CERTAINTY EQUIVALENCE IN THE CONTINUOUS–TIME PORTFOLIO–CUM–SAVING MODEL †

by

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Abstract A model of optimal accumulation of capital and portfolio choice over an infinite horizon in continuous time is considered in which the vector process representing returns to investment is a general semimartingale with independent increments and the welfare functional has the 'discounted constant relative risk aversion' form. The following results are proved under slight conditions. If suitable variables are chosen, the sure (i.e. non–random) plans form a complete class. If an optimal plan exists, then a sure optimal plan exists, and conversely an optimal sure plan is optimal. The problem of portfolio choice can be separated from the problem of optimal saving. Conditions are given for the uniqueness of the portfolio plan generating a given returns process and for the uniqueness of an optimal plan.

Key Words Investment, portfolios, independent increments, risk aversion, certainty equivalence, optimisation.

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1. INTRODUCTION

The present article continues the author's recent work [1] on the infinite–horizon portfolio–cum–saving (PS) model with semimartingale investments. The model considers an investor holding divisible assets which can be traded at market prices without transaction costs, and receiving no income other than that derived from these assets – in short, the classical rentier. The investor seeks to maximise welfare, defined as the time integral of expected utility, by his choice of a consumption plan c and a portfolio plan π. Consumption and capital are constrained to be non–negative. A portfolio plan is defined in the main discussion as an adapted process, continuous on the left with limits on the right (*collor*), specifying the proportions of capital assigned to the available assets; in general, these proportions are constrained to be non–negative, but if the market process is continuous there is a variant of the model in which short sales are allowed.

We consider here the class of special cases where the market process – more precisely, the vector semimartingale $X = (x^1,\dots,x^\Lambda)$ representing logarithms of asset returns or prices – is a process with independent increments (PII) with respect to a given filtration, and the utility function has the discounted constant relative risk aversion (CRRA) form – see below, (2.15–17). The object is to give a precise and general formulation of the properties of certainty equivalence (CE) and portfolio separation which are characteristic of the PII/CRRA model and to prove them, as far as possible, by arguments based directly on the structure of the model.

The properties in question may be summarised as follows. We say that a PS plan is *sure* if the ratio $c_t/k_t = \theta_t$ of consumption to capital and the portfolio composition vector $\pi_t = (\pi_t^1,\dots,\pi_t^\Lambda)$ with $\Sigma_\lambda \pi_t^\lambda = 1$ are deterministic functions of t, and that the plan is *invariable* if in addition θ_t and π_t are constant over time. The *First Certainty Equivalence Theorem* asserts that *an optimal sure plan is optimal* – more explicitly, a plan which is optimal in the class of all sure plans is optimal (in the class of all feasible plans). The *Second Certainty Equivalence Theorem* asserts that, *if an optimal plan exists at all, then a sure optimal plan exists* (i.e. a plan exists which is both optimal and sure). The *Complete Class Theorem for Sure Plans* asserts that, *for every plan with finite welfare, there is a sure plan which yields at least the same level of welfare.* Let b denote the coefficient of CRRA and let $\varphi^* = \varphi^*(b)$ be the supremum of the welfare functional. The Complete Class Theorem is proved here for $b > 1$, also for $b < 1$ if $\varphi^*(\beta) < \infty$ for some $\beta \in (0,b)$. A separate version of the theorem is proved for $b = 1$. When true, the Complete Class Theorem implies both of the CE Theorems. The latter are true anyway for all b, though this is fully proved here only in the case of the Second Theorem. If X is a process with stationary independent increments (PSII) then 'sure' may be replaced in the CE Theorems by 'invariable'.

A further important property is that of *Portfolio Separation*, according to which the problem of optimal portfolio selection may be formulated without explicit reference to optimal consumption. Let x^π denote the logarithmic return (or compound interest) process generated by a portfolio plan π and let $b \neq 1$; if π is sure, then x^π is a PII. It is shown that, if an optimal sure plan (θ^*,π^*) exists, then π^* maximises the function

$$\Psi(\pi,T) = (1-b)^{-1}\ell n\, Ee^{(1-b)x^{\pi}(T)} \qquad ...(1.1)$$

among all sure π for each T; conversely, if a sure π^* maximises (1) among all sure π for each T, then there is an optimal sure plan of the form (θ^*,π^*) provided that the distant future is sufficiently discounted. Taking into account the CE Theorems, the problem of choosing an optimal PS plan may therefore be reduced to the problem of choosing an optimal sure portfolio plan, defined as one which maximises $\Psi(\pi,T)$ for each T. Similar results hold if $b = 1$, with (1) replaced by

$$\Psi^{\ell}(\pi,T) = Ex^{\pi}(T). \qquad ...(1.2)$$

These results greatly simplify the investigation of the existence (or non–existence) and characterisation of optimal plans in the PII/CRRA model. For reasons of space, these topics are deferred to a separate paper [3], but a brief trailer is in order. For a sure π, the function $\Psi(\pi,.)$, if finite, may be represented by a certain integral – related to the Lévy–Khinchin formula – involving π and the characteristics of the PII–semimartingale X. Using this representation and the sufficient conditions for optimality in [1], a separate proof of the First CE Theorem is given, thus providing an alternative justification for the restriction of portfolio choice to sure π. Results on the existence and characterisation of optimal sure portfolio plans are easily obtained if the characteristics of X are assumed to satisfy some conditions of smoothness and non–degeneracy, as well as inequalities which ensure in particular that Ψ is finite for all T and all sure π. Then it is shown that a sure π^* is optimal, in the sense that it maximises $\Psi(\pi,T)$ among all sure π for each T, iff for each t the vector π^*_t maximises the derivative $\psi(\pi_t,t) = (\partial/\partial t)\Psi(\pi,t)$. In case the restriction $\pi \geq 0$ (no short sales) applies, it is found that $\psi(.,t)$ is for each t a strictly concave function on a unit simplex $\mathcal{S}$ – see (5.1) below – and that $\psi(.,.)$ is continuous. The existence of a maximum of $\psi(.,t)$ on $\mathcal{S}$ at some unique $\pi^*(t)$ and the continuity of $\pi^*(.)$ and $\psi(\pi^*(.),.)$ then follow directly; moreover conditions characterising $\pi^*(t)$ are obtained by elementary concave programming. This argument does not apply in the case of continuous X with short sales permitted, but existence and characterisation of an optimum are obtained very simply from conditions for a maximum of Ψ or of ψ provided that the covariance matrix of $X_T - X_S$ for $S < T$ is always positive definite. If the characteristics of X are not sufficiently smooth, then typically the class of admissible portfolio plans must be extended beyond the continuous, and even the collor, functions if an optimal sure portfolio plan is to exist.

It is, of course, well known that in particular models of the PII/CRRA type which have been studied in detail – notably discrete–time models and models driven by Brownian motion – optimal plans have the properties that the ratio of consumption to capital and the portfolio composition are independent of both the external state and the current capital stock. The contribution of the present paper is, first, to formulate the properties of certainty equivalence and portfolio separation quite generally, as regards both their content and the class of PII–semimartingales considered, and secondly to give proofs based on the structure of the model rather than obtaining the results as corollaries of conditions of optimality derived by applying some general technique of optimisation. The present procedure involves extra work at the outset, but yields new insights and results – notably the complete class theorem – and greatly simplifies the treatment of existence and characterisation of optima.

The rest of this paper is arranged as follows. We begin in Section 2 by recalling some definitions and formulae from [1] and setting out some additional preliminaries. This Section also contains some results, which seem to be new in part, on the uniqueness of the portfolio plan generating a given returns process and on the uniqueness of an optimal plan. Proofs of the certainty equivalence and complete class theorems – independent of [1] apart from some preliminary results – are given in Section 3. Section 4 derives the optimal consumption plan for a given sure π and formulates the principle of portfolio separation. An alternative method of proving the First CE Theorem is indicated. The special case where X is a PSII is briefly considered in Section 5. Up to this point it is assumed that $b \neq 1$, and Section 6 reviews the modifications needed in the case of logarithmic utility. A postscript considers how far the present methods and results can be extended to cases where the characteristics of X are not sufficiently smooth to permit the existence of an optimum with a collor portfolio plan.

2. THE MODEL

The PS model considered here is a special case of that in [1], and unless otherwise stated the definitions and assumptions stated there continue to apply. In order to keep the present exposition more or less self–contained, we shall recall the main features of the model; additional conditions are distinguished by writing *here*.

There is given a time domain $\mathscr{T} = [0,\infty)$, a complete probability space $(\Omega, \mathscr{A}, P)$ with a filtration $\mathfrak{A} = (\mathscr{A}_t;\ t \in \mathscr{T})$ satisfying the 'usual conditions' of right continuity and completeness, where $\mathscr{A} = \mathscr{A}_\infty$; also $\mathscr{A}_0 = \mathscr{A}_{0-}$ is generated by the P–null sets, so that an $\mathscr{A}_0$–measurable variable is a.s. constant. The following conventions apply to processes and functions unless we state or imply otherwise. Scalar processes take finite real values, while vector processes are families of scalar processes with a finite number $\Lambda \geq 1$ of components, or equivalently $\mathfrak{R}^\Lambda$–valued functions of (ω,t). Unless otherwise stated, all processes considered are *here* assumed , or may easily be shown to be, adapted and either corlol or collor, hence at least optionally measurable. If processes z and z' are indistinguishable, we write $z \equiv z'$ and treat them as identical. For a scalar process, say z, $z > 0$ means $z(\omega,t) > 0$ for all (ω,t), and $z \geq 0$ means $z(\omega,t) \geq 0$ for all (ω,t), while similar notation for vector processes means that the condition applies to each component. The terms positive, negative, increasing, decreasing have their strict meaning throughout, but $\uparrow$, $\downarrow$ mean non–decreasing, non–increasing. Semimartingales and their components will by definition be finite on $\mathscr{T}$ and corlol and will almost surely not jump at $t=0$, so that for stochastic integrals we have $\int_{[0,T]} = \int_{(0,T]}$, which we usually write as $\int_0^T$. The concepts of semimartingale and PII are always defined relative to $\mathfrak{A}$. All PII considered *here* are assumed or may be shown to be semimartingales, so that usually we say simply 'PII' rather than 'PII–semimartingale' etc. Definitions and properties of such processes are given in detail in [4] and [5] and an excellent survey appears in [6].

As in [1], a finite number of assets (or securities) indexed by $\lambda = 1,\ldots,\Lambda$ are assumed to be available at all times. For each λ there is given a semimartingale x^λ with $x^\lambda(\omega,0) = 0$ called the *log–returns* or *compound interest process* for λ, and the formula $z^\lambda = e^{x^\lambda}$ defines a positive semimartingale called the *returns* or *price process* for λ. The vector $X = (x^1,\ldots,x^\Lambda)$ is called the *market log–returns process.* Decompositions of x^λ are written

$$x^\lambda = M^{\lambda c} + V^{\lambda c} + M^{\lambda d} + V^{\lambda d} \qquad \ldots(2.1)$$

where $M^{\lambda c}$, $M^{\lambda d}$ are continuous and compensated jump martingales respectively, $V^{\lambda c}$, $V^{\lambda d}$ are continuous and discontinuous processes of finite variation; all these processes vanish at $t=0$, and in general only $M^{\lambda c}$ is uniquely defined. We denote by $\langle \mathfrak{M}^c \rangle_T$ the matrix whose elements are the 'angle brackets' $\langle M^{\lambda c}, M^{\ell c}\rangle_T$, $\lambda, \ell = 1,\ldots,\Lambda$, and write $\langle M^{\lambda c}, M^{\lambda c}\rangle$ as $\langle M^{\lambda c}\rangle$ or $\langle x^{\lambda c}\rangle$. It is assumed *here* that X is a vector PII relative to the filtration $\mathfrak{A} = (\mathscr{A}_t)$, i.e. for each S, the increments $X_T - X_S$ for $T \geq S$ are independent of $\mathscr{A}_S$. In this case, the $\langle \mathfrak{M}^c \rangle_T - \langle \mathfrak{M}^c \rangle_S$ are *deterministic* non–negative definite, symmetric matrices.

A portfolio plan π or *π–plan* is defined as a vector process with components π^λ which is adapted collor for $t > 0$ and satisfies

$$\Sigma_\lambda \pi^\lambda(\omega,t) = 1 \qquad \ldots(2.2)$$

for all (ω,t). The vector $\pi(0)$ may be defined arbitrarily; usually we take

$\pi(0) = \pi(0+)$. We denote by Π^0 the set of all portfolio plans and by Π^+ the subset satisfying $\pi \geq 0$, or explicitly

$$0 \leq \pi^\lambda(\omega,t) \leq 1 \qquad \text{...(2.3)}$$

for all (ω,t) and each λ. The set of all π which are *admissible* in a particular problem is denoted by Π, and for reasons explained in [1] we assume that $\Pi = \Pi^+$ if X has jumps, while both the cases $\Pi = \Pi^0$ and $\Pi = \Pi^+$ are considered if X is continuous. We write simply Π, or omit to specify the admissible set, when it does not matter which case is considered.

Given a portfolio plan $\pi \epsilon \Pi$, the *portfolio returns process* z^π generated by π is defined as the unique semimartingale satisfying the equation

$$z^\pi(T) = 1 + \int_0^T \Sigma_\lambda \pi^\lambda(t) e^{-x^\lambda(t-)} de^{x^\lambda(t)}, \qquad \text{...(2.4)}$$

and the definition of π adopted above ensures that $z^\pi(T)$ and $z^\pi(T-)$ are defined and positive for all $T \epsilon \mathcal{T}$, a.s. – see [1] eq.(2.4–17) for details. Consequently the relation $z^\pi = e^{x^\pi}$ defines a semimartingale x^π on $\mathcal{T}$ called the *portfolio log–returns process*, or simply the *compound interest process* generated by π. The change–of–variables formula yields

$$\int_0^T e^{-x^\lambda(t-)} de^{x^\lambda(t)} = x_T^\lambda + \tfrac{1}{2}\langle x^{\lambda c}\rangle_T + \Sigma_{t \leq T}[e^{\Delta x^\lambda(t)} - 1 - \Delta x_t^\lambda], \qquad \text{...(2.5)}$$

the sum on the right converging absolutely for all T, a.s. Using this equation we may calculate x_T^π explicitly as

$$\begin{aligned} x^\pi(\omega,T) &= x_T^\pi \\ &= \textstyle\int \Sigma_\lambda \pi^\lambda \, dM^{\lambda c} \\ &\quad + \textstyle\int \Sigma_\lambda \pi^\lambda \, dV^{\lambda c} + \tfrac{1}{2}\int \Sigma_\lambda \pi^\lambda \, d\langle M^{\lambda c}\rangle - \tfrac{1}{2}\int \Sigma_\lambda \Sigma_\ell \pi^\lambda \pi^\ell \, d\langle M^{\lambda c}, M^{\ell c}\rangle \\ &\quad + \textstyle\int \Sigma_\lambda \pi^\lambda \, dM^{\lambda d} \\ &\quad + \Sigma_{t \leq T}[\Delta x^\pi - \Sigma_\lambda \pi^\lambda \, \Delta M^\lambda], \end{aligned} \qquad \text{...(2.6)}$$

the sum over t in the last line converging absolutely for all T, a.s; here

$\int = \int_0^T$, all variables and angle brackets on the right of the equation should have the subscript t, and

$$\Delta x_t^\pi = \ell n\left[\Sigma_\lambda \pi_t^\lambda \, e^{\Delta x^\lambda(t)}\right]. \qquad ...(2.7)$$

The first and third lines of (6) represent local martingales, the second and fourth processes of finite variation, so that the equation gives a decomposition of x^π analogous to that of x^λ in (1).

It is assumed that the investor has an initial capital $K_0 > 0$ and no 'outside income'. Given $\pi \in \Pi$ and x^π, a π–feasible *consumption plan in natural units*, or $\bar{c}$*–plan*, is defined *here* as an (adapted) positive corlol process $\bar{c} = \bar{c}(\omega,t)$ such that a.s. the equation

$$\int_0^T \frac{d\bar{k}(t)}{\bar{k}(t-)} = \int_0^T e^{-x^\pi(t-)} de^{x^\pi(t)} - \int_0^T \frac{\bar{c}(t)}{\bar{k}(t-)} dt, \qquad \bar{k}(0) = K_0, \qquad ...(2.8)$$

has a semimartingale solution $\bar{k}$ which is positive on $\mathcal{T}$; this solution is unique and is called the *capital plan in natural units* corresponding to $\bar{c}$.[1] On introducing new processes $c = c(\omega,t)$, $k = k(\omega,t)$ by

$$c(t) = \bar{c}(t)e^{-x^\pi(t)}, \quad k(t) = \bar{k}(t)e^{-x^\pi(t)}, \qquad ...(2.9)$$

the equation of accumulation (8) becomes simply

$$k(T) = K_0 - \int_0^T c(t)dt. \qquad ...(2.10)$$

The condition $\bar{k} > 0$ is equivalent to the two conditions $c > 0$ and

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[1] The definition *here* is more restrictive than that in [1], where a $\bar{c}$–plan was required to be a.s. locally integrable instead of corlol. However, nothing is lost as far as the search for an optimum is concerned, since Theorem 1 of [1] implies that under present assumptions the shadow price process $y^* = (\bar{c})^{-b} e^{(1-b)x^\pi} q$ defined by an optimal plan is a local martingale, so that an optimal $\bar{c}$–plan must in fact be a semimartingale. For simplicity we have also specified *here* that $\bar{c} > 0, \bar{k} > 0$ rather than $\bar{c} \geq 0, \bar{k} \geq 0$, but this property also must hold at an optimum.

$$\int_0^\infty c(t)dt \le K_0 \quad \text{a.s.,} \qquad \text{...(2.11)}$$

which no longer involve x^π or k. A *consumption plan in standardised units,* or *c–plan,* can therefore be defined directly as a positive corlol process $c = c(\omega,t)$ satisfying (11) a.s., and then k can be defined by (10) if required. We denote by $\mathscr{C}$ the set of all c–plans. Every $\bar{c}$ which is feasible for some π can be obtained by specifying a pair (c,π) and writing $\bar{c} = c \cdot e^{x^\pi}$, and one can take $\mathscr{C} \times \Pi$ as the set of feasible *plans* for the PS model. For a given plan, the *consumption ratio plan* $\theta = \theta(\omega,t)$ is defined *here* by

$$\theta_T = c_T/k_T = \bar{c}_T/\bar{k}_T\ , \qquad \text{...(2.12)}$$

hence, using $c = -\dot{k}$,

$$k_T = K_0 \cdot \exp\{-\textstyle\int_0^T \theta(t)dt\}, \qquad c_T = \theta_T K_T = K_0 \theta_T \cdot \exp\{-\int_0^T \theta(t)dt\}. \quad \text{...(2.13)}$$

Conversely, *any* positive corlol process θ defines a c–plan by way of (2.13). The advantage of this method (which was not used in [1]) is that *no separate integral constraint need be specified*; in fact, (11) holds as an equality iff $\int_0^T \theta_t dt \to \infty$ with T. Clearly, a plan can also be specified as a pair (θ,π), which we often denote *here* by $f = (\theta,\pi)$, the corresponding feasible set being $\mathscr{F} = \Theta \times \Pi$. A plan in the form $f = (\theta,\pi)$ *can be defined separately for disjoint (ordinary or stochastic) time intervals,* subject to the minor inconvenience that θ is by definition right continuous, π left continuous. Thus we sometimes work with the right continuous version $f_+ = (\theta,\pi_+)$, or with the left continuous version $f_- = (\theta_-,\pi)$. Then, if f^0 and f^1 are two plans and T is a finite stopping time, a new plan ${}^0f^{1T}$ may be defined whose right and left continuous versions are defined for $t \ge 0$ by

$$\begin{aligned} {}^0f^{1T}_{t+} &= f^1_{t+}\, I_{\{t\ge T\}} + f^0_{t+}\, I_{\{t<T\}}\ , \\ {}^0f^{1T}_{t-} &= f^1_{t-}\, I_{\{t>T\}} + f^0_{t-}\, I_{\{t\le T\}}\ ; \end{aligned} \qquad \text{...(2.14)}$$

this new plan is called 'f^0 before T, f^1 after t'. Suitable French terminology might be 'f^0 avant le five–o'clock' etc. (If $T = t = 0$, we set ${}^0f^{1o}(0+) = {}^0f^{1o}(0-) = f^1(0+)$; in future we shall omit such pedantic qualifications). Sometimes we also consider θ_- or π_+ separately. In the sequel, we usually label processes corresponding to different plans with suitable superscripts, sometimes without special comment. Note that, for a given plan, the variables π_T, c_T, k_T, $\bar{c}_T$, $\bar{k}_T$, θ_T at a given time T *may depend on the whole observed past before* T, indeed c_T, $\bar{c}_T$, $\bar{k}_T$, θ_T may also depend on jumps occurring at T.

Following the usual neo–classical approach, it is assumed in [1] that the investor seeks to maximise a functional of the form

$$\bar{\varphi}(\bar{c}) = E\int_0^\infty \bar{u}[\bar{c}(\omega,t);\ \omega,t]dt, \qquad \text{...(2.15)}$$

where $\bar{u}$ is called the *utility function* and $\bar{\varphi}$ the *welfare functional.* Denoting by $\bar{u}' = \partial\bar{u}/\partial\bar{c}$ the marginal utility function, the 'discounted CRRA' form assumed *here* specifies

$$\bar{u}'[\bar{c}(\omega,t);\ \omega,t] = \bar{c}(\omega,t)^{-b}q(t), \qquad b > 0, \qquad \text{...(2.16)}$$

where the *discount density* $q(t)$ is a (deterministic) positive corlol function of finite variation on compacts of $\mathcal{T}$ (hence a semimartingale). Thus we may set

$$\bar{u}(\bar{c}_t,t) = (1-b)^{-1}\,\bar{c}_t^{1-b}q_t \qquad \text{if } b \neq 1;$$

$$\bar{u}(\bar{c}_t,t) = (\ell n\ \bar{c}_t)q_t \qquad \text{if } b = 1. \qquad \text{...(2.17)}$$

When X is a PSII we adopt the usual assumption that $q(t) \propto e^{-rt}$ with r constant. Using the transformation (9), the functional (15) may be rewritten for $b \neq 1$ as

$$\varphi(c,\pi) = (1-b)^{-1}E\int_0^\infty c(\omega,t)^{1-b}\,e^{(1-b)x^\pi(\omega,t)}q(t)dt. \qquad \text{...(2.18)}$$

For brevity we often write

$$\eta^{\pi}(t) = e^{(1-b)x^{\pi}(t)}, \qquad \text{...(2.19)}$$

similarly η^* when $\pi = \pi^*$. It will be convenient to take (18) as the basis of the present discussion, and for the time being to consider only cases with $b \neq 1$. When written in the form (18), the domain of the welfare functional is taken to be $\mathscr{C}\times\Pi$, and its supremum on this set is denoted φ^* or $\varphi^*(b)$. If φ^* is finite (and only then) an element (c^*,π^*) for which $\varphi(c^*,\pi^*) = \varphi^*$ is called *optimal*, and the problem of optimal saving and portfolio choice is to select such an element if one exists. Similarly, we say that c^* is π^**–optimal* if $\varphi(c,\pi^*)$ attains a finite maximum on $\mathscr{C}$ at c^*. Given two plans with finite welfare, we say that the one is *better than/at least as good as* the other if the value of welfare for the one is greater than/at least as great as the welfare for the other. The terminology extends in an obvious way to optima and welfare rankings relative to subsets of $\mathscr{C}\times\Pi$, $\mathscr{C}$, and Π. If $b < 1$, so that φ is positive, finiteness of φ^* requires that there should be no plan with $\varphi(c,\pi) = \infty$; but if $b > 1$, so that φ is negative, it is permissible to have $\varphi = -\infty$ for some plans but not for all.

When a plan is specified in the form $f = (\theta,\pi)$, we write $\varphi(f) = \varphi(\theta,\pi)$ instead of $\varphi(c,\pi)$ and speak of an optimal plan $f^* = (\theta^*,\pi^*)$ etc. The functional (18) can then be rewritten with c replaced by its formula from (13), and then it is usually convenient to set $K_0 = 1$. Further, given a plan f and stopping times $0 \leq T < \tau \leq \infty$, we define a random variable $\phi_{[T,\tau)}(\omega,f)$, called *(rescaled) welfare during* $[T,\tau)$, by

$$\phi_{[T,\tau)}(f) = (1-b)^{-1}\int_{[T,\tau)} (c_t/k_T)^{1-b}(\eta_t^{\pi}/\eta_T^{\pi})(q_t/q_T)dt$$

$$= (1-b)^{-1} \int_{[T,\tau)} \theta_t^{1-b} \cdot \exp\{-(1-b)\int_T^t \theta_s ds\} \cdot \eta_t^\pi / \eta_T^\pi)(q_t/q_T)dt. \qquad ...(2.20)$$

If $\tau = \infty$, we write $\phi_T(f)$ or $\phi(\omega,T;f)$ and call this (*rescaled*) *welfare after* T. If $f = {}^0f^{1T}$ is defined as in (14), then

$$\phi(f) = \phi_{[0,T)}(f^0) + (k_T^0)^{1-b} \eta_T^0 q_T \phi_{[T,\infty)}(f^1). \qquad ...(2.21)$$

For any f and T we also write

$$\Phi(\omega,T;f) = \Phi_T(f) = E^T\phi_T(f), \quad E^T = E(./\mathscr{A}_T), \qquad ...(2.22)$$

and the convention $K_0 = 1$ ensures that $\Phi_0(f) = E\phi_0(f) = \varphi(f)$. It is easily seen that a plan f^* is optimal (in a given set $\mathscr{T}$) iff, for each $T \in \mathscr{T}$, $\Phi_T(f^*) \geq \Phi_T(f)$ a.s. for each f.

Special importance attaches to sure plans. We say that π, c, (c,π) are *sure* if π_t, c_t, (c_t,π_t) are equal (almost surely) to non–random functions on $\mathscr{T}$, and we denote by Π^s, Π^{os}, Π^{+s}, $\mathscr{C}^s$ etc. the corresponding subsets of Π, Π^o, Π^+, $\mathscr{C}$ etc. Note that θ is sure iff c_t is sure. Again, we say that π is *invariable* if it is sure and there is a vector $\check{\pi}$ such that $\pi_t = \check{\pi}$ for all t; that c (or equivalently θ) is *invariable* if it is sure and there is a constant $\check{\theta} > 0$ such that $\theta_t = \check{\theta}$ for all t; and that the plan (c,π) or (θ,π) is invariable if both components are invariable. The corresponding sets are written Π^i, Π^{oi}, Π^{+i}, $\mathscr{C}^i$ etc. An element (c^*,π^*) of $\mathscr{C}^s \times \Pi^s$ is called an *optimal sure plan* if the functional φ attains a (finite) maximum on $\mathscr{C}^s \times \Pi^s$ at that point. This concept is to be distinguished from a *sure optimum*, which is an element of $\mathscr{C}^s \times \Pi^s$ at which φ attains a finite maximum on $\mathscr{C} \times \Pi$. In the same way, we speak of an *optimal invariable plan* and an *invariable optimum*. Expressions such as optimal sure π–plan, sure optimal π–plan, optimal sure c–plan are used analogously in cases where a separate concept of optimality for π or c is defined.

UNIQUENESS CONDITIONS.

To conclude this Section, we consider briefly some uniqueness and convexity properties of plans. First, we seek conditions under which distinct π generate distinct x^{π}; it is well known that this is not the case generally even in one–period models. Let π^o, π^1 be elements of Π generating x^o, x^1. Let M^{oc}, M^{1c} be the continuous martingale parts of x^o, x^1 – i.e. the terms given by the first line on the right of (6) – and let Δx^o_t, Δx^1_t be the jump terms given by (7). We write $\delta\pi = \pi^1 - \pi^o$, $\delta x = x^1 - x^o$, $\delta M^{\pi c} = M^{1c} - M^{oc}$, etc. Suppose $\delta x \equiv 0$, i.e. $\delta x_t = 0$ on $\mathcal{T}$ a.s. This implies $\delta M^{\pi c} \equiv 0$, see [4] 2.23, hence, a.s. for each pair $S < T$ from $\mathcal{T}$,

$$0 = \langle \delta M^{\pi c}\rangle_T - \langle \delta M^{\pi c}\rangle_S = \int_S^T \Sigma_\lambda \Sigma_\ell \, \delta\pi^\lambda_t \delta\pi^\ell_t \, d\langle M^{\lambda c}, M^{\ell c}\rangle_t. \qquad ...(2.23)$$

Since $\delta\pi$ is collor, the following condition is sufficient to ensure that $\delta\pi \equiv 0$: For each pair $S < T$,

$\langle \mathfrak{M}^c\rangle_T - \langle \mathfrak{M}^c\rangle_S$ *is a positive definite symmetric matrix.* ...(2.24)

Alternatively, uniqueness may be inferred from assumptions about the jumps of X. Consider the 'square brackets' process $[\delta x]$ defined by

$$[\delta x]_T = \langle \delta M^{\pi c}\rangle_T + \Sigma_{t \le T}(\delta \Delta x_t)^2. \qquad ...(2.25)$$

The hypothesis $\delta x \equiv 0$ implies $[\delta x] \equiv 0$, and since the first term on the right vanishes, so does the second; using (7) and $\Sigma_\lambda \delta\pi^\lambda_t = 0$, this implies

$$0 = \Sigma_{t \le T}\Big[\Sigma_\lambda \delta\pi^\lambda_t \{e^{\Delta x^\lambda(t)} - 1\}\Big]^2, \qquad T \in \mathcal{T}, \text{ a.s.} \qquad ...(2.26)$$

Letting μ denote the integer–valued random measure associated with the jumps of X, see [4] 3.22 or [5] II.1.16, this may be written as

$$0 = \int_{[0,T]} \int_\xi \Big[\Sigma_\lambda \delta\pi^\lambda_t \{e^{\xi^\lambda} - 1\}\Big]^2 \mu(d\xi, dt), \qquad T \in \mathcal{T}, \text{ a.s.,} \qquad ...(2.27)$$

where the inner integral is taken over the 'jump vectors' $\xi = (\xi^1, \ldots \xi^\Lambda)$ in $\mathfrak{R}^\Lambda \setminus \{0\}$. Since the double integral vanishes, so does its compensator, and taking into account that $\delta\pi$ is predictable we may replace μ in (27) by its compensator F, see [4] 3.15, [5] II.1.8; note that F is *deterministic* since X is a PII. Now F can be factorised in the form $F(d\xi,dt) = f(d\xi,t)dG(t)$, where $f(\cdot,t) = f_t(\cdot)$ is for each t a non–negative ('Lévy') measure on $\Re^\Lambda$ satisfying $f_t(\{0\}) = 0$ and $\int_{\Re^\Lambda}\{1+\|\xi\|^2\}f_t(d\xi) < \infty$, while G is a non–decreasing, right continuous function on $\mathcal{T}$ with $G(0) = 0$. We leave aside any components of X corresponding to fixed discontinuities or to a singular part of G, and assume that

G *is absolutely continuous on* $\mathcal{T}$ *with*

$$g(t) = dG(t)/dt > 0 \quad \text{a.a. } t\in\mathcal{T} \qquad ...(2.28)$$

and $$f_t\left[\Re^\Lambda\backslash\{0\}\right] > 0 \quad \text{a.a. } t\in\mathcal{T} \qquad ...(2.29)$$

This assumption ensures that the probability of a (moving) discontinuity of X occurring during any interval (S,T] is positive. Now $\mu(d\xi,dt)$ can be replaced in (27) by $f(d\xi,t)g(t)dt$, and then (27) leads to

$$0 = \Sigma_\lambda \delta\pi_t^\lambda(e^{\xi^\lambda} - 1) \quad \text{for } f_t\text{–almost all } \xi\in\Re^\Lambda, \text{ a.a. } t\in\mathcal{T}. \qquad ...(2.30)$$

The equation above says that the vector $\delta\pi_t$ must be orthogonal in $\Re^\Lambda$ to the (variable) vector $e^\xi - 1$; in addition, since $\Sigma_\lambda \delta\pi_t^\lambda = 0$, $\delta\pi_t$ must be orthogonal to 1. Under the further assumption that

the support of f_t *is* Λ*–dimensional*, a.a. $t\in\mathcal{T}$, ...(2.31)

it is seen that we must have $\delta\pi_t = 0$ a.s. for a.a. t, indeed $\delta\pi = 0$ on $\mathcal{T}$ a.s. since $\delta\pi$ is required to be collor. The substance of (31) is roughly that the distribution of ΔX_t conditional on $\Delta X_t \neq 0$ (when it exists) is not concentrated on any set of $\Re^\Lambda$ generating a proper linear subspace. Actually, it is enough if the support of f_t generates a $(\Lambda-1)$–dimensional linear subspace not containing 1 – a useful remark, since it yields a jump condition for portfolio uniqueness in case there is a riskless asset. Uniqueness conditions can also be given for cases where X has fixed discontinuities.

Note next that x^π is a concave function of π in the following sense. With π^0, π^1 and x^0, x^1 as above, let $\tilde{\pi}^\alpha$ be defined a.s. for $t \in \mathcal{T}$ by

$$\tilde{\pi}^\alpha_t = \alpha\pi^1_t + (1-\alpha)\pi^0_t, \qquad 0 < \alpha < 1; \qquad \ldots(2.32)$$

obviously $\tilde{\pi}^\alpha$ is in Π and generates some x^α. Referring to (6–7), it is seen that for given $S < T$ from $\mathcal{T}$ the function assigning to each α the random variable $x^\alpha_T - x^\alpha_S$ is concave in α. More precisely, the variables in the first and third lines on the right of (6) are linear in α, as are the first two terms in the second line; the term $-\frac{1}{2}\int_S^T \Sigma_\lambda \Sigma_\ell \tilde{\pi}^{\alpha\lambda} \tilde{\pi}^{\alpha\ell}\, d\langle M^{\lambda c}, M^{\ell c}\rangle$ is a.s. defined and finite, and it is concave in α because $\langle \mathfrak{M}^c\rangle_T - \langle \mathfrak{M}^c\rangle_S$ is non–negative definite; and the last line of (6) is also a.s. defined and finite and is concave by (7) because the log function is convave. It follows that

$$x^\alpha_T - x^\alpha_S \geq \alpha(x^1_T - x^1_S) + (1-\alpha)(x^0_T - x^0_S), \quad 0 < \alpha < 1, \qquad \ldots(2.33)$$

a.s. for $S < T$. (The use of α both as a number and as an index should cause no confusion).

The inequality (33) *is strict with positive probability if* $x^0 \not\equiv x^1$ *on* $(S,T]$ *and either* (24) *or both* (28) *and* (31) *are satisfied.* Let us briefly verify the following form of this assertion, which will be needed below: if $x^0 \not\equiv x^1$, there is a $T \in \mathcal{T}$ and a set $A \in \mathcal{A}_T$ with $PA > 0$ such that

$$x^\alpha_t > \alpha x^1_t + (1-\alpha)x^0_t \quad \text{for } t \geq T,\ \omega \in A,\ 0 < \alpha < 1. \qquad \ldots(2.34)$$

It is only necessary to prove the inequality for $t=T$, since (33) does the rest. Now, if $x^0 \not\equiv x^1$, there is some $T \in \mathcal{T}$ and $C = \{\omega: \delta x(\omega,T) \neq 0\}$ with $PC > 0$. It follows from (6) and the 'local character' of the stochastic integral, [7] VIII.23, that there is a predictable set $H = \{(\omega,t): t \leq T \ \& \ \delta\pi(\omega,t) \neq 0\}$ whose sections H_ω are non–empty for P–almost all $\omega \in C$, hence have positive Lebesgue measure since $\delta\pi$ is collor. Under (24), the term

$-\frac{1}{2}\int_{H_\omega}\Sigma_\lambda\Sigma_\ell\tilde{\pi}^{\alpha\lambda}\tilde{\pi}^{\alpha\ell}\, d\langle M^{\lambda c},M^{\ell c}\rangle$ is strictly concave for $\omega\in C$, and taking into account the concavity of the remaining terms in (6) this implies (34) with $A = C$ a.s. Alternatively, under (28) and (31), we may take $A = \{\omega: \Sigma_{t\leq T}(\delta\Delta x_t)^2 > 0\}$. To see this, note that if $\omega\in A$, then $\Delta x_t^1 \neq \Delta x_t^o$ for some $t \leq T$, and it follows from (7) and the strict concavity of the log function that $\Delta x_t^\alpha > \alpha\Delta x_t^1 + (1-\alpha)\Delta x_t^o$; taking into account that all terms in (6) are concave this implies $x_T^\alpha > \alpha x_T^1 + (1-\alpha)x_T^o$ a.s. for $\omega\in A$, and of course $A \subset C$. It remains to check that $PA > 0$; indeed, arguing as in (25–31) above, $PA = 0$ would imply $\delta\pi_t \equiv 0$ for $t \leq T$, contrary to assumption.

We next consider uniqueness properties of *optimal* plans. First an optimal $\bar{c}^*$ is unique. Indeed, let $(\bar{c}^o,\pi^o)$ and $(\bar{c}^1,\pi^1)$ be optimal, so that $\bar{\varphi}(\bar{c}^o) = \bar{\varphi}(\bar{c}^1) = \varphi^*$. Let $\bar{c}^\alpha = \alpha\bar{c}^1 + (1-\alpha)\bar{c}^o$, $0 < \alpha < 1$. It may be shown, see [1] (4.3–5), that there is a $\pi^\alpha \in \Pi$ such that c^α is π^α–feasible, i.e. π^α generates an x^α such that $c^\alpha = \bar{c}^\alpha e^{-x^\alpha} \in \mathscr{C}$; it suffices to set

$$\pi_t^{\alpha\lambda} = [\alpha\pi_t^{1\lambda}\bar{k}_t^1 + (1-\alpha)\pi_t^{o\lambda}\bar{k}_t^o]/[\alpha\bar{k}_t^1 + (1-\alpha)\bar{k}_t^o]. \qquad ...(2.35)$$

(Note that $\tilde{\pi}^\alpha$ as defined in (32) is not suitable). It then follows from the strict concavity of $(1-b)^{-1}\bar{c}^{1-b}$ that

$$(1-b)^{-1}[\bar{c}_t^\alpha]^{1-b} \geq (1-b)^{-1}[\alpha(\bar{c}_t^1)^{1-b} + (1-\alpha)(\bar{c}_t^o)^{1-b}] \qquad ...(2.36)$$

with strict inequality if $\bar{c}^1(\omega,t) \neq \bar{c}^o(\omega,t)$, hence from (15) and (17) that

$$\bar{\varphi}(\bar{c}^\alpha) \geq \alpha\bar{\varphi}(\bar{c}^1) + (1-\alpha)\bar{\varphi}(\bar{c}^o)^{1-b} = \varphi^* \qquad ...(2.37)$$

with strict inequality unless $\bar{c}^1 \equiv \bar{c}^o$; since $\bar{\varphi}(\bar{c}^\alpha) \leq \varphi^*$ by optimality, the assertion follows.

For the processes c^*, π^*, x^* associated with an optimal plan, the results concerning uniqueness are less clear–cut. If (c^o,π^*) and (c^1,π^*) are optimal (with the *same* π^*), then $c^o \equiv c^1$; this follows immediately from the convexity of the set $\mathscr{C}$ and inequalities like (36) and (37), with $\bar{c}$ replaced by c and $\bar{\varphi}(\bar{c})$ by $\varphi(c,\pi^*)$ as defined in (18). In the same way, a c–plan which is π–optimal for an arbitrary fixed π is unique. Now let (c^0,π^0,x^0) and (c^1,π^1,x^1) be two optimal plans. By the preceding argument, we may write

$$c^0e^{-x^0} \equiv c^1e^{-x^1} \equiv \bar{c}^*, \qquad \text{...(2.38)}$$

where the identity defines $\bar{c}^*$. Obviously $c^0 \equiv c^1$ iff $x^0 \equiv x^1$. Now define $\tilde{\pi}^\alpha$ as in (32), let $\tilde{\pi}^\alpha$ generate x^α, and define processes c^α, $\tilde{\bar{c}}^\alpha$ by

$$c_t^\alpha = \alpha c_t^1 + (1-\alpha)c_t^0, \qquad \tilde{\bar{c}}_t^\alpha = c_t^\alpha e^{x^\alpha(t)}; \qquad \text{...(2.39)}$$

then $c^\alpha \in \mathscr{C}$, so $\tilde{\bar{c}}^\alpha$ is $\tilde{\pi}^\alpha$–feasible. Omitting the variables (ω,t) for brevity, we calculate

$$\begin{aligned} \tilde{\bar{c}}^\alpha &= c^\alpha e^{x^\alpha} \geq [\alpha c^1 + (1-\alpha)c^0]e^{\alpha x^1+(1-\alpha)x^0} \\ &= \bar{c}^*[\alpha e^{(1-\alpha)(x^0-x^1)} + (1-\alpha)e^{\alpha(x^1-x^0)}] \geq \bar{c}^*; \end{aligned} \qquad \text{...(2.40)}$$

the first inequality follows from (33) above, the equalitites from (38–39), and the last inequality from the convexity of the exponential function. Since $\bar{c}^*$ is optimal, the inequalities must in fact be equalities for $t \in \mathscr{T}$ a.s., and $(c^\alpha, \pi^\alpha, x^\alpha)$ is optimal for each α. Now assume either (24) or both (28) and (31). Then if $x^0 \not\equiv x^1$ there is a $T \in \mathscr{T}$ and an event $A \in \mathscr{A}_T$ with $PA > 0$ satisfying the strict inequality (34); but then the first inequality in (40) is also strict, contrary to the result obtained previously, implying that in fact $x^0 \equiv x^1$. It then further follows that $\pi^0 \equiv \pi^1$.

This discussion may be summed up by

PROPOSITION 1: *Uniqueness of Optimal Plans.*

If (c^*,π^*) is optimal, then

(i) for any optimal plan (c^o,π^o), we have $\bar{c}^o \equiv \bar{c}^*$;

(ii) for any optimal plan (c^o,π^*), we have $c^o \equiv c^*$;

(iii) for any optimal plan (c^*,π^o), we have $x^o \equiv x^*$.

(iv) if (24) holds, then for any optimal plan (c^o,π^o) we have $c^o \equiv c^*$, $x^o \equiv x^*$, $\pi^o \equiv \pi^*$; similarly if (28) and (31) hold.

3. CERTAINTY EQUIVALENCE

This Section gives direct proofs of the certainty equivalence properties. The proofs make use of some additional assumptions which are set out after the statement of the Theorems. We continue to assume $b \neq 1$.

THEOREM 1: *First Certainty Equivalence Theorem.*

An optimal sure plan is optimal.

THEOREM 2: *Second Certainty Equivalence Theorem.*

If an optimal plan exists, a sure optimal plan exists.

THEOREM 3: *Completeness of the Class of Sure Plans.* Given any plan f^o with $\varphi(f^o)$ finite, there exists a sure plan $f^\diamond$ with $\varphi(f^\diamond)$ finite such that $\varphi(f^o) \leq \varphi(f^\diamond)$.

Theorems 1 and 2 are immediate consequences of Theorem 3, which is proved below. Indeed, Theorem 3 implies that $\sup \varphi(f)$ taken over all sure plans is at least as great as $\sup \varphi(f)$ taken over all plans with $\varphi(f)$ finite; write this temporarily as $\alpha \geq \beta$. If an optimal plan f^* exists, then $\varphi(f^*) = \beta$ is finite by definition and of course $\alpha \leq \beta$, hence $\alpha = \beta$, and if $\varphi(f^\diamond) \geq \varphi(f^*)$ then $\alpha \geq \varphi(f^\diamond) \geq \varphi(f^*) = \beta = \alpha$ implies that $f^\diamond$ is optimal, proving Theorem 2. On the other hand, if an optimal sure plan f^* exists, then α is finite by definition and $\varphi(f^*) = \alpha \geq \beta \geq \varphi(f^*)$, and Theorem 1 follows.

In case $b < 1$, the proof of Theorem 3 given here requires a uniform finite supremum (u.f.s.) condition to the effect that $\varphi^*(\beta) < \infty$ for some $\beta \in (0,b)$ – see (vii) below for details. It is shown below that Theorem 2 is true without this condition. Theorem 1 can also be proved without the condition – see Section 4 below and [3] – but is obtained here only as a corollary of Theorem 3.

Some additional conditions are also imposed on $(\Omega,\mathfrak{A})$ in this Section. It is assumed that Ω is the space of all adapted corlol functions $\omega = \zeta_t(\omega)$ from $\mathcal{T}$ to some vector space, for example some $\mathbb{R}^N$, each function satisfying $\zeta_0(\omega) = 0$. Then, for each T, one can define for each ω the functions

$$^T\omega = \{\zeta_t(\omega)\colon t\leq T\}, \qquad \omega^T = \{\zeta_t(\omega)-\zeta_T(\omega)\colon t\geq T\},$$

and we assume that $\mathcal{A}_T = \sigma\{\zeta(t); t\leq T\}$, augmented so that $\mathfrak{A}$ is right continuous. Informally, $^T\omega$ represents the past of all variables which the investor observes, ω^T the future; the structure of the vector space is immaterial, except that its dimension should be at least Λ. Each ω can be identified with the pair $(^T\omega,\omega^T)$, Ω with the corresponding product set $^T\Omega \times \Omega^T$, $^T\Omega$ can be equipped with the filtration $(\mathcal{A}_t; t\leq T)$ and with the restriction ^TP of P to $\mathcal{A}_T$. However, we need not distinguish explicitly between sets of $^T\Omega$ and corresponding sets of Ω, between ^TP and P etc. In particular, given an element $\omega^\diamond \in \Omega$, the 'singleton' $\{^T\omega^\diamond\} \subset {}^T\Omega$ may be identified with the subset of Ω comprising all elements of the form $(^T\omega^\diamond,\omega^T)$ with ω^T ranging through Ω^T. Note that $\{^T\omega^\diamond\} \in \mathcal{A}_T$ because the functions are adapted, so that in fact $\mathcal{A}_T$ is generated by these sets. Also, if $\omega^\diamond$ and ω are elements of Ω, then $(^T\omega^\diamond,\omega^T)$ is another element; in other words, the future of one history can be grafted onto the past of another to make a new history. The assumptions stated in Section 2 continue to apply, in particular X is a vector PII relative to the filtration $\mathfrak{A} = (\mathcal{A}_t)$. It would no doubt simplify the formulation slightly to take for Ω the space of corlol functions from $\mathcal{T}$ to $\mathbb{R}^\Lambda$ issuing from the origin and for X the canonical process. However, while the assumption that logarithms of asset prices define a PII relative to all information available about the past is a reasonable formulation of an efficient markets hypothesis, the assumption that the investor has no information other than that derived from the observation of past asset prices is not only invariably false but might seem unduly to pre–empt the conclusions of the argument.

PROOF OF THEOREM 3.

The idea of the proof is to replace the given plan f^{o}, which at any given time T may depend on the whole observed past, successively by plans f^{n} converging to a limit $f^{\diamond}$ independent of the past and satisfying $\varphi(f^{o}) \leq \varphi(f^{n}) \uparrow \varphi(f^{\diamond})$. We write the given plan as $f^{o} = (\theta^{o}, \pi^{o})$ with corresponding processes c^{o}, $x^{o} = x^{\pi^{o}}$, distinguishing other plans and their associated processes by appropriate superscripts.

(i) The replacement procedure is as follows. Given an arbitrary plan f, we consider the right continuous version f_{+}, choose a fixed time $T \in \mathcal{T}$ and an element $\omega^{\diamond}$ in Ω, and consider replacing the 'future' $f_{+}^{T} = (f_{t+};\, t \geq T)$ of f_{+} by the process $f_{+}^{\diamond T}$ with variables

$$f^{\diamond T}(\omega, t+) = f({}^{T}\omega^{\diamond}, \omega^{T}, t+), \quad t \geq T; \qquad \text{...(3.1)}$$

the whole new plan is called ${}^{T}f^{\diamond T}$, or 'f before T, $f^{\diamond T}$ after T' – (cf. 2.14). Note that the replacement variable at T is constant, more precisely that

$$f^{\diamond T}(\omega, T+) = f(\omega^{\diamond}, T+) \quad \text{for } all\ \omega \qquad \text{...(3.2)}$$

– or equivalently for *all* ${}^{T}\omega$ (not just a.s.). The independence of the increments of X implies that the distribution of the variable

$$\phi(\omega, T;\, {}^{T}f^{\diamond T}) = \phi({}^{T}\omega^{\diamond}, \omega^{T}, T;\, f) \qquad \text{...(3.3)}$$

– representing rescaled welfare after T, cf. (2.20) – is independent of $\mathcal{A}_{T}$. For the conditional expectation at T we therefore have

$$\Phi(\omega, T;\, {}^{T}f^{\diamond T}) = \Phi({}^{T}\omega^{\diamond}, \omega^{T}, T;\, f) = \text{constant}, \qquad \text{...(3.4a)}$$

except perhaps for ω in a P–null set $N_{T} \in \mathcal{A}_{T}$; this may also be written as

$$\Phi({}^{T}\omega, T;\, {}^{T}f^{\diamond T}) = \Phi({}^{T}\omega^{\diamond}, T;\, f) = \text{const} = E\Phi_{T}({}^{T}f^{\diamond T}), \qquad {}^{T}\omega \notin N_{T}. \quad \text{...(3.4b)}$$

So far, $\omega^\diamond$ has been an arbitrary fixed element. Now, there must be a subset B_T in $\mathscr{A}_T$ with $PB_T > 0$ such that, for $\omega^\diamond \in B_T$,

$$\Phi(^T\omega^\diamond,T;\, f) \geq E\Phi(^T\omega,T;\, f) \;=\; E\Phi_T(f). \qquad \text{...(3.5)}$$

Choosing any $\omega^\diamond \in B_T$ and replacing f by $^Tf^{\diamond T}$ as above, it follows from (4b) and (5) that the new plan is at least as good as the original after T, hence is at least as good overall (i.e. after 0). Further, since the distribution of (3) is independent of $\mathscr{A}_T$, we may in fact replace $\omega^\diamond \in B_T$ by an *arbitrary* $\omega^\diamond$ in Ω (except perhaps for $\omega^\diamond$ in a P–null set of $\mathscr{A}_T$, which we may again call N_T) without altering the value of (4). So far, then, we have

$$E\Phi_T(f) \leq E\Phi_T(^Tf^{\diamond T}) = \Phi(\omega,T;{}^Tf^{\diamond T}) = \bar{\Phi}, \qquad \text{...(3.6)}$$

where $\bar{\Phi} = \bar{\Phi}(T,f)$ is some constant, for all ω and $\omega^\diamond$, except perhaps for ω and $\omega^\diamond$ in a P–null set N_T of $\mathscr{A}_T$.

(ii) We now apply the replacement procedure iteratively to the given plan f^O. For $n = 1,2,\ldots$, let

$$\delta_j^n = j2^{-n}, \quad j = 0,1,\ldots,n2^n,$$

so that for each n the δ_j^n define a dyadic subdivision of $[0,n]$ and successive subdivisions are refining. At the first step, set $n = 1$, $j = 2$, $T = \delta_2^1 = 1$, and carry out the replacement to $f = f^O$ described under (i), choosing $\omega^\diamond$ outside a P–null set N_2^1. This yields a new plan f_2^1 which is at least as good as f; briefly, the new plan coincides with f for $t \geq 1$, the variables $f_2^1(t+)$ are constant with respect to the past before 1, in particular $f_2^1(1+)$ is constant. Now set $n = 1$, $j = 1$, $T = \delta_1^1 = \frac{1}{2}$, and at the second step carry out replacements to f_2^1 to obtain a new plan f_1^1, noting that since $\omega^\diamond$ may be chosen outside a P–null set N_1^1 we may choose the same $\omega^\diamond$ at both stages. Also, since the variables $f_2^1(t+)$ for $t \geq 1$ are already constant with respect to the past before 1 we may (making changes on a null set if need be) assume that the transition from f^1_2 to f^1_1 alters only the variables $f^1_2(t+)$ with $\frac{1}{2} \leq t < 1$. Next, at $n = 1, j = 0, T = \delta^1_0 = 0$, there are no changes of substance to be made, since there is no 'pre–history' and $\mathscr{A}_0$ is generated by the null sets; at most, we make changes on a null set to ensure first that the same $\omega^\diamond$ is used throughout and second that $f(\omega,0+)$ is constant for all ω, not just a.s. This done, we rename f^1_0 as f^1 and proceed to the next sequence of replacements, starting at $n = 2, j = 4, T = \delta^2_4 = 2$ and with f^1 as the process to be altered. Letting j vary backwards until $j = 0$ is reached, we define a new plan f^2, then on repeated iterations new plans f^n, noting that the same element $\omega^\diamond$ may be chosen throughout as the basis for replacements. Two further remarks are in order. First, the passage from $f^n_j(t+)$ to $f^n_{j-1}(t+)$ may be assumed to alter only variables with $\delta^n_{j-1} < t \leq \delta^n_j$; thus the passage from f^{n-1} to f^n amounts to making separate alterations, always using the same $\omega^\diamond$, on $[n,\infty)$ and on each of the intervals $(\delta^n_{j-1}, \delta^n_j]$. Second, if T appears as a point of subdivision at stage N, then it appears again for each $n > N$, and for each $\omega \in \Omega$ we have

$$f^n(\omega,T+) = f^N(\omega,T+) = f^N(\omega^\diamond,T+) = f^O(\omega^\diamond,T+). \qquad ...(3.7)$$

(iii) It follows that, as $n \to \infty$, we have, for all ω,

$$f^n(\omega,t+) \longrightarrow f(\omega^\diamond,t+) \qquad ...(3.8)$$

for all t which appear as δ^n_j for some (j,n). Of course the 'variables' $f(\omega^\diamond,t+)$ are in fact constant for each t and define the right continuous version of a sure plan $f^\diamond = (\theta^\diamond,\pi^\diamond)$ satisfying

$$f^\diamond(\omega,t) = f^O(\omega^\diamond,t) \quad \text{on } \mathscr{T} \text{ a.s.} \qquad ...(3.9)$$

Now (8) says that we have for each ω a sequence of corlol functions converging on a dense set to a corlol limit – the same for each ω – from which it follows that the functions converge to this limit on $\mathscr{T}$ a.s., together with their left limits. Since $\varphi(f^n) \leq \varphi(f^{n+1})$ for each $n = 0,1,\ldots$, the Theorem will be proved if we show that

$$\lim \varphi(f^n) \leq \varphi(f^\diamond) \qquad \ldots(3.10)$$

with $\varphi(f^\diamond)$ finite.

(iv) Let $x^n = x^{\pi^n}$, $x^\diamond = x^{\pi^\diamond}$, and consider the convergence properties of x^n as $n \to \infty$, $\pi_t^n \to \pi_t^\diamond$ on $\mathcal{T}$ a.s.; (here we revert to the left continuous version of π). Bear in mind that $\pi^\diamond$ is sure but $x^\diamond$ is not. Suppose first that $\Pi = \Pi^+$, so that $0 \leq \pi^{\lambda n} \leq 1$. In this case one can apply the dominated convergence theorem (d.c.t.) for stochastic integrals – see [4] 2.72–4, [5] I.4.31–44 – to the formula (2.6) with $\pi = \pi^n$ to show that $x_t^n \to x_t^\diamond$ in probability, uniformly on compacts, i.e. for each $T < \infty$,

$\sup_{t \leq T} |x_t^n - x_t^\diamond| \longrightarrow 0$ in probability as $n \to \infty$,

and selecting subsequences if necessary we may assume that

$$x_t^n \to x_t^\diamond \quad \text{a.s.,} \qquad \ldots(3.11)$$

first for $t \leq T$ and then, on letting $T = 1,2,\ldots$, on the whole of $\mathcal{T}$.

Similarly $\eta_t^n \to \eta_t^\diamond$ a.s.

To justify the application of the d.c.t. in more detail, consider the four lines on the right of (2.6) one by one, setting $\pi = \pi^n$. The first and third lines are straightforward, taking as 'dominated integrands' the bounded predictable functions π^λ and applying [4] 2.73. The second line contains only Stieltjes integrals of the bounded integrands π^λ and $\pi^\lambda\pi^\ell$ with respect to deterministic integrators of finite variation, and the passage to the limit follows from [4] 2.72, i.e. essentially from the ordinary d.c.t. The last line may be rewritten as

$$\Sigma_{t \leq T}\left[\ell n(\Sigma_\lambda \pi_t^{\lambda n} e^{\Delta x^\lambda(t)}) - \Sigma_\lambda \pi_t^{\lambda n} \Delta x_t^\lambda\right] + \Sigma_{t \leq T}\left[\Sigma_\lambda \pi_t^{\lambda n} \Delta V_t^\lambda\right] \qquad \ldots(3.12)$$

and we take as 'integrands' the summands in square brackets. The passage to the limit in the second term follows from [4] 2.72 because $\Sigma_{t\leq T} \Delta V_t^\lambda$ converges absolutely on $\mathscr{T}$ for each λ, a.s. The first term is non–negative because $\Sigma\pi^\lambda e^{\Delta x^\lambda} \geq e^{\Sigma\pi^\lambda \Delta x^\lambda}$ (convexity inequality). Also, using $\ell n\, z < z-1$, $0 \leq \pi^\lambda \leq 1$ and $\Sigma\pi^\lambda = 1$, it is bounded above, uniformly with respect to π^n, by the non–negative sum $\Sigma_{t\leq T} \Sigma_\lambda \left[e^{\Delta x^\lambda(t)} - 1 - \Delta x_t^\lambda\right]$, which converges absolutely on $\mathscr{T}$ a.s. according to (2.5), and the passage to the limit again follows from [4] 2.72.

If $\Pi = \Pi^o$, X continuous, only the first two lines in (2.6) need be considered. The π^n are no longer uniformly bounded by constants. However, each value of $\pi^n(\omega,t)$ is also a value of $\pi^o(\omega',t)$ for some ω', and since π^o is left continuous it is locally bounded, so that there exist (predictable, bounded) stopping times $\chi_m \uparrow \infty$ and constants $\alpha_m \uparrow \infty$ as $m \uparrow \infty$ such that $|\pi^\lambda(\omega,t)| \leq \alpha_m$ for all λ and ω when $t \leq \chi_m$; consequently the same is true if π^λ is replaced by $\pi^{\lambda n}$, for arbitrary n. It now follows that, for each m, $(x_t^n - x_t^\diamond)I\{t \leq \chi_m\}$ tends to zero in probability, uniformly on compacts, as $n \to \infty$, and letting $m = 1,2,\ldots$ we again conclude that (11) holds on $\mathscr{T}$ a.s.

(v) Consider next the convergence of the c^n as $n \to \infty$, $\theta^n \to \theta^\diamond$. Here it is convenient to consider left continuous versions, since these are locally bounded. Changing the values of the χ_m and α_m if necessary, we may assume that, for each m, $\theta^o(\omega,t-) \leq \alpha_m$ for $t \leq \chi_m$ a.s., and since all values of $\theta^n(\omega,t-)$ are values of $\theta^o(\omega',t-)$ for some ω' we may assume that $\theta^n(\omega,t-) \leq \alpha_m$ for all $t \leq \chi_m$ and all n, a.s. But then the (ordinary) d.c.t. gives

$$\int_0^t \theta^n(\omega,s-)ds \longrightarrow \int_0^t \theta^\diamond(\omega,s-)ds \quad \text{for } t \leq \chi_m \text{ a.s.,}$$

and inside the integral signs we may replace θ_{s-} by θ_s. Referring to (2.13) it is seen that

$c^n(\omega,t\pm) \longrightarrow c^\diamond(\omega,t\pm)$ a.s., ...(3.13)

first for $t \le \chi_m$ and then, letting $m \to \infty$, on $\mathcal{T}$; moreover it follows from (2.13) that $c^n_{t-} \le \alpha_m$ for $t \le \chi_m$.

(vi) To simplify the notation, write the integrand in (2.18) as

$(1-b)U_t q_t = c_t^{1-b} e^{(1-b)x\pi(t)} q(t) = \bar{c}_t^{1-b} q_t > 0$ ...(3.14)

with suitable superscripts when the plans are f^o, f^n, $f^\diamond$ or simply f. The argument under (iv) and (v) shows that $U^n_t \to U^\diamond_t$ on $\mathcal{T}$ a.s., and if $b > 1$ we may conclude from Fatou's Lemma that

$\liminf_n (1-b)E \int_0^\infty U^n_t q_t dt \ge (1-b)E \int_0^\infty U^\diamond_t q_t dt,$ ...(3.15)

which is (10). The inequality $\varphi(f^\diamond) > -\infty$ follows from (10), taking into account that $\varphi(f^o) > -\infty$ and that $\varphi(f^n) \uparrow$ with n. This completes the proof for $b > 1$.

(vii). If $b < 1$, $U_t > 0$, an additional condition is apparently needed in order to prove (10). The simplest is a condition similar to one used in [2] S.3, where its properties are discussed in more detail. Assume that

$\int_0^\infty q(t)dt = 1,$ ...(3.16)

so that $d\mu(\omega,t) = dP(\omega,t)q(t)dt$ defines a probability measure on $(\Omega\times\mathcal{T},\mathcal{O})$, where $\mathcal{O}$ denotes the optional sets, and consider the space $\mathcal{L}_1 = \mathcal{L}_1(\Omega\times\mathcal{T},\mathcal{O},\mu)$ of real–valued, μ–integrable optional processes. Assume further that there is some $\beta\in(0,b)$ such that $\varphi^*(\beta) < \infty$, or explicitly

$$\sup E \int_0^\infty \bar{c}_t^{1-\beta} q_t dt = \sup \int_{\Omega\times\mathcal{T}} \left[(1-b)U^f\right]^{(1-\beta)/(1-b)} d\mu < \infty, \qquad ...(3.17)$$

the supremum being taken over all $\bar{c}$–plans, or equivalently over $(U^f : f\in\mathfrak{F})$. We call (16)–(17) a *uniform finite supremum* (*u.f.s.*) *condition*; it implies the ordinary finite supremum condition $\varphi^*(b) < \infty$ by a standard property of moments. From (17) it follows that the functions $(U^f; f\in\mathfrak{F})$ are uniformly μ–integrable, hence form a weakly sequentially compact set in $\mathcal{L}_1$. In particular, the sequence (U^n) may be assumed, selecting a subsequence if necessary, to converge weakly to some function $U^*\in\mathcal{L}_1$, implying $\int U^n d\mu \to \int U^* d\mu < \infty$. But we already know that $U_t^n \to U_t^\diamond$ on $\mathcal{T}$ a.s., so that $U^* \equiv U^\diamond$, implying (10).||

REMARK 1. Various conditions which are sufficient for (17) to hold can be given, for example as follows. Using $\bar{c} = ce^{x^\pi}$, then Hölder's inequality, then (2.11) with $K_o = 1$ we obtain

$$E\int_o^\infty \bar{c}_t^{1-\beta} q_t dt = E\int_o^\infty \{c_t e^{x^\pi(t)}\}^{1-\beta} q_t dt$$

$$\leq \left[E\int_o^\infty c_t dt\right]^{1-\beta} \cdot \left[E\int_o^\infty \{e^{(1-\beta)x^\pi(t)} q_t\}^{1/\beta} dt\right]^\beta dt$$

$$\leq \left[E\int_o^\infty \{e^{(1-\beta)x^\pi(t)} q_t\}^{1/\beta} dt\right]^\beta, \qquad \text{...(3.18)}$$

so that it is enough if the supremum of the last expression taken over all $\pi\in\Pi$ is finite for some $\beta\in(0,b)$, and a fortiori if

$$\int_o^\infty \left[\sup_\pi Ee^{(1-\beta)x^\pi(t)/\beta}\right] q_t^{1/\beta} dt < \infty. \qquad \text{...(3.19)}$$

In particular, if X is a PSII and $q(t) \propto e^{-rt}$ with $r > 0$, the supremum in (19) need only be calculated at $t = 1$, and provided that it is finite for some $\beta < b$ then Theorem 3 is true if r is large enough. Bounds on expectations like that appearing in (19) will be considered in [3].

PROOF OF THEOREM 2 (without u.f.s. condition).

Suppose that the plan f^o considered in the proof of Theorem 3 is actually optimal. Then $\bar{c}^o = c^o e^{x^0}$ is the unique optimal consumption plan in natural units. Since the replacement procedure shows that $\varphi(f^o) \leq \varphi(f^n)$ for each n, it follows that in fact $\varphi(f^o) = \varphi(f^n)$ for each n, hence by uniqueeness

$\bar{c}^o \equiv \bar{c}^n$. But $\bar{c}^n \to \bar{c}^\diamond$ on $\mathcal{T}$ a.s. by (11) and (13), and since $\bar{c}^\diamond$ is defined by the sure plan $f^\diamond$ the result follows.||

REMARK 2. The proof of Theorem 2 can be simplified if conditions are in force which imply that a pair (θ^o, π^o), or equivalently (c^o, π^o), defining an optimal plan is unique – see Proposition 1 above. Let $f^o = f = (\theta,\pi)$ be the unique optimal plan, choose T and carry out the replacement of f_+^T by $f_+^{\diamond T}$ as in step (i) above. Since welfare is not diminished, it follows from uniqueness that the revised plan coincides with the original for all $t \geq T$ a.s. In particular, for $t = T$ we have

$$\theta(\omega,T) = \theta(\omega^\diamond,T), \quad \pi(\omega,T+) = \pi(\omega^\diamond,T+), \quad \text{a.s.} \qquad \ldots(3.20)$$

so that in fact $\theta(\cdot,T)$ and $\pi(\cdot,T+)$ were a.s. non–random all along. Since we may carry out the 'replacement' for all T in a dense set of $\mathcal{T}$ with the same $\omega^\diamond$, it follows from the fact that $\theta(\omega^\diamond,t)$ and $\pi(\omega^\diamond,t+)$ are corlol that $\theta(\omega,\cdot)$ and $\pi(\omega,\cdot)$ are a.s. equal to non–random functions on $\mathcal{T}$. The details of the subdivision and convergence arguments are thus not required.

4. PORTFOLIO SEPARATION

In this Section we consider further the relationship between consumption and portfolio choice in an optimal sure plan. The results of the preceding Section serve to justify the restriction to sure plans but play no further direct part. The special definition of Ω as a function space is not used again. We still assume $b \neq 1$. X is always a PII. Unless specified, the precise definitions of Π and Π^S do not matter.

Two crucial simplifications occur when sure plans are considered. First, if π is sure, the compound interest process x^π defined by (2.6–7) is a PII, as is the process $\eta^\pi = e^{(1-b)x^\pi}$; this follows readily from the fact that X is a PII. (If X is a PSII and π is invariable, then x^π and η^π are PSII). Secondly, if c is sure, the functional (2.18) can be written as

$$\varphi(c,\pi) = (1-b)^{-1} \int_0^\infty c_t^{\,1-b} \cdot (E\eta_t^\pi) \cdot q(t)dt. \qquad ...(4.1)$$

Note that, for sure π, the expectation appearing here is a value of the Laplace Transform of a PII.

Let π be initially a given (not necessarily sure) element of Π and consider the *necessary* conditions for an element c^π to be π–optimal in $\mathscr{C}^S$. Obviously $\varphi(c^\pi,\pi)$ must be finite, and (to avoid waste of resources) we must have

$$\int_0^\infty c_t^\pi \, dt = K_0. \qquad ...(4.2)$$

Bearing in mind that c^π is corlol, the form of the maximand (1) implies that we must have $c_t^\pi > 0$ for *all* t, and then the Euler condition for a maximum of (1) subject to the constraint (2) is

$$(c_t^\pi)^{-b} \, E\eta_t^\pi \, q_t = \text{constant} = (c_0^\pi)^{-b} > 0. \qquad ...(4.3)$$

Note that $E\eta_t^\pi = Ee^{(1-b)x^\pi(t)}$ is always positive because X is finite (Fatou's Lemma). Also, since c^π and q are corlol, the same must be true of $E\eta_t^\pi$ by (3). Defining

$$\mathfrak{N}^\pi = \mathfrak{N}(\pi,q) = \int_0^\infty [E\eta_t^\pi q_t]^{1/b}dt, \qquad \text{...(4.4)}$$

we get from (2), (3) and then (1) that

$$c_0^\pi/K_0 = 1/\mathfrak{N}^\pi, \qquad c_t^\pi = c_0^\pi [E\eta_t^\pi q_t]^{1/b}, \qquad \text{...(4.5)}$$

$$\varphi(c^\pi,\pi) = K_0^{1-b}(\mathfrak{N}^\pi)^b/(1-b). \qquad \text{...(4.6)}$$

Obviously these calculations make sense only if

$$\mathfrak{N}^\pi < \infty. \qquad \text{...(4.7)}$$

Suppose conversely that π is sure and is such that $\mathfrak{N}^\pi < \infty$. Then the positive function $E\eta_t^\pi$ is finite for almost all t, and since this function is corlol (because x^π is corlol) it is finite for *all* t. Further, the process η^π is a PII, from which it follows easily that $(\eta_t^\pi/E\eta_t^\pi;\ t\in\mathscr{T})$ is a martingale, so $E\eta^\pi$ must be a semimartingale. If c^π is defined by (5), it follows that c^π is a positive semimartingale satisfying the equality (2) and so is in $\mathscr{C}$. Moreover (6) is satisfied so that $\varphi(c^\pi,\pi)$ is finite, and then a standard sufficiency argument based on the concavity of φ with respect to c (π fixed) shows that c^π is π–optimal in $\mathscr{C}^S$. An example of such an argument is the proof of Theorem 1 of [2]. We shall not set out the details, but rather note that this theorem yields a stronger assertion, namely that c^π is π–optimal in $\mathscr{C}$. Indeed, the theorem implies (allowing for differing definitions) that the assertion follows if it is shown that $\varphi(c^\pi,\pi)$ is well defined as an integral and finite, that (2) holds, and that the shadow price process y defined by c^π is a martingale. The first two conditions are satisfied by construction, and with a functional (2.18) we have

$y_t = (c_t^\pi)^{-b}\eta_t^\pi q_t$, or, using (3)

$$y_t = y_0\eta_t^\pi/E\eta_t^\pi, \quad y_0 = (c_0^\pi)^{-b}, \quad \eta_t^\pi = e^{(1-b)x^\pi(t)}. \qquad \ldots(4.8)$$

As previously noted, $\eta^\pi/E\eta^\pi$ is a martingale, which proves the assertion. Proposition 1(ii) shows that c^π is in fact the *unique* π–optimal element of $\mathscr{C}$. To sum up so far, we have

PROPOSITION 2: *π–optimal c–plans.*

For arbitrary $\pi \in \Pi$, a π–optimal element c^π of $\mathscr{C}^S$ exists iff the integral $\mathfrak{N}^\pi = \mathfrak{N}(\pi,q)$ defined by (4) is finite; then c^π is defined explicitly by (5) and the corresponding value of the welfare functional is given by (6). If π is sure, then c^π is the unique π–optimal element of $\mathscr{C}$.

The preceding argument shows that the problem of constructing an optimal sure plan (c^*,π^*) can be considered in two stages, the first being to choose a π^* to maximise $\mathfrak{N}^\pi/(1-b)$ on Π^S if possible, the second to choose c^* in $\mathscr{C}^S$ to maximise $\varphi(c,\pi^*)$. For the first stage to be well defined it is necessary, if $b < 1$ that $\mathfrak{N}^\pi < \infty$ for all $\pi \in \Pi^S$, if $b > 1$ that $\mathfrak{N}^\pi < \infty$ for some $\pi \in \Pi^S$. In this formulation, the first–stage problem makes no explicit reference to consumption, but the existence of a solution still depends on the discount density q. Going a step further we note that,

if π^* maximises $\mathfrak{N}^\pi/(1-b)$ on Π^S,

then π^* maximises $(1-b)^{-1}E\eta_t^\pi$ on Π^S for each $T \in \mathscr{T}$,

or equivalently π^* maximises

$(1-b)^{-1}E(\eta_T^\pi/\eta_S^\pi) = (1-b)^{-1}E\eta_T^\pi/E\eta_S^\pi$ for each pair $S < T$ from $\mathscr{T}$.

This follows from (4), taking into account that all the η^π for $\pi \in \Pi^S$ are PII and that a sure portfolio plan can be chosen separately on each interval (S,T]. Conversely, if π^* maximises $(1-b)^{-1}E\eta_T^\pi$ on Π^S for each T, one can always find some q which decreases fast enough far out so that

$\mathfrak{N}(\pi^*,q) < \infty$, and for any such q it is clear that π^* maximises $\mathfrak{N}(\pi,q)/(1-b)$ on Π^S and that an optimal sure plan of the form (c^*,π^*) exists, c^* being defined by (5) with $\pi = \pi^*$. On the strength of these remarks we can define an optimal sure π, without further reference to consumption or discount, as one which maximises $(1-b)^{-1}E\eta_t^\pi$ on Π^S for each T simultaneously. In the sequel [3], it will be slightly more convenient to consider the function

$$\Psi(\pi,T) = (1-b)^{-1} \ln E\eta^\pi(T), \qquad T\in\mathcal{T},\ \pi\in\Pi^S, \qquad \ldots(4.9)$$

and to define the *problem of optimal sure portfolio choice* as the problem of choosing (if possible) a $\pi^* \in \Pi^S$ such that $\Psi(.,T)$ attains a (finite) maximum on Π^S at $\pi = \pi^*$ for each $T\in\mathcal{T}$. A solution is called an *optimal sure portfolio plan*, or simply an optimal sure π. With this terminology, the discussion can be summed up in the simple but fundamental

THEOREM 4: *Principle of Portfolio Separation.* Let $\pi^*\in\Pi^S$. For a given PII X and fixed $b \neq 1$, the following properties are equivalent:

(i) π^* maximises $\Psi(\pi,T)$ on Π^S for each $T\in\mathcal{T}$;

(ii) π^* maximises $\Psi(\pi,T)-\Psi(\pi,S)$ on Π^S for each pair $S < T$ from $\mathcal{T}$.

(iii) For every discount density q such that $\mathfrak{N}(\pi^*,q) < \infty$, π^* maximises $\mathfrak{N}(\pi,q)/(1-b)$ on Π^S.

(iv) For every discount density q such that $\mathfrak{N}(\pi^*,q) < \infty$, an optimal plan (c^*,π^*) exists, c^* being defined as in (4.5) with $\pi = \pi^*$, $\mathfrak{N}^\pi = \mathfrak{N}(\pi^*,q)$.

To review progress so far, we have replaced the original PS problem of choosing an optimal pair $(\bar{c},\pi)$, first by the problem of choosing an optimal pair (c,π), then by the problem of choosing an optimal sure pair (c,π) or (θ,π). If this problem has a solution (c^*,π^*), then c^* is π^*–optimal and

π^* is optimal sure, i.e. π^* maximises $\Psi(\pi,T)$ among all sure π for each T. Conversely, if π^* is an optimal sure portfolio plan, then for every discount density q which makes $\mathfrak{N}(\pi^*,q)$ converge there is a sure c^* such that (c^*,π^*) is optimal, and an explicit formula for c^* can be given. As will be shown in [3], an explicit formula can also be given for $\Psi(\pi,T)$ when π is sure, and with its aid an alternative proof of the First C.E. Theorem which is independent of Section 3 above can be given. Of course, only this Theorem is needed to allow results on the existence and characterisation of an optimal PS plan to be obtained by solving the corresponding problem for an optimal sure portfolio plan. The role of the Second C.E. Theorem is to guarantee that no cases of existence are omitted if only sure plans are considered, and in particular to simplify the construction of examples of non–existence.

In conclusion, we indicate the points which remain to be established in order to complete the alternative proof of Theorem 1. It has to be shown that a plan (c^*,π^*), where π^* is optimal sure and c^* is defined as in (4.5), is optimal in $\mathscr{C}\times\Pi$. According to Theorem 1 of [1], it is sufficient for optimality if c^* is π^*–optimal in $\mathscr{C}$ (which has already been shown) and if for each asset λ the process $y^\lambda = y^*\cdot e^{x^\lambda - x^*}$ is a supermartingale in case $\Pi = \Pi^+$, a martingale in case $\Pi = \Pi^o$; here y^* is given by (8), with $\pi = \pi^*$, $c = c^*$, $x^\pi = x^*$. Using (8) and (5) we have

$$y_t^\lambda = y_o^*\cdot e^{x^\lambda(t)-bx^*(t)}/Ee^{(1-b)x^*(t)} \qquad \text{...(4.10)}$$

where $y_o^* = (c_o^*)^{-b} = [K_o/\mathfrak{N}(\pi^*,q)]^{-b}$, and of course the y^λ are PII. Hence

$$Ey_t^\lambda = y_o^*\cdot Ee^{x^\lambda(t)-bx^*(t)}/Ee^{(1-b)x^*(t)}, \qquad \text{...(4.11)}$$

and the appropriate supermartingale (or martingale) property follows easily if it is shown that the ratio of expectations on the right of (11) is finite and non–increasing (or constant), or equivalently that the difference

$$\ell n\ Ee^{x^{\lambda}(t)-bx^{*}(t)} - \ell n\ Ee^{(1-b)x^{*}(t)} \qquad ...(4.12)$$

is finite and non–increasing (or constant). These facts will be verified in [3] for the various cases which arise.

5. STATIONARY INCREMENTS

In this Section we review briefly some complements to the results of Sections 3–4 which are obtainable when X is a PSII and $q(t) = e^{-rt}$ with some real r. We assume $b \neq 1$ and set $K_0 = 1$. For brevity, we consider only the case $\Pi = \Pi^+$, and assume that $\Psi(\pi,T)$ is finite on $\mathcal{T}$ for all $\pi \in \Pi^s$. In addition, each proposition stated here is subject to the conditions of the theorem which it complements.

If π is an *invariable* portfolio plan, then π has the form $\pi_t \equiv \check{\pi}$, $t \in \mathcal{T}$, where $\check{\pi}$ is an element of the simplex

$$\mathcal{S} = \{\check{\pi} \in \mathbb{R}^\Lambda : \check{\pi} \geq 0 \text{ and } \Sigma_\lambda \check{\pi}^\lambda = 1\}, \qquad ...(5.1)$$

and conversely each such $\check{\pi}$ defines an element of Π^i. For $\pi \in \Pi^{+i}$, define

$$\psi(\pi_1) = \Psi(\pi,1); \qquad ...(5.2)$$

then obviously

$$\Psi(\pi,T) = T\psi(\pi_1) \qquad T \in \mathcal{T}, \qquad ...(5.3)$$

and x^π is a PSII. It follows from an explicit formula for Ψ to be derived in [3] that $\psi(.)$ is a finite concave function on $\mathcal{S}$, and further that for any *sure* π we have

$$\Psi(\pi,T) = \int_0^T \psi(\pi_t)dt, \qquad T \in \mathcal{T}. \qquad ...(5.4)$$

Then we have the following complement to Theorem 3.

Proposition 3. If $\pi^0 \in \Pi^s$, then for each $T \in \mathcal{T}$ there exists a $\pi^\diamond \in \Pi^i$ such that

$$\Psi(\pi^0,T) \leq \Psi(\pi^\diamond,T). \qquad ...(5.5)$$

Proof. Fix T and set

$$\check{\pi}_T^\diamond = (1/T)\int_0^T \pi_t^0 \, dt, \qquad \pi_t^\diamond = \check{\pi}_T^\diamond \text{ for all } t. \qquad ...(5.6)$$

Since ψ is concave on $\mathcal{S}$ we have

$$(1/T)\int_0^T \psi(\pi_t^0)\, dt \leq \psi(\check{\pi}_T^\diamond), \qquad ...(5.7)$$

and taking into account (3), with $\pi = \pi^{\diamond}$, $\check{\pi}_1 = \check{\pi}^{\diamond}_T$, this implies (5).||

It is tempting to go further and claim that, for each sure π^0, there exists an invariable $\pi^{\diamond}$ such that $\Psi(\pi^0,t) \leq \Psi(\pi^{\diamond},t)$ for all t, but this apparently is not true without some further assumption. Again, the 'obvious' extension of Theorem 3, namely that, for each sure $f^0 = (\theta^0,\pi^0)$ with $\varphi(f^0)$ finite there is an invariable $f^{\diamond}$ such that $\varphi(f^0) \leq \varphi(f^{\diamond}) < \infty$, is not true without reservation – though it is true, rather trivially, when an optimum exists. We omit further details and go on to results which complement the Certainty Equivalence Theorems.

PROPOSITION 4. An optimal invariable plan is optimal.

PROOF. In view of Theorem 1, it is enough to show that, if (c^*,π^*) is optimal among invariable plans, then it is optimal among sure plans. Again, it is enough to show that, if (c^*,π^*) is optimal among all invariable plans (c,π) such that c is π–optimal in $\mathscr{C}^i$, then it is optimal among sure plans. Now, if c is invariable, then $c_t = c_0 e^{-\theta t}$ for some $c_0 > 0$ and $\theta > 0$, and π–optimality in $\mathscr{C}^i$ implies $\int c\cdot dt = 1$, hence $c_0 = \theta$. On the other hand, if π is invariable, so that $\pi_t = \pi_1$ for each t and x^π is a PSII, then

$$(E\eta^\pi_t q_t)^{1/b} = (E\eta^\pi_1 e^{-r})^{t/b} \stackrel{\text{def}}{=} e^{-n^\pi t},$$

and since $E\eta^\pi_1 = \Psi(\pi,1) = T\psi(\pi_1)$ our assumptions ensure that these expressions are finite for all t. Next, if c is π–optimal in $\mathscr{C}^i$, then by definition $\varphi(c,\pi)$ is finite, and substituting in (4.1) we have

$$\varphi(c,\pi) = (1-b)^{-1}\int_0^\infty (\theta e^{-\theta t})^{(1-b)} e^{-bn^\pi t}dt = \theta^{1-b}/(1-b)[bn^\pi+(1-b)\theta]$$

with $bn^\pi+(1-b)\theta > 0$. A final necessary condition for π–optimality in $\mathscr{C}^i$ is that the above expression has a (finite) maximum at some $\theta > 0$. Writing $c = c^\pi$, $\theta = \theta^\pi$ for the maximising values, we have

$$\theta = n^\pi, \quad \varphi(c^\pi,\pi) = 1/(1-b)(n^\pi)^b.$$

Reference to the discussion following (4.1), or direct calculation, shows that the conditions which have been derived as necessary for $c = c^{\pi}$ to be π–optimal in $\mathscr{C}^{i}$ are also sufficient (given that π is invariable) for c^{π} to be π–optimal in $\mathscr{C}^{s}$. In particular, we have $\varphi(c^{\pi},\pi) = \mathfrak{N}^{\pi}/(1-b)$, where $\mathfrak{N}^{\pi}$ is defined as in (4.4). It therefore only remains to show that, if (c^*,π^*) is optimal among invariable plans, then π^* maximises $\mathfrak{N}^{\pi}/(1-b)$ among sure π. Now π^* maximises $\mathfrak{N}^{\pi}/(1-b)$ among invariable π by definition, so π^* maximises $\Psi(\pi,1)$ on Π^{i}. But then the vector π_1^* maximises $\psi(\check{\pi})$ on $\mathscr{S}$, and it follows – see (4) – that π^* maximises $\Psi(\pi,T)$ for each T on Π^{s}.||

PROPOSITION 5. If an optimal plan exists, an invariable optimal plan exists.

PROOF. In view of Theorem 2 it is enough to show that, if a sure optimal plan exists, an invariable optimal plan exists. Now, if (c^*,π^*) is sure optimal, it is optimal sure, so (4.1–7) apply, and π^* maximises $\mathfrak{N}^{\pi}/(1-b)$ on Π^{s}. But then π^* maximises $\Psi(\pi,T)$ among all sure π for each T, which in view of Proposition 3 implies that π_t^* is constant on $[0,T]$ for each T, hence constant on $\mathscr{T}$. Thus π^* is invariable and by definition c^* is π^*–optimal in $\mathscr{C}^{s}$. It is easily checked that this implies (using the notation of the preceding Proposition) that $c^* = n^*e^{-n^*t}$ with $n^* > 0$.||

6. LOGARITHMIC UTILITY

If $b = 1$ in (2.17), the welfare functional in standardised units (when defined) takes the form

$$\varphi(c,\pi) = E \int_0^\infty (\ell n\, c_t + x_t^\pi) q_t dt \qquad ...(6.1)$$

in place of (2.18). We review the theory of Sections 3–4 case briefly, noting only the main alterations. For simplicity, we assign the value $-\infty$ to undefined integrals. This time it is clear from the outset that we may without loss consider separately the problem of *optimal consumption in standardised units*

$$\max \int_0^\infty E(\ell n\, c_t) q_t dt: \; c \in \mathscr{C}, \qquad ...(6.2)$$

and the problem of *optimal portfolio choice*

$$\max \int_0^\infty E(x_t^\pi) q_t dt: \; \pi \in \Pi, \qquad ...(6.3)$$

provided that these problems are well defined, and the question of certainty equivalence can be considered separately for the two problems. Regarding (2), we have

PROPOSITION 6. If $c \in \mathscr{C}$, the deterministic function Ec defined by $(Ec)_t = E(c_t)$ is in $\mathscr{C}^S$ and is at least as good as c, i.e.

$$\int_0^\infty E(\ell n\, c_t) \cdot q_t dt \le \int_0^\infty \ell n(Ec_t) \cdot q_t dt. \qquad ...(6.4)$$

The inequality is strict if both sides are finite unless $c \equiv Ec$.

PROOF. Ec inherits from c the properties defining a c–plan. If both sides of (4) are finite, the asserted inequality follows from the strict concavity of the logarithm. If the left side is $+\infty$, consider the integrals on $[0,T]$ and go to the limit.||

It follows from this proposition that we may without loss replace (2) by the problem

$$\max \int_0^\infty (\ell n\, c_t) q_t dt: \; c \in \mathscr{C}^S. \qquad ...(6.5)$$

Setting $K_0 = 1$, it is found that necessary conditions for $c^* \in \mathscr{C}^S$ to be optimal are

$$(c_t^*)^{-1} q_t = (c_0^*)^{-1} q_0 > 0 \quad \text{for all } t,$$

$$K_0 = 1 = \int_0^\infty c_t^* \, dt = (c_0^*/q_0) \cdot \int_0^\infty q_t dt < \infty,$$

$$\int_0^\infty (\ell n\, c_t^*) q_t dt = \int_0^\infty [\ell n\, q_t - \ell n(\int_0^\infty q_s ds)] q_t dt,$$

and these calculations make sense only if

$$\int_0^\infty q_t dt < \infty \qquad \text{and} \qquad \int_0^\infty |\ell n\, q_t| q_t dt < \infty. \qquad ...(6.6)$$

Conversely, if (6) is satisfied, the preceding formulae allow an optimal c^* to be constructed.

Turning to (3), we assume, to avoid tedious complications, that Ex_t^π is defined and finite for all π and T and further that the functions $V^{\lambda c}$ and $\langle M^{\lambda c}, M^{\ell c} \rangle$ are absolutely continuous with derivatives v_t^λ, $\sigma_t^{\lambda \ell}$, the 'instantaneous covariance matrix' $[\sigma_t^{\lambda \ell}]$ being non–negative definite for every t. We have

Proposition 7. Let $\Pi = \Pi^+$. If $\pi \in \Pi$, the deterministic function $E\pi$ defined by $(E\pi)_t = E(\pi_t)$ is in Π^S. For each $T \in \mathscr{T}$,

$$Ex_T^\pi \le E(x_T^{E\pi}). \qquad ...(6.7)$$

Proof. If $\pi \in \Pi^+$, the expectation $E\pi_t$ is defined and finite and inherits from π the properties defining a non–negative portfolio plan. Consider the formula for x^π in (2.6) and compare it with the corresponding expression for $x^{E\pi}$. Since π is predictable and $0 \le \pi \le 1$, the integrals in the first and third lines of (2.6) are (true) martingales with zero expectation, so that these lines can be left aside in proving (7). For given (ω, t), the integrands in the second line are finite concave functions of the vector $\pi(\omega,t)$, while in the fourth line each summand is a finite concave function of $\pi(\omega,t)$. Then (7) follows from Jensen's inequality. ||

COROLLARY. The inequality (7) is strict for T in some interval $[\tau,\infty)$ if π is not deterministic and either (2.24) or both (2.28) and (2.31) are satisfied.

It follows from Proposition 7 that we may without loss replace (3) by the problem

$$\max \int_0^\infty (Ex_t^\pi)q_t dt: \ \pi \in \Pi^S \qquad \text{...(6.8)}$$

if $\Pi = \Pi^+$. Theorem 3 is in this case an immediate consequence of Propositions 6 and 7, and in its turn implies Theorems 1 and 2. Next we define, in place of (4.9), the function

$$\Psi^\ell(\pi,T) = Ex^\pi(T), \qquad \pi \in \Pi^S, \ T \in \mathcal{T}. \qquad \text{...(6.9)}$$

A necessary condition for π^* to be a solution of (8) is that $\Psi^\ell(.,T)$ attains a finite maximum on Π^S at $\pi = \pi^*$ for almost all T, or equivalently for all T because of right continuity. Conversely, this condition is sufficient if

$$\int_0^\infty (Ex_t^*)q_t dt, \qquad x^* = x^{\pi^*}, \qquad \text{...(6.10)}$$

is defined and finite, and this can always be achieved by a suitable choice of q. Accordingly we may define the *problem of optimal sure portfolio choice* simply as the problem of choosing (if possible) a $\pi^* \in \Pi^S$ such that (9) attains a finite maximum at π^* for each T. The statement corresponding to Theorem 4 which is valid in the present case is contained in the preceding discussion and need not be set out separately.

As regards the uniqueness assertions of Proposition 1, these remain valid, but can be strengthened as follows. First, (ii) may be replaced by

(ii)′ If (c^*,π^*) and (c^o,π^o) are optimal, then $c^o \equiv c^*$;

from which it follows, by Proposition 1(i), that (iii) can be replaced by

(iii)′ If (c^*,π^*) and (c^o,π^o) are optimal, then $x^o \equiv x^*$.

So much for the case $\Pi = \Pi^+$.

If $\Pi = \Pi^0$ (X continuous) the proof of Proposition 7 does not work as stated, first because $E\pi_t$ may be undefined for some π and t, and then because the integral in the first line of (2.6) is in general only a *local* martingale. A possible way out is to restrict Π to elements of Π^o which are uniformly bounded on compacts, i.e. such that, for each π and T, there is an α such that, a.s., $|\pi(\omega,t)| \leq \alpha$ for all $t \leq T$.

Alternatively, Theorem 3 may be proved along the lines of Section 3. Briefly, if (6) above is assumed, it is enough to consider $\hat{\varphi}(\pi) = \int_0^\infty Ex_t^\pi q_t dt$ as the functional instead of $\varphi(c,\pi)$ and to apply to it a simplified version of the replacement procedure. A u.f.s. condition of the form

$$\sup \{\int_0^\infty E|x_t^\pi|^\beta \cdot q_t dt : \pi \in \Pi^o\} < \infty \quad \text{for some } \beta > 1 \qquad \text{...(6.11)}$$

ensures that $\hat{\varphi}$ is defined for all π and that the step corresponding to (vii) of Section 3 is valid. In fact, $|x_t^\pi|^\beta$ can be replaced in (11) by $(x_t^{\pi+})^\beta + |x_t^{\pi-}|$, where x^+, x^- denote the positive and negative parts of x. We omit further details. Once Theorem 3 is proved, Theorems 1 and 2 follow, and the problem of optimal sure portfolio choice can be formulated as above. Once again, Theorem 2 is valid without the u.f.s. condition. The uniqueness conditions of Proposition 1 as amended above remain valid under (6), but only the first part of (iv) applies and its proof requires a stopping argument.

Finally, Theorem 1 can be proved by the method outlined in Section 4 without the function space set–up or a u.f.s. condition. Suppose that (c^*,π^*) is optimal sure and set $K_o = 1$, $q_o = 1$. Then c^* must satisfy the conditions preceding (6.6), and we may further set $\int q \cdot dt = 1$, so that $c^* \equiv q$. Obviously $\varphi(c^*,\pi^*)$ is finite, (4.2) is satisfied and $Ex^*(t)$ is defined and finite for all t. The shadow price processes (4.8) and (4.10) are replaced by

$$y_t^* = (c_t^*)^{-1} q_t = (c_o^*)^{-1} q_o = y_o^*, \qquad y_t^\lambda = y_o^* \cdot e^{x^\lambda(t) - x^*(t)}, \qquad \ldots(6.12)$$

and y_t^* is trivially a martingale so that c^* is π^*–optimal in $\mathscr{C}$. It remains to show that y^λ is a supermartingale if $\Pi = \Pi^+$, a martingale if $\Pi = \Pi^o$ – or equivalently, since x^λ and x^* are PII, that Ey_t^λ is non–increasing in the former case, constant in the latter; we shall return to these points in [3]. This completes our survey of the theory for logarithmic utility.

7. POSTSCRIPT

The requirement that a portfolio plan be collor, although intuitively appealing, has the drawback that the existence of an optimum is liable to be ruled out unless the characteristics of the PII X are sufficiently smooth with respect to time. This topic will be pursued more fully in [3]; here we give a short informal discussion of points relevant to the results proved above.

To begin with an example, assume CRRA utility with $b \neq 1$ and exponential discounting, $\Pi = \Pi^+$, and suppose first that $X = \bar{\bar{\mathfrak{X}}}$ is a Brownian motion with drift and covariance relative to $\mathfrak{A}$. With a sufficiently high rate of discount, there is an optimal plan (c^*, π^*) with $\pi^*_t = \check{\pi}^B$ for all t, where $\check{\pi}^B$ is some element of $\mathscr{S}$ – see (5.1). Now let $X = \mathfrak{X} + \bar{\bar{\mathfrak{X}}}$, where $\bar{\bar{\mathfrak{X}}}$ is as before and $\mathfrak{X} = \Delta X_\tau I_{\{t \geq \tau\}}$, $\tau > 0$ being a fixed time and ΔX_τ an $\mathscr{A}_\tau$–measurable random vector independent of $\mathscr{A}_{\tau-}$. Intuitively, it is clear that under suitable conditions of integrability we should have an 'optimal plan' with π^* of the form

$$\pi^*_t \;=\; \check{\pi}^\tau I_{\{t=\tau\}} \;+\; \check{\pi}^B I_{\{t \neq \tau\}}, \qquad \ldots(7.1)$$

where $\check{\pi}^\tau$ maximises $(1-b)^{-1} E e^{(1-b)\Delta X^\pi(\tau)}$ among all $\check{\pi} \in \mathscr{S}$; but, accidents apart, such a plan is not collor. To state the point another way: the best portfolio to hold when $t \neq \tau$ is the usual 'Brownian' portfolio $\check{\pi}^B$, the best to hold at τ is the usual 'single–period' portfolio $\check{\pi}^\tau$, but during $(0,\tau)$ there is no latest time for switching, and during (τ,∞) no earliest time for switching back.

More generally, if X is a PII with a finite or countable set $J = (\tau_m)$ of fixed discontinuities, X can be represented as the sum $\mathfrak{X} + \bar{\mathfrak{X}}$ of two PII, where $\mathfrak{X} = \int I_J dX$ and $\bar{\bar{\mathfrak{X}}}$ has no fixed discontinuities, see [4] p.91. An optimum with π collor is liable not to exist in cases where both components are present (unless $\bar{\bar{X}}$ is constant in a neighbourhood of each τ_m) or where J has a finite point of accumulation in $\mathscr{T}$. It might thus appear that the theory developed in this paper in empty in such cases. However, the main results, in particular Theorems 1–4, generalise without much difficulty if the set Π of admissible portfolio plans is extended to allow processes of the form

$$\pi_t = \Sigma_m \breve{\pi}_m I_{\{t=\tau_m\}} + \bar{\bar{\pi}}_t I_{\{t \notin J\}}, \qquad \text{...(7.2)}$$

where $\bar{\bar{\pi}}$ is a collor portfolio plan as previously defined and, for each m, $\breve{\pi}_m$ is an $\mathscr{A}_{\tau_m -}$–measurable random vector satisfying $\Sigma_\lambda \breve{\pi}_m^\lambda(\omega) = 1$ a.s., also $\breve{\pi}_m^\lambda \geq 0$ for each λ a.s. if short sales are forbidden. (Some changes are needed in the conditions for uniqueness and strict concavity given in Section 2.) As a matter of common sense it may be questioned whether, by allowing as 'feasible' a plan involving an instantaneous portfolio switch at the same instant τ as the market jumps we do not violate the requirement that the investor should not be able to observe the jump ΔX_τ when selecting the vector π_τ; however, all is well formally since π_τ is $\mathscr{A}_{\tau-}$–measurable whereas ΔX_τ is independent of $\mathscr{A}_{\tau-}$, and in practice an 'instantaneous' switch might be approximated by switching at nearby times or by means of options or other contingent contracts.

As will be seen in [3], there are also cases *without* fixed discontinuities where, because of insufficient smoothness of the characteristics of X, no optimum with a collor π exists, but an optimum does exist if in the definition of a portfolio plan we replace 'adapted collor' by 'predictable and locally bounded'; in particular, it can happen that no optimal sure collor π exists but that there is an optimal sure measurable π which is bounded on compacts of $\mathcal{T}$. In conclusion, we review briefly the implications of such a revised definition for the preceding discussion.

A portfolio plan is now defined explicitly as a locally bounded, predictable vector process satisfying (2.2) for all (ω,t); a non–negative π must in addition satisfy (2.3) for all (ω,t) and λ. The definitions of Π^o, Π^+, Π are revised accordingly. The theory in [1] goes through virtually unchanged with the wider definitions, so that Section 2 above stands, up to and including the definitions of sure and invariable plans, apart from amendments needed to take account of the possible non–existence of the limits $\pi(\pm)$. The discussion of portfolio separation for sure plans in Section 4 and the alternative proof of Theorem 1 are also practically unchanged.

Turning to the discussion of uniqueness at the end of Section 2, we assume here that the functions $V^{\lambda c}$ and $\langle M^{\lambda c},M^{\ell c}\rangle$ as well as G are absolutely continuous on $\mathcal{T}$. Under (2.24), or under (2.28) and (2.31), $x^o \equiv x^1$ now implies only that $\delta\pi_t = 0$ for *almost* all t a.s., (not $\delta\pi \equiv 0$); conversely, $\delta\pi_t = 0$ for a.a. t a.s. implies $\delta x \equiv 0$ without the stated assumptions. The concavity and strict concavity properties of x^π stated in Section 2 remain valid, but some changes are needed in the proofs. The assertions in Proposition 1 concerning the uniqueness of optimal $\bar{c}^*$, c^*, and x^* then stand with the same proof, but the assertion concerning π^* needs an amendment relating to null sets.

Passing to Section 3, it is found that with the new definition of a portfolio the proofs of Theorem 3 breaks down at step (iii), and consequently also at step (iv). The proof of Theorem 2 without the u.f.s. condition is similarly affected. However, if some condition is in force which implies that an optimal x^* is unique, so that an optimal c^* or θ^* is also unique, the procedure outlined in Remark 2 of Section 3 shows that c^* or θ^* is in fact sure, even if the corresponding inference for π^* is no longer valid. It follows that an optimal plan (c^*,π^*) is one which maximises the functional $\varphi(c,\pi)$ in the form (4.1) with c sure. The problem of proving Theorem 2 is then reduced to showing that, if there is a π^* in Π which maximises $\Psi(\pi,T)$ among all $\pi\in\Pi$ for each T, then there is such a π^* in Π^S. This problem no longer involves the interdependence between consumption and portfolio planning and can be attacked by the methods to be considered in [3].

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