure defined on $\underline{A}$ by $dQ^n/dP^n = y^\Lambda(\chi_n)$. Then Q^n is a martingale measure for $\tilde{Z}$ up to χ_n - in other words, the process $(\tilde{z}^\lambda(t\wedge\chi_n);\ t\in\underline{I})$, where $\tilde{z}^\lambda = z^\lambda/z^\Lambda = y^\lambda/y^\Lambda$, is a uniformly integrable $(Q^n,\underline{A}^n)$-martingale.

Remark. The significance of this proposition lies largely in what it does not say. Since the shadow price y^Λ will typically not be uniformly integrable to infinity - cf. the example in Section 6 - it cannot be asserted that there is a single martingale measure for asset prices up to infinity even if the y^λ are all (true) martingales. If $\Pi = \Pi^+$, the y^λ will not in general even be local martingales unless $\pi^* > 0$; however, if $\pi^\Lambda(t) > 0$ on $\underline{I}$ a.s., i.e. if security Λ is always held, the argument leading to Proposition 8 goes through except that the equality in the fourth line of (4) is replaced by $\leq$, showing that $(\tilde{z}^{\lambda}_{t\wedge\chi})$ is a local supermartingale. These points indicate serious limitations of the change-of-measure technique in the analysis of infinite-horizon models of market equilibrium when investors have diverse opinions and portfolios.

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