Tranching

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Abstract

The structure of securitization deals, referred to as "tranching", is standard. In those transactions, claims on cash flows generated by the collateral are split into several classes of notes, at least 3 and possibly more than 5. Each class is called a tranche and has absolute priority in the cash flows over the more junior ones. Typically, investors with increasing sophistication acquire tranches with decreasing seniority.

This paper offers a model where such a slicing of claims into a stack of several debt like contracts arises endogenously as a value maximizing arrangement. It also predicts the relationship between the seniority of tranches and the sophistication of their acquirers.

It considers the situation of an issuer of asset-backed securities facing heterogeneous financial institutions. The institutions differ in their abilities to screen the collateral and retail the securities. Tranching induces good screeners to specialize on junior tranches to save retail costs, leaving senior tranches to good retailers. This may boost the price of junior tranches by increasing information collection, and improve the liquidity of senior tranches by mitigating the Winner's Curse.

The number of tranches, driven by the structure of the buy side, is arbitrary, and whether the sell side has private information or not about the collateral is irrelevant.

1 Introduction

1.1 Definition

Most securitization deals are structured the same way, referred to as "tranching". Claims on cash flows generated by the portfolio of loans or notes are split into several classes of notes, or "tranches", with varying seniorities and absolute priorities. For instance, the generic structure of Collateralized Debt Obligations (CDOs) is described as follows in Fabozzi and Goodman (2001):

> "The securities issued by the CDO are tranched into rated and unrated classes. The rating of each class is primarily determined through the priority of interest in the cash flows generated by the collateral. The senior notes are typically rated AAA to A (...) and have the highest priority on cash flows. The mezzanine classes are typically rated BBB to B (...) and have a claim on cash flows that is subordinate to the senior notes. The subordinated notes/equity of the CDO are generally unrated and are the residual of the transaction."

The simplest structures consist thus in an equity tranche plus two classes of notes, while deals featuring up to 4 or 5 classes are fairly common[1]. Figure 1 illustrates tranching.

[Figure 1 about here]

1.2 A Stylized Fact

Moreover, casual observation suggests that institutions with increasing sophistication buy tranches with decreasing seniority. This is well acknowl-

[1] see also Fabozzi and Goodman (2001) for examples of such deals.

edged by practitioners. Detailed evidence is uneasy to collect because of commercial secrecy, but Table 1 provides a suggestive piece of evidence supporting this point. It summarizes the distribution by account type for the CDOs arranged by a top tier US investment bank through financial years 2000-2001.

Table 1

	CDO Rated Note	CDO Equity
Bank/CP Conduit	80%	17%
Insurance/Pension	16%	50%
Money Manager/Hedge Fund	2%	19%
Other	2%	14%

The left-hand column displays each account type's stake in the rated-or most senior-tranches, while the right-hand one features this distribution for the most junior ones, either unrated notes or equity tranches.

Of course, banks are not intrisically unsophisticated, but they act as pure retailers in such deals. They redistribute quickly the tranches to more dispersed investors. Their investment decisions are based mainly upon rating agencies reports and a brief check of the legal aspects of the deal.

Conversely, insurance companies and pension funds buy and hold these assets to meet their commitments towards policyholders or beneficiaries. They exert a much more important screening effort (comprehensive due diligence, interviews with the collateral manager) before buying.

Hedge funds and money managers are likely to be the most "sophisticated" type of institutions, in the sense that they are endowed with the most important quantitative research departments and aim at exploiting subtle arbitrage opportunities.

Table 1 shows that the share of banks in tranches is divided by more than 4, while insurance companies and hedge funds shares are multiplied by 3 and

10 respectively as seniority decreases. Thus, the average sophistication of acquirers, defined as their screening abilities and efforts, decreases with the seniority of tranches.

1.3 Motivation

This paper addresses the design of securitization transactions as well as this relationship between seniority and sophistication.

The simplest tranching, with only two tranches, splits claims on cash flows the very same way as a mix of debt and equity. Since Modigliani-Miller (1958) irrelevancy result, an enormous literature has offered a number of rationales for the relevancy of this mix. But surprisingly, whether those rationales also predict more generally more tranches as an optimal design has been hardly investigated. One exception discussed below is Winton (1995). It is worth filling this gap. Indeed, tranching with two tranches is a stylized representation of real arrangements, hardly observed in the real world, while the slicing of cash flows into a larger number of layers is a very widespread risk sharing arangement in practice. It prevails not only in securitization but also in reinsurance and corporate finance[2].

To derive tranching, Winton (1995) extends the Townsend (1979)-Gale Hellwig (1985) model of standard debt in presence of state verification costs. He considers the case where an entrepreneur has to tap several wealth constrained investors to fund a project. If agents are assumed to be risk neutral, the optimal outside financing consists in a stack of debt contracts with varying absolute seniority, because it minimizes the duplication of verification costs. This contribution captures more than 2 tranches as an optimal design. However, the duplication of verification costs, which is the essential imper-

[2] This point is detailed in Section 4.

fection driving tranching in this model, does not seem to be important in securitization. In practice, investors coordinate and rely upon a single audit firm hired by the special purpose vehicle to verify loans payoffs[3]. The situation investigated here is the very different one of an issuance game with *ex ante* private information, more likely to fit securitization transactions. Bernardo and Cornell (1997) provide empirical evidence of private information on the buy side in an auction for collateralized mortgage obligations.

The main intuition may be outlined as follows. The model studies the situation of an issuer of asset-backed securities facing heterogeneous financial institutions. The institutions differ in their abilities to screen the collateral and retail the securities. Tranching is the efficient way to induce good screeners to specialize on junior tranches in order to save retail costs, hence leaving senior tranches to good retailers. This has two potential benefits. It may boost the price of junior tranches by spurring information collection, and improve the liquidity of senior tranches by mitigating the Winner's Curse.

1.4 Organization of the Paper

For expositional clarity, Section 2 presents a particular case of the model featuring only two types of financial institutions, "sophisticated" and "unsophisticated". Tranching with two tranches arises as the value maximizing splitting of future cash flows.

Section 3 copes with the general economy, introducing more degrees of sophistication on the buy side. The optimal structured financing is a tranching featuring possibly more than 2 tranches, which is consistent with empirical evidence.

The related literature on private information and security design, as well

[3] Costly state verification fits better the situation of a small firm without a very formal reporting.

as extensions of the model, are discussed in Section 4.

Section 5 concludes.

Proofs are in Appendix 1.

2 Tranching With Two Tranches

2.1 The Model

The practical situation I have in mind is the primary market for a securitization deal. An investment bank structures the deal and sells over the counter blocks of securities to financial institutions.

More precisely, there are three dates, $t = 0, 1, 2$. All agents are risk neutral and do not discount future cash flows.

At date 1, the management of an investment bank, simply referred to as "the bank" or "the issuer" henceforth, raises cash against claims on some collateral paying off at date 2. At $t = 0$, the bank structures the deal so as to maximize date 1 expected proceeds. I shall describe the collateral, the issuance process and the buy side.

The collateral

The collateral is a given portfolio of loans or notes. Its date 2 payoff is the realization of a real random variable $\widetilde{L}$ whose support is within $[0, L_F]$. L_F is the sum of principals and interests owed by borrowers to the originator(s) of the portfolio. The reason why the originator appeals to securitization to raise fresh money is unmodelled. In practice, securitization is often motivated by regulatory arbitrage (see e.g. Donahoo Shaffer 1991).

The collateral may be either "good" or "bad". Let F_G and F_B denote the cumulative distribution functions of the collateral's payoff conditionally to its type being good or bad respectively. For $T \in \{G; B\}$, let also

$$\overline{F}_T = 1 - F_T$$
$$L_T = \int \overline{F}_T$$

$\overline{F}_T$ is the survival function of payoffs conditionally to the portfolio being of type T, L_T their conditional expected value.

A good portfolio is better than a bad portfolio because it is preferable in the sense of hazard rate stochastic dominance:

$$\frac{\overline{F}_G}{\overline{F}_B} \text{ is increasing}$$

with the convention that this ratio is infinite when the denominator is 0. Note that hazard rate stochastic dominance is a sufficient condition for first order stochastic dominance, so that $L_G \geq L_B$[4]. Of course, hazard rate stochastic dominance holds for instance if a portfolio of the good type does not default.

In this version of the model, the issuer has no private information about the collateral. The case where she knows its type is dealt with in Subsection 4.3. Here, the issuer puts a *prior* $q \in (0, 1)$, common knowledge, on the collateral being good.

The issuance process

In order to maximize date 1 expected proceeds, the issuer is allowed to split claims on cash flows generated by the collateral into any arbitrary number m of securities $(P_i)_{i \in [1,m]}$. The structure of the transaction is constrained as follows:

- Each security P_i has to be a piecewise differentiable, increasing function of the total payoff L.

[4] A non proved conjecture is that my results hold under first order stochastic dominance only.

- $\forall L \in [0, L_F], \sum_{i=1}^{m} P_i(L) = L$

The strong assumption is monotonicity. It stems from unmodeled moral hazard reasons (see e.g. Biais and Mariotti 2001, Harris and Raviv 1989, Innes 1990, Nachman and Noe 1994)[5].

Once designed at date 0, securities are marketed through simultaneous first price sealed bid auctions at date 1.

This mechanism, in particular simultaneity, fits over-the-counter deals, where the issuer has a bilateral and secret relationship with each potential investor for each security and deals eventually at the best price[6]. As precised later, the findings are robust to alternative auction mechanisms.

The buy side

The buy side is made of several financial institutions, sometimes simply referred to as "investors" throughout the paper. Before bidding for a given security, a financial institution has two tasks to perform, first screening the collateral and then finding a retail clientele for the security. Retailing is costly and screening is imperfect. I shall now describe the screening technology and the retail process.

Screening technology. Financial institutions are endowed with an imperfect screening technology. When the collateral is good, a financial institution with degree of sophistication $k \in [k_1, 1)$ learns it privately with probability k when exerting screening. Thus, k measures her screening skills. From now on, a financial institution is referred to as "unsophisticated" when she

[5] The argument is as follows. If some securities are locally decreasing, it is easy for the holders of increasing claims to lend money to the issuer and then share with her the profit from reporting a higher payoff to the holder of decreasing claims.

[6] Most securitization deals are over-the-counter. Why it is so is beyond the scope of this paper, which does not address optimal mechanism design, but rather optimal security design as a response to this given mechanism.

has the minimal skills k_1, as "sophisticated" otherwise. Conditionally to the collateral being good, financial institutions receive independent signals.

Retail process. It is assumed that financial institutions cannot hold the securities they bid successfully for, but have to retail them shortly after $t = 1$ in case of successful bids[7]. Because it takes time, the acquisition of a retail clientele has to take place before date 1. This acquisition process is costly. Viewed at date 0, the expected private cost that a financial institution with sophistication k has to sink to acquire retail customers for a given security P is

$$c(k) \times P_G$$

where P_G is the expected payoff from the security conditionally to the collateral being good and $c(.)$ such that $\frac{1-k}{k} \times c(k)$ is non decreasing with respect to k. Moreover, $c(k_1)$ is normalized to 0 without loss of generality but to simplify further discussions.

Appendix 2 motivates this specification with an elementary search modelling of the customers acquisition process. The broad idea is that sophisticated institutions must find sufficiently financially educated customers to whom explain what they are doing, hence a more tedious customers acquisition as sophistication increases.

In this Section I solve for a particular case of the model featuring only two classes of investors indexed by $i \in \{1, 2\}$.

Class 1 is made of n_1 unsophisticated investors (hence with sophistication k_1). Class 2 is made of n_2 sophisticated investors with common screening skills $k_2 > k_1$. This structure of the buy side is common knowledge.

n_1 and n_2 are supposed to be larger than 2. This is a technical assump-

[7] "Retail" is not in a strict legal sense. For instance, an insurance company holds the security, but still has to raise policyholders money to finance it.

tion. My results hold without it but bidding strategies and expected profits have simpler forms in this case. In what follows, I consider only symmetric equilibria where investors within one class behave identically.

The timing of the issuance may now be summarized as follows.

At $t = 0$, the bank structures the deal and presents it to investors.

Between 0 and 1, each investor compares the expected profits and costs from bidding for each security. If the expected outcome is nonpositive for each security, she does not participate in the deal. Otherwise, she screens the collateral, acquires the retail customers for the securities she plans to bid for, then submits sealed bids for those securities.

At $t = 1$, securities are allocated to the highest bidders. Ties are settled with a fair lottery.

The fact that investors decide whether to participate or not before screening is not important. It may stem from an unmodelled fixed screening cost such as analysts compensation. The crucial part of this structure is of course the sunk retail cost. It drives participation in the auctions because sophisticated investors have no time and money to waste retailing vast cash flows for which their informational advantage does not pay off.

In the following, I take two steps to derive tranching with two tranches. First, next Subsection studies the properties of the auction game for a given, arbitrary security P. Then, Subsection 2.3 uses these properties to derive tranching.

2.2 The Auction Game For a Security

This Subsection deals with the auction for a given security P.

Because of the sunk retail cost, it is not clear at this stage whether a sophisticated investor (within class 2) finds it worth bidding for P . This question is addressed in two steps. Proposition 1 derives the bidding strategies and bidders expected profits in the auction game when one and two classes participate in the auction for P. This enables to characterize the securities for which class 2 finds it worth bidding (Proposition 2).

Then, the conditions under which the bank is better off with sophisticated investors participating are derived (Proposition 3).

Let first

$$P_T = \int_0^{L_F} P' \overline{F}_T$$

the expected payoff of the security conditionally to the type $T \in \{G, B\}$.

Let also

$$K_1 = (1 - k_1)^{n_1}$$

$$K_2 = (1 - k_1)^{n_1} \times (1 - k_2)^{n_2}$$

K_1 and K_2 are the respective probabilities that no investor among class 1 and classes 1 and 2 respectively has learned a good collateral is good during screening.

Because of a discrete common value, the auction game does not meet the sets of sufficient conditions stated in Wilson (1977) or Milgrom and Weber (1982) for an unique pure strategies Nash equilibrium to exist. However, I obtain an unique mixed strategies equilibrium close to solutions obtained by Campbell and Levin (2000) in a setup with binary common value and two heterogeneous bidders.

Proposition 1

i) If investors of class 1 *only participate in the auction, there exists* (P_1^1, P_2^1) *such that*

$$P_B < P_1^1 < P_2^1 < P_G$$

and the auction game admits a unique mixed strategies equilibrium such that

- *Each investor bids* P_1^1 *if she has not learned that the collateral is good.*
- *If she knows the collateral is good, she mixes bids* b *over the support* $\left[P_1^1, P_2^1\right]$ *with the following cumulative distribution*

$$F(b) = \frac{1-k_1}{k_1}\left(\left(\frac{P_G - P_1^1}{P_G - b}\right)^{\frac{1}{n_1-1}} - 1\right)$$

- *Each investor's expected profit is*

$$\frac{(1-q)\, qK_1}{1-q+qK_1}\,(P_G - P_B)\,\frac{k_1}{1-k_1}$$

ii) If all the investors participate in the auction, there exists $\left(P_1^2, P_2^2, P_3^2\right)$ *such that*

$$P_B < P_1^2 < P_2^2 < P_3^2 < P_G$$

and the auction game admits a unique mixed strategies equilibrium such that

- *Each investor bids* P_1^2 *if she does not know that the collateral is good.*
- *If she knows it is good, a member of class* 1 *mixes bids* b *over the support* $\left[P_1^2, P_2^2\right]$ *with the following cumulative distribution*

$$F_1(b) = \frac{1-k_1}{k_1}\left(\left(\frac{P_G - P_1^2}{P_G - b}\right)^{\frac{1}{n_1+n_2-1}} - 1\right)$$

- *If she knows it is good, a member of class* 2 *mixes bids* b *over the support* $\left[P_1^2, P_3^2\right]$ *with the following cumulative distribution*

$$\begin{cases} F_2(b) = \frac{1-k_2}{k_2}\left(\left(\frac{P_G-P_1^2}{P_G-b}\right)^{\frac{1}{n_1+n_2-1}} - 1\right) & for \quad b \in \left[P_1^2, P_2^2\right] \\ F_2(b) = \frac{1-k_2}{k_2}\left(\left(K_1\frac{P_G-P_1^2}{P_G-b}\right)^{\frac{1}{n_2-1}} - 1\right) & for \quad b \in \left[P_2^2, P_3^2\right] \end{cases}$$

- *The expected profit of a member of class* $i \in [1, 2]$ *is*

$$\frac{(1-q)\,qK_2}{1-q+qK_2}\,(P_G - P_B)\,\frac{k_i}{1-k_i}$$

Proof. See Appendix 1. ■

The expected profits are expected profits at date 0, before screening takes place and when the decision whether to participate has to be made. The properties of these equilibria are rather intuitive. Figure 2 illustrates bidding strategies.

[Figure 2 about here]

First, non informed investors, whatever their class, bid very conservatively: They bid the lower bound of informed investors mixtures supports. This lower bound is actually the security's expected payoff conditionally to no participating investor having learned the collateral is good. As a result, uninformed investors bid succesfully in this case only and break even on average. The fact that uninformed bidders do not make profits is reminiscent of Wilson (1992). Of course, uninformed bids decrease as participation increases: None learning that the portfolio is good is worse news when more numerous and more sophisticated investors have exerted screening.

If she knows the collateral is good, an investor bids more aggressively if she belongs to class 2. This is because she has a higher informational advantage over her competitors. Unsophisticated investors bid successfully less often, but with a higher conditional expected profit. However, sophisticated investors have a higher unconditional expected profit since $\frac{k_2}{k_1} \times \frac{1-k_1}{1-k_2} > 1$.

Let us now derive the conditions under which sophisticated investors, rationally anticipating the retail costs and these equilibria, decide to bid for P.

Proposition 2

Let

$$t_2 = \frac{1}{1 - \frac{1-q+qK_2}{(1-q)qK_2} \times \frac{1-k_2}{k_2} c(k_2)}$$

where $K_2 = (1-k_1)^{n_1} \times (1-k_2)^{n_2}$.

Investors of class 2 *participate in the auction for* P *only if*

$$\frac{P_G}{P_B} \geq t_2$$

Proof. See Appendix 1. ■

The denominator of t_2 is nonnegative if, all else equal, $c(k_2)$ is sufficiently small. This is the only interesting case dealt with from now on.

An investor within class 2 finds it worth bidding for P only if the expected profit from her informational advantage overcomes the expected hassle from distributing the security to the public. This is true only if the informational sensitivity of the security, simply measured by the ratio $\frac{P_G}{P_B}$, is sufficiently high.

The last point which remains to be addressed before deriving tranching is the derivation of the conditions under which the participation of sophisticated investors in the auction for P makes the bank better or worse off.

Proposition 3

i) All else equal, for q *large enough, the bank is better off if class* 2 *does not participate in the auction.*

ii) All else equal, for n_2 *or* k_2 *large enough, the bank is better off if class* 2 *participates in the auction.*

Proof. See Appendix 1. ■

The two competing effects of a participation of class 2 are transparent from Figure 2 :

- The lower bound of bids support decreases as class 2 participates. Investors who do not learn that the collateral is good are more concerned by being sold a lemon if they bid succesfully: This increased Winner's Curse leads them to bid more conservatively.

- The upper bound of bids support increases as class 2 participates. Investors are better informed on average and more numerous, hence a fiercer competition for informed investors who bid more aggressively.

If q is sufficiently high, the "Winner's Curse" effect prevails because the event that the collateral is good but none learns it, very costly to the bank, is more likely. Conversely, the fiercer competition among informed investors is the prevailing effect for k_2 or n_2 large enough other things equal.

2.3 Tranching

It turns out from Propositions 2 and 3 that there are two polar cases in which designing two securities may make the bank better off.

Assume first that, all else equal, sophisticated investors are sufficiently sophisticated (k_2 close enough to 1), so that the bank is better of encouraging them to participate in the issuance. If the collateral as a whole is not very informationally sensitive:

$$\frac{L_G}{L_B} < t_2$$

Then unfortunately, sophisticated investors do not participate in a wholesale. The bank maximizes her expected proceeds by isolating the most sensitive cash flows in a junior tranche, more appealing for sophisticated institutions.

Assume conversely that, all else equal, q is so close to 1 that the bank is better off limiting the participation of sophisticated investors in the issuance.

If the collateral is very informationally sensitive:

$$\frac{L_G}{L_B} \geq t_2$$

sophisticated institutions find it worth bidding though. The bank maximizes her expected proceeds by isolating the least sensitive cash flows in a senior tranche which sophisticated institutions will neglect, thereby increasing the liquidity of those non sensitive cash flows by reducing the Winner's Curse for retail institutions.

Borrowing the terminology of Axelson (2002), I term the former case "sensitization" of cash flows and the latter "immunization"[8]. This is formalized in the two following Propositions.

Proposition 4 *(Tranching to sensitize)*

Assume

1. *Class 2 is highly sophisticated (k_2 large), so that bank's expected revenue is larger if she participates in the issuance.*

2. *The portfolio is not information sensitive*

   $$\frac{L_G}{L_B} < t_2$$

   so that any investor within class 2 would not be willing to participate in a wholesale.

Under such circumstances, the bank maximizes her expected revenue by issuing a junior, equity like, and a senior, debt like, tranche.

The face value T of the senior tranche is

$$T = \min \left\{ x \in [0, L_F] \, s.t. \int_x^{L_F} \overline{F}_G > t_2 \int_x^{L_F} \overline{F}_B \right\}$$

[8] Axelson (2002) is commented later.

Sophistication decreases with seniority in the sense that investors from class 2 bid only for the junior tranche.

Proof. See Appendix 1. ■

Proposition 5 *(Tranching to immunize)*

Assume

1. q *is very close to* 1*, so that the bank's expected revenue is smaller if class* 2 *participates in the issuance.*

2. *The portfolio is informationally sensitive*

$$\frac{L_G}{L_B} \geq t_2$$

so that any investor within class 2 *would be willing to participate in a wholesale.*

Under such circumstances, the bank maximizes her expected revenue by issuing a junior, equity like, and a senior, debt like, tranche.

The face value T *of the senior tranche is*

$$T = \max\left\{x \in [0, L_F]\, s.t. \int_0^x \overline{F}_G \leq t_2 \int_0^x \overline{F}_B\right\}$$

Sophistication decreases with seniority in the sense that investors from class 2 bid only for the junior tranche.

Proof. Similar to proof of Proposition 4. ■

Interestingly, both the "sensitization" and "immunization" rationales for tranching seem relevant in practice. It is indeed commonplace among CDOs arrangers to classify deals into two categories, balance sheet transactions and arbitrage transactions. In balance sheet transactions, an originator mandates the arranger for removing the collateral from her balance sheet, in general for regulatory reasons, at the lowest cost. Tranching aims mainly in this case at tailoring liquid senior tranches, appealing for retail institutions. This has a flavor of the immunization story. In arbitrage transactions, the arranger is rather mandated by some sophisticated investors who want to gain exposure on a given risk, e.g. corporate borrowers in a given emerging country. The arranger builds the collateral and then tranches so as to leverage the exposure of her sophisticated customers. This has a flavor of the sensitization story.

Another remark is that in most deals in practice, the senior tranches have a very large face value compared to the junior ones. In this setup, this corresponds to the situation where good and bad collaterals have fairly similar cumulative distributions up to a threshold close to L_F, above which F_B becomes much larger than F_G. This is consistent with a situation where the collateral is a well diversified portfolio of loans and the expected defaults on bad portfolios are not too large compared to those on good portfolios.

At this point, it is worth commenting the assumption that institutions only choose to bid or not, taking their sophistication as given. This is only meant to simplify the exposition. All my results hold under the alternative, maybe more satisfactory, assumption that a sophisticated investor can choose to participate with the degree of sophistication k_1 of her unsophisticated competitors. In this case, the bank has to decide whether she is better off with sophisticated investors using their full skills or not, and then tranches to control for the degree of sophistication with which they participate.

3 Tranching With More Tranches

As stressed in the Introduction, the scope of the existing literature on security design is very much restricted to stylized tranchings with only two tranches, similar to the one derived in former Section.

This setup admits a natural extension to more degrees of sophistication among investors, hence an arbitrarily large number of tranches. This is consistent with evidence on securitization. This Section copes with this general model.

The model outlined in Section 2 is extended as follows. The number of different classes of financial institutions is no more 2, but $N > 1$. Each class $i \in [1, N]$ is characterized by

- a number of institutions $n_i > 1$
- their common sophistication k_i

  Sophistication of class i increases with respect to i in the sense that $(k_i)_{i \in [1,N]}$ is nondecreasing.

The other features of the model are unchanged. However, one point which was immaterial with 2 classes has to be made explicit now. The auctions format is still first price sealed bid, but it is assumed that investors announce publicly and sequentially the securities for which they bid, class i being the i^{th} mover. As detailed in Proposition 7, this ensures that the entry game for each security, and hence the model, has an unique subgame perfect equilibrium, with the i least sophisticated classes bidding for a given security[9]. Simultaneous participation decisions would yield multiple equilibria, including this one but also less interesting ones.

[9] $i \in [1, N]$ depending of course upon the informational sensitivity of the security.

The derivation of tranching takes exactly the same steps as in Section 2: I first solve for the (unique) equilibrium for the auction game for a given security (Subsection 3.1) and then derive tranching (Subsection 3.2).

3.1 The Auction Game For a Security

This section addresses the auction for a given security P. The equivalent of Propositions 1, 2, 3 are derived in this more general setup.

Let

$$K_1 = 1$$
$$K_i = \prod_{j=1}^{i-1} (1 - k_j)^{n_j} \text{ for } i \in [2, N+1]$$

For $i \geq 2$, K_i is the probability that no investor within the $i-1$ least sophisticated classes learns that a good collateral is good. As in Section 3, the following Proposition first characterizes the mixed strategies equilibrium of the auction game.

Proposition 6

Assume all the classes do participate in the auction for P.

There is one unique equilibrium with mixed strategies.

There exists $(P_i)_{i\in[1,N+1]}$ a strictly increasing series within interval (P_B, P_G) such that

- *If an investor does not know that the collateral is good, she bids P_1.*
- *If an investor belonging to class i knows that the collateral is good, she mixes her bids over $[P_1, P_{i+1}]$.*

$\forall j \in [1, N], \forall i \in [j, N]$, *the cumulative distribution of bids of an investor belonging to class i over $[P_j, P_{j+1}]$ is*

$$F_i(b) = \frac{1-k_i}{k_i}\left(\left(K_j \frac{P_G - P_1}{P_G - b}\right)^{\frac{1}{(\sum_{l=j}^{N} n_l)-1}} - 1\right)$$

- *The expected profit of a member of class $i \in [1, N]$* is

$$\frac{(1-q)\, qK_{N+1}}{1-q+qK_{N+1}} (P_G - P_B) \frac{k_i}{1-k_i}$$

Proof. See Appendix. ■

This is a straightforward extension of Proposition 1. P_1, the lower bound of the mixture as well as the bid of uninformed investors, is the expected payoff from P conditionally to none having learned that the collateral is good. Aggressivity of perfectly informed bids increases with sophistication in the sense that their upper bound increases with sophistication. This is again because more sophisticated investors have a higher informational advantage over the rest of the bidders. For a given class i, expected profits from any bid within the support of mixture are equal, proportional to $\frac{k_i}{1-k_i}$. Thus, expected profits increase with sophistication. Mixed bidding strategies are illustrated in Figure 3.

[Figure 3 about here]

Interestingly, it is straightforward to check that the expected profit for any individual within class i

$$\frac{(1-q)\, qK_{N+1}}{1-q+qK_{N+1}} (P_G - P_B) \frac{k_i}{1-k_i}$$

is equal to the one she would obtain in a second price sealed bid auction. Indeed, for such a mechanism, bidding strategies consist obviously in bidding P_G when learning that the security is good and P_1 otherwise, so that an investor makes a nonnegative profit only when she is the only one informed.

This revenue equivalence contrasts with the well known ranking of seller expected revenues obtained by Milgrom and Weber (1982):

First price sealed bid $\leq$ Second price sealed bid

This ranking order is driven by the "linkage principle". A mechanism is all the more efficient in terms of seller's revenue because the price is linked with a large quantity of private information. The equivalence obtained here relies crucially upon the fact that the signal has perfectly informative realizations.

The following Proposition is the extension of Proposition 2 : It derives the conditions under which class i finds it worth bidding for P.

Proposition 7

i) $\forall i \in [2, N]$, *if class* i *participates in the auction for* P*, then any class* $j \leq i$ *also participates.*

ii) $\forall i \in [2, N]$, *let*

$$t_i = \frac{1}{1 - \frac{1-q+qK_{i+1}}{q(1-q)K_{i+1}} qc_i \frac{1-k_i}{k_i}}$$

The number of classes I *participating in the auction for* P *verifies*

$$I = \max\{i \in [1, N]\, s.t. P_G \geq t_i P_B\}$$

Proof. See Appendix 1. ■

$(t_i)_{i \in [2,N]}$ being nondecreasing, the extent of participation in the auction for P increases with its informational sensitivity, $\frac{P_G}{P_B}$. This is illustrated in Figure 4.

[Figure 4 about here]

As in Section 2, the last step before analyzing optimal structuring by the bank is to investigate the circumstances under which she is better off encouraging a large participation.

Proposition 8

Let $i \in [1, N-1]$

i) All else equal, for q *large enough, the bank is worse off if the* $i+1$ *least sophisticated classes participate in the auction rather than only the* i *least sophisticated.*

ii) All else equal, for n_{i+1} *or* k_{i+1} *large enough, the bank is better off if the* $i+1$ *least sophisticated classes participate in the auction rather than only the* i *least sophisticated.*

Proof. Similar to proof of Proposition 3. ■

The result is driven by exactly the same forces as in Proposition 3.

As in the "2 classes" modelling, Propositions 6, 7, 8 prepared the ground for tranching.

3.2 Tranching

The reasons why tranching may create value are exactly the same as in Section 2. For brievity, the "sensitization" case only is outlined.

Proposition 9

Assume that the bank is better off maximizing participation in the issuance but that the portfolio is not very informationally sensitive in the following sense:

$$\exists i \in [2, N] \, s.t. \frac{L_G}{L_B} < t_i$$

The bank maximizes her expected proceeds by splitting the claims into several tranches (at most N*).*

Seniority decreases with sophistication in the sense that class 1 *participates in the auction for all the tranches, while a class* $i > 1$ *participates only in the issuance of the most junior tranches. The number of tranches for which a given class participates decreases with her sophistication.*

Proof. See Appendix 1. ■

This implies that the average sophistication of the successful bidder for a given tranche decreases with its seniority, because (i) more and more sophisticated investors bid as tranches become junior; (ii) sophisticated investors are more likely to learn that a good portfolio is good; (iii) in this case they bid more aggressively.

Under this general formulation, it is barely possible to give more precise predictions about the number of tranches and the number of classes bidding for each tranche. In Appendix 1, a numerical example, with 3 classes, is solved. It shows that three configurations may actually be obtained, depending on the value of the parameters:

- Three tranches with one, two and three classes bidding for the senior, mezzanine and junior tranche respectively.
- Two tranches with three classes bidding for the junior tranche and either one or two classes bidding for the senior one.

These findings are depicted in Figure 5.

[Figure 5 about here]

An interpretation is as follows. If, all else equal, k_2 is sufficiently close to k_1, then it is possible to have this class bidding for all the cash flows. As k_2 increases, her participation requirements become too high, so that a senior tranche has to be "sacrificed", namely left to unsophisticated institutions. As k_2 gets closer to k_3, it becomes impossible to design a mezzanine tranche for this class and only one junior tranche is left for sophisticated institutions.

4 Discussion

4.1 Related Literature

A strand of literature deals with this broad idea that security design is a control for informational sensitivity. More specifically, the "sensitization" motive for tranching is reminiscent from Boot and Thakor (1993), while the "immunization" motive is related to Gorton and Pennacchi (1990).

Boot and Thakor (1993) obtain two tranches, namely a risk free tranche and residual equity, as an optimal response to a lemons problem. They address pooling equilibria of an issuance game plagued by adverse selection. Interestingly, sophistication is endogenous in their setup. Competitive investors may buy information about the quality of an issued claim. Like in Grossman and Stiglitz (1980), prices reveal noisily private information, which makes information acquisition all the less profitable because there are many informed investors. The marginal investor stops buying information when it stops being profitable. By isolating the most information sensitive part of her assets, the issuer of a good claim maximizes her expected proceeds because it makes information acquisition more profitable. This is achieved through isolating the deterministic part of the cash flows in a risk free tranche. There are two important differences with the model outlined here.

First, informed traders demand is observed and reveals their information, hence the free riding problem. This problem is mitigated by the presence of liquidity traders making systematic losses. This fits for instance the analysis of an equity issuance in a large exchange. But free riding plays no role in primary securitization transactions involving over the counter or sealed bids deals.

Second, the issuer is informed. In securitization, it is not clear why the arranger should have an important informational advantage over investors. Regarding mortgage backed securities for instance, DeMarzo (2001) stresses that the most important risk is not default, covered by agencies, but prepayment. Assessing this risk involves a sophisticated analysis of the yield curve and borrowers behavior that specialized institutions are likely to carry out more efficiently than the arranger.

Gorton and Pennacchi (1990) use the fact that riskless (hence informationally insensitive) debt cannot be used by sophisticated institutions to make trading profits at the expense of uninformed investors in order to endogenize the creation of bank deposits as means of payment. The immunization story outlined here has the same flavor. In this case indeed, the gain from tranching is also a smaller liquidity premium on senior tranches. This is because unsophisticated investors are reassured absent the threat of sophisticated investors.

This paper extends those contributions in two ways. First, deriving both effects out of a unique model emphasizes that they are the two sides of the same coin. Second, a more realistic splitting of cash flows into several tranches is predicted.

Axelson (2002) obtains both "immunization" and "sensitization" as possible optimal strategies in an auction game. The scope of the paper is very different. Axelson sticks to security designs featuring only two tranches and has homogeneous investors, but models explicitly the fact that the collateral is a pool of assets. He considers a situation where an issuer may retain claims on some future cash flows, at a cost, or auction them off. If the number of assets is large compared to the number of investors, the issuer is better off pooling them, issuing debt against the pool and retaining equity. This is the immunization strategy. In the opposite case, maximal sensitization, namely issuing equity like securities backed by each individual asset, is optimal.

4.2 Retained Tranche

The originator of the collateral typically retains the equity tranche, or at least a significant stake in it in most securitization deals. In this model, if investors of class N, the most sophisticated, are shareholders of the originating bank, retained equity emerges. Now, why should shareholders be more "sophisticated" than other investors? Because inside shareholders are better informed about the bank's risks than outside investors. They have a privileged access to the "soft" information upon which credit decisions heavily rely (see e.g. Rajan 1992). Banks are more likely to have inside shareholders with high skills in financial analysis than less heavily regulated firms: This is required by the regulator in many countries.

It may also be that the shareholders are only the second most sophisticated investors. In this case, the most sophisticated is the provider of a hedge on the securitized portfolio.

It is worth pointing out that this explanation for retained equity contrasts sharply with the rationalizations usually exhibited in setups where the deal is arranged by the owner-manager of a project, namely signaling and incentive considerations.

Retaining risk as a signaling device is of course irrelevant here, for the issuer has no information to reveal. The case of an informed issuer is discussed in next Subsection.

Moral hazard is not in the scope of this paper but is likely to plague securitization deals. Indeed, originating banks remain in charge of managing securitized portfolios. Efficient credit risk management entails important administrative costs (monitoring of borrowers, proactive recovery policy, appropriate seizure of collateral). Thus, the bank is not likely to "behave" if she has no incentives to do so. In a setup with continuous outcomes and risk neutrality[10], Innes (1990) exhibits retained equity as the optimal incentive compatible arrangement.

4.3 Informed Issuer

As claimed in the Introduction, whether the issuer has an informational advantage or not over investors is irrelevant in this paper. The model presented in Sections 3 and 4 addressed the situation of an uninformed issuer. Proposition 10 states that tranching still prevails if the issuer is informed in this model.

Proposition 10

In the model developed in Sections 3 *and* 4*, assume that the bank knows the type of the collateral but cannot credibly communicate it to investors. Propositions* 1 *through* 9 *still hold. In particular, tranching occurs under the same circumstances.*

Proof. See Appendix 1. ■

Of course, this result stems from the modelling choice made here, positing that the deal is structured by a penniless agent. There is no room for a signaling retention by the informed party. It is thus sufficient to check that an informed good issuer is still better off tranching while mimicked by a bad one, since all that changes in the game is that she computes her expected proceeds conditionally to this information.

If retention is feasible at some cost, the model becomes a particular case of DeMarzo and Duffie (1999). DeMarzo and Duffie capture the debt equity mix as an optimal response to a lemons problem. They exhibit debt issuance

[10] and the same restriction to monotonous payoffs as here.

and retained equity as a signaling equilibrium in an issuance game plagued by adverse selection.

4.4 Tranching in Reinsurance and Corporate Finance

This Subsection comments briefly the analogies and differences between the structures of securitization and reinsurance deals and the right hand side of corporations balance sheets.

Like banks, insurance companies also have the ability to sell their outstanding claims in a secondary market, the reinsurance market. Interestingly, the way insurance firms finance catastrophe losses[11] may be viewed as a tranching as well, and features a similar link between seniority and sophistication. Catastrophe events are typically covered by non proportional reinsurance. For a given event, the insurance company bears the first losses up to a threshold ("priority" in reinsurance terminology). This is similar to retained equity for moral hazard reasons. Then reinsurers, who can be considered to be more sophisticated risk managers than retail insurers given their large research divisions and their worldwide exposure, finance the excess losses up to a specified extent. Once the reinsurance layers are pierced, shareholders of the insurance company bear of course the excess losses up to the value of equity. If net of reinsurance losses exceed equity, namely in case of bankruptcy, policyholders bear the residual losses. This is summarized in Figure 6.

[Figure 6 about here]

By reading the balance sheet of any corporation[12], one can easily con-

[11] *id est* losses with a low frequency and a high severity.

[12] including financial institutions.

clude that tranching is also a very common feature of corporate finance. This point is documented by Fama (1990). He shows that most contracts within US corporations are debt like. Broadly, financially sophisticated agents (institutional investors) hold claims subordinated to those of less sophisticated agents (labor, suppliers). Moreover, unsophisticated investors are protected not only by the high seniority of their claims but also by their short maturity. A major difference with securitization though is the fact that commercial banks do not act as pure retailers like in securitization deals. They perform monitoring and restructuring in case of financial distress. In spite of those sophisticated tasks, they hold very senior claims, bank loans being typically senior to public debt for instance.

The corporate finance literature suggests that the seniority of bank claims stems precisely from *ex post* imperfections (moral hazard, incompleteness of contracts) once the project has been financed and the firm is a going concern (see e.g. Aghion and Bolton 1992 or Dewatripont and Tirole 1994). Diamond (1993) and Park (2000) predict that banks should get senior claims under such circumstances. Rajan and Winton (1995) predict conditional seniority at the optimum, implemented by taking collateral or not as interim information accrues. Thus, corporate governance issues seem a good candidate to explain why sophisticated investors may have senior claims in the balance sheet of industrial firms. Such issues are less likely to play a role in securitization because a special purpose vehicle is not a firm: Each investor's *ex post* decision (selling shares) does not create strong externalities for the others.

5 Conclusion

This paper offers a model in which slicing claims into several tranches arises endogenously as a value maximizing arrangement. It contributes to the existing literature on security design focusing on more stylized versions of this structure. The possible benefits of tranching, namely reassuring unsophisticated investors or attracting sophisticated ones, have been put forward independently by Gorton and Pennacchi (1990) and Boot and Thakor (1993). Here, those effects coexist and compete because they are the two sides of the same coin.

An interesting route for future research consists in relaxing the assumption that financial institutions behave in a non cooperative manner. More precisely, their interaction could be modified in two ways. First, the deals I have in mind involve a relatively small number of specialists. Thus there is room for collusion, in particular if their interaction is to be repeated over several issuances. Second, even if those institutions bid competitively, they may coordinate their complementary screening and retailing skills. I have in mind that the best screeners should specialize in bidding for the collateral, then retain an equity tranche to signal its quality and sell the senior one to good retailers. Such intermediation chains, with "good screeners" being close to the initial seller and "good retailers" close to the final buyer, are observed in practice. The typical example is an insurance company holding share of a mutual fund whose business consists in picking shares of hedge funds.

6 Appendix 1: Proofs

6.1 Proof of Proposition 1

i) Let

$$P_1^1 = \frac{qK_1P_G + (1-q)\,P_B}{qK_1 + 1 - q}$$

$$P_2^1 = \left(1 - (1-k_1)^{n_1-1}\right) P_G + (1-k_1)^{n_1-1}\, P_1^1$$

P_1^1 is the expected value of the security conditional to the fact that no investor has learned that the portfolio is good.

A Nash equilibrium is necessarily such that, if an investor learns that the portfolio is good, she bids $b > P_1^1$ only if the additional expected profit from winning more often equates the opportunity cost from paying too much when no other investor is informed. Formally,

$$\left(k_1F\left(b\right) + 1 - k_1\right)^{n_1-1} \times \left(P_G - b\right) = \left(1-k_1\right)^{n_1-1}\left(P_G - P_1^1\right)$$

This yields the mixture claimed in the Proposition.

It rests to check that an investor who does not learn anything about the portfolio has no hope for any expected profit if she bids strictly above P_1^1. Indeed, for such a bid b, the expected outcome π is

$$\pi = \frac{(1-k_1)\,q}{1-qk_1} \times \frac{K_1}{1-k_1}\left(P_G - P_1^1\right) - \frac{1-q}{1-qk_1}\left(b - P_B\right)$$

This stems from Bayes' formula. The first term is the expected profit if the portfolio is good, $\frac{K_1}{1-k_1}\left(P_G - P_1^1\right)$, multiplied by the according probability while the second one is the expected loss if it is bad, weighted by the according probability. Rearranging yields

$$\pi = \frac{1-q}{1-qk_1}\left(P_1^1 - b\right) < 0$$

An investor has a nonnegative expected profit only if informed about the security. In this case, any bid within the mixture's support provides the same expected profit $\frac{K_1}{1-k_1}\left(P_G - P_1^1\right)$. It is thus straightforward to compute the expected profit for an investor:

$$\frac{qK_1}{1-k_1} \times k_1 \times \left(P_G - P_1^1\right) = \frac{q(1-q)K_1}{1-q+qK_1} \times \frac{k_1}{1-k_1}\left(P_G - P_B\right)$$

ii) Let

$$P_1^2 = \frac{qK_2P_G + (1-q)\,P_B}{qK_2 + 1 - q}$$
$$P_2^2 = \left(1 - (1-k_1)^{n_1+n_2-1}\right)P_G + (1-k_1)^{n_1+n_2-1}\,P_1^2$$
$$P_3^2 = \left(1 - K_1\,(1-k_2)^{n_2-1}\right)P_G + K_1\,(1-k_2)^{n_2-1}\,P_1^2$$

Again, P_1^2 is the expected value of the security conditional to the fact that no investor has learned that the portfolio is good.

The proof that it is not optimal for an uninformed investor, whatever her class, to bid strictly above P_1^2 is the very same as in i).

At the equilibrium, an informed investor should be indifferent between bidding b above P_1^2 and bidding P_1^2.

If b is in the interval over which both classes mix,

- This is actually the case for an investor of class 1 if

$$\begin{aligned}&\begin{pmatrix}(k_1F_1(b) + 1 - k_1)^{n_1-1} \times (k_2F_2(b) + 1 - k_2)^{n_2} \\ -(1-k_1)^{n_1-1}(1-k_2)^{n_2}\end{pmatrix}(P_G - b) \\ = \;& (1-k_1)^{n_1-1}(1-k_2)^{n_2}\left(b - P_1^2\right)\end{aligned}$$

- This is actually the case for an investor of class 2 if

$$\begin{aligned}&\begin{pmatrix}(k_1F_1(b) + 1 - k_1)^{n_1} \times (k_2F_2(b) + 1 - k_2)^{n_2-1} \\ -(1-k_1)^{n_1}(1-k_2)^{n_2-1}\end{pmatrix}(P_G - b) \\ = \;& (1-k_1)^{n_1}(1-k_2)^{n_2-1}\left(b - P_1^2\right)\end{aligned}$$

  Combining this two equations and rearranging yields the distributions over $\left[P_1^2, P_2^2\right]$ claimed in the Proposition.

If b is in the interval over which only investors belonging to class 2 mix, this is actually the case if

$$(k_2 F_2(b) + 1 - k_2)^{n_2-1}(P_G - b) = K_1 (1-k_2)^{n_2-1}\left(P_G - P_1^2\right)$$

The right hand side is the expected profit from any bid within $\left[P_1^2, P_2^2\right]$. This yields the distribution claimed in the Proposition.

An investor has a nonnegative expected profit only if informed about the security. In this case, any bid within the mixture's support provides the same expected profit. It is thus straightforward to compute the expected profit for an investor within class $i \in [1,2]$:

$$\frac{qK_2}{1-k_2} \times k_2 \times \left(P_G - P_1^2\right) = \frac{q(1-q)K_2}{1-q+qK_2} \times \frac{k_2}{1-k_2}(P_G - P_B)$$

¥

6.2 Proof of Proposition 2

An investor belonging to class 2 is willing to participate in the auction for P only if the expected profit at least offsets expected retail costs:

$$\frac{(1-q)\,qK_2}{1-q+qK_2}(P_G - P_B)\frac{k_2}{1-k_2} \geq c(k_2)\,P_G$$

Rearranging yields Proposition 2.¥

6.3 Proof of Proposition 3

The issuance game is a zero sum game: The expected loss for the bank is the sum of each investor's expected profit. If only class 1 participates, it amounts to

$$\frac{(1-q)\,qK_1}{1-q+qK_1}(P_G - P_B)\,n_1\frac{k_1}{1-k_1}$$

While it is

$$\frac{(1-q)\,qK_2}{1-q+qK_2}\left(P_G-P_B\right)\left(n_1\frac{k_1}{1-k_1}+n_2\frac{k_2}{1-k_2}\right)$$

if investors from both classes bid.

Thus, the bank is better off if both classes bid if

$$\frac{(1-q)+qK_1\left(1-k_2\right)^{n_2}}{(1-q)+qK_1}>\frac{n_1\frac{k_1}{1-k_1}+n_2\frac{k_2}{1-k_2}}{n_1\frac{k_1}{1-k_1}}\left(1-k_2\right)^{n_2}$$

This inequality holds if, other things equal, n_2 or k_2 are large enough. It does not hold if, *ceteris paribus*, q is large enough.¥

6.4 Proof of Proposition 4

Clearly, the only parameter driving the expected loss the bank faces for a given security P is the number of classes bidding for P. Thus, the bank's program amounts to design a security $P_2(L)$ such that

i) both classes bid for $P_2(L)$

ii) class 1 only bids for $L-P_2(L)$

iii) the sum of expected losses on both auctions is minimal.

Let

$$l_1=\frac{(1-q)\,qK_1}{1-q+qK_1}n_1\frac{k_1}{1-k_1}$$
$$l_2=\frac{(1-q)\,qK_2}{1-q+qK_2}\left(n_1\frac{k_1}{1-k_1}+n_2\frac{k_2}{1-k_2}\right)$$

$l_i\left(\overline{F}_G-\overline{F}_B\right)dL$ is the bank's expected loss when i classes participate in the auction for an infinitesimal tranche. By assumption, $l_1>l_2$.

The bank solves for the following program

$$\min_{P_2(.)} l_1\int_0^{L_F}\left(1-P_2'\right)\left(\overline{F}_G-\overline{F}_B\right)+l_2\int_0^{L_F}P_2'\left(\overline{F}_G-\overline{F}_B\right)$$

subject to

$$\forall L\in\left[O,L_F\right],P_2'(L)\in[0,1] \qquad (1)$$

$$\int_0^{L_F} P_2' \left(\overline{F}_G - t_2\overline{F}_B\right) \geq 0 \tag{2}$$

Ignoring (1), the Lagrangian of the program (I actually maximize the opposite objective) is linear with respect to P_2'. Denoting μ the multiplicator of (2), the coefficient of P_2' is

$$\overline{F}_B \left((l_1 - l_2 + \mu) \left(\frac{\overline{F}_G}{\overline{F}_B} - 1 \right) + \mu (1 - t_2) \right)$$

$\frac{\overline{F}_G}{\overline{F}_B}$ is increasing, so that P_2' is first 0 then 1 and P_2 is necessarily equity like, possibly with face value 0 if $\frac{\overline{F}_G}{\overline{F}_B}$ does not reach sufficiently high values. ¥

6.5 Proof of Proposition 6

For $i \in [1, N]$, let

$$N_i = \sum_{j=i}^{N} n_j$$

The series $(P_i)_{i\in[1,N+1]}$ defining bounds of mixture supports is defined as follows:

$$\begin{aligned} P_1 &= P_G - \frac{(1-q)}{1 - q(1 - K_{N+1})} (P_G - P_B) \\ \forall i &\in [1, N], P_{i+1} = \left(1 - K_i (1 - k_i)^{N_i - 1}\right) P_G + K_i (1 - k_i)^{N_i - 1} P_1 \end{aligned}$$

As in the proof of Proposition 1, P_1 is the updated expected payoff of $P(.)$ conditionally to no investor learning the portfolio is good.

- The proof that any uninformed investor makes no expected profit by bidding above P_1 is the very same as in the proof of Proposition 1.

- Mixed strategies of informed investors are derived by recursion on the intervals $[P_j, P_{j+1}]$.

1. Assume first that $b > P_1$ is included in the support of mixture of each class. A Nash equilibrium requires that an informed investor from class i is indifferent between bidding b and P_1. This yields

$$\forall i \in [1, N], \begin{bmatrix} \frac{\prod_{l=1}^{N} (k_l F_l(b)+1-k_l)^{n_l}}{k_i F_i(b)+1-k_i} \\ -\frac{1}{1-k_i} K_{N+1} \end{bmatrix} (P_G - b) = \frac{1}{1-k_i} K_{N+1} (b - P_1)$$

Thus

$$\forall (i,j) \in [1, N]^2, \frac{k_i F_i(b) + 1 - k_i}{1 - k_i} = \frac{k_j F_j(b) + 1 - k_j}{1 - k_j}$$

Hence

$$\forall i \in [1, N], F_i(b) = \frac{1 - k_i}{k_i} \left(\left(\frac{P_G - P_1}{P_G - b} \right)^{\frac{1}{N-1}} - 1 \right)$$

And any investor within class 1 may actually bid b only if $b \in [P_1, P_2]$.

2. Now, for $j \in [2, N]$, assume that mixed strategies of informed agents are as claimed in Proposition 5 over $[P_1, P_j]$ and let us derive them over $[P_j, P_{j+1}]$.

Necessarily, the $j-1$ first classes do not bid over $[P_j, P_{j+1}]$. An informed investor belonging to class $i \geq j$ is willing to bid $b > P_j$ only if it doesn't bring her expected profit down. This yields

$$\forall i \in [j, N], \frac{\prod_{l=j}^{N} (k_l F_l(b) + 1 - k_l)^{n_l}}{k_i F_i(b) + 1 - k_i} (P_G - b) = \frac{1}{1-k_i} K_{N+1} (P_G - P_1)$$

The right-hand side is the expected profit stemming from any bid below P_j.

This equation yields

$$\forall (i,l) \in [j, N]^2, \frac{k_i F_i(b) + 1 - k_i}{1 - k_i} = \frac{k_l F_l(b) + 1 - k_l}{1 - k_l}$$

Reinjecting yields

$$F_i(b) = \frac{1-k_i}{k_i}\left(\left(K_j \frac{P_G - P_1}{P_G - b}\right)^{\frac{1}{N_j - 1}} - 1\right)$$

and any investor within class j can actually bid b only if b is below P_{j+1}.

An investor has a nonnegative expected profit only if informed about the security. In this case, any bid within the mixture's support provides the same expected profit. It is thus straightforward to compute the expected profit for an investor within class $i \in [1, N]$:

$$\frac{q(1-q)K_{N+1}}{1-q+qK_{N+1}} \times \frac{k_i}{1-k_i}(P_G - P_B)$$

¥

6.6 Proof of Proposition 7

i) For any subset S of $[1, N]$, let

$$K_S = \prod_{s \in S} (1 - k_s)^{n_s}$$

Let us now define

$$I = \{i \in [1, N] \text{ s.t. class } i \text{ participates in the auction for } P\}$$

And

$$m = \max I$$

From a straightforward adaptation of Proposition 6, the expected profit for an investor within class m is

$$\frac{(1-q)\,qK_I}{1-q+qK_I}(P_G - P_B)\frac{k_m}{1-k_m}$$

and because she participates, necessarily

$$\frac{P_G}{P_B} \geq \frac{1}{1 - \frac{1-q+qK_I}{q(1-q)K_I} qc\left(k_m\right)\frac{1-k_m}{k_m}}$$

so that necessarily as well

$$\forall l \in [1, m], \frac{P_G}{P_B} \geq \frac{1}{1 - \frac{1-q+qK_{I\cap[1,l]}}{q(1-q)K_{I\cap[1,l]}} qc(k_l) \frac{1-k_l}{k_l}}$$

because the right hand side is smaller ($K_{I\cap[1,l]} \geq K_I$ and $k_l \leq k_m$).

It is easy to see that this ensures $l \in I$. This is because of sequentiality. This condition states indeed that it is worth participating for class l if it is the most sophisticated class bidding for P. This is sufficient to make participation a dominant strategy. Indeed, if more sophisticated classes find it worth participating given that she participates, then it is worth for her too because $\frac{1-k}{k}c(k)$ is increasing.

Hence $I = [1, m]$.

ii) is a straightforward consequence of i).¥

6.7 Proof of Proposition 9

For $i \in [1, N]$, let

$$l_i = \frac{q(1-q)K_{i+1}}{1-q+qK_{i+1}} \left(\sum_{j=1}^{i} n_j \frac{k_j}{1-k_j} \right)$$

$l_i \left(\overline{F}_G - \overline{F}_B \right) dL$ is the bank's expected loss when the i least sophisticated classes participate in the auction for an infinitesimal tranche. $(l_i)_{i\in[1,N]}$ is decreasing by assumption.

If a bank issues a given security P, the only parameter driving the expected price of this security is the number of classes participating in its auction. Thus, the bank designs at most N securities $(P_i)_{i\in[1,N]}$ such that the i least sophisticated classes bid for P_i so as to solve

$$\min_{(P_i)_{i\in[1,N]}} \sum_{i=1}^{N} l_i \int_0^{L_F} P_i' \left(\overline{F}_G - \overline{F}_B \right)$$

subject to

$$\forall i \in [2, N], \int_0^{L_F} P_i' \left(\overline{F}_G - t_i \overline{F}_B\right) \geq 0 \tag{1i}$$

$$\forall i \in [1, N], \forall L \in [0, L_F], P_i'(L) \in [0, 1] \tag{2i}$$

$$\forall L \in [0, L_F], \sum_{i=1}^{N} P_i'(L) = 1 \tag{3}$$

Substituting

$$P_1' = 1 - \sum_{i>1} P_i'$$

And ignoring $(2i)$, the Lagrangian has pointwise the following coefficient for P_i' $(i > 1)$:

$$\overline{F}_B \left[(l_1 - l_i + \mu_i) \left(\frac{\overline{F_G}}{\overline{F_B}} - 1 \right) + \mu_i (1 - t_i) \right]$$

where μ_i is the multiplicator of constraint $(1i)$.

$\frac{\overline{F_G}}{\overline{F_B}}$ increases with L by virtue of hazard rate stochastic dominance, thus P_1 is necessarily debt like (potentially with a zero face value).

Pointwise optimization on the remaining equity like claim with at least 2 classes participating in each auction yields recursively Proposition 8.¥

6.8 An Example

Assume there are 3 classes of investors ($N = 3$).

The good portfolio pays off 1 almost surely. The bad portfolio's payoff is uniformly distributed over $[0, 1]$.

Assume q and degrees of sophistication are such that, with the notation adopted in the proof of Proposition 8 :

$$l_1 = 3$$

$$l_2 = 2$$

$$l_3 = 1$$

Let

$$s_i = \frac{1}{t_i}$$

for $i = 2, 3$. Assume $s_3 < \frac{1}{2}$, so that tranching is relevant.

Let $[x_i, x_{i+1}]$ denote the $i + 1^{th}$ tranche designed by the bank.

The bank solves the following program

$$\min_{(x_i)_{0 \le i \le 3}} -3x_0^2 + x_1^2 + x_2^2 + x_3^2$$

subject to

$$0 = x_0 \le x_1 \le x_2 \le x_{3=1}$$

$$x_2 \ge 1 - 2s_3$$

$$x_1 + x_2 \ge 2 - 2s_2$$

Thus,

1. If

$$s_2 - s_3 \ge \frac{1}{2}$$

Then

$$x_2 = 1 - 2s_3$$

$$x_1 = 0$$

2. If

$$s_2 - s_3 \le \frac{1}{2}$$

And

$$s_2 \ge 2s_3$$

Then

$$x_2 = 1 - 2s_3$$

$$x_1 = 1 - 2\left(s_{2-}s_3\right)$$

3. If

$$s_2 - s_3 \leq \frac{1}{2}$$

And

$$s_2 \leq 2s_3$$

Then

$$x_1 = x_2 = 1 - s_2$$

In case 1, two tranches are issued. Classes 1 and 2 bid for the senior one, all classes bid for the junior one.

In case 2, three tranches are issued. Class 1 bids for the senior one, classes 1 and 2 bid for the mezzanine one, all classes bid for the junior one.

In case 3, two tranches are issued. Class 1 bids for the senior one, all classes bid for the junior one.¥

6.9 Proof of Proposition 10

Clearly, if the bank knows her portfolio is bad, the best she has to do is to behave as if it were good in order not to reveal this information.

If the bank knows her portfolio is good, the only difference with the modeling featuring uninformed banks is that she computes her expected losses for a given security conditionally to this information plus the fact that she's mimicked by bad banks. Hence, only Propositions 3 and 8 are potentially altered in this situation. They are actually unchanged. Let us check it for Proposition 3.

From the good bank's point of view, if only class 1 participates in the auction for $P(.)$, her expected loss is now

$$K_1 \left(P_G - P_1^1\right) \left(n_1 \frac{k_1}{1-k_1} + 1\right)$$

If investors from both classes 1 and 2 participate, it equates

$$K_2 \left(P_G - P_1^2\right) \left(n_1 \frac{k_1}{1-k_1} + n_2 \frac{k_2}{1-k_2} + 1\right)$$

so that the bank is better off as two classes bid for $P(.)$ if

$$\frac{(1-q) + qK_1 (1-k_2)^{n_2}}{(1-q) + qK_1} > \frac{1 + n_1 \frac{k_1}{1-k_1} + n_2 \frac{k_2}{1-k_2}}{1 + n_1 \frac{k_1}{1-k_1}} (1-k_2)^{n_2}$$

The sufficient conditions for this inequality holding or not are thus similar to these stated in Proposition 2 in the uninformed bank's case.

Showing that Proposition 8 also holds is very similar.¥

7 Appendix 2

This Appendix offers an elementary modelling of the retail process taking place between dates 0 and 1.

There is a potential retail clientele available to financial institutions. This potential retail clientele is made of a vast number of individuals with unit wealth, so that institutions face no financial constraint.

Retailing a security means dividing it into unitary shares and acquiring a customer for each share, namely finding an individual willing to purchase it in case of a successful bid. For simplicity only, individuals are risk neutral and the institution makes them take-it-or-leave-it offers.

Individuals differ in their financial education, a variable that the financial institution cannot observe. If the financial institution has learned that the collateral is good, she needs to explain it to each individual so that she accepts the fair price. This explanation comes at a cost c. An individual can be convinced by this explanation only if she is sufficiently financially educated. All that the financial institution knows is $\Phi(k)$, the proportion of individuals who are sufficiently financially educated to understand screening exerted with a degree of sophistication at least equal to k. Before screening, the expected retail cost is then for an institution with sophistication k:

$$qk\frac{c \times P_G}{\Phi(k)} \equiv c(k) P_G$$

It is the probability to come across a good collateral multiplied by the cost to convince a sufficient number of individuals to purchase the security. Thus the assumption that $\frac{1-k}{k}c(k) = qc\frac{1-k}{\Phi(k)}$ increases with respect to k amounts to assume that $\frac{1-k}{\Phi(k)}$ is increasing. In words, the distribution of financial education within the population must have sufficiently thin tails. More precisely, this distribution has to be dominated by the uniform distribution in the sense of hazard rate stochastic dominance.

It would be straightforward to assume that individuals are risk averse, or have outside options whose yield increases with financial education in order to give them a more realistic nonnegative profit.

References

[1] Aghion, Philippe; Bolton, Patrick, "An Incomplete Contracts Approach to Financial Contracting", *Review of Economic Studies,* v59, n3 (July 1992): 473-94

[2] Axelson, Ulf, "Security Design With Investor Private Information", working paper (2002)

[3] Bernardo, Antonio E.; Cornell, Bradford, "The Valuation of Complex Derivatives by Major Investment Firms: Empirical Evidence" (in Shorter Papers), *Journal of Finance*, Vol. 52, No. 2. (Jun., 1997), 785-798

[4] Biais, Bruno; Mariotti, Thomas, "Strategic Liquidity Supply and Security Design", working paper (2001)

[5] Boot, Arnoud W. A.; Thakor, Anjan V., "Security Design", *Journal of Finance,* v48, n4 (September 1993): 1349-78

[6] Campbell, Colin M.; Levin, Dan, "Can the Seller Benefit from an Insider in Common-Value Auctions?", *Journal of Economic Theory,* v91, n1 (March 2000): 106-20

[7] DeMarzo, Peter; Duffie, Darrell, "A Liquidity-Based Model of Security Design" *Econometrica,* v67, n1 (January 1999): 65-99

[8] DeMarzo, Peter, "The Pooling and Tranching of Securities: A Model of Informed Intermediation", working paper (2001)

[9] Dewatripont, Mathias; Tirole, Jean, "A Theory of Debt and Equity: Diversity of Securities and Manager-Shareholder Congruence", *Quarterly Journal of Economics,* v109, n4 (November 1994): 1027-54

[10] Diamond, Douglas W., "Seniority and Maturity of Debt Contracts", *Journal of Financial Economics,* v33, n3 (June 1993): 341-68

[11] Donahoo, Kathleene K.; Shaffer, Sherrill, "Capital Requirements and the Securitization Decision", *Quarterly Review of Economics and Business,* v31, n4 (Winter 1991): 12-23

[12] Fabozzi, Frank G. ; Goodman, Laurie S., "Investing in Collateralized Debt Obligations", Frank J. Fabozzi Associates (2001)

[13] Fama, Eugene F, "Contract Costs and Financing Decisions", *Journal of Business,* v63, n1 (Part 2, January 1990): S71-91

[14] Gale, Douglas; Hellwig, Martin, "Incentive-Compatible Debt Contracts: The One-Period Problem", *Review of Economic Studies,* v52, n4 (October 1985): 647-63

[15] Gorton, Gary; Pennacchi, George, "Financial Intermediaries and Liquidity Creation", *Journal of Finance*, Vol. 45, No. 1. (Mar., 1990): 49-71

[16] Grossman, Sanford J.; Stiglitz, Joseph E., "On the Impossibility of Informationally Efficient Markets", *American Economic Review,* v70, n3 (June 1980): 393-408

[17] Harris, Milton; Raviv, Artur, "The Design of Securities", *Journal of Financial Economics,* v24, n2 (October 1989): 255-87

[18] Innes, Robert D., "Limited Liability and Incentive Contracting with Exante Action Choices", *Journal of Economic Theory*, v52, n1 (October 1990): 45-67

[19] Milgrom, Paul R.; Weber, Robert J., "A Theory of Auctions and Competitive Bidding", *Econometrica,* v50, n5 (September 1982): 1089-1122

[20] Modigliani, Franco; Miller, Merton H. "The Cost of Capital, Corporation Finance and the Theory of Investment", *The American Economic Review*, Vol. 48, No. 3. (Jun., 1958), pp. 261-297.

[21] Nachman, David C.; Noe, Thomas H., "Optimal Design of Securities under Asymmetric Information", *Review of Financial Studies,* v7, n1 (Spring 1994): 1-44

[22] Park, Cheol, "Monitoring and Structure of Debt Contracts", *Journal of Finance,* v55, n5 (October 2000): 2157-95

[23] Rajan, Raghuram G., "Insiders and Outsiders: The Choice between Informed and Arm's-Length Debt", *Journal of Finance*, v47, n4 (September 1992): 1367-400

[24] Rajan, Raghuram; Winton, Andrew, "Covenants and Collateral as Incentives to Monitor", *Journal of Finance,* v50, n4 (September 1995): 1113-46

[25] Townsend, Robert M., "Optimal Contracts and Competitive Markets with Costly State Verification", *Journal of Economic Theory v21*, n2 (Oct. 1979): 265-93

[26] Wilson, Robert, "A Bidding Model of Perfect Competition ", *Review of Economic Studies* v44, n3 (Oct. 1977): 511-18

[27] Wilson, Robert, "Strategic Analysis of Auctions", *Handbook of game theory with economic applications.* Volume 1. (1992): 227-79

[28] Winton, Andrew, "Costly State Verification and Multiple Investors: The Role of Seniority", *Review of Financial Studies,* v8, n1 (Spring 1995): 91-123

Figure 1

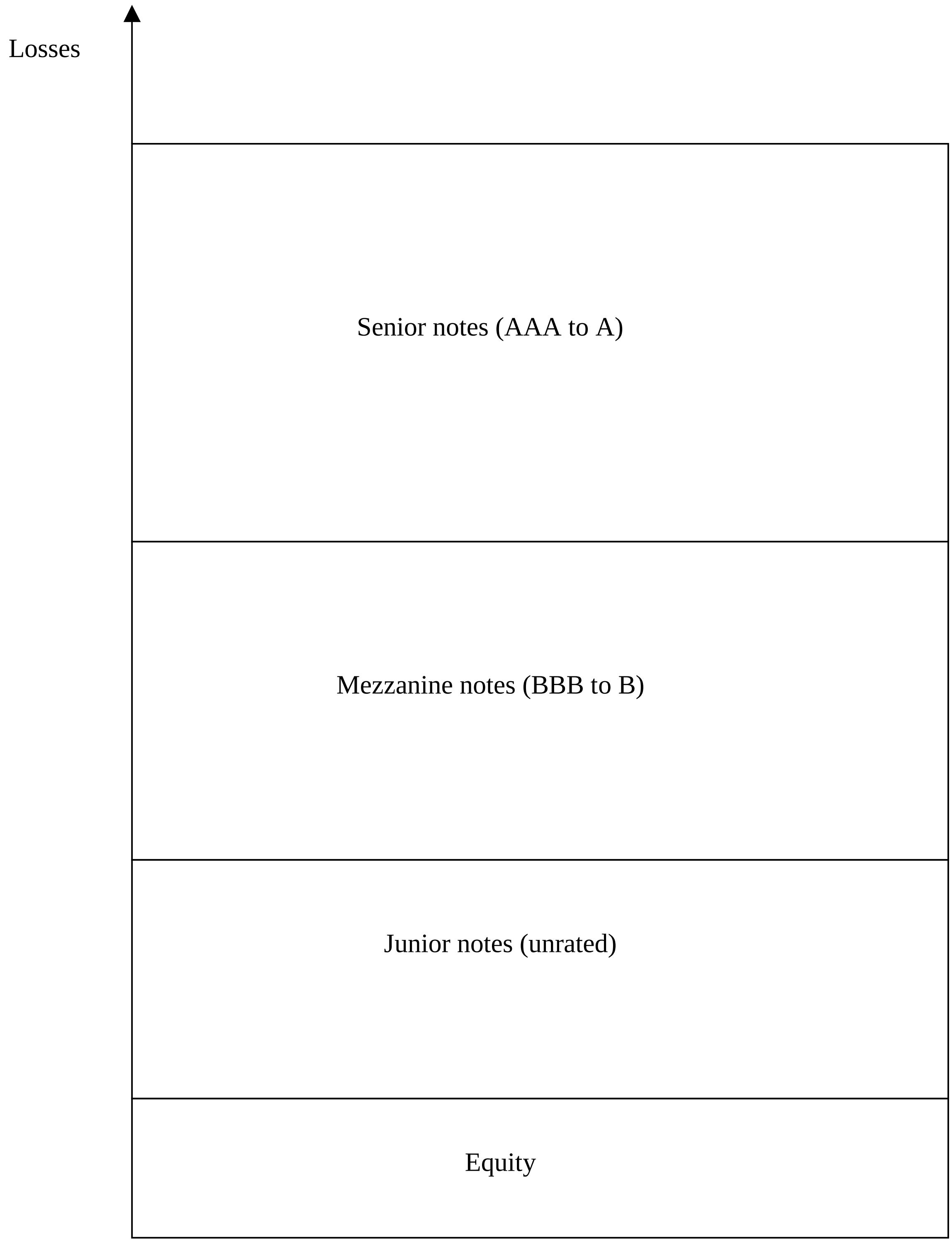
Losses
Senior notes (AAA to A)
Mezzanine notes (BBB to B)
Junior notes (unrated)
Equity

Figure 2
Bidding strategies in the case of 2 classes

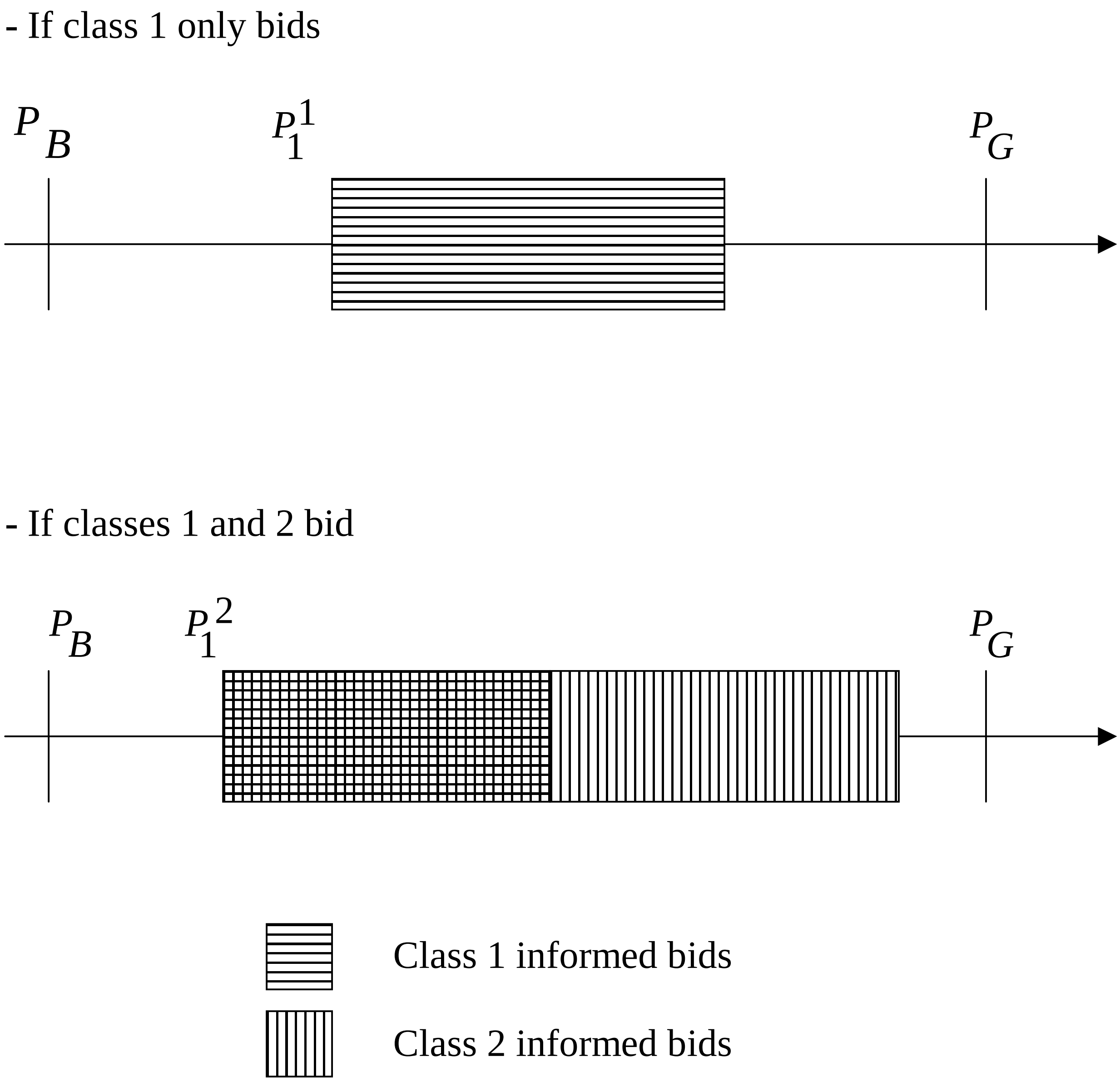

Figure 3

Bidding strategies in the case of 4 classes

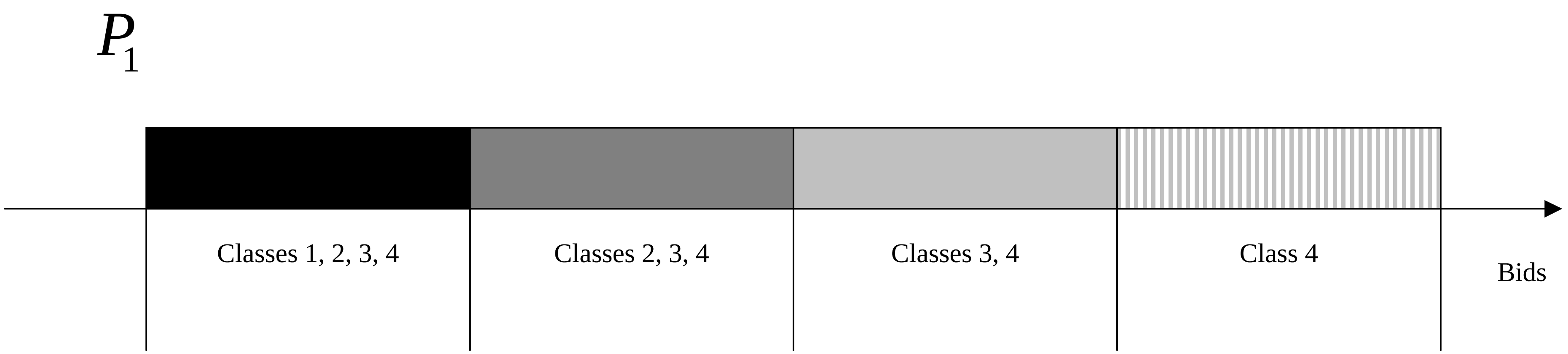

Figure 4
Participation
and informational sensitivity of P
(with 4 classes)

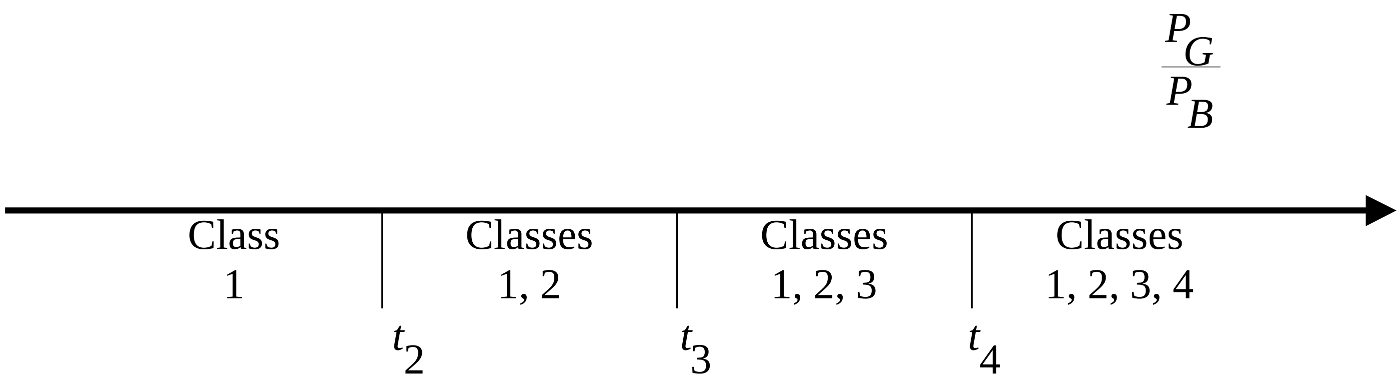

Figure 5
A numerical example with 3 classes

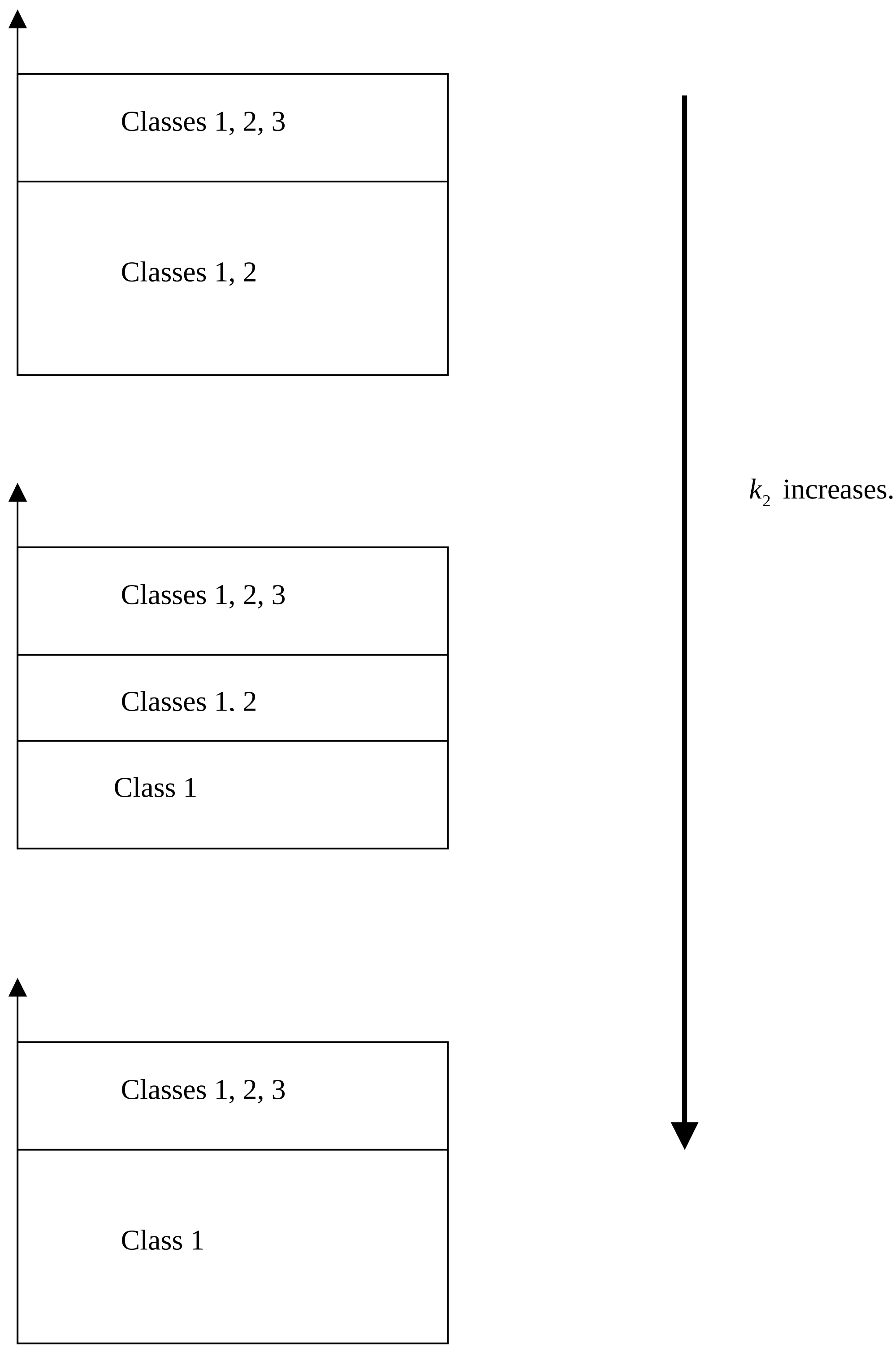

Figure 6
Non proportional reinsurance