

**Continuous-Time Optimal Stochastic Growth:
Local Martingales, Transversality and Existence**

By

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CONTINUOUS—TIME OPTIMAL STOCHASTIC GROWTH:
Local martingales, Transversality and Existence †

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Abstract †*

The present work deals with optimal planning in continuous time, infinite horizon, stochastic neo-classical one-sector models of economic growth (or decline). In the main model, called the Standard Model, the influence of risk is represented in an abstract way by the measurability of production and utility with respect to a general filtration, while the equation of accumulation is written as a random ordinary differential equation. We also consider a model in which depreciation, technological progress, population and impatience are modelled as general semimartingales and the equation of accumulation may be written as a stochastic differential equation, and show that this can be represented as a special case of the Standard Model

Consider a ‘star’ plan for which welfare is finite (and remains finite if consumption is reduced by a small proportion) Let $y = y(\omega, t)$ denote the ‘shadow price’

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process, defined as the product of marginal utility and (compound) marginal return to capital, and let $J^* = J^*(\omega, t)$ be the return-discounted capital process, both calculated along the star plan. It is not hard to show that the star plan is optimal if y is a martingale and $E\{y(t)J^*(t)\} \rightarrow \infty$ as $t \uparrow \infty$, but in general these conditions are not necessary. The process y associated with an optimum may be only a *local* martingale 'reduced' by some sequence $(\chi_n) \uparrow \infty$ of stopping times, and then the form of the transversality condition needs to take into account the form of this sequence. We seek conditions which are necessary and sufficient for optimality, and for which the reducing sequences have significant economic interpretations. It is shown that the star plan is optimal iff (i) y is a local martingale reduced by some sequence (ν_n) of *depletion times*, i.e. first downcrossing times for J^* , which may be unbounded and even take the value ∞ , and (ii) $y(\omega, \infty) = 0$ in states ω where $J^*(\omega, t)$ is bounded away from zero. Alternatively, the plan is optimal iff (i) y is a local martingale reduced by some sequence (χ_n) of *bounded* stopping times and (ii) $E\{y(\chi_n)J^*(\chi_n)\} \rightarrow 0$. In particular, the bounded times may be chosen as *price times* $\varphi_n \wedge n$, where φ_n are first exceedence times for y . In case there is a deterministic function J^- such that $J^*(\omega, t) > J^-(t) > 0$ for all t , a.s., the bounded times may be chosen as *clock times* $(t_n \uparrow \infty)$, so that y is a true martingale; this holds in particular if the 'propensity to consume out of capital' process c^*/k^* is bounded.

Conditions are also given for the existence of an optimum based on a lower closure property of the set of consumption plans and on weak compactness of a maximising utility sequence in a suitable L_1 space. For negative and bounded utilities, and typically also for positive utilities, a finite supremum of the welfare functional implies existence.

A separate Chapter reviews some consequences of assuming that the marginal productivity of capital at $k = 0$ is infinite, in place of a Lipschitz-type condition adopted in previous chapters.

Note: An early version of much of the present ms was completed as long ago as 1986 as an extension of work on optimal saving with risk; it was accepted for publication but put aside in favour of further work on portfolio problems. The formulation and methods of proof adopted here follow broadly the lines of the author's papers on saving and portfolio choice, but the presence of a production function with diminishing marginal productivity of capital gives rise to complications which have not been fully dealt with in publications. It has therefore seemed worthwhile to correct, substantially extend and update the text. Some later developments in growth theory, such as endogenous growth, have had to be left aside, but it is hoped that our approach, if not all the details, will prove to be more generally relevant.

Parts of the present material are to be included in a monograph in book form on the optimal accumulation of capital in continuous-time, stochastic neo-classical growth models, which will in particular investigate the structure of the optimal consumption function in systems driven by Brownian motions.

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Contents

Abstract	
Chapter 1.	INFORMAL REVIEW
Chapter 2.	THE STOCHASTIC NEO-CLASSICAL GROWTH MODEL
	(A) Deterministic Model
	(B) Stochastic Model
Chapter 3.	THE STANDARD MODEL, CONDITIONS FOR OPTIMALITY AND SUFFICIENCY THEOREM
	(A) The Standard Model
	(B) Conditions for Optimality and Sufficiency Theorem
Chapter 4	FEASIBLE PLANS
	(A) Justification of Definitions
	(B) Properties of Feasible Sets
Chapter 5	DIRECTIONAL DERIVATIVES
Chapter 6.	THE SUPERMARTINGALE PROPERTY
Chapter 7.	NECESSARY CONDITIONS INVOLVING PRICE TIMES
Chapter 8.	NECESSARY CONDITIONS INVOLVING DEPLETION TIMES
Chapter 9.	EXISTENCE OF AN OPTIMUM
Chapter 10.	LIPSCHITZ <i>versus</i> INADA
	(A) Survey and Example
	(B) Definitions concerning Plans
	(C) Properties of the Set of Extended Plans
	(D) Conditions for Optimality
Figures 1–4	
References	

1 INFORMAL REVIEW

The present work addresses the problem of optimal accumulation, or optimal consumption, in a continuous time, infinite horizon, stochastic neo-classical growth model, with concave utility and production functions and a welfare functional whose supremum is finite. The version of the model mainly considered, to be called the *Standard Model* (SM), is specified in detail in Chapters 3–4. Briefly, there is given the time domain $\mathcal{T} = [0, \infty)$, a complete probability space (Ω, \mathcal{A}, P) with a right continuous filtration $\mathfrak{A} = (\mathcal{A}_t; t \in \mathcal{T})$, where $\mathcal{A} = \mathcal{A}_\infty$ and $\mathcal{A}_0 = \mathcal{A}_{0-}$ is generated by the P -null sets (so that an \mathcal{A}_0 -measurable variable is a.s. constant on Ω). \mathfrak{A} represents the planner's information structure and P his beliefs. The equation of accumulation, representing the relation between consumption c , capital k and investment \dot{k} – all expressed in suitable 'standard' units – is defined for each random element ω by an ordinary differential equation (o.d.e.) of the form

$$\dot{k}(\omega, t) = f[k(\omega, t); \omega, t] - c(\omega, t) \quad \dots (1.1)$$

with the initial condition $k(\omega, 0) = K_0 > 0$. The problem of optimal accumulation is to maximise a welfare functional of the form

$$\varphi(c) = \Phi(k) = E \int_0^\infty u[c(\omega, t); \omega, t] q(\omega, t) dt = \int u[c(\omega, t); \omega, t] d\mu(\omega, t) \quad \dots (1.2)$$

subject to (1), the consumption plan $c = c(\omega, t)$ and corresponding capital plan $k = k(\omega, t)$ being constrained to be non-negative and to satisfy conditions of 'progressive' measurability – roughly speaking, observability at each time T of the history during $[0, T]$; see Chapter 3, fn.1 and Chapter 4(A). The product uq is here called 'utility', the split between 'felicity' u and 'impatience density' $q > 0$ being chosen for convenience, in particular so that $d\mu = q \cdot dt \cdot dP(\omega)$ defines a finite (usually unitary) measure on (ω, t) -sets called impatience measure. For fixed (ω, t) ,

the felicity function u and the production function f are concave and sufficiently smooth in their first arguments, with u increasing,¹ possibly unbounded (in one direction or both) and satisfying $u'(0;t) = \infty$, while f satisfies $f(0;t) = 0$, and $f'(0;t)$, $f'(\infty;t)$ are subject to conditions of integrability which ensure that solutions of (1) are unique – see (3.2–4) and Chapter 4(A); however, in Chapter 10, we review the theory under the traditional ‘Inada’ assumption that $f'(0;t) = \infty$. For fixed values of their first arguments, u and f define general (progressive) processes. In case production is proportional to capital for each (ω, t) , say in the form $f(k; \omega, t) = k \cdot r(\omega, t)$ where r is a given process, as in [F1], we refer to a model of *saving* or *linear production* rather than a *growth* model; (see Chapter 3, fn. 6 for a wider definition).

In the Standard Model, risk is introduced in an abstract, general form by way of the random element ω , the main restrictions being that the structure of information as well as the form of dependence of f , u and q on ω be exogenous, and that capital and consumption plans k and c as well as corresponding production plans $F_k(\omega, t) = f[k(\omega, t); \omega, t]$ and felicity plans $U_c(\omega, t) = u[c(\omega, t); \omega, t]$ be observable (progressive). However, various models of accumulation (or growth, decay, depletion etc), in which particular exogenous sources of risk are modelled explicitly as random processes – possibly with a constraint represented by a stochastic differential equation (s.d.e) – can be transformed by suitable changes of variable into special cases of the SM; see Chapter 2.

Reverting to the SM, let a distinguished ‘star’ plan, defined by positive processes $c^* = c^*(\omega, t)$ and $k^* = k^*(\omega, t)$, be given. Let

$$r = r(\omega, t) = f'[k^*(\omega, t); \omega, t], \quad v = v(\omega, t) = u'[c^*(\omega, t); \omega, t]q(\omega, t) \quad (1.3)$$

denote the marginal product of capital and marginal utility processes (or plans) along the star plan, and write

¹ The terms *positive*, *negative*, *increasing*, *decreasing* have their strict meaning throughout. The symbols \uparrow, \downarrow and \uparrow, \downarrow denote strict and weak monotonicity.

$$R = R(\omega, T) = \int_0^T r(\omega, t) dt, \quad (1.4)$$

so that R is the (star) marginal log-return to capital process, e^R is the marginal return or compound interest process and e^{-R} the discount process. Next, using abridged notation, define processes

$$y = ve^R, \quad J^* = k^*e^{-R}, \quad g^* = c^*e^{-R}, \quad G^*(T) = \int_0^T g^*(\omega, t) dt; \quad (1.5)$$

y is called the shadow price of capital process, J^* the discounted (star) capital process (or plan), while g^* and G^* represent (star) discounted and cumulative discounted consumption. All these processes are positive on \mathcal{F} . Note that $yJ^* = vk^*$ may be interpreted as the (star) 'value-of-capital' process. If an alternative plan (c, k) is given, we write

$$J = ke^{-R}, \quad g = ce^{-R}, \quad G(T) = \int_0^T g(\omega, t) dt, \quad \delta J = J - J^*, \quad \delta G = G - G^*. \quad (1.6)$$

Conditions characterising the star plan as optimal may be stated in three parts:

(i) A finite welfare condition, namely that $\varphi(c^*)$ be finite, (and, for a necessary condition, that $\varphi([1-\alpha]c^*)$ be finite for small $\alpha > 0$). In this Chapter, we take the finite welfare condition as read.

(ii) That $y = y(\omega, t)$ be a local martingale 'reduced' by a sequence (χ_n) of stopping times, where $\chi_n(\omega) \uparrow \infty$ a.s. as $n \uparrow \infty$. Thus, for each n , the process

$$y^n = y^n(\omega, t) = y[\omega, t \wedge \chi_n(\omega)], \quad (1.7)$$

representing 'y stopped at χ_n ', is a uniformly integrable martingale.²

² *Notes on stopping times and martingales.* A random variable $\chi = \chi(\omega)$ is a *stopping time* (relative to \mathcal{A}) if it takes values in $[0, \infty]$ and, for each $t \in \mathcal{T}$, the event $\{\omega: \chi(\omega) \leq t\}$ is in \mathcal{A}_t . The σ -algebra of events at (i.e. not later than) χ is defined by $\mathcal{A}_\chi = \{A \in \mathcal{A}: \forall t \in \mathcal{T}, A \cap \{\chi(\omega) \leq t\} \in \mathcal{A}_t\}$

All processes are taken, without special mention, to be progressively measurable (hence adapted) with respect to \mathcal{A} and defined on \mathcal{F} , unless otherwise specified; see Chapters 3 and 4. Thus a process is a *martingale* if $E|y(t)| < \infty$ for each $t \in \mathcal{T}$ and, for each pair $t < T$ from \mathcal{T} , we have $E^t y(T) = y(t)$ a.s.; here E^t means $E(\cdot / \mathcal{A}_t)$. *Super and sub martingales* are defined in the same way with the preceding equality replaced by \leq, \geq respectively. A martingale is *uniformly integrable* (u.i.) iff there is a 'closing' variable $y(\infty)$ such that, for each $t \in \mathcal{T}$, $y(t) = E^t y(\infty)$. A process y is said to be *reduced* by a stopping time χ if $y(t \wedge \chi)$ is a u.i. martingale, and then the closing variable is $y(\chi)$. The process y is a *local martingale* if there is a sequence $(\chi_n; n = 0, 1, \dots)$ with $\chi_n(\omega) \uparrow \infty$ a.s. such that each χ_n reduces y , and then (χ_n) is a

(iii) A transversality condition at infinity, say in the form

$$E\{I\{\omega: \chi_n(\omega) < \infty\} y[\omega, \chi_n(\omega)] J^*[\omega, \chi_n(\omega)]\} \rightarrow 0 \text{ as } n \rightarrow \infty, \quad \dots (1.8)$$

$$y(\omega, \infty) = 0 \text{ if } \chi_n(\omega) = \infty \text{ for some } n. \quad \dots (1.9)$$

It is easily shown, using the concavity assumptions of neo-classical models, that it is *sufficient* for optimality if y is a true martingale and transversality holds in the form

$$E\{y(T) J^*(T)\} = E\{v(T) k^*(T)\} \rightarrow 0 \text{ as } T \rightarrow \infty. \quad \dots (1.10)$$

For typical proof procedures see [F1], which considers a model of optimal saving; [B1M], which considers a growth model but does not explicitly use martingales; also [F5], Brownian growth model; and Chapter 3 below. Untutored intuition might suggest that these conditions should also be *necessary*, and this is indeed the case in a discrete-time version of the growth model, see [F2] and remarks below. *However, in continuous time with an infinite horizon, standard necessity proofs (by whatever technique) yield only that in general y is a local martingale, and then the transversality condition must also be adjusted to refer to convergence along a reducing sequence.*

The mere information that a price process is a local martingale does not provide an adequate characterisation. We wish to identify suitable reducing times, not merely to know that they exist, and if possible to select times with an interesting economic interpretation. Going a step further, the possibility suggests itself of defining a continuous, increasing family of stopping times such that every sequence from this family which tends a.s. to infinity is a reducing sequence for y — so that the family defines a random time change, such that y is transformed into a true martingale and

reducing sequence for y , see [DM] VI 27 e.s. A given process may admit various reducing sequences; here we consider only sequences with $\chi_n > 0$ a.s. for $n > \text{some } n_0$. Martingales (true or local) may be ‘modified’ so that sample functions are a.s. continuous on the right with limits on the left (*coriol*), and sometimes this property will be assumed without special mention. A non-negative local martingale is a supermartingale. All martingales and supermartingales considered will be non-negative on $[0, \infty)$; the left limit $y(\omega, \infty)$ then exists a.s. However, if y is a martingale, the process obtained by adjoining $y(\infty)$ to y may not be a martingale on $[0, \infty]$. In particular, it may happen that $y(\omega, t) > 0$ on $[0, \infty)$ but $y(\infty) = 0$; cf. fn. 3

the transversality condition is stated in terms of (transformed) 'clock' times. A related question concerns the form of the transversality condition, e.g. whether it involves convergence of an expectation as in (10) or convergence pathwise. We return to these ideas later. Here we note that various reducing sequences χ_n can be chosen to yield necessary and sufficient conditions of the form (i), (ii), (iii) above. It is always permissible to choose *price times*, i.e. to set $\chi_n = \varphi_{i(n)}$ with $i(n) \uparrow \infty$, the price time φ_i being defined by

$$\varphi_i = \varphi(\omega, i) = i \wedge \inf\{t \in [0, \infty): y(\omega, t) > i\} \text{ for } 0 \leq i < \infty \quad (1.11)$$

These times are bounded, so that (1.9) is redundant. It is also always permissible to choose *depletion times*, i.e. to set $\chi_n = \nu_{i(n)}$ with $i(n) \uparrow K_0$, the depletion time ν_i being defined by

$$\nu_i = \nu(\omega, i) = \inf\{t \in [0, \infty): K_0 - J^*(\omega, t) > i\} \text{ for } 0 \leq i < K_0 \quad (1.12)$$

if such a number exists, and $\nu(\omega, i) = \infty$ in case $J^*(\omega, t) \geq K_0 - i$ for all t . We shall also need to consider *consumption times* τ_i , defined by

$$\tau_i = \tau(\omega, i) = \inf\{t \in [0, \infty): G^*(\omega, t) > i\} \text{ for } 0 \leq i < \infty \quad (1.13)$$

if such a number exists, and $\tau(\omega, i) = \infty$ in case $G^*(\omega, \infty) \leq i$. Further details of these times are given below and in Chapter 3. Finally, it is of interest to find conditions under which y is a (true) martingale (i.e. a local martingale reduced by a sequence $T_n \uparrow \infty$ of 'clock' times).

These remarks define the main programme of the present work, namely to give proofs of sufficiency and necessity for conditions of optimality of the 'local martingale and transversality' type for the growth model, with special reference to local martingales reduced by price, depletion or clock times. Proofs are also given of the existence of an optimum, extending procedures for a model of saving developed in [F1].

* * *

I have considered local martingale and transversality conditions for optimality in continuous time, and related time changes, in earlier papers, mainly in the setting of a

model of optimal saving with risk [F1], with perhaps the additional complication of a problem of portfolio choice [F3,4], or the presence of an exogenous income or an indivisible investment opportunity [F4]. The later papers refer on important points to the results of the earlier. It is useful to give more or less self-contained proofs for the growth model, both because of its importance in economic theory and because the presence of a production function leads to complications not found in the saving model. (However, some analogous complications do arise in the model of saving with exogenous income, and certain arguments used in connection with that model in [F4] were borrowed from an earlier draft of the present paper.)

As in the earlier papers, the main method of proof used here is a generalised calculus-of-variations argument, which has a clear economic interpretation and yields results by relatively elementary methods, with only minimal assumptions about the stochastic environment (filtration and exogenous processes). It may be useful to review some of the intuitive considerations underlying the martingale and transversality conditions, indicating sources of differences (a) between discrete and continuous time versions and (b) between growth and saving models. The remarks which follow do not form part of the formal development of the SM, which is resumed in Ch.3.

(a) *Discrete vs Continuous Time* Starting with necessary conditions in the *discrete* time growth model, the argument yielding the supermartingale property for y is familiar. Briefly, if we had $S < T$ and, with positive probability, $y_S < E^S y_T$, the utility from consuming a marginal unit δJ_S of discounted capital at S would be less than the expected utility at S from investing that unit at S and consuming the (random) return at T . Thus welfare would be increased by deferring consumption, contrary to optimality; hence $y_S \geq E^S y_T$ a.s., and y is a supermartingale.

Replacing δJ_S by $-\delta J_S$, if feasible, yields a submartingale inequality. However, the argument is not quite symmetrical, since it is straightforward to spend what has

previously been saved, but in the presence of random returns it may not be possible to save up what has previously been spent. The imposition of additional constraints to deal with this difficulty can be avoided as follows. We consider a variation $\delta J = J - J^*$ with δJ_t always negative for $t > 0$ and calculate the derivative $D\Phi$ in the 'direction' δJ as

$$D\Phi = -\lim_{T \rightarrow \infty} E \sum_0^T [y_t (\delta J_t - \delta J_{t-1})] = \lim_{T \rightarrow \infty} E \sum_0^{T-1} [\delta J_t (y_{t+1} - y_t) - y_T \delta J_T], \quad (1.14)$$

see [F2] eq.(3.5); note the 'integration by parts'. Departing slightly from the argument in [F2], we may decompose the supermartingale y as $M - V$, where $M \geq 0$ is a martingale and V is a predictable non-decreasing process with $V(0) = 0$, see [RW] II 54. Then, for each t , we may replace $E\{\delta J_t (y_{t+1} - y_t)\}$ by $E\{\delta J_t E^t[y_{t+1} - y_t]\}$ and then y by $M - V$. Since M is a martingale and V is predictable, this reduces to $-E\{\delta J_t (V_{t+1} - V_t)\} \geq 0$, and we also have $-y_T \delta J_T \geq 0$. However optimality requires $D\Phi \leq 0$, hence $D\Phi = 0$, yielding $V \equiv 0$, $y \equiv M$. The transversality condition follows on setting $J \equiv 0$, $\delta J = -J^*$.

In *continuous* time, the derivation of the supermartingale property is in principle similar, except that one has to consider an *interval* of saving followed by an *interval* of spending, and then go to limits (from the right) as the lengths of the intervals contract; see Chapter 6. Attempting to imitate the subsequent discrete-time argument, one finds that the ('Doob') decomposition $y = M - V$ of the positive supermartingale yields only a *local* martingale M , reduced by some sequence $(\chi_n) \uparrow \infty$ of bounded stopping times, and a predictable, non-decreasing process V with $V(0) = 0$. see [DM] VII.12-13. Choosing a variation with $\delta J < 0$ for $t > 0$, the directional derivative is obtained as

$$-\lim_{n \rightarrow \infty} E \int_0^{\chi_n} y(t) d\delta J(t) = \lim_{n \rightarrow \infty} E \left\{ -\int_0^{\chi_n} \delta J(t) [dV(t) - dM(t)] + [V(\chi_n) - M(\chi_n)] \delta J(\chi_n) \right\} \quad (1.15)$$

on integrating by parts. The (local) martingale integral vanishes, leaving an expression which is ≥ 0 , hence $= 0$, again yielding $V \equiv 0$ and $y \equiv M$. In this case one cannot obtain the transversality condition in the form (8) by setting $\delta J = -J^*$ unless free disposal of capital is allowed, and a rather tedious limiting argument is needed. See Chapter 7.³

(b) *Growth vs Saving.* We now compare briefly the variational arguments in the *continuous-time growth* and *saving* models. In both cases, the process $G^* = (G^*(t); t \in \mathcal{T})$ is continuous and (strictly) increasing, so that $\tau(\omega, i)$, if finite, is the (strict, unique) upcrossing time by $G^*(\omega, t)$ of the level $i \in (0, \infty)$; thus

$$G^*[\omega, \tau(\omega, i)] = i \quad \text{for } 0 \leq i < G^*(\omega, \infty). \quad (1.16)$$

The family $\tau = (\tau_i; 0 \leq i < \infty)$ defines a process which is adapted, continuous and increasing while finite, and as such may be considered as the *time change* associated with G^* , called here the change from 'clock time' to 'consumption time'.⁴ Conversely,

³ In [F2] fn.1, I suggested that the crucial difficulty in imitating the discrete-time procedure was the failure of the integration by parts in the formula corresponding to (14) above. The present discussion shows that this is at best misleading: the crucial difference arises from the Doob decomposition, which in the continuous-time case yields only a *local* martingale in general.

Note that, even if y is a true martingale, it need not be uniformly integrable. For example, in the saving model with $u'(c) = c^{-b}$, $b > 0$, $q(t)$ non-random and a log-returns process R with independent increments we obtain, subject to some conditions of integrability, $y(t) = y(0) \cdot \exp\{(1-b)R(t)\} / E[\exp\{(1-b)R(t)\}]$. This is a martingale on $[0, \infty)$ but in general is not u.i.; if $b \neq 1$ and R is Brownian motion with drift, we have $y(\infty) = 0$ a.s. The case $b = 1$ (logarithmic utility), yielding $y(t) \equiv y(0)$ even without independence of increments, is exceptional. See [F1] S.1 for details.

⁴ *Notes on time changes.* Suppose that a family $\chi = (\chi_i)$ of stopping times is defined for an index i taking values in a real interval $\mathcal{J} = [0, I)$ with $I \leq \infty$, such that $\chi = (\chi(\omega, i))$, regarded as a process, is right continuous and non-decreasing and takes values in $[0, \infty]$, with $\chi(0) = 0$ and $\chi(i) \uparrow \infty$ as $i \uparrow I$. Let \mathcal{A}_i denote the σ -algebra of events at χ_i ; the family $\tilde{\mathcal{A}} = (\mathcal{A}_i; i \in \mathcal{J})$ is right continuous. Then χ (with $\tilde{\mathcal{A}}$) can be regarded as defining a *time change*. If $\xi = (\xi_t; t \in \mathcal{T})$ is an \mathcal{A} -process, its *transform* under χ is the $\tilde{\mathcal{A}}$ -process $\tilde{\xi} = (\tilde{\xi}_i; i \in \mathcal{J})$, where $\tilde{\xi}_i = \xi(\chi_i)$; if ξ admits an a.s. limiting variable ξ_∞ we set $\tilde{\xi}_i = \xi_\infty$ when $\chi_i = \infty$, and also define the limiting variable $\tilde{\xi}_I = \xi_\infty$. If no limiting variable exists, its place may be taken by a suitable 'variable at infinity'.

In particular, if Ψ is a right continuous, non-decreasing process, with $\Psi(0) = 0$ and $\Psi(\infty) \leq \infty$, a time change χ may be defined by

G^* defines the time change associated with τ . The time changes are mutually inverse in the sense that $G^*[\tau(i)] = i$ for $0 \leq i < G^*(\infty)$ – see eq.(16) – and $\tau[G^*(t)] = t$ for $0 \leq t \leq \infty$. See Chapters 3 and 6 for further details.

Let \hat{G}^*, \hat{y} denote the transforms under τ of the processes G, y , satisfying

$$\hat{G}^*(i) = G^*(\tau_i), \hat{y}(i) = y(\tau_i) \quad \text{for } 0 \leq i < G^*(\infty), \tau_i < \infty, \quad \dots(1.17)$$

and set

$$\hat{G}^*(i) = G^*(\infty), \hat{y}(i) = 0 \quad \text{if } i \geq G^*(\infty), \tau_i = \infty. \quad \dots(1.18)$$

Carrying out the ‘save now, spend later’ argument sketched above, but using variations which start and end at specified consumption times (at least in the limit as the variations become small), it is found that \hat{y} is a supermartingale (with respect to the transformed filtration) and hence, using the inverse transform and optional stopping, that y is a supermartingale (with respect to the original filtration). Making changes on null sets if necessary, the processes \hat{y} and y are corlol, and the left limit $y(\infty) = \hat{y}[G^*(\infty -)]$ exists a.s.

The stochastic model of optimal *saving* considered in [F1] could be transformed into a stochastic cake-eating problem, with K_0 as the cake and a depletion equation

$$K_0 - G^*(\omega, t) = J^*(\omega, t), \quad 0 \leq t < \infty, \text{ a.s.} \quad \dots(1.19)$$

$\chi(i) = \inf\{t: \Psi(t) > i\}$ for $i \in \mathcal{J}$, setting $\chi(i) = \infty$ if $\Psi(\infty) \leq i$. Conversely, given a time change χ , we can define Ψ as an inverse time change by $\Psi(t) = \inf\{i: \chi(i) > t\}$ for $t \in \mathcal{S}$.

In all cases considered here, Ψ will be continuous and χ right continuous; if Ψ is strictly increasing, then χ is continuous, (including continuity at values of i for which $\chi(i) = \infty$).

In the sequel, we shall encounter situations where there is a corlol process $y > 0$ admitting a limiting variable $y(\infty)$, and a time change χ , such that *every sequence* $(\chi_i) = (\chi_{i(n)})$ with $\chi_i \uparrow \infty$ a.s. reduces y ; we then say simply that χ reduces y . In this case, for each $i = i(n) < I$, $y^i = (y(t \wedge \chi_i); t \in \mathcal{S})$ is a u.i. \mathcal{A} -martingale, which implies (by the Stopping Theorem) that $\tilde{y}^i = (\tilde{y}(j \wedge i); j \in \mathcal{J})$ is a u.i. $\tilde{\mathcal{A}}$ -martingale, so that $\tilde{y} = (\tilde{y}_i : i \in \mathcal{J})$ is an $\tilde{\mathcal{A}}$ -martingale, which need not be u.i. (A converse proposition applies if χ is defined by a strictly increasing process Ψ , but if Ψ is only non-decreasing the paths $i \mapsto \chi(i)$ may ‘jump across’ intervals of \mathcal{S} , and then martingale properties of \tilde{y} imply nothing about y on these intervals.) For more on time changes, see [DM] VI 56, [Mey1] Ch.IV, [J] Ch X, [EM], [EW].

In this case the concepts of consumption times and depletion times coincide, and these times need be considered only for $0 \leq i < K_0$. The variations yielding the supermartingale inequality for \hat{y} can be reversed to yield the submartingale inequality, so that \hat{y} is a martingale and a transversality condition is obtained in the form

$$E\{\hat{y}(i)[K_0 - \hat{G}^*(i)]\} \rightarrow 0 \text{ as } i \uparrow K_0 \quad \dots(1.20)$$

Equivalently, y is a local martingale reduced by some (indeed by any) sequence

$\tau_i = \nu_i$ with $i \uparrow K_0$, and

$$E\{y(\tau_i)[K_0 - G^*(\tau_i)]\} \rightarrow 0 \text{ as } i \uparrow K_0. \quad \dots(1.21)$$

This implies, using Fatou's Lemma, that

$$\begin{aligned} y(\omega, \infty) &= 0 \text{ in case } K_0 - G^*(\omega, \infty) > 0, \\ &\text{i.e. in case } \tau_i(\omega) = \infty \text{ for some } i < K_0, \end{aligned} \quad \dots(1.21a)$$

so that both (1.8) and (1.9) are satisfied. However, *we cannot use optional stopping to conclude that y is a true martingale*, because in general the times τ_i are not bounded and \hat{y} is not uniformly integrable, cf. [DM] VI.10, [Mey1] VI.13–14

The new complication arising in the case of the *growth* model is that, in the presence of production, a cake no longer shrinks by precisely the amount that is eaten. While G^* is still a strictly increasing process, we now have $K_0 - J^* \leq G^*$ in consequence of the law of diminishing returns, see Chapter 3; also $K_0 - J^*$ need no longer be monotonic or converge to a limiting variable at infinity. Consequently the definitions of depletion times and consumption times are no longer equivalent. The 'save now, spend later' argument, using the transformation to consumption time, still works with some complications, showing first \hat{y} and then y to be supermartingales (and hence that the a.s. limiting variable $y(\infty)$ exists).

However, the 'spend now, save later' variations on intervals starting and ending at consumption times τ_i are in general infeasible, and it is necessary to consider variations defined by depletion times ν_i . Since $K_0 - J^*$ is not monotonic, it does not define a time change; however, if Γ is the non-decreasing process which 'fills in the

troughs' of $K_0 - J^*$, we still have

$$\nu(\omega, i) = \inf\{t \in (0, \infty) : \Gamma(\omega, t) > i\} \text{ for } 0 \leq i < K_0 \text{ if } i < \Gamma(\omega, \infty), \quad \dots (1.12a)$$

and $\nu(\omega, i) = \infty$ otherwise; see also (3.20) and (3.23–26). The family

$\nu = (\nu_i; 0 \leq i < K_0)$ defines a process which is adapted, right continuous and

increasing while finite, and as such may be regarded as the time change associated

with Γ , called the change to 'depletion time'. Conversely, Γ defines the time change

associated with ν . (Note that ν may have discontinuities, 'jumping over' time intervals

on which Γ is constant; thus Γ is inverse to ν in the sense that $\Gamma[\nu(i)] = i$ for

$0 \leq i < \Gamma(\infty)$, but $\nu[\Gamma(t)] = t$ only for t in the (finite) range of ν .) We define the

transforms $\check{\Gamma}, \check{y}$ of Γ, y under ν by

$$\check{\Gamma}(i) = \Gamma(\nu_i), \quad \check{y}(i) = y(\nu_i), \quad 0 \leq i < K_0, \quad \nu_i \leq \infty,$$

and of course $\Gamma(\nu_i) = K_0 - J^*(\nu_i)$ if $\nu_i < \infty$. See Figure 1. Since y is a super-

martingale, it follows by optional stopping that \check{y} is also a supermartingale (for the

transformed filtration), so that the function $E\{\check{y}(i)\} = E\{y(\nu_i)\}$ is non-increasing

in i . Conversely, a rather complicated construction shows that the 'troughs' of $K_0 - J^*$

may be neglected in calculating the directional derivatives for certain variations, so

that, loosely speaking, one can use the time change defined by Γ and a suitable

variation with $\delta J \leq 0$ to show that $y(0) \leq E\{y(\nu_i)\}$ for each i , hence that $E\{y(\nu_i)\}$ is

actually constant. Thus \check{y} is a martingale, and y is a local martingale reduced by any

sequence (ν_i) with $i \uparrow K_0$. The transversality condition then holds in the form

$$\begin{aligned} E\{\check{y}(i)[K_0 - \check{\Gamma}(i)]\} &= E\{y(\nu_i)[K_0 - \Gamma(\nu_i)]\} \\ &= E\{I\{\nu_i < \infty\}y(\nu_i)J^*(\nu_i)\} \rightarrow 0 \text{ as } i \uparrow K_0. \end{aligned} \quad \dots (1.22)$$

As above, this implies that

$$y(\omega, \infty) = 0 \text{ if } K_0 - \Gamma(\omega, \infty) > 0,$$

$$\text{i.e. if } \nu_i(\omega) = \infty \text{ for some } i < K_0,$$

$$\text{i.e. if } \inf\{J^*(\omega, t) : t \in \mathcal{S}\} > 0, \quad \dots (1.22a)$$

so that both (1.8) and (1.9) are satisfied. The stated local martingale condition for y

together with *either* (22) *or* (22a) is also sufficient for optimality; see Chapters 3 and 8 for details (However, the martingale condition for \check{y} together with (22), while necessary, is in general not sufficient because it says nothing about martingale properties of y during t -intervals on which Γ is constant.)

Some assumptions which imply that it is necessary for optimality that y be a *true* martingale and transversality hold in the form (10) are stated at the end of Chapter 3. Briefly, these conditions hold if there is a deterministic function J -satisfying

$$J^*(\omega, t) \geq J^*(t) > 0, \quad 0 \leq t < \infty, \text{ a.s.} \quad (1.23)$$

In particular, (23) is satisfied if the 'propensity to consume process' $c^*/k^* = g^*/J^*$ is bounded above by a constant, say $c^*/k^* < \theta^* < \infty$ on \mathcal{F} a.s

* * *

A word about related literature on the necessary conditions for optimality is in order. Essentially our programme is to extend the treatment of martingale and transversality conditions initiated in [F1] to the continuous-time stochastic growth model, which to my knowledge has not been done elsewhere. The paper closest to the present one is [RW], which considers a two-sector stochastic growth model that is essentially an extension of the Brownian version of our model. This paper derives necessary conditions for optimality by dynamic programming, and brings in a (pointwise) transversality condition only as part of a set of *sufficient* conditions used to verify the optimality of proposed solutions. It also gives a brief survey of some earlier work on neo-classical growth models which need not be repeated here.

Widening the catchment area to (continuous-time, stochastic) control models generally and considering first the 'local' conditions for optimality, it is found that our 'classical' calculus of variations approach, leading directly to martingale properties of

the shadow price process, is not usual in stochastic control.⁵ Of course, such properties can be derived as corollaries of necessary conditions obtained by alternative methods such as dynamic programming, stochastic maximum principles or duality, although often they are not spelled out. The numerous ways in which terminology, assumptions, methods and the form of results vary makes it difficult to give systematic comparisons and I shall not attempt a survey here. For some general techniques which could be applied (subject to various differences in assumptions) to our stochastic growth model, see for instance [Bi], [BiM], [CHL], [FR], [FS], [KS], [Kr], [MB] and references given in these works; also, for related deterministic methods, [AC], [AK], [Be], [BIM], [Ha], [Sh] [Ta], [Ye].⁶

Turning to the necessity of the transversality condition at infinity in continuous-time models, the literature on this point concentrates on deterministic problems, so that the questions on which we focus here – concerning localisation, the form of the condition (in terms of expected value or pointwise) etc do not arise. As is well known, there are infinite-horizon control problems in which the condition is not necessary for optimality, so that strictly speaking necessity cannot be taken for granted even in a deterministic version of our model – say, in the form $y(t)J^*(t) = v(t)k^*(t) \rightarrow 0$. However, there are necessity results for deterministic models with features similar to ours (concave utility and production, impatience, finite welfare at and near the optimum); see for example [AK], [MB], [Mi], [K1], [Ta], which also give further references. Of course, necessity for the deterministic version of our model follows from the stochastic results to be proved here; there are some differences

⁵ For an interesting alternative ‘classical’ approach to stochastic calculus of variations, using Hilbert space methods rather than martingales, see [Ya].

⁶ Most techniques of continuous-time stochastic control are designed primarily for finite-horizon problems, the ‘infinite-horizon case’, if considered at all, being treated as in some sense a limiting case, using essentially the same concepts and methods. In a model of the type considered here but with a finite horizon, the shadow price process y is typically a uniformly integrable martingale, which is replaced in the infinite horizon case by a local (possibly true) martingale.

from the assumptions considered in the literature but it would be tedious to spell out the details. I am not aware of necessity results for transversality in a continuous-time, stochastic growth model (as distinct from results for savings models mentioned earlier), so in that respect the work presented here is new ⁷

* * *

The rest of the paper proceeds as follows. Chapter 2 outlines a model of optimal growth – called the Stochastic Neo-classical (SNC) Model – in which exogenous sources of risk are modelled explicitly as semimartingales and the equation of accumulation may be written as a stochastic differential equation (s.d.e), and shows how this version can be represented as a special case of the SM. The formal discussion of the SM begins in Chapter 3. This chapter defines the model, sets out the basic definitions relating to stopping times and martingale and transversality conditions and proves a general form of the Sufficiency Theorem. Chapter 4 proves some technical points relating to the definition and properties of feasible plans. Chapters 5–8 deal in detail with conditions for optimality, in particular necessary conditions. Chapter 5 characterises an optimum as a (star) plan such that the directional derivatives of the welfare functional are non-positive in all directions. Chapter 6 proves the super-martingale property of y by generalised calculus-of-variations methods, using variations of the consumption plan. Chapter 7 begins with a short proof, along lines sketched above, that y is a local martingale, in particular that y is reduced by price times, and

⁷ There is also a literature on transversality in discrete-time deterministic growth models, which I shall not review here. As regards stochastic versions of such models, there are long-standing results for models with uncertainty defined by a stationary process, where the transversality condition is a straightforward generalisation of the deterministic case, specifically that ‘the present expected value of the output (and input) at period t decreases to zero as t goes to infinity’, [Z1] p 174, see also references cited there and [Z2]. As was mentioned above, a similar result for a model without stationarity assumptions was obtained in [F2] using directional derivatives and martingale arguments. A somewhat more general result using directional derivatives was obtained recently in [K2], (mention being made of [Z1–2] but not of [F2]); that paper also gives references to deterministic discrete-time publications whose results it generalises.

also shows that the transversality condition holds in the form (1.8) with $\chi_n = \rho_{1(n)}$. Chapter 8 derives the necessity of the local martingale and transversality conditions with depletion times, a more interesting but more difficult result. Conditions for y to be a true martingale and the corresponding transversality relation are obtained as a corollary to the results of Chapters 7–8.

The last two chapters take up separate topics. Chapter 9 extends to the Standard Model the proofs of the existence of an optimum given in [F1] for the saving model. Finally, Chapter 10 reviews the consequences of restricting the *Generalised Lipschitz Condition*

$$\int_0^T |f'(k; \omega, t)| dt < \infty \quad \text{for all } (\omega, T), \quad \dots (1.24)$$

which in Chapters (1–9) is imposed for $0 \leq k \leq \infty$, to the interval $0 < k \leq \infty$, and assuming the traditional *Inada* condition

$$f'(0; \omega, t) = \infty \quad \text{for all } (\omega, t) \quad \dots (1.25)$$

The latter assumption has the advantage that it allows useful production functions, such as Cobb–Douglas; but it is unrealistic and leads to technical complications because multiple solutions for the o.d.e (1.1) can arise on the axis $k = 0$. This discussion appears to be new

2. THE STOCHASTIC NEO-CLASSICAL GROWTH MODEL

Among the features of the Standard Model are the abstract formulation of risk, which is introduced through the random element ω and the filtration, and the appearance of the constraint on accumulation as a random ordinary differential equation (rather than, as might be expected, a stochastic differential equation). Before turning to a formal analysis of the Standard Model, it will be instructive to consider a model – to be called the *Stochastic Neo-Classical* (SNC) Growth Model – which is a stochastic version of the traditional continuous–time neo–classical model of optimal growth, the exogenous influences of labour–augmenting technological progress, population growth, depreciation of capital and time preference being modelled explicitly as semi-martingales. This is a convenient setting in which to illustrate the relationship between deterministic and stochastic formulations and between ordinary and stochastic (differential or integral) equations of accumulation. We also show how the SNC model can be represented as a special case of the Standard Model. This Chapter is by way of an intermezzo and is rather informal. It is included largely to motivate the study of the Standard Model, but also to provide a link with further work on the detailed structure of optimal plans, in the special case where the exogenous processes are independent Brownian motions, see [F5–6]; we refer to this case as the *Brownian Neo-Classical* (BNC) Model. The formal development of the theory for the Standard Model is independent of this Chapter.

(A) *Deterministic Model.*

We begin with a review of the deterministic model, following with some changes the standard work by Arrow and Kurz [AK]. Briefly, the problem of optimal growth is to choose a function $\tilde{C}(t) \geq 0$ from a suitable class so as to maximise a welfare functional

$$\varphi = \int_0^{\infty} \tilde{U}[\tilde{C}(t)]\Pi(t)\exp\{-\rho(t)\}dt, \quad (2.1)$$

subject to the condition that the solution $\tilde{K}(t)$ of the o.d.e.

$$d\tilde{K}(t)/dt = \Psi[\tilde{K}(t),L(t)] - \dot{\gamma}(t)\tilde{K}(t) - \tilde{C}(t)\Pi(t) \quad (2.2)$$

with $\tilde{K}(0) = K_0 > 0$ be defined and non-negative for all $t \geq 0$. Here $\tilde{C}(t)$ represents consumption per head of the population, $\tilde{K}(t)$ the total capital stock and $\Psi[\tilde{K}(t),L(t)]$ total gross output, all expressed in 'natural' units. The population – or equivalently, with a constant proportion of employment, the labour force – is $\Pi(t) = \exp\{\pi(t)\}$, while the 'effective' labour force is $L(t) = \exp\{\pi(t)+\beta(t)\}$, where $\beta(t)$ represents labour-augmenting technological progress. The derivative $\dot{\gamma}(t)$ is usually taken to be positive and to represent the rate of depreciation of capital, but other possibilities are open (e.g. appreciation due to discovery of mineral deposits). The function ρ is usually assumed to be positive and to represent 'inter-generational weighting', but negative values may be considered. The four functions

$$\eta = \beta, \gamma, \pi, \rho, \quad \eta = \eta(t), \quad (2.3)$$

are defined for $0 \leq t < \infty$ with $\eta(0) = 0$, and (initially) assumed to be of class C^1 . It is usual in the case of certainty to assume that the functions are linear, say $\eta(t) = m_{\eta}t$, where the m_{η} are (usually positive) constants, but we adopt a more general notation in order to avoid repetitions later on. We further introduce the abbreviations

$$\begin{aligned} x(t) &= x = -(\gamma + \beta + \pi) \\ v(t) &= v = (1-b)\beta + \pi - \rho \\ w(t) &= w = (b-1)\gamma + b\pi - \rho \end{aligned} \quad (2.4)$$

where b is the constant appearing in (8) below, and note the relations

$$v + x = -(\gamma + b\beta + \rho), \quad v + (1-b)x = w. \quad (2.5)$$

Next, the production function $\Psi(K,L)$, representing gross output in natural units, is defined for $K \geq 0, L \geq 0$ with $\Psi(0,L) = \Psi(K,0) = 0$ and is homogeneous of degree one, concave and C^2 (i.e. twice continuously differentiable, including one-sided

limits on the axes); and for $\kappa > 0, L > 0$ it is increasing and strictly concave in each variable separately. We write

$$\psi(\kappa) = \Psi(\kappa, 1) \text{ with } \psi'(\kappa) > 0 > \psi''(\kappa), \quad a(\kappa) = \psi(\kappa)/\kappa, \quad \dots(2.6)$$

and further assume

$$(a) \ 0 < \psi'(0) < \infty, \quad (b) \ \psi'(\infty) = 0, \quad \dots(2.7)$$

$\psi'(0), a(0), \psi'(\infty)$ and $a(\infty)$ being defined as one-sided limits while $\psi(0) = 0$. Note that (a) departs from the traditional 'Inada' condition $\psi'(0) = \infty$ adopted in [AK].

The function \tilde{U} , defined for consumption $c \geq 0$, has the 'CRRA' form

$\tilde{U}'(c) = c^{-b}$ with $b > 0$, so that

$$\tilde{U}(c) = (1-b)^{-1} c^{1-b} \quad \text{if } b \neq 1, \quad \dots(2.8a)$$

$$\tilde{U}(c) = \ln c \quad \text{if } b = 1. \quad \dots(2.8b)$$

We now introduce 'intensive' variables \bar{k} and \bar{c} , also (for later reference)

'standardised' variables \underline{k} and \underline{c} , by writing

$$\begin{aligned} \bar{k}(t) &= \tilde{K}(t)e^{-\pi(t)-\beta(t)} = \tilde{K}(t)e^{x(t)+\gamma(t)} = \underline{k}(t)e^{x(t)}, \\ \bar{c}(t) &= \tilde{C}(t)e^{-\beta(t)} = \tilde{C}(t)e^{x(t)+\pi(t)+\gamma(t)} = \underline{c}(t)e^{x(t)}. \end{aligned} \quad \dots(2.9)$$

By the linear homogeneity of Ψ ,

$$\Psi[\tilde{K}(t), L(t)] = \Psi[\tilde{K}(t), e^{\pi(t)+\beta(t)}] = e^{\pi(t)+\beta(t)} \psi[\bar{k}(t)]. \quad \dots(2.10)$$

Thus the average and marginal products of capital, in intensive or natural units, are

$$\begin{aligned} a_t &\doteq a[\bar{k}(t)] = \psi[\bar{k}(t)]/\bar{k}(t) = \Psi[\tilde{K}(t), L(t)]/\tilde{K}(t), \\ \psi'_t &\doteq \psi'[\bar{k}(t)] = d\psi[\bar{k}(t)]/d\bar{k}(t) = \partial\Psi[\tilde{K}(t), L(t)]/\partial\tilde{K}(t) \end{aligned} \quad \dots(2.11)$$

On differentiating $\bar{k}(t)$ in (9) and using (10), the o.d.e. (2) is easily transformed into

$$d\bar{k}(t)/dt = \psi[\bar{k}(t)] + \dot{x}(t)\bar{k}(t) - \bar{c}(t), \quad \bar{k}(0) = K_0. \quad \dots(2.12)$$

Note that this is equivalent to

$$\underline{k}(T) = \bar{k}(T)e^{-x(T)} = K_0 + \int_0^T \{\psi[\bar{k}(t)] - \bar{c}(t)\}e^{-x(t)}dt \quad \dots(2.13)$$

or

$$d\underline{k}(t)/dt = \psi[\underline{k}(t)e^{x(t)}]e^{-x(t)} - \underline{c}(t), \quad \underline{k}(0) = K_0. \quad \dots(2.13a)$$

We sometimes write

$z(t) = \ln \bar{k}(t)$ or simply $z = \ln \bar{k}$, $z_0 = \ln K_0$, also

$$A(z) = a(\bar{k}), \quad M(z) = \psi'(\bar{k}) \quad \dots(2.14)$$

If $\bar{k}(t) > 0$ for all t , we may divide (12) by $\bar{k}(t)$ and integrate to obtain

$$z(T) = z_0 + \int_0^T \{A[z(t)] - \bar{c}(t) \cdot e^{-z(t)}\} dt + x(T) \quad \dots(2.15)$$

In intensive units, the functional (1) assumes the form

$$\varphi = (1-b)^{-1} \int_0^{\infty} \bar{c}(t)^{1-b} e^{\nu(t)} dt \quad \text{if } b \neq 1; \quad \dots(2.16a)$$

$$\varphi = \int_0^{\infty} [\ln \bar{c}(t) + \beta(t)] e^{\pi(t) - \rho(t)} dt \quad \text{if } b = 1 \quad \dots(2.16b)$$

The relations (12) or (15) and (16), together with the constraints $\bar{c}(t) \geq 0$, $\bar{k}(t) \geq 0$, are usually taken as the starting point for detailed analysis. As is well known, the main advantages of this formulation, in case the functions β, γ, π, ρ are *linear*, are twofold:

(i) An optimal plan (if one exists) may be determined by specifying consumption as a (timeless) function of capital only – say, in the form

$$\ln \bar{c}(t) = H[z(t)] \quad \text{or simply } \ln \bar{c} = H(z) \quad \text{or } \bar{c}/\bar{k} = \exp\{H(z) - z\} = \theta(z), \quad \dots(2.17)$$

noting that along an optimum both $\bar{c}(t)$ and $\bar{k}(t)$ must be (strictly) positive. Thus

(15) may be replaced by

$$z(T) = z_0 + \int_0^T \{A[z(t)] - \theta[z(t)]\} dt + x(T) \quad \dots(2.18)$$

The problem of optimal growth may then be restated as the choice of a positive function $\theta(z)$, defined for $z \in \mathfrak{R}$, to maximise (16) with $\bar{c}(t) = \theta[z(t)] \cdot e^{z(t)}$, where $z(t)$ satisfies (2.18). In this formulation, the constraints $c \geq 0$ and $\bar{k} \geq 0$ no longer appear.

(ii) Under some restrictions on the parameters, the values of $\bar{k}(t)$ and $\bar{c}(t)$ along an optimal plan tend to steady state values as $t \rightarrow \infty$.

(B) *Stochastic Model*

We turn to the formulation of the SNC model. The time domain, probability space, filtration and definitions relating to measures and processes are as in Chapters 1 and 3–4. The definitions and properties of the functions U and Ψ remain unchanged. The variables $\eta = \beta, \gamma, \pi, \rho$ are now introduced as general – possibly discontinuous – semimartingales $\eta = \eta(\omega, t)$ with $\eta(0) = 0$. The abbreviations (2.4–5) stand. The functional (1) is replaced by its expectation, which is still called φ . Its domain is a class \mathcal{E} of progressive processes $\tilde{C} = \tilde{C}(\omega, t) \geq 0$, to be defined more precisely later

The processes η are, in general, not differentiable pathwise in the classical sense, so that the equation of accumulation (2) cannot simply be reinterpreted as a random (pathwise) o d e. We consider two possible reformulations

The first, which generalises the approach adopted in [F1], is to replace (2) by the equivalent *ordinary integral equation*

$$\begin{aligned} \tilde{K}(T) = & \exp\left[-\gamma(T) + \int_0^T \{\Psi[\tilde{K}(t), L(t)]/\tilde{K}(t)\} dt\right] \\ & \times \left\{ K_0 - \int_0^T \left[\tilde{C}(t) \cdot \exp\{\pi(t) + \gamma(t) - \int_0^t \{\Psi[\tilde{K}(\tau), L(\tau)]/\tilde{K}(\tau)\} d\tau\} \right] dt \right\}, \end{aligned} \quad (2.19)$$

and to reinterpret this as a pathwise relation among the processes \tilde{K} , \tilde{C} , Π , γ , (\tilde{C} being the ‘control’, \tilde{K} the ‘solution’)

The second approach, which generalises the formulation adopted in [F5] for the Brownian model, is to replace (2) by an *s d e.*, say

$$d\tilde{K}_t = \{\Psi[\tilde{K}_t, L_t] - \tilde{C}_t e^{\pi(t)}\} dt + \tilde{K}_{t-} d\mathcal{L}(e^{-\gamma})_t \quad (2.20a)$$

with $\tilde{K}(0) = K_0 > 0$ – or, more properly, the *stochastic integral equation*

$$\tilde{K}_T = K_0 + \int_0^T \{\Psi[\tilde{K}_t, L_t] - \tilde{C}_t e^{\pi(t)}\} dt + \int_0^T \tilde{K}_{t-} d\mathcal{L}(e^{-\gamma})_t. \quad (2.20b)$$

Here $\mathcal{L}(e^{-\gamma})$ is the ‘mart–log’ of $e^{-\gamma}$, i.e.

$$d\mathcal{L}(e^{-\gamma})_t = e^{-\gamma(t-)} de^{\gamma(t)}, \quad (2.21)$$

which reduces to $-\dot{\gamma}(t)$, as in (2), in case γ is pathwise C^1 .¹ Note that (21) is a

¹ *Note on martingale logarithms and exponentials.* Given a semimartingale ξ with $\xi(t)$

generalised Doléans integral equation, so that for a suitable progressive $\tilde{C} \geq 0$ the (unique, semimartingale) solution satisfies

$$\tilde{K}_T = e^{-\gamma(T)} \left[K_0 + \int_0^T e^{\gamma(t-)} \{ \Psi[\tilde{K}_t, L_t] - \tilde{C}_t e^{\pi(t)} \} dt \right], \quad (2.22)$$

taking into account that $\Delta \mathcal{L}(e^{-\gamma}) > -1$ always, cf. [Ja] T.6.8. To check explicitly that (22) satisfies (20a), write $\delta = e^{-\gamma}$, let ξ denote the term in long brackets in (22), differentiate $\xi \cdot \delta$ using the product rule for semimartingales², replace $\xi(t-)$ in the resulting expression by $\tilde{K}(t-)e^{\gamma(t-)}$ using (22), then substitute from (21) and simplify. Note also that $e^{\gamma(t-)}$ may be replaced by $e^{\gamma(t)}$ under the Lebesgue integral sign in (22) because the set $\{t: \gamma(\omega, t) \neq \gamma(\omega, t-)\}$ is at most countable, a.s.

It turns out that the two formulations (19) and (20) are essentially equivalent. Briefly, if \tilde{K} satisfies (19) on \mathcal{S} a.s., then clearly $\tilde{K}e^\gamma$ is absolutely continuous. Taking the derivative of $\tilde{K}e^\gamma$, substituting in the resulting expression from (19) for the term in curly brackets and simplifying yields the o.d.e.

$$d(\tilde{K}_t e^{\gamma(t)})/dt = \{ \Psi[\tilde{K}_t, L_t] - \tilde{C}_t e^{\pi(t)} \} e^{\gamma(t)} \quad (2.23)$$

with $\tilde{K}(0) = K_0 > 0$, or equivalently (22).³ Conversely, if \tilde{K} satisfies (22), then $\tilde{K}e^\gamma$ is absolutely continuous and differentiation yields (23). Thus, provided conditions are

and $\xi(t-)$ positive on \mathcal{S} , a.s., the 'martingale logarithm' (or mart-log) of ξ is the semimartingale $\mathcal{L}[\xi]$ defined by $\mathcal{L}[\xi](T) \doteq \int_0^T (1/\xi(t-)) d(\xi(t))$ with $\mathcal{L}(0) = 0$. Inversely, $\xi(T)/\xi(0) = \mathcal{E}[\mathcal{L}(\xi)](T)$ where \mathcal{E} is the 'martingale exponential', i.e. $\xi(T)/\xi(0)$ is the (unique, semimartingale) solution of the equation

$$\eta(T) = 1 + \int_0^T \eta(t-) d\mathcal{L}(\xi)(t).$$

² Note on differentiation of products (integration by parts). Given two semimartingales ξ and δ , the general product rule is (in abridged notation)

$$d(\xi \cdot \delta) = \xi \cdot d\delta + \delta \cdot d\xi + d[\delta, \xi]$$

see [DM] VIII.18. If ξ is absolutely continuous, the square bracket process vanishes.

³ Strictly speaking, the pathwise derivative of $\tilde{K}e^\gamma$ in (23) is defined only up to null Lebesgue sets. However, if for given \tilde{C} the (semimartingale) solution $\tilde{K}e^\gamma$ of (23) is uniquely defined on \mathcal{S} a.s., we may fix the derivative process by choosing that version which satisfies the equation identically. Then, since \tilde{C} will be chosen as a progressive process, the same will be true of $d(\tilde{K}e^\gamma)/dt$. Similarly for $d\tilde{k}(t)/dt$ in (24).

satisfied which ensure the existence and uniqueness of solutions of (23) on \mathcal{I} a.s., it follows that a given semimartingale \tilde{K} satisfies (19) iff it satisfies (22). Similarly for solutions defined only on some stochastic interval $\llbracket 0, \tau \rrbracket$. The stochastic problem of optimal growth (in natural units) can therefore be defined provisionally as the choice of a progressive process $\tilde{C} = \tilde{C}(\omega, t) \geq 0$ from a suitable class so as to maximise φ , subject to the condition that the (semimartingale) solution $\tilde{K} = \tilde{K}(\omega, T)$ of (23), and hence of (19) or (22), be a.s. uniquely defined and non-negative for all $t \geq 0$.

We again introduce variables \bar{k} , \bar{c} and \underline{k} , \underline{c} as in (9) – this time as processes – note that equation (10) remains valid, and consider the formulation and properties of the stochastic model in standardised and intensive forms, starting with the *equation of accumulation*

Standardised Form Substituting the expressions for \underline{k} and \underline{c} into (23), using (10) and simplifying yields,

$$d\underline{k}(t)/dt = \psi[\underline{k}(t)e^{x(t)}]e^{-x(t)} - \underline{c}(t) \quad \dots (2.24)$$

with $\underline{k}(0) = K_0$, or, in integral form,

$$\underline{k}(T) = K_0 + \int_0^T \{\psi[\underline{k}(t) \cdot e^{x(t)}]e^{-x(t)} - \underline{c}(t)\}dt, \quad \dots (2.25)$$

in agreement (pathwise) with the corresponding deterministic expressions (13–13a) If $\underline{k}(t) > 0$ for all t , we may divide (24) by $\underline{k}(t)$ and integrate to obtain

$$\ln \underline{k}(T) = \ln K_0 + \int_0^T \{a[\underline{k}(t)e^{x(t)}] - \underline{c}(t)/\underline{k}(t)\}dt. \quad \dots (2.26)$$

Intensive Form. Replacing \underline{k} by $\bar{k} \cdot e^{-x}$, \underline{c} by $\bar{c} \cdot e^{-x}$, also writing $z = \ln \bar{k} = \ln \underline{k} + x$, (25) and (26) become

$$\bar{k}(T)e^{-x(T)} = K_0 + \int_0^T \{\psi[\bar{k}(t)] - \bar{c}(t)\}e^{-x(t)}dt. \quad \dots (2.27)$$

$$z(T) = z_0 + \int_0^T \{A[z(t)] - \bar{c}(t) \cdot e^{-z(t)}\}dt + x(T), \quad \dots (2.28)$$

again in agreement with the corresponding deterministic forms (13) and (15).

Alternatively, we may multiply both sides of (27) by $e^{x(T)}$ and differentiate according to the product rule for semimartingales (see fn.2). Denoting the r.h.s. of (27) temporarily by ξ , we obtain the s.d.e.

$$d\bar{k}_t = e^{x(t-)}d\xi_t + \xi_{t-}de^{x(t)} = e^{-\Delta x(t)}[\psi(\bar{k}_t) - \bar{c}_t]dt + \bar{k}_{t-}e^{-x(t-)}de^{x(t)}, \quad \dots (2.29)$$

using (27) again to evaluate $d\xi_t$ and ξ_{t-} . Also, (29) yields

$$\bar{k}_t - \bar{k}_{t-} = \Delta\bar{k}_t = \bar{k}_{t-}e^{-x(t-)}\Delta e^{x(t)} = \bar{k}_{t-}(e^{\Delta x(t)} - 1), \quad \dots (2.30)$$

hence $\bar{k}_t = \bar{k}_{t-}e^{\Delta x}$. In case \bar{k}_t and \bar{k}_{t-} are always positive, we may divide (29) by \bar{k}_{t-} and, using (30) and the definition of the mart-log, obtain

$$\begin{aligned} (\mathcal{L}\bar{k})_T &= \int_0^T \{[\psi(\bar{k}_t) - \bar{c}_t]/\bar{k}_t\}dt + \mathcal{L}(e^x)_T \\ &= \int_0^T \{a(\bar{k}_t) - \bar{c}_t/\bar{k}_t\}dt + \mathcal{L}(e^x)_T \end{aligned} \quad \dots (2.31)$$

This equation looks neat, but bear in mind that, in general,

$$\mathcal{L}[e^x]_T = x(T) + \frac{1}{2}\langle\langle x, x \rangle\rangle_T^c + \sum_{t \leq T} [e^{\Delta x(t)} - 1 - \Delta x(t)], \quad \dots (2.32)$$

and x is here the sum of three processes, see (4). Actually the form (28) based on natural logarithms turns out to be more useful, particularly in case \bar{c}/\bar{k} is specified as a function of \bar{k} only, see (2.41) below.

So far, we have not specified precisely the class of *admissible consumption processes* \tilde{C} (or \bar{c} or \underline{c}). Actually, it makes little difference for the characterisation (as distinct from the existence) of an optimum whether we require consumption to be progressively measurable or optional or even corlol and adapted (or, if the processes η are continuous, predictable or even continuous and adapted) To be specific, we define an (admissible) consumption plan in standardised form as a non-negative, progressive process $\underline{c} = \underline{c}(\omega, t)$ such that the solution \underline{k} of (25) is a.s. uniquely defined and non-negative for $0 \leq t < \infty$, and denote by $\underline{\mathcal{C}}$ the set of all such processes. (This definition implies that $\int_0^T \underline{c}(\omega, t)dt < \infty$ for all $T < \infty$, a.s.). The set $\bar{\mathcal{C}}$ of consumption plans in intensive form is then the image of $\underline{\mathcal{C}}$ under $\underline{c} \mapsto \bar{c} = \underline{c}e^x$, or equivalently the set of non-negative, progressive processes \bar{c} such that a (semimartingale) solution \bar{k} of (27) is uniquely defined and a.s. non-negative for $0 \leq t < \infty$. Similarly for \tilde{C} .

If now we rewrite the functional φ in standardised and intensive units we have, for $b \neq 1$,

$$\begin{aligned}\varphi &= (1-b)^{-1} E \int_0^{\infty} \underline{c}(t)^{1-b} e^{w(t)} dt \\ &= (1-b)^{-1} E \int_0^{\infty} \bar{c}(t)^{1-b} e^{v(t)} dt,\end{aligned}\tag{2.33}$$

and when necessary we write $\varphi(\underline{c})$ or $\bar{\varphi}(\bar{c})$ to distinguish the two forms. In case $b = 1$, we have $v = w = \pi - \rho$, and the functional becomes

$$\begin{aligned}\varphi &= E \int_0^{\infty} [\ln \underline{c}(t) - \gamma(t) - \pi(t)] e^{\pi(t) - \rho(t)} dt \\ &= E \int_0^{\infty} [\ln \bar{c}(t) + \beta(t)] e^{\pi(t) - \rho(t)} dt.\end{aligned}\tag{2.34a}$$

The expected integrals in (33) and (34a) are defined as Lebesgue integrals (To avoid tedious reservations in the case $b = 1$ we set $\varphi = -\infty$ if the integrals have the form $+\infty + (-\infty)$.) In (34a) it is natural to drop in each expression the terms not involving consumption, and to maximise simply

$$\begin{aligned}\varphi(\underline{c}) &= E \int_0^{\infty} [\ln \underline{c}(t)] e^{\pi(t) - \rho(t)} dt \quad \text{on } \underline{\mathcal{C}}, \text{ or} \\ \bar{\varphi}(\bar{c}) &= E \int_0^{\infty} [\ln \bar{c}(t)] e^{\pi(t) - \rho(t)} dt \quad \text{on } \bar{\mathcal{C}}.\end{aligned}\tag{2.34b}$$

The resulting problems are equivalent to one another, and to the problem in natural units, if the terms which are dropped are integrable, which we always assume.

The *stochastic problem of optimal growth* (or decline) can now be briefly defined. We consider φ (or $\bar{\varphi}$) as an integral functional on $\underline{\mathcal{C}}$ (or $\bar{\mathcal{C}}$) and assume directly, or infer from stated assumptions, that it possesses a finite supremum φ^* (i.e. that $\varphi(\underline{c}) \leq \varphi^* < \infty$ for all $\underline{c} \in \underline{\mathcal{C}}$ and $\varphi(\underline{c}) > -\infty$ for some $\underline{c} \in \underline{\mathcal{C}}$). Then the problem expressed in standardised (or intensive) units is to find a \underline{c} (or \bar{c}) which attains this supremum, if such an element exists. For brevity we omit conditions ensuring the convergence of the integral defining φ and the existence of a maximum; see the analogous discussion of these questions in the case of the Standard Model. It is usually convenient to work with this formulation, leaving aside the original problem in natural units. Clearly, the transformed models can also be interpreted as models of optimal saving with diminishing returns to capital driven by two (dependent) semimartingales, namely v and x in the intensive form, w and x in the standardised form.

We now review briefly some properties of the SNC model in standardised and in

intensive units. The model in *standardised units* can obviously be regarded as a special case of the Standard Model.⁴ On setting $c = \underline{c}$, $k = \underline{k}$, a comparison between (1.2) and (33) gives, for $b \neq 1$,

$$\begin{aligned} u[c(t);t]q(t) &= (1-b)^{-1} \underline{c}(t)^{1-b} e^{w(t)} = (1-b)^{-1} \bar{c}(t)^{1-b} e^{v(t)}, \\ u'[c(t);t]q(t) &= \underline{c}(t)^{-b} e^{w(t)} = \bar{c}(t)^{-b} e^{v(t)+x(t)} \end{aligned} \quad (2.35)$$

The split between u and q is in general not unique, and can be chosen for convenience, e.g. in applying the existence criteria in Chapter 9. Thus, if e^W is m -integrable, one can take (up to a scaling constant)

$$(1-b)u[c(t);t] = \underline{c}(t)^{1-b}, \quad c = \underline{c}, \quad q(t) = e^{w(t)}, \quad (2.36)$$

whereas if e^V is m -integrable one can take

$$(1-b)u[c(t);t] = \bar{c}(t)^{1-b}, \quad c = \bar{c}, \quad q(t) = e^{v(t)}. \quad (2.37)$$

In the case $b = 1$, the remarks in the preceding paragraph apply to the two forms of the functional in (34a-b), with $(1-b)^{-1} \underline{c}^{1-b}$ and $(1-b)^{-1} \bar{c}^{1-b}$ replaced by $\ln \underline{c}$ and $\ln \bar{c}$, noting that $v = w = \pi - \rho$ since $b = 1$.

Further, a comparison of (24) with (1.1) gives

$$f[k(t);t] = \psi[\underline{k}(t)e^{x(t)}]e^{-x(t)} = \psi[\bar{k}(t)]e^{-x(t)}, \quad (2.38)$$

so that the average and marginal products of capital are invariant to the choice of units,⁵ i.e.

⁴ If the processes η are sample differentiable, the SNC model in intensive units can, of course, be treated directly as a special case of the Standard Model with $c = \bar{c}$, $k = \bar{k}$, (2.10) corresponding to (1.1) etc

⁵ The variety of units in which economic variables are expressed makes it difficult to devise a coherent terminology. It seems most convenient always to use the same words for concepts which correspond to one another under the various transformations but to qualify these words by specifying the units of measurement. Thus the symbols k , c , f , u , q , $uq = v$ appearing in the Standard Model are called capital, consumption, production, felicity (or instant utility), impatience density and utility in *standard* units, (or, more precisely, 'when goods are denominated in standard units', since q is not measured in units of goods; we adopt similar 'abus de langage' in other cases). The symbols \tilde{K} , \tilde{C} , $\tilde{\Psi}$, \tilde{U} , $e^{-(\pi+\rho)}$, $\tilde{U} \cdot e^{-(\pi+\rho)}$ appearing in the SNC Model may be given the same names but are in *natural* units, (although there is some ambiguity here because of the appearance of $\Pi = e^\pi$, which is the reason why ρ has been given a

$$\begin{aligned}
f[k(t);t]/k(t) &= \psi[\bar{k}(t)]/\bar{k}(t) = \Psi[\tilde{K}(t),L(t)]/\tilde{K}(t), \\
f'[k(t);t] &= \psi'[\bar{k}(t)] = \partial\Psi[\tilde{K}(t),L(t)]/\partial\tilde{K}(t).
\end{aligned}
\tag{2.39}$$

Necessary and sufficient conditions for optimality in 'local martingale and transversality' form for the SNC model can be obtained directly from the results stated in Chapters 1 and 3(B). Briefly, conditions for $(\underline{c}^*, \underline{k}^*)$ or (\bar{c}^*, \bar{k}^*) to be optimal may be obtained – writing $c = \underline{c}$, $k = \underline{k}$, using (2.4–5) and (2.9), then (1.3–10), dropping the stars and abridging the notation – by setting

$$\begin{aligned}
R(T) &= \int_0^T \psi'[\bar{k}(t)]dt, \\
y(T) &= \underline{c}(T)^{-b} \cdot e^{w(T)+R(T)} = \bar{c}(T)^{-b} \cdot e^{x(T)+v(T)+R(T)}, \\
J(T) &= \underline{k}(T) \cdot e^{-R(T)} = \bar{k}(T) \cdot e^{-x(T)-R(T)}, \\
y(T)J(T) &= \underline{k}(T)\underline{c}(T)^{-b} \cdot e^{w(T)} = \bar{k}(T)\bar{c}(T)^{-b} \cdot e^{v(T)}.
\end{aligned}
\tag{2.40}$$

For the Brownian case⁶, more detailed statements of these formulae are given in [F5]; see also Chapter 3, eq. (3.47) et seq. Some existence results for this case are obtained in Chapter 9.

Of course, the main reason for being interested in a 'standardised' version of the stochastic growth model, with the equation of accumulation written as an o.d.e., is

separate name in the text) Again, the symbols \bar{k} , \bar{c} , $\psi(\bar{k})$, $(1-b)^{-1}\bar{c}^{1-b}$, e^v , $(1-b)^{-1}\bar{c}^{1-b} \cdot e^v$ and \underline{k} , \underline{c} , $\psi(\underline{k}e^x)e^{-x}$, $(1-b)^{-1}\underline{c}^{1-b}$, e^w , $(1-b)^{-1}\underline{c}^{1-b} \cdot e^w$ refer to *intensive* and *standardised* units respectively, again with the same names (although these are not entirely satisfactory). The welfare functional should perhaps be written φ , $\tilde{\varphi}$, $\bar{\varphi}$, $\underline{\varphi}$ in the four cases, but no confusion arises if φ is used throughout and called simply 'welfare'. Reverting to the SM, the variables J , j , g , G defined in (1.5–6) and (3.8–9) are in *discounted* units. The term 'reduced', used in [F1] instead of 'discounted', is now reserved for its meaning in connection with local martingales.

⁶ In the BNC model, the processes $\eta = \beta, \gamma, \pi, \rho$ – cf (2.3) – are assumed to have the form

$$\eta(\omega, t) = \alpha_\eta t + \sigma_\eta B_\eta(\omega, t), \tag{*}$$

where α_η and $\sigma_\eta > 0$ are constants and the B_η are a set of independent standard Brownian motions. The linear combinations defined as in (2.4) are then Brownian motions with drift which may again be written in the form (*), with η replaced by x , v or w . See [F5] eqs. (1.7) e s for details (with slightly different notation)

that it makes available the methods and results presented in our other Chapters.⁷ These in turn have certain advantages as regards precision of results and connection with economic intuition, although it is difficult to make exact claims without going into details of alternative procedures. As was mentioned in Chapter 1, the local martingale property of the shadow price process can be obtained by various methods (dynamic programming, maximum principle, duality), but so far as I am aware the precise characterisation of reducing sequences and corresponding transversality conditions, in particular as regards necessity, have not been derived by these methods. The o.d.e. formulation also offers advantages in proving the existence of solutions of equations and the existence of an optimal plan, due to the availability of simpler and more general theorems. Regarding the economic basis of conditions for optimality used in the Standard Model, see the related discussion in [F4].

As to the formulation in intensive units, it has been mentioned above that in the case of certainty this yields some nice properties. How far can analogous results be obtained in the stochastic case, with the 'intensive' equation of accumulation written as an s.d.e.? I offer only a few informal remarks, without proof.

(i) If the η are processes with stationary independent increments (p.s.i.i.) and \mathcal{Q} is the 'internal history' of these processes⁸, intuition suggests (and it should not be hard to prove) that an optimal plan, if one exists, may be generated by specifying consumption as a (time and state invariant) function of capital only, as in (17). In this case the equation of accumulation (28) appears in the form

⁷ It is worth noting that, in the simple model considered here, the possibility of transforming from s.d.e. to corresponding random o.d.e. formulations extends to equations driven by general (not necessarily continuous) semimartingales, unlike the situations usually considered in the literature on such transformations, see [Do] and [Su], also [Li] for some recent references. Moreover we may identify solutions of s.d.e.s *pathwise* (up to null sets) with solutions of corresponding o.d.e.s, which simplifies discussion of the existence of solutions.

⁸ More precisely, the smallest filtration making the processes optional (and possessing the properties specified in Chapter 3).

$$z(\omega, T) = z_0 + \int_0^T \{A[z(\omega, t)] - \theta[z(\omega, t)]\} dt + x(\omega, T), \quad (2.41)$$

which agrees pathwise with (18).

(ii) In case the processes η are independent Brownian motions with drift, it is shown in [F5-6] that, if an optimal plan is generated by a consumption function $H(z)$ as in (2.17) and certain restrictions on parameter values are satisfied, then the local martingale condition for optimality may be replaced by the requirement that H satisfy a certain second order o.d.e. of the form $H'' = F(H, H', z)$, while the transversality condition may be replaced by certain boundary conditions for $H'(z)$ and $\theta(z) = \exp\{H(z) - z\}$ as $|z| \rightarrow \pm\infty$. In case the η are general p.s.i.i., preliminary work suggests that the o.d.e. is to be replaced by a suitable integro-differential equation (and of course there are different restrictions on parameters).

(iii) If the η are Brownian motions with drift and \mathcal{A} is the internal history, then standard methods for controlled one-dimensional Markov processes show that, for suitable parameter values, the distributions of the random variables $\bar{k}(\omega, t)$ and $\bar{c}(\omega, t)/\bar{k}(\omega, t)$ in an optimal plan generated by a control of the form (17) converge weakly to steady state (i.e. invariant) distributions, see [Ma]. Results of this type have been obtained for a related model by Merton [Mer], and I have derived detailed results for the BNC model in as yet unpublished work. The question whether there is convergence to a steady state distribution for some discontinuous p.s.i.i. appears to be open; discussion of relevant techniques may be found in [GM] Ch.7, [EK] Ch.4-9.

The properties discussed here depend heavily on our assumptions about the functions ψ and U and on the η being p.s.i.i. If these assumptions are varied, there is of course a large class of (Markov) models for which an optimal control can be obtained in closed-loop form, say $lnc = H(ln k, t)$, but the advantages of an intensive rather than a standardised formulation tend to be lost. On the other hand, the conditions for optimality obtained via the Standard Model are very robust.

(A) *The Standard Model*

This Chapter gives a more detailed specification of the SM, followed by definitions needed for statements of conditions for optimality, statement and proof of the main Sufficiency Theorem, and a review of the forms of the local martingale and transversality conditions which correspond to alternative choices of stopping times.

The definitions and assumptions in this Chapter are maintained throughout the paper unless otherwise stated. Let $\mathcal{T} = [0, \infty)$ be the time domain with Borel sets \mathcal{B} and Lebesgue measure λ and let (Ω, \mathcal{A}, P) be a complete probability space with a filtration $\mathcal{A} = (\mathcal{A}_t; t \in \mathcal{T})$ satisfying the 'usual conditions' of right continuity and completeness, where $\mathcal{A} = \mathcal{A}_{\infty}$ while $\mathcal{A}_0 = \mathcal{A}_{0-}$ is generated by the P-null sets. We define the products $\mathcal{S} = \Omega \times \mathcal{T}$, $\Sigma = \mathcal{A} \times \mathcal{B}$, $m = P \times \lambda$ (not necessarily completed) and write $s = (\omega, t)$, $dm(s) = dP(\omega)dt$, $dt = d\lambda(t)$. Statements which hold apart from null sets of \mathcal{T} , Ω , \mathcal{S} are qualified by 'a.a.t.', 'a.s.', 'a.e.' respectively, the measures considered being λ , P , m or equivalent measures. In the product space \mathcal{S} we consider the σ -algebras \mathcal{H} , \mathcal{O} , \mathcal{P} of progressive, optional and predictable sets, as well as the corresponding classes of processes.¹ All processes considered are assumed, or may easily be shown to be, at least progressively measurable; (however, for the sufficiency proof in this Chapter it is enough if all processes are adapted to \mathcal{A}).²

¹ \mathcal{H} comprises (ω, t) sets H such that, for each $T \in \mathcal{T}$, the subset $H \cap \{\Omega \times [0, T]\}$ belongs to $\mathcal{A}_T \times \mathcal{B}_T$. \mathcal{O} is generated by intervals of the form $]\sigma, \tau[$ where σ, τ are stopping times, or equivalently by the right continuous, adapted processes. \mathcal{P} is generated by intervals $]\sigma, \tau]$ and sets $A \times \{0\}$ with $A \in \mathcal{A}_0$, or equivalently by the left continuous, adapted processes. We have $\mathcal{A} \times \mathcal{B} \supseteq \mathcal{H} \supseteq \mathcal{O} \supseteq \mathcal{P}$.

² The following conventions apply unless we state or imply otherwise. The terms *positive*, *negative*, *increasing*, *decreasing* have their strict meaning throughout. The symbols \uparrow , \downarrow and $\uparrow\uparrow$, $\downarrow\downarrow$ denote weak and strict monotonicity respectively. Measures are by definition non-negative. Random variables are finite, or occasionally extended, real-valued, \mathcal{A} -measurable functions. For a random variable ζ , $\zeta > 0$ means $\zeta(\omega) > 0$ a.s. and $\zeta \geq 0$ means $\zeta(\omega) \geq 0$ a.s., while for a (progressive) process $\xi > 0$

On the measurable space $(\mathcal{S}, \mathcal{H})$ we consider, in addition to m , a finite measure μ with the same null sets, defined by

$$d\mu(s) = d\mu(\omega, t) = q(\omega, t)dP(\omega)dt = q(s)dm(s) \quad \dots(3.1)$$

where q is a positive, finite, \mathcal{H} -measurable and m -integrable function, i.e.

$E \int_0^\infty q(\omega, t)dt < \infty$; μ is called the *impatience measure*, q the *impatience density*. It will be convenient (except occasionally in examples) to normalise q so that μ is a probability measure on $(\mathcal{S}, \mathcal{H})$. We denote by $\mathcal{L}_1 = \mathcal{L}_1(\mathcal{H}, \mu)$ the space of (classes of similar) \mathcal{H} -measurable, μ -integrable processes $\xi = \xi(s) = \xi(\omega, t)$ on \mathcal{S} with the norm $\int |\xi| d\mu$. When the domain is not specified, μ -integrals are taken over \mathcal{S} .

The *production function* $f = f(k; \omega, t) = f(k; s)$ is defined on $[0, \infty] \times \mathcal{S}$ and takes values in $[-\infty, \infty]$.³ Considered as a function of all its variables, f is supposed to be $\mathcal{B}_{[0, \omega]} \times \mathcal{H}$ measurable. For fixed s , it is assumed that f is continuous and (weakly) concave in k and satisfies $f(0; s) = 0$;⁴ in general, no continuity is assumed

means $\xi(s) > 0$ a.e. and $\xi \geq 0$ means $\xi(s) \geq 0$ a.e.; however, if a process is specified (possibly after changes on a null set) to be continuous on the right with left limits (*corlol*), then $\xi > 0$ is taken to mean $\xi(t) > 0$ and $\xi(t-) > 0$ on \mathcal{T} a.s. For any process ξ , we set $\xi(0-) = \xi(0)$. *Finite variation* means finite variation on compacts. Non-decreasing and finite variation processes satisfy $\xi(0) = 0$ unless otherwise stated. For $t \in \mathcal{T}$, E_t means $E(\cdot / \mathcal{A}_t)$; similarly E^ν , $E^{\nu-}$ for a stopping time ν . Martingales (true or local), non-decreasing and finite variation processes are finite on \mathcal{T} and are taken (subject to context) to be corlol; similarly for supermartingales ξ with $t \rightarrow E\xi(t)$ right continuous. Finally, if a given set Ξ of processes is specified, we identify elements ξ_1 and ξ_2 which are *indistinguishable*, i.e. which satisfy $\xi_1(\omega, t) = \xi_2(\omega, t)$ for all $t \in \mathcal{T}$ a.s., and write $\xi_1 \equiv \xi_2$. We call two processes *similar* if they differ only on an m -null set of \mathcal{H} , and sometimes identify with one another processes which satisfy a given set of conditions and are similar; see also Chapter 4, fn.1 and Chapter 6. Note that similarity as defined here is weaker than the relation 'is a modification of' often considered in martingale theory, see [DM] IV.6. Explicitly, progressive processes ξ_1 and ξ_2 are modifications of one another iff $\xi_1(\omega, t) = \xi_2(\omega, t)$ a.s. for each $t \in \mathcal{T}$, whereas they are similar iff $\xi_1(\omega, t) = \xi_2(\omega, t)$ a.s. for a.a.t, or equivalently for a.a.t, a.s.

³ Note the use of small capitals. These are often (but not exclusively) used to distinguish capital, consumption etc considered as real variables k, c etc from corresponding functions of time or processes k, c etc

⁴ The assumption $f(0; \omega, t) = 0$ may be replaced by $f(0; \omega, t) \geq 0$ to allow for an exogenous random income. It is then necessary to replace (3) by $\partial(k) = [f(k) - f(0)]/k$ and to make consequential changes elsewhere. Also, if the property that an optimum has positive capital is to stand it should be assumed that, for every $T \in \mathcal{T}$, there exists a positive random variable $h_T = h_T(\omega)$ such that

with respect to t . The *marginal product function* $f'(k; s)$, $f' = \partial f / \partial k$, is also defined on $[0, \infty] \times \mathcal{S}$ with values in $[-\infty, \infty]$ and is continuous and non-increasing in k for each s , with $f'(0)$ and $f'(\infty)$ defined as one-sided limits; (however, continuity of f' is not needed for the Sufficiency Theorem or in Chapters 4 and 9). Clearly f' inherits the conditions of measurability assumed for f . Except in Chapter 10 it is assumed that, for each $k \in [0, \infty]$,

$$\int_0^T |f'(k; \omega, t)| dt < \infty \quad \text{for each } (\omega, T) \in \mathcal{S}, \quad \dots (3.2)$$

and since $f' \downarrow$ in k it is enough if (8) holds for $k = 0$ and $k = \infty$; these conditions operate as generalised Lipschitz conditions for the o.d.e. (1.1) – see Chapter 4(A) – and models in which they all hold will accordingly be called *Lip* models. Now let

$$\mathfrak{D}(k; \omega, t) = f(k; \omega, t)/k, \quad 0 < k < \infty, \quad (\omega, t) \in \mathcal{S}, \quad \dots (3.3)$$

denote the *average product function*; since f is concave, $\mathfrak{D} \downarrow$ in k and we may define $\mathfrak{D}(0)$ and $\mathfrak{D}(\infty)$ as one-sided limits. Also, for $0 \leq k \leq \infty$, write

$$\mathfrak{D}(k; \omega, t) = \int_0^T \mathfrak{D}(k; \omega, t) | dt, \quad (\omega, T) \in \mathcal{S}. \quad \dots (3.3a)$$

We have, for each (ω, t) ,

$$f'(0) = \mathfrak{D}(0) \geq \mathfrak{D}(k) \geq f'(k) \geq f'(\infty) = \mathfrak{D}(\infty); \quad \dots (3.4)$$

thus, under (2), the integrals (3a) are defined and finite and $f'(0)$ is finite with $f' / \mathfrak{D} \uparrow 1$ as $k \downarrow 0$

Next, the *felicity* (or *instant utility*) function $u = u(c; \omega, t) = u(c; s)$ is defined on $[0, \infty] \times \mathcal{S}$ and takes values in $[-\infty, \infty]$. Considered as a function of all its variables, it is $\mathcal{B}_{[0, \omega]} \times \mathcal{H}$ measurable. For fixed s , u is continuous, concave and

$$P^T[f(0; t) = 0 \text{ for } T \leq t < T+h_T] > 0, \text{ a.s.}$$

(Cf. [F4] Section 2B and App.B. on the case of linear production with an exogenous income) The theory of Chapters 3–9 then extends with more or less routine changes; in particular T replaces ∞ as the upper limit of integration in (4.9) and the inequality is required to hold for all $T \in \mathcal{T}$, a.s. On the other hand, allowing $f(0; t) \leq 0$, as in the case of reparations or debt repayments, creates more serious problems. These variants will not be pursued here.

increasing in c . As usual, the function u may be chosen arbitrarily from a collection of functions differing only as to scale and origin. The *marginal felicity function* $u'(c, s)$, $u' = \partial u / \partial c$, is defined on $[0, \infty] \times \mathcal{S}$ with values in $[0, \infty]$, and for each s is continuous and non-increasing in c with $u'(0) = \infty$; (however, diminishing marginal felicity is not needed for the Sufficiency Theorem, nor is marginal felicity as such needed in Chapters 4 and 9).

We now state the fundamental definitions concerning plans. Let the initial capital $K_0 > 0$ be given. A *consumption plan*, or *c-plan*, is a process $c = c(\omega, t)$ which is progressive, a.e. non-negative, a.s. locally integrable (i.e. Lebesgue integrable on finite intervals of \mathcal{T}), and such that the o.d.e. (1.1) with initial conditions $k(\omega, 0) = K_0$ has a progressive, a.s. unique and non-negative solution $k = k(\omega, t)$ on $\mathcal{S} = \Omega \times \mathcal{T}$. More explicitly, it is required that a.s. the o.d.e. (1.1) possesses a unique local solution $k(\omega, \cdot)$ through $(K_0, 0)$ which can be uniquely continued to the whole of \mathcal{T} , the solution remaining finite and non-negative, while $k(\cdot, \cdot)$ is progressive. The process k is called the *capital plan*, or *k-plan*, corresponding to c , and the pair (c, k) is simply a *plan*. The process $F = F_k$ defined by

$$F_k(s) = F_k(\omega, t) = f[k(\omega, t); \omega, t] \quad s = (\omega, t) \in \mathcal{S} \quad \dots(3.5)$$

is the *production plan*. In the same way, c defines a *felicity plan* $U = U_c$ by

$$U_c(s) = U_c(\omega, t) = u[c(\omega, t); \omega, t] \quad s = (\omega, t) \in \mathcal{S} \quad \dots(3.6)$$

Also associated with (c, k) are the processes

$$f'[k(\omega, t); \omega, t], \quad \partial[k(\omega, t); \omega, t], \quad u'[c(\omega, t); \omega, t], \quad \dots(3.6a)$$

called the *marginal product*, *average product* and *marginal felicity plans*.⁵ These definitions need some justification as regards existence, uniqueness and measurability; this is postponed to Chapter 4(A).

⁵ We often abridge the notation in various ways, e.g. omitting ω or t or both, writing $f(k_t)$ or simply $f(k)$ rather than $f[k_t; t]$, similarly $f'(k_t)$, $\partial(k_t)$, $\partial(k)$ or even ∂_t when k is given, also $u(c_t)$ or $u(c)$ rather than $u[c_t; t]$, similarly $u'(c_t)$ or $u'(c)$.

We denote by \mathcal{C} and \mathcal{K} the set of all c and k plans respectively and by $\Pi = \mathcal{C} \times \mathcal{K}$ the set of plans (c, k) . Any one of these sets may be regarded as the domain of the *welfare functional* φ or Φ defined by (1.2), and it is always assumed, or inferred from additional conditions, that for each plan the positive part of the integral (1.2) is finite, so that the functional is well defined. It is further assumed or inferred that the supremum φ^* of the functional is finite. Corresponding elements c^* and k^* are called *optimal* if $\varphi(c^*) = \Phi(k^*) = \varphi^*$, and the *problem of optimal accumulation* is to choose an optimal pair (c^*, k^*) if one exists.

The main differences between this model and the model of saving considered in [F1] are the introduction of the concave production function f and (less important) the definition of q as a process and the slightly stronger assumption of product measurability for u . Also, we shall consider here only the case of an infinite horizon, the changes needed to allow for a random horizon being much the same as in the saving model.⁶

⁶ The equation of accumulation and the welfare functional considered in [F1] (1.2–6) and (1.8) may be obtained from the present specification of the Standard Model in (at least) two ways. One way is to set $\bar{f} \equiv 0$ and to replace c, k, K_0 in the present notation by $g, K-G, K$ in the notation of [F1]. The compound interest process R defined in (3.7) will then also vanish, and the symbols g, G and $k = J$ in the present notation – see (3.8–9) – will have the same meaning as g, G and $k \cdot \exp(-x)$ in [F1].

The alternative is to replace $f(k; \omega, t)$ formally in (1.1) by a linear expression $k \cdot r(\omega, t) = k \cdot dR(\omega, t)/dt$ where R is a *given* process, so that the equation becomes

$$\dot{k}(t) = k(t)r(t) - c(t) \quad \dots(i)$$

which may be rewritten as

$$K_0 - k(T)e^{-R(T)} = \int_0^T c(t)e^{-R(t)} dt \quad \dots(ii)$$

If R is not sample differentiable, then only (ii) is regarded as properly defined and (i) is just shorthand. With this formulation, the symbols c, k, g, G have the same meaning here as in [F1], while R, K_0, J and j here mean the same as $x, K, k \cdot \exp(-x)$ and $d(k \cdot \exp(-x))/dt$ in [F1]. Formulae such as (3.11–12) remain valid if one sets $f(k) - kf'(k) = 0$, yielding $K_0 - J(T) = G(T)$ as in [F1]. When reference is made here to ‘linear production’ or ‘the saving model’ we shall generally have the second procedure in mind; this avoids pedantic distinctions between a linear version of the present model and the model of saving considered in [F1].

(B) *Conditions for Optimality and Sufficiency Theorem.*

We now introduce some concepts and formulae which are needed throughout the discussion of the conditions for optimality, then state the conditions and prove their sufficiency. Let (c^*, k^*) be a distinguished plan, and define the corresponding *star marginal product plan* r and *star marginal log-return plan* R by

$$r(\omega, t) = f'[k^*(\omega, t); \omega, t], \quad R(\omega, T) = \int_0^T r(\omega, t) dt; \quad (3.7)$$

note that R is a s. finite on \mathcal{S} by virtue of (2) (Cf. (1.3–4); we do not now assume a priori that $c^* > 0$ and $k^* > 0$, but of course only such a plan can be optimal.) We use R to express any given plan (c, k) in *discounted units*; explicitly, we write

$$J(\omega, T) = k(\omega, T)e^{-R(\omega, T)}, \quad j(\omega, t) = dJ/dt = [k(t) - r(t)k(t)]e^{-R(t)}, \quad (3.8)$$

$$G(\omega, T) = \int_0^T c(\omega, t)e^{-R(\omega, t)} dt, \quad g(\omega, t) = dG/dt = c(t)e^{-R(t)}, \quad (3.9)$$

and refer to J, j, g, G as the *discounted capital, investment, consumption and cumulative consumption plans*. In particular, we define the discounted plans

J^*, j^*, g^*, G^* corresponding to k^* and c^* , as well as the *shadow price process*

$$y(\omega, T) = u'[c^*(\omega, T); \omega, T]q(\omega, T)e^{R(\omega, T)}; \quad (3.10)$$

all these processes are progressive – see Chapter 4(A). Obviously y is positive and finite if $c^* > 0$.

Using (8–9) and abridging the notation, the equation $\dot{k} = f(k) - c$ becomes

$$\dot{j}_t = -g_t + [f(k_t) - k_t f'(k_t^*)]e^{-R(t)} = -g_t + \{\partial[J_t e^{R(t)}] - r_t\}J_t; \quad (3.11)$$

the second equality follows trivially from $\partial(k) = f(k)/k$ when $k_t > 0$ and is still true if $k_t = 0$ and $\partial(0) = f'(0)$. On integrating, one has

$$K_0 - J_T = G_T - \int_0^T [f(k_t) - k_t r_t]e^{-R(t)} dt = G_T - \int_0^T \{\partial[J_t e^{R(t)}] - r_t\}J_t dt. \quad (3.12)$$

Thus a progressive process g defines a plan iff it is a e. non-negative, a s. locally integrable, and such that the o.d.e. (11) with initial condition $J(0) = K_0$ – or equivalently the integral equation (12) – has a progressive, a s. unique and non-negative solution J .

Given (c^*, k^*) and another plan (c, k) , we often write (omitting ω and t)

$$\delta c = c - c^*; \quad \delta u = u(c) - u(c^*); \quad \delta u \leq \delta c \cdot u'(c^*) = \delta c \cdot u'^*, \quad \dots (3.13)$$

$$\delta k = k - k^*; \quad \delta f = f(k) - f(k^*); \quad \delta c + \delta k = \delta f \leq \delta k \cdot f'(k^*) = \delta k \cdot f', \quad \dots (3.14)$$

the inequalities being due to the concavity of u and f . Similarly, we write $\delta g, \delta j$ etc and obtain

$$\delta j = -\delta g + [\delta f - r \cdot \delta k] e^{-R} \leq -\delta g \quad \dots (3.15)$$

$$\delta J_T = -\delta G_T + \int_0^T [\delta f - r \cdot \delta k] e^{-Rt} dt \leq -\delta G_T \quad \dots (3.16)$$

$$\delta u \cdot q \leq \delta c \cdot u'^* q = \delta g \cdot y \leq -\delta j \cdot y \quad \dots (3.17)$$

$$-c^* u'^* q = -g^* y \leq \delta g \cdot y \leq -\delta j \cdot y \quad \dots (3.18)$$

where (15–16) are derived from (8–9) with (14), and (17–18) are derived from (9–10) with (13) and (15).

We turn now to some fundamental properties and definitions concerning the ‘star’ plan. Let $j = j^*, g = g^*$ etc. in (11–12) and note that $f(k) \geq kf'(k)$ for all k because f is concave and $f(0) = 0$; thus

$$j^*(\omega, t) \geq -g^*(\omega, t), \quad K_0 - J^*(\omega, T) \leq G^*(\omega, T). \quad \dots (3.19)$$

Obviously the sample functions of G^* are non-decreasing, but those of $K_0 - J^*$ are in general not monotonic or even (as far as I can see) of bounded variation on \mathcal{I} , nor need they converge as $T \rightarrow \infty$. However, one can define a non-decreasing *depletion process* Γ , which ‘fills in the troughs’ of $K_0 - J^*$, by

$$\Gamma(\omega, T) = \sup\{K_0 - J^*(\omega, t); t \leq T\}, \quad (\omega, T) \in \mathcal{E}; \quad \dots (3.20)$$

clearly Γ is adapted and absolutely continuous, hence progressive, and $0 \leq \Gamma \leq K_0$ since $J^* \geq 0$ and $J^*(0) = K_0$. Since G^* and Γ are non-decreasing, the random variables $G^*(\infty)$ and $\Gamma(\infty)$ exist as a.s. limits. Typical sample functions of $J^*, G^*, K_0 - J^*$ and Γ are depicted in Figure 1.

Associated with (c^*, k^*) are three families of stopping times relative to \mathfrak{A} . The *consumption times* τ_i are defined for $i \geq 0$ by

$$\tau_i = \tau(\omega, i) = \inf\{t \in \mathcal{T} : G^*(\omega, t) > i\} \quad \dots (3.21)$$

with $\tau(\omega, i) = \infty$ in case $G^*(\omega, \infty) \leq i$. Clearly $\tau_i > 0$ a.s. for $i > 0$, and

$$G^*(\omega, \tau(\omega, i)) = i \quad \text{if } 0 \leq i < G^*(\omega, \infty). \quad \dots (3.22)$$

The *depletion times* ν_i are defined for $0 \leq i < K_0$ by

$$\nu_i = \nu(\omega, i) = \inf\{t \in \mathcal{T} : K_0 - J^*(\omega, t) > i\} = \inf\{t \in \mathcal{T} : \Gamma(\omega, t) > i\}, \quad \dots (3.23)$$

with $\nu(\omega, i) = \infty$ in case $J^*(\omega, t) \geq K_0 - i$ for all t , i.e. in case $\Gamma(\omega, \infty) \leq i$. Using the definitions (in particular the continuity) of J^* and Γ we have, a.s.,

$$J^*(\nu_i) = K_0 - \Gamma(\nu_i) = K_0 - i \quad \text{if } \nu_i < \infty, \quad \dots (3.24)$$

$$\nu_i = \infty \quad \text{for some } i < K_0 \quad \text{iff } \inf_{t \in \mathcal{T}} J^*(t) > 0, \quad \text{i.e. iff } \Gamma(\infty) < K_0, \quad \dots (3.25)$$

$$\nu_i \uparrow \infty \quad \text{when } i \uparrow K_0 \quad \text{iff } J^*(t) > 0 \quad \text{for all } t \in \mathcal{T}. \quad \dots (3.26)$$

Obviously (19) implies $\tau_i \leq \nu_i$ a.s., cf. Figure 1. Note that the times τ_i and ν_i are predictable because G^* and Γ are absolutely continuous.

The *price times* \wp_i^0 and \wp_i are defined by

$$\wp_i = i \wedge \wp_i^0, \quad \wp_i^0 = \wp_i^0(\omega, i) = \inf\{t \in \mathcal{T} : y(\omega, t) > i\}, \quad \dots (3.27)$$

with $\wp_i^0 = \infty$ if $y(t) \leq i$ for all t . The economic meaning of \wp_i^0 is clear, while the upper bound is introduced for technical reasons which will become apparent in Chapter 7. Note that $\wp_i > 0$ when $i > y(0)$. Also, for fixed ω , $\wp(\omega, i) \uparrow \infty$ when $i \uparrow \infty$ provided that $y(\omega, t)$ is bounded on compacts of \mathcal{T} ; (this condition is satisfied a.s. if y is a right continuous supermartingale)

The families (τ_i) , (ν_i) and (\wp_i) may be regarded as processes, and as such are easily seen to be adapted, with sample functions which are a.s. non-decreasing and right continuous, (provided, in the case of \wp , that y is right continuous); thus the processes define time changes relative to \mathfrak{A} . (See Chapters 1 and 6–8 for further details.)

CONDITIONS FOR OPTIMALITY: GENERAL FORM

The following conditions are *sufficient* for (c^*, k^*) to be optimal:

(a) $\varphi(c^*)$ is finite ... (3.28)

and there exists a non-decreasing sequence (χ_n) of stopping times, with $\chi_n > 0$ for $n > \text{some } n_0$, such that

(b) $\chi_n(\omega) \uparrow \infty$ a.s. when $n \uparrow \infty$, ... (3.29)

(c) for each n , the stopped process $y^n(\omega, t) = y[\omega, t \wedge \chi_n(\omega)]$ is a uniformly integrable martingale, ... (3.30)

(d) $\lim_n E\{ I\{\omega: \chi_n(\omega) < \infty\} y[\omega, \chi_n(\omega)] J^*[\omega, \chi_n(\omega)] \} = 0$, ... (3.31)

(e) $y(\omega, \infty) = 0$ if $\chi_n(\omega) = \infty$ for some n , a.s. ... (3.32)

If the χ_n are finite, (d) and (e) may be replaced by the single condition

(d') $\lim_n E\{ y[\omega, \chi_n(\omega)] J^*[\omega, \chi_n(\omega)] \} = 0$ (3.33)

The preceding conditions are also *necessary* for optimality if (a) is replaced by

(a') there is an $\alpha_0 \in (0, 1)$ such that $\varphi(c^* - \alpha c^*) > -\infty$ for $0 \leq \alpha \leq \alpha_0$. (3.34)

Note that, since we assume throughout that φ satisfies a finite supremum condition, (a) is equivalent to $\varphi(c^*) > -\infty$. The reason for considering (a') will appear in Chapter 5, see eq (5.6). We shall call (a) or (a'), depending on the context, the *finite welfare (FW) condition*, distinguishing (a) as the *weak*, (a') as the *strong* FW condition if necessary; but sometimes we shall take finite welfare as read. Conditions (b) and (c) together assert that y is a (right continuous) local martingale reduced by the sequence (χ_n) and are together called the (local) *martingale condition*, while (d) and (e), or (d'), comprise the *transversality condition (at infinity)*. Note that (b) and (c) require y to be finite and imply the existence of the finite limiting variable $y(\infty)$ by martingale convergence. The conditions also imply $c^* > 0$ a.e., since $c^*(t) = 0$ would imply $y(t) = \infty$, contrary to the martingale property; hence also $k^* > 0$ on \mathcal{S} a.s. to ensure feasibility. Also they imply that $y(t) > 0$ on \mathcal{S} a.s., since a

non-negative local martingale cannot hit zero at a finite time ρ without remaining at zero thereafter, [DM] VI.17, which by (10) would imply $c^*(t) = \infty$ for $t > \rho$, contrary to feasibility. Further, (c) implies $E y(\chi_n) = y(0)$ for each n .

SUFFICIENCY THEOREM.

The Conditions (28–32) are sufficient for optimality.

PROOF. Let (c^*, k^*) satisfy the Conditions. We have to show that, for any plan (c, k) , we have $\varphi(c) - \varphi(c^*) = \int \delta u \cdot q \cdot dm \leq 0$. It may be assumed that $\varphi(c) > -\infty$, and then $\delta u \cdot q$ is m -integrable. For brevity, write

$$y_t^n = y^n(t) = y(t \wedge \chi_n) \quad \text{and} \quad i_t^n = I\{(\omega, t): t < \chi_n\},$$

and note that for each t these variables are \mathcal{A}_t -measurable because the χ_n are stopping times. We have

$$\begin{aligned} E \int_0^{\chi_n} \delta u_t \cdot q_t \, dt &\leq -E \int_0^{\chi_n} y_t \delta j_t \, dt = -E \int_0^{\chi_n} y_t^n \delta j_t \, dt \\ &= -E \int_0^\infty y_t^n \delta j_t i_t^n \, dt = -E \int_0^\infty [E^t y(\chi_n)] \delta j_t i_t^n \, dt \\ &= -\int_0^\infty E\{[E^t y(\chi_n)] \delta j_t i_t^n\} \, dt = -\int_0^\infty E\{E^t[y(\chi_n) \delta j_t i_t^n]\} \, dt = -\int_0^\infty E\{y(\chi_n) \delta j_t i_t^n\} \, dt \\ &= -E \int_0^{\chi_n} y(\chi_n) \delta j(t) \, dt \\ &= -E\{I\{\chi_n < \infty\} y(\chi_n) \int_0^{\chi_n} \delta j_t \, dt\} = -E\{I\{\chi_n < \infty\} y(\chi_n) \delta J(\chi_n)\} \\ &\leq E\{I\{\chi_n < \infty\} y(\chi_n) J^*(\chi_n)\}. \end{aligned} \tag{3.35}$$

The first inequality follows from (17), which together with the fact that $\delta u \cdot q$ is m -integrable also justifies the interchanges in the order of integration in what follows. The next equality expresses the fact that $y_t = y_t^n$ for $t < \chi_n$, and the third step is just a change of notation. The next equality uses the fact that y_t^n is a uniformly integrable martingale by (30), so that $y_t^n = E^t y_\infty^n = E^t y(\chi_n)$; we pass to the sixth term by Fubini's Theorem, then to the seventh by virtue of the fact that δj_t and i_t^n are \mathcal{A}_t -measurable, then to the eighth by $EE^t = E$. To pass to the ninth term, we apply Fubini again and use the definition of i_t^n . Note that, in this term, the values of $y(\chi_n)$ with $\chi_n = \infty$, which by (32) must equal zero, multiply only finite numbers. In the tenth

and eleventh terms we accordingly restrict the expectation to $\{\omega: \chi_n(\omega) < \infty\}$ and then replace the time integral of δj on $[0, \chi_n)$ by $\delta J(\chi_n)$. Since $-\delta J(\chi_n) = J^*(\chi_n) - J(\chi_n) \leq J^*(\chi_n)$, the last inequality follows. By (31), the final expression in (35) tends to zero with n , whereas the first term in the first line tends to $\varphi(c) - \varphi(c^*)$. \square

REMARK I. The fact that y_t^n in the third term of (35) can be replaced by $y(\chi_n)$ in the ninth is a variant of a standard result on integration of a martingale with respect to a non-decreasing function, [El] 7.16; a direct proof has been given to make the argument self-contained and to allow for the possibility that $J^*(\infty)$ and $J(\infty)$ do not exist as a.s. limits, or, if they exist, that $\delta J(\infty)$ may have the form $\infty - \infty$. If the χ_n are a.s. finite, or if (as in the case of linear production) J^* and J are non-increasing and hence bounded on \mathcal{S} so that δJ is of finite variation on $[0, \infty]$, the proof can be shortened

Indeed, we may in these cases write the ninth term in (35) as

$-E\{y(\chi_n)[J(\chi_n) - J^*(\chi_n)]\}$, and since this is $\leq E\{y(\chi_n)J^*(\chi_n)\}$ we can replace

Conditions (d–e) in the set of sufficient conditions by (d').

* * *

To conclude this Chapter, we consider alternative forms of the Conditions for Optimality which correspond to particular choices of the stopping times, namely as price, depletion or clock times. These allow certain simplifications of statements and proofs. If, as is often the case in applications, one is interested only in *sufficiency*, one can choose any of these times, and it may be convenient to make stronger assumptions than those required for the Sufficiency Theorem; (see the discussion of clock times below). As to *necessity*, it will be shown later that, if (c^*, k^*) is optimal (and satisfies (a')), the Conditions (b–c) and (d') can be satisfied by choosing a sequence of price times (Chapter 7), while (b–e) can be satisfied by choosing a sequence of depletion times (Chapter 8). A sequence of clock times may be chosen if the optimum satisfies additional conditions. Some further details of this programme follow.

CONDITIONS IN 'PRICE TIME' FORM.

Consider a sequence of the form $\chi_n = \rho_i$, $i = i(n) \uparrow \infty$ as $n \uparrow \infty$, where the ρ_i are price times defined as in (27). These times are bounded, so the *sufficient* conditions are (a–c) and (d'). The *necessity assertion to be proved* is that, for an optimal plan (c^*, k^*) satisfying (a'), the process y is a local martingale reduced by some (indeed by any) sequence of price times (ρ_i) with $i \uparrow \infty$, and further that $E\{y(\rho_i)J^*(\rho_i)\} \rightarrow 0$. The proof will be completed in Chapter 7.

CONDITIONS IN 'DEPLETION TIME' FORM.

Suppose now that the times (χ_n) are specified to be of the form $\chi_n = \nu_i$, $i = i(n) \uparrow K_0$ as $n \uparrow \infty$, where the ν_i are depletion times defined as in (23); for example, χ_n may be the sequence

$$\chi_n = \nu_i, \quad i = K_0(1 - e^{-n}), \quad n = 0, 1, \dots \quad (3.36)$$

Let us for the moment denote the resulting Conditions (b–e) by (b''–e''). Note that (25) shows that (b'') reduces to $J^*(t) > 0$ on \mathcal{S} a.s., or equivalently $k^*(t) > 0$ on \mathcal{S} a.s. Now (b'') and (c'') state that y is a positive local martingale reduced by (ν_i) , so that $Ey(\nu_i) = y(0)$ for each i . Given (b'') and (c''), we consider alternative forms of the transversality condition. According to (25), (e) is equivalent to the assertion

$$(e_1'') \quad y(\infty) = 0 \text{ if } \inf_{t \in \mathcal{S}} J^*(t) > 0, \text{ i.e. if } \Gamma(\infty) < K_0, \quad (3.37)$$

the weakest condition to be expected on economic grounds; this may be rewritten as

$$(e_2'') \quad y(\infty)[K_0 - \Gamma(\infty)] = 0 \text{ a.s.} \quad (3.38)$$

Using (24), it follows from this equation and the preceding remarks that

$$\begin{aligned} E\{I\{\nu_i < \infty\}y(\nu_i)J^*(\nu_i)\} &= (K_0 - i)E\{I\{\nu_i < \infty\}y(\nu_i)\} \\ &= (K_0 - i)Ey(\nu_i) = (K_0 - i)y_0, \end{aligned} \quad (3.39)$$

which tends to zero when $i \uparrow K_0$, $n \uparrow \infty$. This shows that (d'') follows from (e'') and so can be discarded. Again, since $y(\nu_i)[K_0 - \Gamma(\nu_i)]$ vanishes if $\nu_i = \infty$ and equals $y(\nu_i)J^*(\nu_i)$ otherwise, it follows from (38–39) that

$$(e''_3) \lim_{i \uparrow K_0} E\{y(\nu_i)[K_0 - \Gamma(\nu_i)]\} = 0 \quad \dots(3.40)$$

(We cannot replace $K_0 - \Gamma$ by J^* in this relation, because $J^*(\infty)$ might be undefined, or defined but greater than $K_0 - \Gamma(\infty)$, possibly infinite.) Finally, (40) implies (38); indeed, the limiting variables $y(\infty)$ and $K_0 - \Gamma(\infty)$ are defined, and it suffices to consider any sequence $i(n) \uparrow K_0$ and apply Fatou's Lemma. *Consequently all three forms of (e'') are equivalent in the presence of (b'') and (c'')*. This result may be claimed as a significant advantage of working with conditions involving depletion times.

To sum up, the *sufficient* conditions are (a), $J^* > 0$, (c'') and any one of the forms of (e''). It is easily seen that, if these conditions are satisfied for *some* sequence $\nu_{i(n)} \uparrow \infty$ with $i(n) \uparrow K_0$, then they are satisfied for *any* such sequence. The *necessity assertion remaining to be proved* is that, for an optimal plan (c^*, k^*) satisfying (a'), y is a local martingale reduced by some (indeed by any) sequence of depletion times (ν_i) with $i \uparrow K_0$, and that (e'') holds, say that $y(\infty)[K_0 - \Gamma^*(\infty)] = 0$ a.s. The proof will be completed in Chapter 8.

CONDITIONS IN CLOCK TIME FORM.

Consider the Conditions for Optimality with stopping times of the form $\chi_n = T_n$ for some sequence $T_n \uparrow \infty$ with $T_n \in \mathcal{S}$. In this case, (c) says that y is a (true) martingale and (d') becomes

$$\lim_n E\{y(T_n)J^*(T_n)\} = 0. \quad \dots(3.41)$$

In statements of *sufficient* conditions, it is sometimes convenient to replace (41) by the stronger condition

$$\lim_{T \rightarrow \infty} E\{y(T)J^*(T)\} = 0 \quad \dots(3.42)$$

cf (1.10); (41) does not imply (42) since (41) might hold for some sequences but not for others. Another condition which can replace (41) in a set of sufficient conditions is

$$\int_0^\infty E\{y(T)J^*(T)\}dT < \infty. \quad \dots(3.43)$$

Note that (43) does not imply (42), because convergence of an integral $\int_0^\infty f(t)dt$ with

$f \geq 0$ does not imply $f(t) \rightarrow 0$ as $t \rightarrow \infty$, see [Har] Section 185. On the other hand, (43) does imply (41), because convergence of the integral implies that $f(t)$ is not bounded away from zero for large t .

Turning to *necessary* conditions, it seems that in general optimality implies neither that y is a true martingale nor that transversality holds in any of the forms (41–43); cf. Chapter 1, fn.3. Of course, the results already stated imply that y is always a positive local martingale, and then it is a standard result that y is a true martingale iff $Ey(t)$ is constant for $0 \leq t < \infty$, in particular if y is bounded on each interval $[0, T]$. It can be shown that, if y is a true martingale, then transversality in the form (42) is a necessary condition; this assertion, which apparently is not trivial, follows from arguments presented in Chapter 7 – see Remark II at the end of that Chapter.

Further conditions for y to be a martingale can be obtained from the Conditions in depletion time. Given the necessity of the latter conditions, suppose that for each clock time $T \in \mathcal{T}$ there is a depletion time ν such that $T \leq \nu(\omega)$ a.s. Then it follows from the fact that y is reduced by any sequence of depletion times increasing a.s. to ∞ that y is also reduced by a sequence $T_n \uparrow \infty$ of clock times and so is a true martingale. For example, this situation obtains if there is a deterministic function J^- , (which may w.l.o.g. be taken to be non-increasing and right continuous) satisfying

$$J^*(\omega, T) \geq J^-(T) > 0 \text{ on } \mathcal{T}, \text{ a.s.}; \quad \dots (3.44)$$

see Chapter 8, Remark IV for details, also Figure 4.

In particular, note that by (8), (1.1) and (4) we have (in abridged notation)

$$j^*/J^* = \dot{k}^*/k^* - r = f(k^*)/k^* - c^*/k^* - r \geq -c^*/k^*,$$

hence (omitting the stars from now on)

$$J(\omega, T) \geq K_0 \cdot \exp\left\{-\int_0^T [c(\omega, t)/k(\omega, t)] dt\right\}. \quad \dots (3.45)$$

Thus, if the ‘propensity to consume’ process c/k is bounded, a.s. on \mathcal{T} , by a constant $\theta^+ < \infty$, we may take

$$J^-(T) = K_0 \cdot \exp\{-\theta^+ T\} \quad \dots (3.46)$$

and conclude that y is a true martingale. Then (42) also holds. Clearly the conditions stated are not the weakest possible, but they are the most useful in applications.

The case of the SNC model considered in Chapter 2 deserves special mention. Consider first a set of *sufficient* conditions, with y assumed to be a true martingale and c^* and k^* always positive. Omitting the stars and using (2.33), the finite welfare condition (28) may be written for $b \neq 1$ as

$$\int_0^\infty E\{\bar{c}(T)^{1-b} e^{v(T)}\} dT = \int_0^\infty E\{[\bar{c}(T)/\bar{k}(T)] \bar{k}(T) \bar{c}(T)^{-b} e^{v(T)}\} dT < \infty \quad (3.47)$$

and this coincides with the strong finite welfare condition (34). Next, we have $y(T)J(T) = \bar{k}(T)\bar{c}(T)^{-b} e^{v(T)}$ according to (2.40), so that (43) reads

$$\int_0^\infty E\{\bar{k}(T)\bar{c}(T)^{-b} e^{v(T)}\} dT < \infty \quad (3.48)$$

The integral appearing here was called the *transversality integral* in [F5], and its convergence implies a transversality condition of the form

$$\lim_n E\{\bar{k}(T_n)\bar{c}(T_n)^{-b} e^{v(T_n)}\} = 0 \quad (3.49)$$

for some sequence $T_n \uparrow \infty$, cf. (43).⁷ Condition (48) is convenient for proving sufficiency, not least because of its connection with (47). Explicitly, for $b \neq 1$, if the process

$$\bar{c}(\omega, t)/\bar{k}(\omega, t) = c(\omega, t)/k(\omega, t)$$

is bounded above (on \mathcal{S} , a.s.) by a constant $\theta^+ < \infty$, then (48) implies (47); and if the process is bounded below by some constant $\theta^- > 0$, then (47) implies (48). Thus it is sufficient for optimality if y is a martingale and *either* (47) holds with $\bar{c}/\bar{k} > \theta^- > 0$, *or* (48) holds with $\bar{c}/\bar{k} < \theta^+ < \infty$.⁸

⁷ The statement of the Corollary to Theorem 1 of [F5], p. 1960, contains the incorrect assertion that (48) implies a condition of the form

$$\lim_{T \rightarrow \infty} E\{\bar{k}(T)\bar{c}(T)^{-b} e^{v(T)}\} = 0$$

To correct the argument in [F5], it is enough to prove Theorem 1 with the transversality condition in the form (3.49) above. Specifically, it is enough to insert the words 'for some sequence $T_n \uparrow \infty$ ' before (iii) in the statement of the Theorem and to replace T by T_n in the last line of the proof.

⁸ The coincidence of the ordinary and strong finite welfare conditions also holds for

We now restrict attention to the Brownian version of the model. In [F5–6] we constructed optimal plans determined by logarithmic consumption functions of the form $\bar{c}/\bar{k} = \theta(\ln \bar{k})$, cf. (2.17) above, the function θ being of class C^1 on \mathfrak{R} with θ and θ' satisfying certain finite boundary conditions as $\ln \bar{k} \rightarrow \pm \infty$. The boundary conditions were of two types, called Type 1 and Type 0, defined for different ranges of parameter values, with θ bounded away from zero for Type 1 and θ tending to zero at either ∞ or $-\infty$ for Type 0. Obviously θ is bounded on \mathfrak{R} in both cases, so that the process $\bar{c}(\omega, t)/\bar{k}(\omega, t)$ which the function determines is also bounded. For both Types (including the case $b = 1$) the process y is a martingale by construction and the proofs of sufficiency involve proofs that the transversality integral converges

The new point to be made here – which for the present is advanced as a conjecture – relates to *necessity*. Assume that \mathfrak{A} is the smallest filtration for which the vector Brownian motion is optional, (and, as usual, that (2.7) is satisfied). It appears that every optimal plan in the BNC model can be generated by an invariant log–consumption function of the form (2.17), *and further that this function must satisfy one of the types of boundary condition specified in [F5–6]*. In view of the uniqueness (up to null sets) of optimal plans in this model, and the constructive procedure described in [F5–6], this amounts to the conjecture that optimal plans (as defined) exist only for the ranges of parameter values considered in these papers. Of course, since the function $\theta(\ln \bar{k})$ is in all these cases bounded by some constant, it follows from the argument leading to (46) that y is always a martingale; however, this also results from the construction.

REMARK II. If the finite supremum assumption is dropped, the question arises as to the appropriate definition in the present setting of ‘overtaking’ criteria. New

$b = 1$ if $\int_0^\infty \{E e^{v(T)}\} dT < \infty$. Then it is sufficient for optimality if, in addition to finite welfare, y is a martingale and $\bar{c}^*/\bar{k}^* > \theta_- > 0$.

possibilities suggest themselves, once it is accepted that economic time is not necessarily to be measured by the clock. For example, given a sequence (or more generally a family) of stopping times $(\chi_n) \uparrow \infty$ a.s., a preordering of plans might be defined by the relation 'c* overtakes c along (χ_n) ' iff the integrals appearing in the initial term of (35) are defined for all n and their limit is non-positive. Then if (c^*, k^*) is a plan satisfying $\varphi(c^*) > -\infty$ and Conditions (b)–(e), the proof of sufficiency (with only minor changes) shows that this plan is 'optimal relative to (χ_n) ' in the sense that it overtakes all other plans along (χ_n) . If the χ_n are clock times, the overtaking criterion is essentially the usual one, applied to expected utility. However, if the χ_n are (say) the depletion times associated with the 'star' plan, then each plan chosen as 'star' defines its own preordering – thus giving rise a priori to the possibility that one plan may overtake another according to the one's preordering, while the other overtakes the one according to the other's preordering. These ideas will not be pursued here, but they point to new difficulties with the overtaking approach.

4. FEASIBLE PLANS

This Chapter deals with some technicalities concerning the definition of plans in the Standard Model which were omitted from Chapter 3, and derives some properties of feasible sets which will be needed later. Readers willing to accept the Feasibility Lemma stated at the end of the chapter can skip the rest without loss of continuity.

(A) *Justification of Definitions.*

Referring to the definitions of c and k plans given in Chapter 3(A), it should be checked that the assumptions of the model imply the existence, uniqueness and progressive measurability of the required solutions of (1.1), and further that the 'indirect' functions of $s = (\omega, t)$ such as $F_k(s) = f[k(s); s]$, $U_c(s) = u[c(s); s]$, $f'[k(s); s]$, $\partial[k(s); s]$, $u'[c(s); s]$ derived from a given plan (c, k) are progressive — see (3.5–6a), also (3.7–10) for functions derived from (c^*, k^*) .

Existence and Uniqueness of Solutions. We first consider (1.1) with ω fixed, regarding $c(t)$ as a given non-negative locally integrable function. In order to be able to speak of negative solutions, it is convenient to extend (1.1) — sometimes without special mention — by setting

$$f(k; t) = 0, \quad k < 0, \quad t \in \mathcal{T}. \quad \dots(4.1)$$

The right side of (1.1) is then defined for all real k and $t \in \mathcal{T}$, and is continuous in k for each t and measurable in t for each k . The Carathéodory Theorem, [CL] pp. 42–8, then shows that an absolutely continuous solution $k(t)$ through a given point (\bar{k}, \bar{t}) exists on some right neighbourhood of \bar{t} if there is a rectangle

$$Q = [k_1, k_2] \times [\bar{t}, t_2] \quad \text{with } k_1 < \bar{k} < k_2,$$

and a Lebesgue integrable function $n(t)$ on $[\bar{t}, t_2]$, such that

$$|f(k; t) - c(t)| \leq n(t) \quad \text{for } (k, t) \in Q;$$

this solution can be continued to the boundary of Q . Further, the solution is unique if there is an integrable function $n(t)$ on $[\bar{t}, t_2]$ such that

$$|f(k'';t) - f(k';t)| \leq |k'' - k'|n(t) \quad \text{for } (k'',t) \text{ and } (k',t) \text{ in } Q.$$

Note that we consider solutions of initial value problems and continuations only in the forward direction of time

Under present assumptions, the concavity of f and $f(0;t) = 0$ imply that, for $k \geq 0$,

$$kf'(\infty;t) - c(t) \leq f(k;t) - c(t) \leq kf'(0;t), \quad \dots (4.2)$$

so that, for any real k and $k_1 \leq k \leq k_2$,

$$-|k_1| \cdot |f'(\infty;t)| - c(t) \leq f(k;t) - c(t) \leq |k_2| \cdot |f'(0;t)| \quad \dots (4.3)$$

The concavity of f and the extension (1) also yield

$$|f(k'';t) - f(k';t)| \leq |f'(0;t)| \cdot |k'' - k'|. \quad \dots (4.4)$$

Since $f'(0;t)$ and $f'(\infty,t)$ are locally integrable by (3.2), the local existence and uniqueness follow. Further, a solution $k(t)$ can be continued forward on $[\bar{t}, \infty)$ because no explosion can occur at a finite time; this follows readily from the inequalities (2) together with (1) and the local integrability of $f'(0)$, $f'(\infty)$ and c . For brevity we now confine attention to solutions satisfying the initial condition $k(0) = K_0$.

Progressive Measurability. It remains to check that k and \bar{k} regarded as functions of the pair $s = (\omega, t)$ define progressive processes, and further that the 'indirect' functions F_k , U_c , etc. derived from a given plan (c, k) are progressive. We start with the latter point, taking F_k as an example. Recall that $(\mathcal{S}, \mathcal{H}, \mu)$ is a probability space, and the measurable ('random') variables on this space are the progressive processes. We may regard the function $f(k; s) = \{f_k(s); k \in [0, \infty]\}$ as a 'process on $(\mathcal{S}, \mathcal{H}, \mu)$ ', i.e. a family of variables parameterised by k , and according to the assumptions about f this 'process' is (k, s) -measurable with respect to $\mathcal{B}_{[0, \infty]} \times \mathcal{H}$. It follows from a Measurability Lemma for processes, [Lo] p.503, that, if $k(s)$ is a measurable variable with range in the parameter set $[0, \infty]$, the function $F_k(s) = \{f[k(s); s]; s \in \mathcal{S}\}$ has the same property. In other words, if k is a progressive process, so is F_k . The proofs for other 'indirect' functions on \mathcal{S} rely on similar arguments, together with elementary properties of measurable functions. We further

note that, if k and hence F_k are known to be progressive then, since c is progressive by assumption, this property extends to \hat{k} by virtue of equation (1.1).

Turning to the progressive measurability of k , we recall that a process with continuous sample paths is progressive iff it is adapted, [RW] Ch.II(73 9–10). Thus the function k of (ω, t) obtained by ‘collecting’ the solutions $k(\omega, \cdot)$ will be a progressive process iff, for each t , $k(\cdot, t)$ is a random variable and is \mathcal{A}_t -measurable. Now the latter property can be inferred from the procedure used to prove the Carathéodory Theorem if the approximations to the solution of (1.1) are constructed as processes, i.e. simultaneously for all ω . Rather than rewrite the whole proof for the stochastic case, we shall use the fact that, for fixed ω , a unique solution $k(\omega, \cdot)$ through $(K_0, 0)$ and defined on the whole of \mathcal{T} is already known to exist. In these circumstances the approximating functions may be constructed on an arbitrary fixed interval $[0, \tau]$, and – as noted on p.45 of [CL] – they must converge at least pointwise to the solution $k(\omega, \cdot)$ on this interval. To be explicit, we recall that, for the purpose of constructing approximations, the o.d.e. (1.1) is replaced by the equivalent integral equation

$$k(T) = K_0 + \int_0^T \{f[k(t); t] - c(t)\} dt, \quad T \in \mathcal{T} \quad (4.5)$$

Adapting formula (1.5) of [C1] p.43, the approximations may be defined on $[0, \tau]$, simultaneously for all ω , by

$$k_j(T) = K_0, \quad 0 \leq T \leq \tau/j, \quad (4.6a)$$

$$k_j(T) = K_0 + \int_0^{T-\tau/j} \{f[k_j(t); t] - c(t)\} dt, \quad \tau/j < T \leq \tau. \quad (4.6b)$$

Obviously $k_1 = k_1(\omega, T)$ is defined as a progressive process on $[0, \tau]$, since it is the constant K_0 . For any fixed $j \geq 1$, (6a) defines k_j as a constant on $[0, \tau/j]$, and then (6b) defines k_j as a continuous function up to $2\tau/j$. Further, for each $t \leq \tau/j$ the function of ω defined by $f[k_j(t); t] = f[K_0; t]$ is \mathcal{A}_t -measurable by the assumptions about f , and $c(t)$ has the same property; since $t \leq \tau/j < T \leq 2\tau/j$ implies $\mathcal{A}_t \subseteq \mathcal{A}_T$,

it follows that the integral defining $k_j(T)$ is \mathcal{A}_T -measurable for $T \leq 2\tau/j$. It follows that k_j is defined as a progressive process on $[0, 2\tau/j]$. Now assume (inductive hypothesis) that k_j is so defined on $[0, i\tau/j]$ for $0 < i < j$. Then (6b) defines k_j as a continuous function of t on $(i\tau/j, (i+1)\tau/j]$, knowledge of the integrand being required only for $t \leq i\tau/j$. Further, since $k_j(t)$ is \mathcal{A}_t -measurable for $t < i\tau/j$, the Measurability Lemma cited above shows that $f[k_j(t); t]$ has the same property; consequently all these variables and the corresponding $c(t)$ for $t < i\tau/j$ are \mathcal{A}_T -measurable for $i\tau/j < T \leq (i+1)\tau/j$, so that the integral defining $k_j(T)$ is also \mathcal{A}_T -measurable. Therefore, by induction, (6) defines k_j as a progressive process on $[0, \tau]$. Since $j \rightarrow \infty$ implies $k_j(\omega, t) \rightarrow k(\omega, t)$ on $[0, \tau]$ for all ω (making changes on a null set if need be), it follows that k is also progressive on $[0, \tau]$. It then remains to let $\tau \uparrow \infty$.¹

(B) *Properties of Feasible Sets.*

Sometimes it is useful to rewrite (1.1) as an integral equation in another form. Given a k -plan (or more generally a progressive process k), we define the corresponding average returns plan (process) by

$$\mathfrak{D}(k; \omega, T) = \int_0^T \mathfrak{d}[k(\omega, t); \omega, t] dt, \quad \omega \in \Omega, t \in \mathcal{T}, \quad (4.7a)$$

extending the definition $k\mathfrak{d}(k) = f(k)$ to $k < 0$ if necessary; we also write

¹ A word should be added about the treatment of classes of similar progressive processes. As stated in Chapter 3 fn 2, we sometimes identify with one another processes which belong to the same class and satisfy a given set of conditions. Thus we say that a process ξ satisfying conditions C has property Q if there is a ξ' satisfying C and similar to ξ which has Q . Some care is needed in allowing for the implications of the 'given' conditions C . In general — i.e. in the absence of special restrictions — a c -plan as defined above may be replaced by any similar process, and the same goes for the processes denoted by dk/dt , F_k , U_c , g , j and y . On the other hand, two k -plans, being by definition absolutely continuous, are similar iff their sample functions agree on \mathcal{T} , a.s., so that k -plans are actually classes of indistinguishable processes; the same goes for G , J , Γ etc. Given the existence and uniqueness of progressive solutions of (1.1), k -classes correspond to c -classes. Sometimes a particular choice of c or k , or one of the processes derived from them, is made, and then it is implicit that corresponding choices are made for all other processes. This can imply further restrictions: for example if y , once it has been shown to be a supermartingale, is assumed to be right continuous, the choice of c in its class is correspondingly limited; cf. the Remark in Chapter 6. All this is confusing, but will cause no confusion if ignored.

$$\mathfrak{D}(k; \tau, T) = \mathfrak{D}(k; T) - \mathfrak{D}(k; \tau), \quad 0 \leq \tau \leq T. \quad \dots (4.7b)$$

The integrals $\mathfrak{D}(k; T)$ are finite, cf (3.2–4), so that the o.d.e (1.1) with $k(0) = K_0$ is equivalent to the integral equation

$$K_0 - k(T)e^{-\mathfrak{D}(k; T)} = \int_0^T c(t)e^{-\mathfrak{D}(k; t)} dt. \quad \dots (4.8)$$

It follows that, in the definition of a c-plan, the requirement that $k(T) \geq 0$ on \mathcal{S} a.s. may be replaced by the condition

$$\int_0^\infty c(t)e^{-\mathfrak{D}(k; t)} dt \leq K_0 \quad \text{a.s.} \quad \dots (4.9)$$

Explicitly, a c-plan may be defined as a non-negative, progressive, locally integrable process such that there exists a progressive k which a.s. satisfies (8) for all $T \in \mathcal{S}$ as well as (9)

We now establish some basic properties of the feasible sets \mathcal{C} , \mathcal{K} and $\Pi = \mathcal{C} \times \mathcal{K}$.

PROPERTY (i): $0 \in \mathcal{C}$, therefore \mathcal{C} is not empty

PROOF This follows immediately from (8)–(9) and the results on existence and uniqueness of solutions ||

PROPERTY (ii): If $c_1 \in \mathcal{C}$ and c_2 is a progressive process such that $0 \leq c_2 \leq c_1$ a.e., then $c_2 \in \mathcal{C}$, i.e. there exists k_2 such that $(c_2, k_2) \in \Pi$; moreover $0 \leq k_1(t) \leq k_2(t)$ on \mathcal{S} a.s.

PROOF. We may assume $c_1 \geq c_2$ everywhere. Obviously c_2 is locally integrable, so that $k_2(t)$ can be defined as the unique, progressive solution of the (extended) equation (1.1) through $(K_0, 0)$; it is only necessary to check that $k_2 \geq k_1$. If not, let $D = \{\omega : \exists t \in \mathcal{S}, k_1(t) > k_2(t)\}$ with $P(D) > 0$, and consider a fixed $\omega \in D$. Bearing in mind that both k_1 and k_2 start at K_0 , there must be some interval (τ, T) with $k_1(t) > k_2(t)$ in the interval and $k_1(\tau) = k_2(\tau) \geq 0$ since $k_1 \geq 0$. From (7–8),

$$k_i(\tau) - k_i(T)e^{-\mathfrak{D}(k_i; \tau, T)} = \int_\tau^T c_i(t)e^{-\mathfrak{D}(k_i; \tau, t)} dt, \quad i = 1, 2. \quad \dots (4.10)$$

Now $\mathfrak{D}(k_2; \tau, t) \geq \mathfrak{D}(k_1; \tau, t)$ for $t \in [\tau, T]$ since $\partial \downarrow$ with k ; thus the left-hand side of (10) is greater for $i=2$ than for $i=1$, while the right-hand side is at least as great for $i=1$ as for $i=2$ since $c_1 \geq c_2 \geq 0$, a contradiction which proves the result. ||

PROPERTY (iii): The sets \mathcal{E} and \mathcal{K} are convex

PROOF Let (c^*, k^*) and (c, k) be plans, write $\tilde{c}^\alpha = \alpha c + (1-\alpha)c^*$,
 $k^\alpha = \alpha k + (1-\alpha)k^*$, $c^\alpha = f(k^\alpha) - k^\alpha$ for $0 < \alpha < 1$. By the concavity of f ,

$$c^\alpha \geq \alpha f(k) + (1-\alpha)f(k^*) - [\alpha k + (1-\alpha)k^*] = \tilde{c}^\alpha \geq 0. \quad \dots (4.11)$$

Thus c^α and k^α are both non-negative, hence define a plan since the remaining conditions are certainly satisfied, and it follows that \mathcal{K} is convex. Since $0 \leq \tilde{c}^\alpha \leq c^\alpha$, it follows from Property (ii) that $\tilde{c}^\alpha \in \mathcal{E}$, so that \mathcal{E} also is convex ||

PROPERTY (iv): \mathcal{E} is closed under \liminf , i.e. if (c_n) is a sequence from \mathcal{E} , then $c_* \in \mathcal{E}$, where

$$c_*(\omega, t) = \lim_n \inf_{m \geq n} c_m(\omega, t), \quad (\omega, t) \in \mathcal{E}. \quad \dots (4.12)$$

PROOF. Since $0 \leq \bar{c}_n \doteq \inf_{m \geq n} c_m \leq c_n$ and the \bar{c}_n are obviously progressive, it follows from Property (ii) that $\bar{c}_n \in \mathcal{E}$, and since $\bar{c}_n \uparrow c_*$ we may as well replace c_n by \bar{c}_n from the outset and assume $c_n(\omega, t) \uparrow c_*(\omega, t)$ a.e. Then Property (ii) implies that $k_n(\omega, t) \downarrow$ pointwise. Let $k_*(\omega, t)$ denote the limit process. Clearly c_* and k_* are non-negative and progressive, and it remains to show that c_* is a.s. locally integrable and k_* is the solution of (1.1) corresponding to c_* . It is enough to show that a.s., for each $T \in \mathcal{T}$,

$$\begin{aligned} \int_0^T c_*(t) dt &= \lim_n \int_0^T c_n(t) dt = \lim_n \left\{ \int_0^T f[k_n(t); t] dt - k_n(T) + K_0 \right\} \\ &= \int_0^T f[k_*(t); t] dt - k_*(T) + K_0 < \infty. \end{aligned} \quad \dots (4.13)$$

Now, the first equality follows from monotone convergence and the second from (1.1). The third equality is valid because $k_n(t) \rightarrow k_*(t)$ implies $f[k_n(t); t] \rightarrow f[k_*(t); t]$ by continuity and the latter convergence is dominated on $[0, T]$; this in turn follows from the inequalities

$$k_n(t) f'(\infty; t) \leq f[k_n(t); t] \leq k_n(t) f'(0; t), \quad 0 \leq k_n(t) \leq k_1(t),$$

and the facts that the continuous function $k_1(t)$ is bounded on $[0, T]$ and that $f'(0; t)$, $f'(\infty; t)$ are integrable by (3.2). The same considerations show that all terms are finite. ||

The main results of this Chapter are summed up in the following

FEASIBILITY LEMMA.

(A) Given a progressive, a.e. non-negative and a.s. locally integrable process $c = c(\omega, t)$, the o.d.e (1.1) with initial condition $k(\omega, 0) = K_0$ – or equivalently the integral equation (4.8) with (4.7) – has a.s. a unique local solution $k(\omega, \cdot)$ which can be continued uniquely to the whole of \mathcal{T} , and the function $k = k(\cdot, \cdot)$ on \mathcal{S} is progressive. If $k(\omega, t) \geq 0$ on \mathcal{T} a.s. – or equivalently if the inequality (4.9) holds a.s. – we refer to the processes c and k , or to the pair (c, k) , as plans.

(B) The sets \mathcal{C} and \mathcal{K} of c and k plans have the following properties:

- (i) $0 \in \mathcal{C}$.
- (ii) If $c_1 \in \mathcal{C}$ and c_2 is a progressive process satisfying $0 \leq c_1 \leq c_2$ a.e., then there exists k_2 satisfying $0 \leq k_1(t) \leq k_2(t)$ on \mathcal{T} a.s. such that $(c_2, k_2) \in \Pi$.
- (iii) \mathcal{C} and \mathcal{K} are convex.
- (iv) \mathcal{C} is closed under (pointwise a.e.) passage to the $\lim \inf$.

5. DIRECTIONAL DERIVATIVES

We turn now to a detailed study of conditions characterising an optimum in the Standard Model. Following the procedure of the classical calculus of variations, we begin with a brief review of conditions expressed in terms of directional derivatives. Assume as usual that φ^* is finite, let (c^*, k^*) be a distinguished plan with $\varphi(c^*) > -\infty$ and (c, k) another plan, let $\delta k = k - k^*$, $\delta c = c - c^*$, and for $0 \leq \alpha \leq 1$ write $\tilde{c}^\alpha = c^* + \alpha \delta c$, $k^\alpha = k^* + \alpha \delta k$ and $c^\alpha = f(k^\alpha) - k^\alpha$. Two kinds of directional derivatives may be defined. The first considers the functional φ on \mathcal{E} — see (1.2) — and defines the derivative of φ at c^* in the direction δc by

$$\begin{aligned} D\varphi = D\varphi(c^*, \delta c) &= \lim_{\alpha \downarrow 0} \alpha^{-1} [\varphi(\tilde{c}^\alpha) - \varphi(c^*)] \\ &= \lim_{\alpha \downarrow 0} \alpha^{-1} \int \{u(\tilde{c}) - u(c^*)\} d\mu \end{aligned} \quad (5.1)$$

The second definition considers the same functional, now denoted Φ , on \mathcal{K} and sets

$$\begin{aligned} D\Phi = D\Phi(k^*, \delta k) &= \lim_{\alpha \downarrow 0} \alpha^{-1} [\varphi(k^\alpha) - \varphi(k^*)] \\ &= \lim_{\alpha \downarrow 0} \alpha^{-1} \int \{u(c^\alpha) - u(c^*)\} d\mu \end{aligned} \quad (5.2)$$

By Property (iii) of Chapter 4(B) and its proof, \tilde{c}^α and c^α are in \mathcal{E} , k^α is in \mathcal{K} and $c^\alpha \geq \tilde{c}^\alpha$. The limits in (1) and (2) exist because $\varphi(\tilde{c}^\alpha) - \varphi(c^*)$ and $\Phi(k^\alpha) - \Phi(k^*)$ are concave functions of α on $[0, 1]$. Since u is increasing in c , we have $D\Phi \geq D\varphi$. It is easily shown that (c^*, k^*) is optimal iff $D\varphi(c^*, \delta c) \leq 0$ for all $c = c^* + \delta c \in \mathcal{E}$. Indeed, if (c^*, k^*) is optimal, then $\varphi(c^*)$ is finite by definition and $\varphi(c^*) \geq \varphi(c^* + \delta c)$ for any fixed δc ; hence $[\varphi(c^* + \alpha \delta c) - \varphi(c^*)]/\alpha \leq 0$ for each $\alpha \in (0, 1]$ and it only remains to go to the limit. Conversely, concavity implies $\varphi(c^* + \delta c) - \varphi(c^*) \leq D\varphi(c^*, \delta c)$ and the assertion follows. A similar argument shows that (c^*, k^*) is optimal iff $D\Phi(k^*, \delta k) \leq 0$ for all $k = k^* + \delta k \in \mathcal{K}$.

Provided that it is permissible to differentiate under the integral sign in (1) and (2), we obtain

$$D\varphi = \int \delta c \cdot u'(c^*) d\mu = E \int_0^{\infty} \delta c \cdot u'(c^*) q dt; \quad (5.3)$$

$$D\Phi = \int [\delta k \cdot f'(k^*) - \delta \bar{k}] u'(c^*) d\mu = E \int_0^{\infty} [\delta k \cdot f'(k^*) - \delta \bar{k}] u'(c^*) q dt. \quad (5.4)$$

Using the notation (3.7–10) and (3.15–16), these formulae may also be written

$$D\varphi = E \int_0^{\infty} \delta g \cdot y dt = E \int_0^{\infty} y \cdot d\delta G; \quad D\Phi = -E \int_0^{\infty} \delta j \cdot y dt = -E \int_0^{\infty} y \cdot d\delta J, \quad (5.5)$$

from which it is incidentally clear that the proof of the Sufficiency Theorem amounts to showing that $D\Phi \leq 0$. The differentiation is easily justified without further assumptions if $\varphi(c) = \Phi(k) > -\infty$, but the necessity argument below requires formulae (3–5) to hold for every feasible variation. To ensure this,

we assume that, if c^ is optimal, there is an $\alpha_0 \in (0,1)$ such that*

$$\varphi(c^* - \alpha c^*) > -\infty \quad \text{for } 0 \leq \alpha \leq \alpha_0, \quad (5.6)$$

i.e. c^* satisfies the Strong Finite Welfare Condition (3.34). This condition is actually satisfied for all plans (not just optima) if $u \geq 0$, and for all plans with $\varphi > -\infty$ if u has one of the forms considered in the SNC model. Now, since u and f are concave, $\alpha^{-1}[u(\tilde{c}^\alpha) - u(c^*)]$ and $\alpha^{-1}[u(c^\alpha) - u(c^*)] \uparrow$ when $\alpha \downarrow 0$, and the monotone limits are $\delta c \cdot u'(c^*)$ and $[f'(c^*)\delta k - \delta \bar{k}] u'(c^*)$; therefore (3) and (4) will follow once it is shown that the integrands in (1) and (2) are bounded below, for all $\alpha \in [0, \alpha_0]$, by a μ -integrable function. Using $c^\alpha \geq \tilde{c}^\alpha \geq (1-\alpha)c^*$ and the monotonicity and concavity of u we have

$$\begin{aligned} \alpha^{-1}[u(c^\alpha) - u(c^*)] &\geq \alpha^{-1}[u(\tilde{c}^\alpha) - u(c^*)] \\ &\geq \alpha^{-1}[u(c^* - \alpha c^*) - u(c^*)] \geq \alpha_0^{-1}[u(c^* - \alpha_0 c^*) - u(c^*)], \end{aligned}$$

and by (6) the μ -integral of the last term is finite.

As a corollary, we may set $\delta c = -c^*$, $c = g = 0$, to obtain

$$D^* \doteq -D\varphi(c^*, -c^*) = E \int_0^{\infty} c^* u'(c^*) q dt = E \int_0^{\infty} y g^* dt < \infty, \quad (5.7)$$

and clearly $-D^*$ is a finite minimum of $D\varphi$ and $D\Phi$ for all feasible variations. The proof that the Conditions for Optimality are necessary seems to depend in an essential way on (7).

In case (5.6) does not apply but $\varphi(c) = \Phi(k) > -\infty$, the argument justifying the differentiation under the integral sign is slightly altered. Consider for example the integrands $\alpha^{-1}[u(\tilde{c}^\alpha) - u(c^*)]$, $0 < \alpha \leq 1$, in (5.1). They ascend as $\alpha \downarrow 0$, and on the domain $\{c \geq c^*\}$ they are non-negative while on $\{c < c^*\}$ they are non-positive and bounded below by the μ -integrable function $u(c) - u(c^*)$. The passage to the limit under the integral sign is therefore justified by monotone convergence in each case and the result for $D\varphi$ follows as above. The argument for $D\Phi$ is similar.

We shall not stop to set out these results as a theorem. The main point needed in the necessity theory is that, if (c^*, k^*) is optimal and satisfies (5.6), then all $D\varphi$ and $D\Phi$ may be calculated as in (2), (3) and (4) and satisfy

$$0 \geq D\Phi \geq D\varphi \geq -D^* \quad \dots(5.8)$$

6. THE SUPERMARTINGALE PROPERTY

The next three Chapters are devoted to proving the *necessity* of the Conditions for Optimality stated in Chapter 3(B). Throughout this argument, (c^*, k^*) will be a given optimum, so that $\varphi(c^*) = \varphi^*$ is finite by definition. The present Chapter is mainly devoted to the proof that y is a supermartingale. Chapter 7 derives the local martingale property of y by an argument based on the Doob decomposition of a supermartingale. This method shows that the price times ρ_1 reduce y , and a proof of the transversality condition (3.33) with $\chi_n = \rho_1$ is then given. Chapter 8 takes up the discussion again from the end of Chapter 6 and derives local martingale and transversality conditions based on the depletion times ν_1 by means of a 'calculus of variations' argument. Conditions for y to be a true martingale and for transversality to hold in the form (1.10) or (3.42) are obtained as corollaries of the results of Chapters 7 and 8.

We first note that, if (c^*, k^*) is optimal, then $c^*(\omega, t) > 0$ a.e., and it may be assumed that $c^* > 0$ everywhere. To see this, suppose that $c^* = 0$ on $H \in \mathcal{H}$ with $\mu(H) > 0$, and let c be another plan with $c > 0$ a.e. (Such plans exist: for example, it follows from (4.9), taking into account (3.2–4), that it is feasible to set $c(t)e^{-\mathcal{D}(\infty, t)} = \theta K_0 e^{-\theta t}$ on \mathcal{T} a.s. with a constant $\theta > 0$.) Since $u'(0) = \infty$, it is found from (5.3) that $D\varphi = \int \delta c \cdot u'(c^*) d\mu = \infty$, contrary to optimality. Further, if $c^* > 0$ a.e., it follows from $f(k; t) = 0$ for $k \leq 0$ and the uniqueness of solutions of (1.1) that $k^* > 0$ on \mathcal{T} a.s., and if $c^* > 0$ on \mathcal{S} then $k^* > 0$ on \mathcal{S} .

If $c^* > 0$ on \mathcal{S} , hence $g^* > 0$, the process G^* can be used to define an invertible time change. Briefly, the time change associated with G^* is the process $\tau = (\tau(i): i \in \mathcal{I})$, where $\tau(i)$ is the \mathcal{A} -stopping time defined by (3.21). Since $G^*(t)$ is (strictly) increasing and absolutely continuous, we have, a.s.,

$$G^*(\tau_1) = i, \quad \tau'(i) = d\tau/di = 1/g^*(\tau_1) \quad \text{if } \tau_1 < \infty, \quad (6.1)$$

i.e. if $i < G^*(\infty)$, and we set $\tau'(i) = 0$ otherwise. Letting $\hat{\mathcal{A}}_i = \mathcal{A}_{\tau(i)}$ we obtain a right continuous filtration $\hat{\mathcal{A}} = (\hat{\mathcal{A}}_i; i \in \mathcal{T})$ and define the σ -algebra $\hat{\mathcal{H}}$ of $\hat{\mathcal{A}}$ -progressive sets, the concepts of $\hat{\mathcal{A}}$ -stopping time, $\hat{\mathcal{A}}$ -martingale, etc. The inverse time change is the process $G^* = (G^*(t); t \in \mathcal{T})$; in particular, each $G^*(t)$ is an $\hat{\mathcal{A}}$ -stopping time and $\hat{\mathcal{A}}_{G^*(t)} = \mathcal{A}_t$, cf [F1] S.5, Lemma. Note that, for fixed ω , the map $t \rightarrow G^*(t) = i$ has an inverse for all $t \geq 0$ defined by $i \rightarrow \tau(i) = t$; however, the map $i \rightarrow \tau(i)$ has an inverse only for $i < G^*(\infty)$.

The transform $\hat{y} = (\hat{y}(i); i \in \mathcal{T})$ of y under τ is defined by setting, for each ω

$$\begin{aligned} \hat{y}(i) &= y(\tau_i) & \text{if } 0 \leq i < G^*(\infty), \\ \hat{y}(i) &= 0 & \text{if } G^*(\infty) \leq i < \infty; \end{aligned} \quad \dots (6.2)$$

this process is $\hat{\mathcal{H}}$ -measurable, in particular $\hat{y}(i)$ is $\hat{\mathcal{A}}_i$ -measurable. The inverse transform $\hat{\hat{y}}$, defined by $\hat{\hat{y}}(t) = \hat{y}[G^*(t)]$, coincides with y , and it may be checked

that, if \hat{y} is altered on a null set of $\hat{\mathcal{H}}$, then $\hat{\hat{y}}$ is altered only on a null set of \mathcal{H} .

For integrals, we have a s.

$$\int_{\tau(i)}^{\tau(I)} y(t)g^*(t)dt = \int_i^I \hat{y}(\Theta)d\Theta, \quad 0 \leq i \leq I < \infty, \quad \dots (6.3)$$

and in particular (5.7) yields

$$D^* = E \int_0^\infty \hat{y}(\Theta)d\Theta = E \int_0^\infty y(t)g^*(t)dt < \infty \quad \dots (6.4)$$

We turn to the

SUPERMARTINGALE PROPERTY FOR SHADOW PRICES.

The process y (suitably altered on null sets) is a supermartingale.

The proof is divided into three parts.

(i) *The average values of $E\hat{y}(i)$ on intervals $[0, h)$, $0 < h < K_0$, are uniformly bounded by D^*/K_0 .*

Choose $h \in (0, K_0)$ and define a plan by setting, for each ω ,

$$\begin{aligned} c(t) &= (K_0/h)c^*(t) & \text{for } 0 \leq t \leq \tau(h), \\ c(t) &= 0 & \text{for } t > \tau(h) \end{aligned} \quad (6.5)$$

Then c is non-negative and locally integrable, and it is progressive since the $\tau(h)$ are \mathcal{A} -stopping times. To check feasibility it is therefore enough, by (4.9), to show that

$$K_0 \geq \int_0^T c(t)e^{-\mathcal{D}(k;t)} dt = \int_0^T g(t)e^{R(t)-\mathcal{D}(k;t)} dt \quad (6.6)$$

for $T \in \mathcal{T}$ as, and in view of (5) it is enough to consider $T = \tau(h)$. Now $c(t) > c^*(t)$ for $t < \tau(h)$ implies $k(t) < k^*(t)$ on this interval – see the proof of Property (ii), Chapter 4(B) – and this in turn implies $\mathcal{D}(k;t) \geq \mathcal{D}(k^*;t) \geq R(t)$ by the concavity of f , see (3.2–4) and (3.7). Since $G^*[\tau(h)] \leq h$, (6) follows from

$$\int_0^{\tau(h)} g(t)e^{R(t)-\mathcal{D}(k;t)} dt \leq \int_0^{\tau(h)} g(t) dt = (K_0/h) \int_0^{\tau(h)} g^*(t) dt \leq K_0. \quad (6.7)$$

Now calculate, from (5.5) and (5),

$$\begin{aligned} D\varphi &= E\left\{ \int_0^{\tau(h)} y(t)g^*(t)[K_0/h-1] dt - \int_{\tau(h)}^{\infty} y(t)g^*(t) dt \right\} \\ &= E\left\{ (K_0/h) \int_0^{\tau(h)} y(t)g^*(t) dt - \int_0^{\infty} y(t)g^*(t) dt \right\}. \end{aligned}$$

On rearranging, noting that $D\varphi \leq 0$ by optimality, and using (3–4) we conclude that

$$(1/h)E \int_0^h \hat{y}(\Theta) d(\Theta) = (1/h)E \int_0^{\tau(h)} y(t)g^*(t) dt \leq D^*/K_0, \quad 0 < h < K_0. \quad (6.8)$$

(ii) \hat{y} is an \mathcal{A} -supermartingale.

Choose $i \geq 0$ and $A \in \mathcal{A}_i$, then I, h, h' and ϵ such that

$$i < i+h \leq I < I+h' \quad \text{and} \quad 0 < \epsilon < h < K_0$$

Now define a plan by setting $g = g^* + \delta g$, $J = J^* + \delta J$ etc with

$$\delta g(t) = \left\{ \begin{array}{ll} -(\epsilon/h)g^*(t) & \tau(i) \leq t < \tau(i+h) \\ 0 & \tau(i+h) \leq t < \tau(I) \\ +(\epsilon/h')g^*(t) & \tau(I) \leq t < \rho \end{array} \right\} \quad (6.9)$$

for $\omega \in A$, where $\rho = \rho(\omega, \epsilon)$ is the smallest solution after $\tau(\omega, i)$ of $J(\omega, t) = J^*(\omega, t)$ if this exists and $\rho = \infty$ otherwise. For $\omega \notin A$, we set $\delta g = 0$ and $\rho = \tau(i)$, so

that ρ is well defined as a stopping time. Also $\delta g(t) = 0$ for $t < \tau(i)$ and for $t > \rho$. Thus g is progressive and $g \geq 0$; also $g(t) \leq g^*(t)$ for $t < \tau(i)$ implies $J(t) \geq J^*(t) \geq 0$ on this interval, and the latter inequalities continue by definition for $t \geq \tau(I)$. It follows that g is feasible. Further, by (9) and the definition of the times τ , we have

$$0 \leq -\delta G(t) \leq -\delta G[\tau(i+h)] \leq \epsilon,$$

and either $\tau(I+h') = \infty$ or $\delta G[\tau(I+h')] = 0$. Since

$$0 \leq \delta J(t) \leq -\delta G(t)$$

by (3.16), it follows that $\rho \leq \tau(I+h')$.

Now consider what happens when $\epsilon \rightarrow 0$. The preceding remarks show that $\delta G(t)$ and $\delta J(t)$ tend to zero uniformly with respect to (ω, t) . We next prove that $\rho(\epsilon) \rightarrow \tau(I+h')$ a.s. for $\omega \in A$. Suppose first that $\tau(I+h') < \infty$, so that $\rho < \tau(I+h')$ and $\delta J(\rho) = 0$. Start with ϵ fixed. By virtue of the mean value theorem and the continuity of f' in κ , one can write (3.15) in the form

$$\begin{aligned} \delta j &= -\delta g + \delta k(r^{\eta_{-I}})e^{-R} = -\delta g + (r^{\eta_{-I}})\delta J, \\ r^{\eta} &= f'(k^* + \eta \delta k), \quad \eta = \eta(\delta k; \omega, t), \quad 0 \leq \eta \leq 1, \end{aligned} \quad \dots(6.10)$$

(where $r^0 = r$). Regarding this as a linear o.d.e. for δJ with $\delta J[\tau(i)] = 0$, solving for $\delta J(T)$ with $T > \tau(i)$ and setting $\delta J(\rho) = 0$, then substituting for δg from (9) and cancelling ϵ one obtains

$$\begin{aligned} (1/h) \int_{\tau(i)}^{\tau(i+h)} g^*(t) e^{-\int_{\tau(i)}^t (r^{\eta_{-I}}) d\Theta} dt \\ = (1/h') \int_{\tau(I)}^{\rho} g^*(t) e^{-\int_{\tau(I)}^t (r^{\eta_{-I}}) d\Theta} dt. \end{aligned} \quad \dots(6.11)$$

Now, when $\epsilon \rightarrow 0$, $\delta J(t) \rightarrow 0$ uniformly, and the same is true of $\delta k(t)$, $r^{\eta}(t) - r(t)$ and the integrals $\int (r^{\eta_{-I}})$ appearing above, because of the bounds (3.2), the continuity of f' and the fact that all intervals of integration are contained in the finite interval $[\tau(i), \tau(I+h')]$. On passing to the limit under the outer integral, it is found that the

left-hand side of (11) tends to

$$(1/h)\{G^*[\tau(i+h)] - G^*[\tau(i)]\} = 1,$$

and a similar calculation on the right shows that we must have $\rho \rightarrow \tau(I+h')$ in order to preserve equality. There remains the possibility that $\tau(I+h') = \infty$. If $\tau(I) = \infty$ also, then $\rho(\epsilon) = \infty$ for all $\epsilon > 0$. If $\tau(I) < \infty$, then $\tau(\Theta) < \infty$ for Θ in some right neighbourhood of I , and the limit on the left of (11) when $\epsilon \rightarrow 0$ is still 1; but the limit on the right cannot be 1 if ρ stays in any interval of the form $[\tau(I), \tau(\Theta))$ with $\Theta < I+h'$ and $\tau(\Theta) < \infty$, so that $\rho \rightarrow \infty$. Thus $\lim \rho(\epsilon) = \tau(I+h')$ in all cases.

Reverting to the main argument, we now evaluate $D\varphi$ from (5.5) to obtain

$$0 \geq D\varphi = \epsilon \int_A dP \left[(1/h) \int_{\tau(i)}^{\tau(i+h)} yg^* dt + (1/h') \int_{\tau(I)}^{\rho} yg^* dt \right], \quad \dots(6.12)$$

then cancel ϵ and let $\epsilon \downarrow 0$; the convergence is dominated because of (5.7), so that in the limit one can replace ρ by $\tau(I+h')$. On transforming the integrals as in (3) and rearranging, this yields

$$\int_A dP \left[(1/h) \int_i^{i+h} \hat{y}(\Theta) d\Theta \right] \geq \int_A dP \left[(1/h') \int_I^{I+h'} \hat{y}(\Theta) d\Theta \right] \quad \dots(6.13)$$

or, writing $\hat{Y}(A, \Theta) = \int_A \hat{y}(\omega, \Theta) dP$,

$$(1/h) \int_i^{i+h} \hat{Y}(A, \Theta) d\Theta \geq (1/h') \int_I^{I+h'} \hat{Y}(A, \Theta) d\Theta. \quad \dots(6.14)$$

For fixed i and $A \in \hat{\mathcal{A}}_i$, these inequalities hold for all I, h and h' chosen as prescribed, so that clearly the left side \uparrow when $h \downarrow 0$ and therefore tends to a limit

$$\begin{aligned} \hat{Y}^0(A, i) &= \lim_{h \downarrow 0} (1/h) \int_i^{i+h} \hat{Y}(A, \Theta) d\Theta \\ &= \sup_{h > 0} (1/h) \int_i^{i+h} \hat{Y}(A, \Theta) d\Theta, \end{aligned} \quad \dots(6.15)$$

and on letting $h' \downarrow 0$ also one has

$$\hat{Y}^0(A, i) \geq \hat{Y}^0(A, I), \quad A \in \hat{\mathcal{A}}_i, \quad i \leq I. \quad \dots(6.16)$$

In particular, on choosing $A = \Omega$, $\hat{Y}(\Omega, \Theta) = E\hat{y}(\Theta)$, it is seen that the average

$(1/h) \int_0^h E\hat{y}(\Theta)d\Theta \uparrow$ when $h \downarrow 0$, and so is bounded above by its limit, i.e. by

$\hat{Y}^0(\Omega, 0)$, which by part (i) above does not exceed the finite number D^*/K_0 . It follows that $E\hat{y}(i) \leq D^*/K_0$ for almost all $i \geq 0$, while the same number defines an upper bound for all values of the indefinite integral

$$(1/h) \int_i^{i+h} \hat{Y}(A, \Theta)d\Theta \quad A \in \hat{\mathcal{A}}_i, \quad i \geq 0, \quad h > 0. \quad (6.17)$$

Now, for fixed i , (17) defines for each $h > 0$ a P -continuous, non-negative measure on $\hat{\mathcal{A}}_i$ bounded by D^*/K_0 , and on letting $h \downarrow 0$ it follows from the existence of the limit (15) that $\hat{Y}^0(A, i)$ has the same properties (Vitali-Hahn-Saks Theorem,

[DuS] III 7.4). Consequently $\hat{Y}^0(A, i)$ is the indefinite P -integral on $\hat{\mathcal{A}}_i$ of an a.s. uniquely defined and non-negative random variable $\hat{y}^0(\omega, i)$, with

$E\hat{y}^0(i) = \hat{Y}^0(\Omega, i) \leq D^*/K_0$, (Radon-Nikodym Theorem, [DuS] III.10.2) It follows

from these facts and (16) that the process $\hat{y}^0 = (\hat{y}^0(i); i \in \mathcal{T})$ is an $\hat{\mathcal{A}}$ -supermartingale

Moreover (15) shows that the non-increasing function $E\hat{y}^0(i)$ is right continuous, so

that one can choose \hat{y}^0 with sample functions which are right continuous with left

limits ('corlol'), see [DM] VI.4; this also implies that \hat{y}^0 is progressive. Note that

$\hat{y}^0(0)$ is a.s. equal to a constant, since we have assumed that $\mathcal{A}_0 = \hat{\mathcal{A}}_0$ is generated only by the P -null sets of Ω .

It remains to verify that $\hat{y}^0 = \hat{y}$ a.e. Briefly, define an $\hat{\mathcal{H}}$ -measurable process \hat{y}_0 by

$$\hat{y}_0(\omega, i) = \liminf_{h \downarrow 0} (1/h) \int_i^{i+h} \hat{y}(\omega, \Theta)d\Theta$$

and write $\hat{Y}_0(A, i) = \int_A \hat{y}_0(i)dP$ for $A \in \hat{\mathcal{A}}_i$. By Fatou's Lemma and (15) we have

$\hat{Y}_0(A, i) \leq \hat{Y}^0(A, i)$ for $A \in \hat{\mathcal{A}}_i$, hence $\hat{y}_0(\omega, i) \leq \hat{y}^0(\omega, i)$ a.s. for each $i \geq 0$, so that

$\hat{y}_0 \leq \hat{y}^0$ a.e. On the other hand, it follows from a standard property of the Lebesgue

integral that, for fixed ω , $\hat{y}_0(\omega, i) = \hat{y}(\omega, i)$ for almost all $i \geq 0$, hence $\hat{y}_0 = \hat{y}$ a.e.,

and the same property applied to (15) with $A = \Omega$ gives $E\hat{y}_0(i) = E\hat{y}(i)$

for a.a. $i \geq 0$. Thus $0 \leq \hat{y} \leq \hat{y}^0$ a.e., and the product integrals of \hat{y} and \hat{y}^0 agree on \mathcal{S} , implying $\hat{y} = \hat{y}^0$ a.e. We may therefore replace \hat{y} by \hat{y}^0 , and henceforth denote the latter process by \hat{y} .

(iii) y is an \mathcal{A} -supermartingale.

The inverse image under τ of the new \hat{y} is now called y ; clearly it is \mathcal{H} -measurable and similar to the original y , and inherits from \hat{y} the property of being corlol. Since $G^*(t)$ is an $\hat{\mathcal{A}}$ -stopping time and $\mathcal{A}_{G^*(t)} = \mathcal{A}_t$, an application of the Stopping Theorem, [DM] VI.10, to \hat{y} yields $\hat{y}[G^*(t)] \geq E[\hat{y}[G^*(T)]/\mathcal{A}_t]$ a.s. for $t \leq T$, or equivalently $y(t) \geq E[y(T)/\mathcal{A}_t]$. In particular, $y(0) \geq E y(t)$ for all t , and $y(0) = \hat{y}[G^*(0)] = \hat{y}(0) \leq D^*/K_0$, proving integrability. This completes the proof that y is an \mathcal{A} -supermartingale. In conclusion we record some properties of y . First, $y(0)$ is a.s. constant. Second, the a.s. limit $y(\infty)$ exists by martingale convergence, but may take the value zero. Third, $y(t) > 0$ on \mathcal{S} a.s. To prove the last point, let φ be the first arrival time of y at the level zero and note that, since y is a right continuous, non-negative supermartingale, $y(t) = 0$ for $t \geq \varphi$ a.s., see [DM] VI.17; but $y = 0$ implies $c^* = \infty$, so that $\varphi < \infty$ with positive probability would contradict feasibility. ||

REMARK. It is an inconvenient feature of this argument that, for technical reasons, we want the supermartingale $y(t) = u'[c^*(t);t]q(t)e^{R(t)}$ to be corlol (perhaps after changes on null sets), yet consumption plans $c(t)$ are not required to be corlol, nor are processes of the form $u'[c(t);t]$ or $q(t)$. However, some mild additional assumptions about u' and q allow (c^*,k^*) to be replaced by a plan (\hat{c}^*,k^*) satisfying $\hat{c}^* = c^*$ a.e., which is still optimal and such that \hat{c}^* , $u'[\hat{c}^*(t);t]$ and $q(t)$ are all corlol.

Specifically, suppose that

for all ω , $u'(c; \omega, t)$ is jointly continuous in c and corlol in t ,¹ while $q(\omega, t)$ is corlol in t . Also, for all (ω, t) , $u'(c; \omega, t)$ is strictly decreasing in c , with

$$u'(\infty; \omega, t) = 0. \quad \dots (6.18)$$

These conditions ensure the existence, for every (ω, t) , of a function $\zeta = (u')^{-1}$ inverse to u' , i.e. such that $\zeta[u'(c); \omega, t] = c$ for $c \geq 0$, and $\zeta[u'; \omega, t]$ is jointly continuous in u' and corlol in t . Now, given the process \hat{y} defined in the preceding proof, we may (omitting ω) define a process \hat{c}^* by

$$\hat{c}^*(t) = \zeta[\hat{y}(t)e^{R(t)}/q(t); t], \text{ or } u'[\hat{c}^*(t); t]q(t)e^{R(t)} = \hat{y}(t).$$

Taking into account (18), \hat{c}^* is corlol since \hat{y} , R and q have this property, and $\hat{c}^* = c^*$ a.e. because $\hat{y} = y$ a.e. It follows that the solution \hat{k}^* of (1.1) obtained on replacing c^* by \hat{c}^* coincides with k^* on \mathcal{S} a.s., so that the process \hat{R} defined from \hat{k}^* as in (3.7) coincides with R . Finally, it is clear that $\varphi(\hat{c}^*) = \varphi(c^*)$, so that \hat{c}^* is optimal.

Incidentally, these arguments show why we work with equivalence classes of 'similar' (progressive) processes y and c rather than the more usual classes defined 'up to modification', see Chapter 3, fn.2 and Chapter 4, fn.1. An alternative approach, which avoids the concept of similarity and somewhat simplifies the proof of the supermartingale property, would be to restrict the choice of c from the outset to corlol elements of \mathcal{C} , and to make assumptions like (18) which ensure that $u'[c(t); t]q(t)e^{R(t)}$ is corlol for any such c . It then follows from (13) and the bound on $(1/h)\int_0^T E\hat{y}(\theta)d\theta$ that, for small $h \in (0, K_0)$, the process $\hat{y}^h(i) = (1/h)\int_0^i \hat{y}(\theta)d\theta$ is a corlol supermartingale. Choosing a sequence $h = h_n \downarrow 0$, it then follows as in (15) that, for given i , $\hat{y}^h(i) \uparrow$ a.s., so that (\hat{y}^h) is a non-decreasing sequence of corlol supermartingales, hence by [DM] VI.18 the limit \hat{y}^+ is corlol. Moreover, it follows as in the

¹ i.e., for given (\bar{c}, \bar{t}) , $u'(\bar{c}; \omega, \bar{t})$ and $u'(\bar{c}; \omega, \bar{t}-)$ exist as limits of $u'(c; \omega, t)$ as $c \rightarrow \bar{c}$, $t \downarrow \bar{t}$ and as $c \rightarrow \bar{c}$, $t \uparrow \bar{t}$ respectively. For brevity we omit reservations about one-sided continuity when c or t is 0 or ∞ .

remarks leading to (17) that \hat{y}^+ is integrable and so is a corlol supermartingale, (and of course it coincides with a corlol version of \hat{y}^0 as defined above). Finally, $\hat{y}^+(\omega, i) = \hat{y}(\omega, i)$ for almost all i , a.s., by the property of the Lebesgue integral cited above; but since both processes are corlol they are indistinguishable. The reason why this approach has not been adopted here is that restricting c to be corlol complicates the discussion of feasibility, in particular Property (B)(iv) of the Feasibility Lemma fails, and the rather simple proof of the existence of an optimum in Chapter 9 is invalidated.

In this Chapter we complete the discussion of the Conditions for Optimality in 'Price Time' form begun in Chapter 3(B). A Corollary is obtained about the form of the transversality condition when the shadow price process is a true martingale

First, recall that (any) corollary, non-negative (c.n.n.) supermartingale, say y , admits a unique 'Doob' decomposition $y = M - V$, where V is a predictable, non-decreasing c.n.n. process with $V(0) = 0$ and $EV(\infty) \leq Ey(0)$, and M is a c.n.n. local martingale, see [DM] VII.8,12,13 or [El] 8.24. It can be shown that the process M is reduced by any sequence of stopping times of the form

$$\rho_i = \rho(\omega, i) = i \wedge \inf\{t: y(\omega, t) > i\} \quad \text{with } 0 \leq i \uparrow \infty^1$$

The times $\rho(i)$ were introduced as price times associated with the star plan in (1.11) and (3.27)

OPTIMALITY THEOREM IN PRICE TIME FORM.

A plan (c^*, k^*) satisfying the Strong Finite Welfare Condition (3.34) or (5.6) is optimal iff, for any sequence of price times (ρ_i) with $i \uparrow \infty$, y is a local martingale reduced by (ρ_i) and $\lim_i E\{y(\rho_i)J^*(\rho_i)\} = 0$.

¹ Consider the decomposition $y = M - V$ of a c.n.n. supermartingale y , with V predictable non-decreasing c.n.n. satisfying $V(0) = 0$ and M a c.n.n. local martingale. For given $i \geq 0$, let

$$\xi_i = i \wedge \inf\{t: y_t \geq i\}, \quad \rho_i = i \wedge \inf\{t: y_t > i\}.$$

It is usually shown, either as part of the proof of the decomposition or as a corollary, that M is reduced by a sequence of times (ξ_i) with $0 \leq i = i_n \uparrow \infty$, see [Mey2] IV.4 bis, [DM] VII.12-13. However, the assertion is also valid for a sequence (ρ_i) , provided that the filtration \mathcal{A} is right continuous (so that the ρ_i are stopping times). Indeed, arguing as in [DM] VII.13, we note that $\rho_i \uparrow \infty$ with i and that the stopped supermartingale $(y(t \wedge \rho_i): t \in \mathcal{T})$ is dominated by the random variable $i \vee y(\rho_i)$. It follows from the Stopping Theorem that this variable is integrable, hence by the Decomposition Theorem (for supermartingales of class (D)) there is a decomposition $y(t \wedge \rho_i) = M^i(t) - V^i(t)$, $t \in \mathcal{T}$, with V^i predictable, non-decreasing c.n.n. satisfying $V^i(0) = 0$ and M^i a uniformly integrable martingale. The uniqueness of the given decomposition $y = M - V$ then implies that $M^i(t) = M(t)$, $V^i(t) = V(t)$ for $0 \leq t \leq \rho_i$, hence that M is a local martingale reduced by the sequence (ρ_i) .

The family (ρ_i) is convenient here because it is right continuous and so defines a time change in the usual sense.

Sufficiency follows from the results of Chapter 3(B). We consider necessity

(i) *Local Martingale Condition*. From previous work we know that, if the star plan is optimal and satisfies (3.34), then, up to null sets, $c^* > 0$, $k^* > 0$ and y is a (strictly) positive corlol *supermartingale* with Doob decomposition $y = M - V$. We need to show that y (and not just M) is reduced by any sequence (ρ_i) with $i \uparrow \infty$. Let (c, k) be any plan such that $\delta k(t) < 0$ on \mathcal{T} a.s., or equivalently $\delta J(t) < 0$, and consider the following calculation:

$$\begin{aligned}
 D\Phi &= -E \int_0^\infty y \cdot \delta j \cdot dt = -\lim_{i \uparrow \infty} E \int_0^{\rho(i)} y \cdot \delta j \cdot dt \\
 &= \lim_i E \int_0^{\rho(i)} (V-M) d\delta J \\
 &= \lim_i E \left\{ \int_0^{\rho(i)} V d\delta J - M(\rho_i) \delta J(\rho_i) \right\} \\
 &= \lim_i E \left\{ - \int_0^{\rho(i)} \delta J(t) dV(t) + [V(\rho_i) - M(\rho_i)] \delta J(\rho_i) \right\}. \quad \dots (7.1)
 \end{aligned}$$

The first line starts with (5.5); the passage to the limit is justified because $\rho(i) \uparrow \infty$ a.s. and the product integral defining $D\Phi$ converges when (c^*, k^*) is optimal — see Chapter 5. The second line replaces $\delta j \cdot dt$ by $d\delta J$ and y by its decomposition. The third line uses the formula for integration of a martingale with respect to a non-increasing process, [E1] 7.16, noting that δJ may be written as the difference between two such (finite) processes on the bounded interval $[0, \rho_i]$, while M is uniformly integrable on this interval; also $\delta J(0) = 0$. The fourth line is obtained on integrating by parts. Now $\delta J < 0$, $dV \geq 0$ and $V - M = -y < 0$, yielding $D\Phi \geq 0$, hence $D\Phi = 0$ by optimality. It follows from Fatou's Lemma that the random variables in braces in the fourth line tend to zero as $i \uparrow \infty$, $\rho(i) \uparrow \infty$, so that $-\int_0^\infty \delta J(t) dV(t) = 0$; but then $-\delta J > 0$, $dV \geq 0$ and $dV(0) = 0$ (right continuity) imply that the process V vanishes identically. Thus $y = M$, a local martingale reduced by (ρ_i) ; this leaves

$$D\Phi = 0 = \lim_i E\{y(\rho_i)[J^*(\rho_i) - J(\rho_i)]\}. \quad (7.2)$$

(ii) *Transversality*. It remains to get rid of the term $J(\rho_i)$ in (2) and so obtain the desired 'condition at infinity'

$$0 = \lim_{i \uparrow \omega} E\{y(\rho_i)J^*(\rho_i)\}. \quad (7.3)$$

This step is trivial in a discrete-time model, where one can choose for J the plan which consumes all capital at $t = 0$, and could be made so in the present model by allowing free instantaneous disposal (as distinct from instantaneous consumption) of a 'lump' of capital. Without such an assumption it is not in general possible to construct a feasible J such that $J(\rho_i) = 0$ a.s. for some fixed i . We therefore need a limiting argument.

The first step is to construct explicitly a family of feasible plans $(J^\theta(\omega, t): \theta \geq 0)$ such that $0 < J^\theta \leq J^*$, $J^\theta \downarrow 0$ as $\theta \rightarrow \infty$ and $dJ^\theta/dt < 0$. We show that, if $\alpha = \alpha(\omega, t)$ is any progressive, positive and pathwise locally integrable process, then

$$J^\theta(T) = J^*(T)e^{-\theta \int_0^T \alpha(t) dt} \quad (7.4)$$

with $\theta > 0$ defines a feasible plan. Indeed, for fixed θ and $J = J^\theta$ we have (abridging the notation, see Chapter 3, fn. 5)

$$0 < J < J^* \quad \text{and} \quad j/J = j^*/J^* - \theta\alpha \quad \text{for } T > 0;$$

using the latter condition together with (3.11), $J = ke^{-R}$ and $\vartheta(k) = f(k)/k$ yields

$$g/J - g^*/J^* = -j/J - j^*/J^* + \vartheta(k) - \vartheta(k^*) = \theta\alpha + \vartheta(k) - \vartheta(k^*) \geq \theta\alpha, \quad (7.5)$$

the inequality being due to $k < k^*$. Since $g^* > 0$ and $\theta\alpha > 0$, it follows that $g > 0$, so that the plan is feasible.

In particular, we may set $\alpha = \vartheta(k^*) - r + 1$ — or, explicitly

$$\alpha(t) = \vartheta[k^*(t); t] - r(t) + 1; \quad (7.6)$$

then $\alpha \geq 1$ since $\vartheta(k^*) = f(k^*)/k^* \geq f'(k^*) = r$. From (4) and (6) we have

$$(d/dt) \ln J^\theta = (d/dt) \ln J^* - \theta\alpha = \dot{k}^*/k^* - r - \theta[\vartheta(k^*) - r + 1],$$

and since $k^* < f(k^*)$, hence $k^*/k^* \leq \mathfrak{D}(k^*)$, we obtain, for $\theta \geq 1$,

$$(d|dt)J^\theta(t) < -\theta, \quad J^\theta(t) \leq K_0 e^{-\theta t} \quad \text{for } t \geq 0 \text{ a.s. if } \theta \geq 1. \quad (7.7)$$

Now replace J by J^θ in (2) and note that, by part (i) of the proof, the resulting equation must hold for each $\theta \geq 1$. Thus (3) will follow if it is shown that, for $\theta \uparrow \infty$, $i \uparrow \infty$,

$$\lim_{\theta} \lim_i E\{y(\varphi_i) J^\theta(\varphi_i)\} = 0. \quad (7.8)$$

Turning to the second step, we regard the stopping times $\varphi_i = \varphi(\omega, i)$ as defined for all real $i \geq 0$ and consider the family (φ_i) as a process. The sample functions are non-decreasing and right continuous with

$$\varphi_i = 0 \text{ a.s. for } i \leq y_0, \quad 0 \leq \varphi_i \leq i \text{ for } y_0 < i < \infty,$$

and an a.s. limit $\varphi_\infty = \infty$. Let

$$\tilde{\mathcal{A}}_i = \mathcal{A}_{\varphi(i)},$$

so that the family $\tilde{\mathcal{A}} = (\tilde{\mathcal{A}}_i)$ is a right continuous filtration, and define

$$\tilde{y}(i) = y(\varphi_i)$$

and (for each fixed $\theta \geq 1$)

$$\tilde{J}^\theta(i) = J^\theta(\varphi_i).$$

Then \tilde{y} and \tilde{J}^θ are positive, right continuous processes on $[0, \infty)$ adapted to $\tilde{\mathcal{A}}$, and it follows from part (i) above and the Stopping Theorem that \tilde{y} is an $\tilde{\mathcal{A}}$ -martingale satisfying $E\tilde{y}(i) = y(0)$, while \tilde{J}^θ is a non-increasing process bounded above by K_0 .

We may define the a.s. limits $\tilde{y}(\infty) = y(\infty)$ and $\tilde{J}^\theta(\infty)$, where obviously $\tilde{J}^\theta(\infty) = 0$ by (7). It follows that the product

$$h^\theta(i) = \tilde{y}(i) \cdot \tilde{J}^\theta(i)$$

defines a sample right continuous (hence corlol), positive $\tilde{\mathcal{A}}$ -supermartingale with $h^\theta(\infty) = 0$. Write

$$H^\theta(i) = E h^\theta(i), \quad 0 \leq i < \infty, \quad H^\theta(\infty-) = \lim_{i \uparrow \infty} H^\theta(i), \quad (7.9)$$

noting that $H^\theta(\infty-)$ exists as a limit because $H^\theta(\cdot)$ is non-increasing, and that $H^\theta(\cdot)$ is right continuous for $i < \infty$ because h^θ is sample right continuous, see

[Mey1] VI.4. Thus

$$y(0)K_0 \geq H^\theta(i) \geq H^\theta(\infty-) \geq H^\theta(\infty) = 0, \quad \dots(7.10)$$

bearing in mind that $0 \leq h^\theta(i) < \tilde{y}(i)K_0$ and $E\tilde{y}(i) = y(0)$ for $0 \leq i < \infty$.

Now the third step. Restrict (ρ_i) to some interval $[i_0, \infty)$ with $i_0 > y(0)$, so that the ρ_i are a.s. positive. On letting $\theta \uparrow \infty$, we have $h^\theta(i) \downarrow 0$ for each $i \geq i_0$, so that (by dominated convergence)

$$\lim_{\theta} H^\theta(i) = 0, \quad i_0 \leq i < \infty;$$

this assertion also holds for $i = \infty$ because of (10). Obviously, the left-hand limits $H^\theta(i-) \downarrow 0$ also as $\theta \uparrow \infty$. Thus, for any sequence $1 \leq \theta \uparrow \infty$, the functions $H^\theta(i)$ are a sequence of positive convex functions on $[i_0, \infty]$, decreasing simply to zero together with the left limits $H^\theta(i-)$; it then follows from a generalisation of Dini's Theorem, [DM] VII.2 (Lemma), that the convergence is uniform. This uniformity justifies the following interchange of limits, which completes the proof of the Theorem:

$$0 = \lim_i \lim_{\theta} H^\theta(i) = \lim_i \lim_n H^\theta(i) \parallel$$

REMARK I. The preceding proof implies the following:

If (c^*, k^*) is optimal and satisfies (3.34), then \tilde{y} is an $\tilde{\mathcal{A}}$ -martingale and

$$\lim_{i \uparrow \infty} E\{\tilde{y}(i)\tilde{J}^*(i)\} = 0. \quad \dots(7.11)$$

Stated informally, this says that the price times (ρ_i) define a time change such that the transformed shadow price process is a true martingale and the transformed transversality condition assumes a form analogous to (1.10); cf. the discussion in Chapter 1. Note however that the conditions in (11) are not sufficient for optimality, since they specify properties of y only at price times. The additional condition that y is an \mathcal{A} -supermartingale is enough to yield sufficiency, since then the stopped process $(y(t \wedge \rho_i); t \in \mathcal{T})$ is a uniformly integrable martingale for each i .

REMARK II. The proof of necessity given above remains valid if the price times (ρ_i) are replaced by any right continuous, non-decreasing family (χ_i) of *finite* stopping times, defined for $0 \leq i < \infty$, such that $\chi(\omega, \cdot) \uparrow \infty$ a.s. with i , each $\chi(\cdot, i)$ reduces y , and

$\chi(\omega, i) > 0$ a.s. on some interval $[i_0, \infty)$. (Cf. the note on time change, Chapter 1, fn. 4.)

In particular, suppose that it is known that y is in fact a true martingale. Then Part (i) of the argument, with $V \equiv 0$ and with \mathcal{F}_i and i both replaced by any sequence $T = T_n \uparrow \infty$, yields

$$D\Phi = 0 = \lim_{T \uparrow \infty} E\{y(T)[J^*(T) - J(T)]\}$$

in place of (2). Part (ii) can be simplified because the second step is trivial – briefly, one can take

$$\chi(i) = i = T, \quad \tilde{\mathcal{A}}_i = \mathcal{A}_T, \quad \tilde{y}(i) = y(T), \quad \tilde{J}^\theta(i) = J^\theta(T) \quad \text{etc.} \quad -$$

and then the third step yields (7.3) in the form

$$0 = \lim_{T \uparrow \infty} E\{y(T)J^*(T)\}, \quad \dots(7.12)$$

in agreement with (1.10) or (3.42). This is an important conclusion, which we state as a

COROLLARY

If an optimal plan satisfies the Strong Finite Welfare Condition and is such that y is a true martingale, then transversality holds in the form (1.10) or (3.42).

We return to the end of Chapter 6, and now complete the discussion of the Conditions for Optimality in 'Depletion Time' form. A condition for y to be a true martingale is obtained as a Corollary. The depletion times $\nu(i)$ were introduced in (1.12) and (3.23), see also (3.36–40).

OPTIMALITY THEOREM IN DEPLETION TIME FORM.

A plan (c^*, k^*) satisfying the Strong Finite Welfare Condition (3.34) or (5.6) is optimal iff

- (i) for some sequence of depletion times (ν_i) with $i \uparrow K_0$, y is a local martingale reduced by (ν_i) , and
- (ii) a.s., $y(\omega, \infty) = 0$ if $\Gamma(\omega, \infty) < K_0$, i.e. if $\inf_{t \in \mathcal{T}} J^*(\omega, t) > 0$

Sufficiency follows from the results of Chapter 3(B). We consider necessity. Once again, we know that, if the star plan is optimal and satisfies (3.34), then, up to null sets, $c^* > 0$, $k^* > 0$, and y is a positive coroll supermartingale. We now seek to show that y is a local martingale reduced by any sequence of depletion times $\nu_i, i \uparrow K_0$ — see (3.23) and (3.36) — and further that the transversality condition holds in one of the equivalent forms (3.37), (3.38) and (3.40). A difficulty arises from the fact that, unlike the saving model, the sample paths of the discounted capital plan J^* need not be everywhere decreasing. To deal with this, we first prove a Depletion Lemma which shows that 'humps' of J^* contribute nothing to the integral $E \int y j^* dt$, i.e. to the total value of investment in discounted units, and can therefore be neglected when calculating certain directional derivatives; this Lemma may be of some independent economic interest. The proof of the sub-martingale inequality for the stopped process $y(t \wedge \nu_1) = y^i(t)$, and hence of the local martingale property for y , is then accomplished by constructing suitable alternative plans $J > 0$ which start and stop at given

depletion times. The transversality condition follows almost immediately.

(i) *Properties of Depletion Times.* The stopping time ν_i is by definition the first (strict) upcrossing time of the level $i \in [0, K_0)$ by the process $K_0 - J^*$, or equivalently by the non-decreasing process Γ which levels off the 'troughs' of $K_0 - J^*$ – see (3.23), also Figure 1. On writing $\check{\mathcal{A}}_i = \mathcal{A}_{\nu(i)}$, it is found that the family $\check{\mathcal{A}} = (\check{\mathcal{A}}_i; 0 \leq i < K_0)$ defines a right continuous filtration.

For fixed ω , let

$$B(\omega) = \{t \in \mathcal{T}: t = \nu(\omega, i) \text{ for some } i \in [0, K_0)\} \quad (8.1)$$

i.e. $B(\omega)$ comprises those times t which are depletion times. Bearing in mind that the sample functions $K_0 - J^*$ are absolutely continuous, it is clear that $K_0 - J^* = \Gamma$ on B , more precisely that $K_0 - J^*(\nu_i) = \Gamma(\nu_i) = i$ if $\nu_i < \infty$ – see (3.24) – and further that $K_0 - J^*(t) = \Gamma(t) > i$ for t in some right neighbourhood of ν_i , while $K_0 - J^*(t) \leq \Gamma(t) \leq i$ for all $t \leq \nu_i$; we say for short that each ν_i is a point of increase of $K_0 - J^*$ and of Γ . It further follows that each t in some right neighbourhood of ν_i is again a first upcrossing time of some level $I > i$ and so belongs to B . Thus $B(\omega)$ is the union of a finite or infinite sequence of disjoint half-open intervals of positive length of the form

$$[\alpha_n(\omega), \beta_n(\omega)), \quad n = 1, 2, \dots, \quad (8.2)$$

where $\beta_n = \infty$ may occur. In particular, $\alpha_1 = \nu(0) \geq 0$, and a strict inequality cannot be ruled out. The set $\{(\omega, t): t \in B(\omega)\}$ is obviously progressive. Further, since for fixed ω we have $\Gamma = K_0 - J^*$ on B while Γ is constant on each complementary interval $[\beta_n, \alpha_{n+1})$, it follows that for a.a. $t \in \mathcal{T}$ the (right) derivative $\dot{\Gamma}(t) = d\Gamma(t)/dt$ is defined and satisfies

$$\dot{\Gamma}(t) = -j^*(t) \text{ for } t \in B, \quad \dot{\Gamma}(t) = 0 \text{ for } t \notin B, \quad (8.3)$$

and these equalities may be assumed to hold for all t ; note that $\dot{\Gamma}(t) = -j^*(t) > 0$ for a.a. $t \in B$.

(ii) *Depletion Lemma.*

LEMMA. Let $0 \leq I < i < K_0$ and $A \in \mathcal{A}_I$, and let $\Delta(\omega, t) = \Delta(t)$ be the indicator function of the set $\mathcal{T}-B(\omega)$; then

$$\int_A \left[\int_{\nu(I)}^{\nu(i)} y(t) j^*(t) \Delta(t) dt \right] dP = 0, \quad (8.4)$$

$$E \left[\int_0^{\nu(0)} y(t) j^*(t) \Delta(t) dt \right] = 0. \quad (8.5)$$

For brevity we shall prove (4), the argument for (5) being similar. Actually, the Lemma would remain true if ν_I were replaced by any stopping time $\chi \leq \nu_i$ and A by a set in \mathcal{A}_χ such that $J^*[\omega, \chi(\omega)] = I$ a.s. for $\omega \in A$; then (5) would be a special case with $i = I = 0$ and \mathcal{A}_I the trivial σ -algebra at $t = 0$.

PROOF. (a) We may assume $\nu_I < \infty$ for $\omega \in A$. We first show that the left side of (4) is not positive. Construct a variation $\delta J = J - J^*$ by setting

$$\delta j(t) = -j^*(t), \quad J(t) = K_0 - \Gamma(t) \quad \text{for } t \in B^c \cap [\nu_I, \nu_i], \quad \omega \in A, \quad (8.6)$$

and $\delta j = 0$, $J = J^*$ otherwise (i.e. J cuts off any humps of J^* which occur between ν_I and ν_i , but otherwise coincides with J^*) To check feasibility, it is enough to show that $J \geq 0$, $g \geq 0$ when $J \neq J^*$. Write $\vartheta = \vartheta_t = f[k_t; t]/k_t$, $k_t = J_t e^{R(t)}$ for short, also ϑ^* when $J = J^*$, cf (3.6a), (3.8) and Chapter 3 fn. 5. If $[\beta, \alpha)$ is any interval of B^c contained in $[\nu_I, \nu_i)$, then $J(t) = J^*(\beta) > 0$ on this interval; also $j(t) = j = 0$ implies $g = [\vartheta - r]J$ by (3.11), and since $J \leq J^*$ we have $\vartheta \geq \vartheta^* \geq r$, hence $g \geq 0$. On computing $D\Phi$ from (5.5) we have the expression on the left of (4), which by optimality must be ≤ 0 .

(b) To establish the opposite inequality, one would like to set $\delta j(t) = \epsilon j^*(t)$ with $\epsilon > 0$ for $t \in B^c \cap [\nu_I, \nu_i)$ if $\omega \in A$, and $\delta j = 0$ otherwise (i.e. J would blow up any humps of J^* between ν_I and ν_i but otherwise coincide with J^*) Unfortunately the corresponding value of g given by $-j + (\vartheta - r)J$ could become negative, and we therefore modify the definition by setting $g = 0$ whenever this would happen. More

precisely, we define J on each component interval $[\beta, \alpha)$ of $B^c \cap [\nu_I, \nu_i)$ as the solution of the o.d.e.

$$j_t = (1+\epsilon)j_t^* \quad \text{if } -(1+\epsilon)j_t^* + (\partial_t - r_t)J_t \geq 0 \quad \dots(8.7a)$$

$$j_t = (\partial_t - r_t)J_t \quad \text{otherwise,} \quad \dots(8.7b)$$

with the initial condition $J(\beta) = J^*(\beta)$; note that in (7b) we have $g = 0$,

$j < (1+\epsilon)j^*$ The existence and uniqueness of the solution $J(t)$ on $[\beta, \alpha)$, indeed on $[\beta, \alpha]$, is fairly obvious. Clearly the solution curve cannot ever be above the path

$$L(t) = J^*(t) + \epsilon[J^*(t) - J^*(\beta)]$$

defined by setting $j = (1+\epsilon)j^*$ on the whole interval – see Figure 2. On the other hand, $J(t)$ cannot ever be below $J^*(t)$, since $J < J^*$ implies $\partial \geq \partial^* \geq r$, hence $j/J \geq \partial^* - r > j^*/J^*$ under (7b) and $j/J = (1+\epsilon)j^*/J > (1+\epsilon)j^*/J^*$ under (7a); thus, to the right of a point where the curves J and J^* separate, we must have $J > J^*$.

The same argument actually shows that we cannot have $J = J^*$ on any interval contained in (β, α) . Consequently

$$J^*(t) < J(t) \leq (1+\epsilon)J^*(t) - \epsilon J^*(\beta), \quad \beta < t < \alpha,$$

with $J(\beta) = J^*(\beta)$, and if $\alpha < \infty$ then $J^*(\alpha) = J^*(\beta)$, which implies $J(\alpha) = J^*(\alpha)$ by continuity. In short, the curves of J and J^* separate at β , J lies above J^* but not above L throughout (β, α) , and rejoins J^* at α if $\alpha < \infty$.

Clearly the variation is feasible since $g \geq 0$ and $J > 0$ by construction, and we have $\delta j \leq \epsilon j^*$ on each interval $[\beta, \alpha)$, hence on $B^c \cap [\nu_I, \nu_i)$; it follows by optimality and (5.5) that

$$0 \geq D\Phi = - \int_A \left[\int_{\nu(I)}^{\nu(i)} y \cdot \delta j \cdot \Delta dt \right] dP \geq -\epsilon \int_A \left[\int_{\nu(I)}^{\nu(i)} y j^* \Delta dt \right] dP. \quad \dots(8.8)$$

COROLLARY. Let $0 \leq I < i < K_0$. Then, for $A \in \mathcal{A}_I^c$,

$$\int_A \left[\int_{\nu(I)}^{\nu(i)} y(t) j^*(t) dt \right] dP \leq 0, \quad \dots(8.9)$$

the inequality being strict unless $\nu(I) = \infty$ a.s. on A .

PROOF of (9). This inequality follows from the preceding Lemma (which shows that the contribution to the time integral made by B^c vanishes) and from the facts that $j^* < 0$ on the interior of B while $y(t) > 0$ ||

(iii) *Local martingale condition.* For each fixed $i \in [0, K_0)$, we define the stopped process

$$y^i = (y^i(t); t \in \mathcal{T}), \quad y^i(t) = y(t \wedge \nu_i) = y[\omega, t \wedge \nu_i(\omega)]; \quad \dots (8.10)$$

we have to show that y^i is a uniformly integrable martingale. Since each $t \wedge \nu_i$ is an \mathcal{A} -stopping time and y is a positive coroll supermartingale, it follows by the Stopping Theorem that y^i has the same properties, [DM] VI 12, and by the Convergence

Theorem, [DM] VI.6, $y^i(t)$ converges a.s. to a finite limiting variable $y^i(\infty) = y(\nu_i-)$.

The supermartingale inequality and predictability of ν_i then yield

$$y(0) = y^i(0) \geq E y^i(t) \geq E y^i(\infty) = E y(\nu_i-) \geq E y(\nu_i), \quad t \in \mathcal{T}. \quad \dots (8.11)$$

In the paragraphs which follow we shall show that

$$y(0) \leq E y(\nu_i); \quad \dots (8.12)$$

this will imply first that y^i is a martingale since $E y^i(t) = y(0)$ for all $t \in \mathcal{T}$, and second that the $y^i(t)$ are uniformly integrable because they are positive and we have

$$y^i(t) \rightarrow y^i(\infty) \text{ a.s. and } E y^i(t) \rightarrow E y^i(\infty), \quad t \rightarrow \infty,$$

see [Mey1] II 21.

PROOF OF (12). For fixed $i \in [0, K_0)$, choose numbers $\theta \in (0, K_0 - i)$ and $h > 0$ and define a new plan $g = g^* + \delta g$, $J = J^* + \delta J$ in three phases (a), (b), (c) – see Figure 3.

(a) The first phase is defined for $0 \leq t < h \wedge \nu_i$; speaking informally, we set $\delta j = -\theta/h$ provided that this leaves $g(t) \geq 0$ and set $g(t) = 0$ otherwise. More precisely, for each $\omega \in \Omega$, $J(t)$ is defined up to $h \wedge \nu_i$ as the solution of the o.d.e.

$$j_t = j_t^* - \theta/h \quad \text{if } -j_t^* + \theta/h + (\partial_t - \mathcal{I}_t)J_t \geq 0 \quad \dots (8.13a)$$

$$j_t = (\partial_t - \mathcal{I}_t)J_t \quad \text{otherwise} \quad (8.13b)$$

with initial condition $J(0) = K_0$. The equation $g = -j + (\partial - \mathcal{I})J$ shows that $g \geq 0$, and clearly $j \leq j^* - \theta/h$, or equivalently $\delta j(t) \leq -(\theta/h)$.

We next check that $J > 0$. Let

$$M(t) = J^*(t) - (\theta/h)t, \quad t < h\wedge\nu_1; \quad (8.14)$$

clearly $M > 0$, since $t < \nu_1$ implies $J^*(t) \geq J^*(\nu_1) = K_0 - i > \theta$, and then $t < h$ implies $\theta > (\theta/h)t$. Now $J(0) = M(0)$, and $J(t) \leq M(t)$ for $0 < t < h\wedge\nu_1$ because $j(t) \leq dM(t)/dt$. Obviously $J(t) > 0$ whenever $J = M$. On the other hand, if T is such that $J(T) = M(T)$ but $J(t) < M(t)$ in some $(T, T+\delta)$, $\delta > 0$, then $J(t)$ remains positive on this interval if δ is small enough, and $J \leq J^*$ implies $\vartheta \geq \vartheta^* \geq r$, hence $j \geq 0$ by (13b); thus J does not decrease when $J < M$ and so remains positive. It further follows from these considerations that, for $0 \leq t < h\wedge\nu_1$,

$$\begin{aligned} M(t) \geq J(t) &\geq \min_{t' < t} M(t') \geq \min_{t' \leq t} J^*(t') - (\theta/h)(h\wedge\nu_1) \\ &\geq K_0 - i - \theta > 0 \end{aligned} \quad (8.15)$$

We further claim that $J(t) = M(t)$ if $t \in B$ and $t < h\wedge\nu_1$. To see this, note first that $\vartheta \geq r$ for all $t < h\wedge\nu_1$ because $J \leq J^*$, so that the condition $-j^* + \theta/h + (\vartheta - r)J \geq 0$ is satisfied whenever $j^* \leq 0$, and in particular for all $t \in B$; thus (13b) shows that J cannot start to fall below M at any interior point of B . More precisely, if T is as above, then for small δ we have $(T, T+\delta) \subseteq (\beta, \alpha)$, where $(\beta, \alpha]$ is one of the component intervals of B^c . Let (T, τ) be the whole of the interval starting at T on which $J < M$; we have to show that $\tau \leq \alpha$ in case $\alpha \leq h\wedge\nu_1$. Now $\alpha < \tau$ would imply $J(\alpha) < M(\alpha)$. On the other hand, we have $J(\alpha) \geq J(T)$ because $J \uparrow$ on (T, τ) , then $J(T) = J^*(T) - (\theta/h)T = M(T)$ by the definition of T , and finally $M(T) > M(\alpha)$ because $T < \alpha$ and because $J^*(T) \geq J^*(\alpha)$ for $T \in [\beta, \alpha]$ by the definition of B . These inequalities yield a contradiction which proves the point.

It follows that the value of J at the end of the first phase satisfies the following conditions:

$$J(h\wedge\nu_1) = J(\nu_1) = J^*(\nu_1) - (\theta/h)\nu_1 = K_0 - i - (\theta/h)\nu_1 \quad \text{if } \nu_1 \leq h \quad (8.16a)$$

$$J(h\wedge\nu_1) = J(h) \geq K_0 - i - \theta \quad \text{if } h < \nu_1 \quad (8.16b)$$

The first line holds because $\nu_i < \infty$ implies $\nu_i \in B$, hence $J(\nu_i) = M(\nu_i)$ and $J^*(\nu_i) = K_0 - i$. The second line follows from (15); on the other hand, if $h < \nu_i$ we also have

$$J(h) \leq M(h) = J^*(h) - \theta, \quad J(h) = J^*(h) - \theta \quad \text{if } h \in B \quad \dots(8.17)$$

(b) In case $h < \nu_i$, a second phase is defined for $h \leq t < \nu_i$ by setting $j = j^*$ if this leaves $g \geq 0$ and setting $g = 0$ otherwise. More precisely, the o.d.e. defining J is given by

$$\dot{j}_t = j_t^* \quad \text{if } -j_t^* + (\partial_t - r_t)J_t \geq 0 \quad \dots(8.18a)$$

$$\dot{j}_t = (\partial_t - r_t)J_t \quad \text{otherwise} \quad \dots(8.18b)$$

with $J(h)$ defined by the terminal value of the first phase. This construction guarantees that $g \geq 0$ and $\delta j \leq 0$.

The proof that $J > 0$ is similar to that under part (a) and need only be sketched. Taking into account the initial condition (16b), it is clear that $J(t) \leq J^*(t) - \theta$ on $[h, \nu_i]$. Now $t < \nu_i$ implies $J^*(t) \geq K_0 - i > \theta$, so that the curve $J^*(t) - \theta$ stays positive. On the other hand, $J \uparrow$ when $J(t) < J^*(t) - \theta$ because $\partial \geq r$, and so J stays positive in this case also. Corresponding to (15) we have, for $h \leq t < \nu_i$,

$$J^*(t) - \theta \geq J(t) \geq \min_{h \leq t' \leq t} J^*(t') - \theta \geq K_0 - i - \theta > 0. \quad \dots(8.19)$$

It can further be shown, as under (a), that $J(t) = J^*(t) - \theta$ if $t \in B \cap [h, \nu_i]$. This implies

$$J(\nu_i) = J^*(\nu_i) - \theta = K_0 - i - \theta \quad \text{if } \nu_i < \infty, \quad \dots(8.20)$$

and this equation also holds if $\nu_i = \infty$ and ν_i is a right end-point of one of the component intervals $[\alpha, \beta]$ of B . More generally, (18a) together with (20) and (19) yield

$$\begin{aligned} \delta J(\nu_i) &= -(\theta/h)\nu_i && \text{if } \nu_i \leq h \\ \delta J(\nu_i) &= -\theta && \text{if } h \leq \nu_i < \infty \\ \delta J(\nu_i) &\geq -\theta && \text{if } \nu_i = \infty \end{aligned} \quad \dots(8.21)$$

(c) If $\nu_i < \infty$, a third phase is defined by

$$j(t) = 0, \quad \nu_i \leq t < \rho \quad \dots (8.22)$$

where ρ is the first time in B such that $J(t) = J^*(t)$ if this exists, with $\rho = \infty$ otherwise. (Figure 3 depicts cases with $\rho < \infty$.) For $t \geq \rho$, we set $J(t) = J^*(t)$.

Feasibility is obvious, since $J(t) = J(\nu_i) = K_0 - i - \theta$ on $[\nu_i, \rho)$ implies $J(t) \geq 0$ and $J(t) \leq J^*(t)$, hence $\partial \geq r$, $g = (\partial - r)J \geq 0$. Explicitly, it follows from (21) and the definition of depletion times that

$$\rho = \nu_I, \quad I = i + (\theta/h)\nu_i \quad \text{if } \nu_i \leq h; \quad \rho = \nu_{i+\theta} \quad \text{if } h \leq \nu_i. \quad \dots (8.23)$$

This completes the construction of J . We now substitute into the formula (5.5) for $D\Phi$ and, noting that $\delta j \leq -\theta/h$ in phase (a), ≤ 0 in phase (b), and $= -j^*$ in phase (c), obtain

$$0 \geq D\Phi \geq E \left[(\theta/h) \int_0^{h \wedge \nu(i)} y dt + \int_{\nu(i)}^{\rho} y j^* dt \right]. \quad \dots (8.24)$$

According to (23), ρ may take one of the two values ν_I and $\nu_{i+\theta}$, and clearly $\nu_I \leq \nu_{i+\theta}$; nevertheless we may replace ρ by $\nu_{i+\theta}$ in the second integral in (24) without disturbing the inequality. This follows from the fact that $j^* < 0$ for a.a. $t \in B$, whereas the Depletion Lemma shows that B^c makes a zero contribution to the integral. On dividing the resulting inequality by θ and rearranging we have

$$E \left[(1/h) \int_0^{h \wedge \nu(i)} y dt \right] \leq E \left[(1/\theta) \int_{\nu(i)}^{\nu(i+\theta)} -y j^* dt \right]. \quad \dots (8.25)$$

We consider separately the two sides of this inequality. The random variables on the left are dominated as $h \downarrow 0$ because

$$(1/h) \int_0^{h \wedge \nu(i)} y dt \leq (1/h) \int_0^h y dt \quad \text{and} \quad E \left[(1/h) \int_0^h y dt \right] \leq y(0), \quad \dots (8.26)$$

the second inequality being due to the supermartingale property of y , and of course $y(0) \leq D^*/K_0$; see Chapter 6. On passing to the limit under E and taking into account the right continuity of y , it is seen that the left side of (25) tends to $y(0)$

On the right side of (25), we may by the Depletion Lemma restrict the time integral to $B \cap [\nu_1, \nu_{i+\theta})$, and on this set $-j^*(t)dt$ may, according to (3), be replaced by $\dot{\Gamma}(t)dt = d\Gamma(t)$; on the other hand, $\dot{\Gamma}(t) = 0$ for $t \notin B$, so the whole expression on the right of (25) is equal to

$$(1/\theta)E \int_{\nu(i)}^{\nu(i+\theta)} y(t)d\Gamma(t) \quad \dots(8.27)$$

Since $\Gamma(t)$ is non-decreasing and absolutely continuous and $\nu(I) = \inf\{t: \Gamma(t) > I\}$, a (pathwise) change of variable gives

$$\int_{\nu(i)}^{\nu(i+\theta)} y(t)d\Gamma(t) = \int_{i \wedge \Gamma(\infty)}^{(i+\theta) \wedge \Gamma(\infty)} y[\nu(I)]dI \leq \int_i^{i+\theta} y[\nu(I)]dI, \quad \dots(8.28)$$

where $y[\nu(I)] = y(\infty)$ in case $I \geq \Gamma(\infty)$, i.e. in case $\nu(I) = \infty$. Now the Stopping Theorem implies that $Ey(\nu_I) \leq Ey(\nu_i)$ for $i \leq I < i+\theta$, and it follows from (28) that (27) cannot exceed $Ey(\nu_i)$. On collecting results and referring to (25) we have $y(0) \leq Ey(\nu_i)$, which completes the proof of (12). ||

(iv) *Transversality.* It follows from $Ey(\nu_i) = y(0)$ and the argument following (25) that equality must hold a.s. in (28), in other words that $y(\infty) = 0$ if $\Gamma(\infty) \leq I \in [i, i+\theta)$, and since i and θ are arbitrary subject to $0 \leq i < i+\theta < K_0$ it follows that, a.s., $y(\omega, \infty) = 0$ if $\Gamma(\infty) < K_0$, which is (3.37). ||

This completes the proof of the Optimality Theorem in Depletion Time Form.

REMARKS.

- I. In Condition (i) of the Theorem, 'some' may be replaced by 'any'.
- II. Condition (ii) of the Theorem — which is (3.37) — may be replaced by one of the conditions (3.38) or (3.40).
- III. If we define processes \check{y} and $\check{\Gamma}$ by

$$\check{y}(\omega, i) = y(\omega, \nu_i(\omega)), \quad \check{\Gamma}(\omega, i) = \Gamma(\omega, \nu_i(\omega)), \quad 0 \leq i < K_0, \quad \dots(8.29)$$

the proof of the Theorem implies the following:

If (c^*, k^*) is optimal and satisfies (3.34), then \check{y} is an \mathfrak{A} -martingale and, a.s.,
 $\check{y}(\omega, i) = 0$ if $i \geq \Gamma(\omega, \infty)$.

$$\begin{aligned} \text{Also } \lim_{i \uparrow K_0} E\{\check{y}(i)[K_0 - \check{\Gamma}(i)]\} &= 0, \\ \text{or } \lim_{i \uparrow K_0} E\{I_{\nu_i < \omega} \check{y}(i)[\check{J}^*(i)]\} &= 0. \end{aligned} \quad (8.30)$$

These conditions are not sufficient for optimality, since they specify properties of y only at depletion times. As in Chapter 7, Remark I, the additional condition that y is an \mathfrak{A} -supermartingale is enough to yield sufficiency, since then the stopped process $(y(t \wedge \nu_i); t \in \mathcal{T})$ is a uniformly integrable martingale for each i .

IV Suppose that the paths of the discounted capital plan J^* are a.s. uniformly bounded away from zero on \mathcal{T} — i.e. that there exists a (deterministic) function $J^-(t)$ on \mathcal{T} such that a.s.

$$J^*(\omega, t) \geq J^-(t) > 0 \quad \text{for all } t \in \mathcal{T}, \quad (8.31)$$

or equivalently

$$K_0 - J^*(\omega, t) \leq K_0 - J^-(t) < K_0 \quad \text{for all } t \in \mathcal{T}$$

Setting

$$\Gamma^-(T) = \sup_{t \leq T} [K_0 - J^-(t)] \quad T \in \mathcal{T}$$

defines a non-decreasing, right continuous function, and we have, a.s.,

$$K_0 - J^*(\omega, t) \leq \sup_{t \leq T} [K_0 - J^*(\omega, t)] = \Gamma(\omega, T) \leq \Gamma^-(T) < K_0, \quad t \in \mathcal{T}.$$

Thus $K_0 - \Gamma^-(T)$ satisfies the defining property of $J^-(T)$, and we may from now on assume w.l.o.g. that J^- is chosen non-increasing and right continuous, hence that $\Gamma^- = K_0 - J^-$. See Figure 4, which incorporates this assumption.

Now define, for $0 \leq i < K_0$,

$$\nu^-(i) = \nu_i^- = \inf\{t \in \mathcal{T} : \Gamma^-(t) > i\}$$

if such a time exists, and set $\nu^-(i) = \infty$ in case $\Gamma^-(t) \geq i$ for all t ; clearly $\nu^-(\cdot)$ is non-decreasing and right continuous in i , with

$$0 \leq \nu^-(i) \uparrow \infty \text{ as } i \uparrow K_0.$$

Bearing in mind that $\nu_i(\omega) = \nu(\omega, i)$ satisfies $\Gamma[\omega, \nu_i(\omega)] = i$ if this equation has a finite solution and $\nu_i(\omega) = \infty$ otherwise, we have

$$\nu(i) \leq \nu(\omega, i) \quad \text{a.s.}$$

Now, for each $T \in \mathcal{T}$ there is an $i = i_T \in [0, K_0)$ such that

$$T = \nu(i_T), \quad \text{hence } T \leq \nu(\omega, i_T) \quad \text{a.s.}$$

It has been shown above that, for each i , the stopped process $y^i = \{y(t \wedge \nu_i); t \in \mathcal{T}\}$ is a uniformly integrable martingale. Thus, for $t \leq T$ we have (in abridged notation)

$$y(t) = E^t y[\nu(i_T)] = E^t E^T y[\nu(i_T)] = E^t y(T) \quad \text{a.s.},$$

showing that y is a (true) martingale.

Taking into account the Corollary in Chapter 7, we have the following

COROLLARY

If an optimal plan satisfies the Strong Finite Welfare Condition and there exists a function J^- on \mathcal{T} satisfying (8.31) a.s., then y is a true martingale and transversality holds in the form (1.10) or (3.42). W.l.o.g., the function J^- may be taken to be non-increasing and right continuous.

Once the Feasibility Lemma is established, the main general results on the existence of an optimum are similar to those in the case of the saving model – see [F1] S.3 – but we give a slightly different proof of the Existence Lemma and some new applications to the SNC model.

First some reminders about convergence in L_1 spaces with respect to a unitary measure. Consider as an example the space $\mathcal{L}_1 = \mathcal{L}_1(\mathcal{H}, \mu)$ of (classes of similar) \mathcal{H} -measurable, μ -integrable real-valued processes $\xi = \xi(s) = \xi(\omega, t)$ on \mathcal{S} with the norm $\|\xi\| = \int |\xi| d\mu$. Recall that $d\mu(s) = q(s)dm(s)$, see eq.(3.1), where $dm(s) = dm(\omega, t) = dP(\omega)dt$. In this Chapter we assume that μ is a unitary measure. A sequence (ξ_n) from \mathcal{L}_1 is said to *converge weakly* to $\xi \in \mathcal{L}_1$ if $\int \xi_n h d\mu \rightarrow \int \xi h d\mu$ for every $h \in \mathcal{L}_\infty(\mathcal{H}, \mu)$. An equivalent condition is that the norms $\int |\xi_n| d\mu$ are uniformly bounded and that $\int_H \xi d\mu = \lim_n \int_H \xi_n d\mu$ for every $H \in \mathcal{H}$, see [DuS] IV.8.7. If (ξ_n) converges weakly to ξ in \mathcal{L}_1 , then for each integer n there is an integer m_n , a set of integers $j = 1, \dots, J_n = J(n)$ and a set of non-negative numbers $(\alpha_{1n}, \dots, \alpha_{J_n})$ satisfying $\sum_{j=1}^{J_n} \alpha_{jm} = 1$ such that $m_{n+1} > m_n + J_n$ and $\|\xi - \sum_{j=1}^{J_n} \alpha_{jm} \xi_{m_n+j}\| \rightarrow 0$ as $n \rightarrow \infty$, see [Be] pp.54 and 91, also [DuS] V.3.14. More briefly, we shall say that, if $(\xi_n) \rightarrow \xi$ weakly in \mathcal{L}_1 , there is a sequence $(\tilde{\xi}_m)$ of convex combinations,

$$\tilde{\xi}_m = \sum_{j=1}^{J_m} \alpha_{jm} \xi_{m+j}, \quad \sum_{j=1}^{J_m} \alpha_{jm} = 1, \quad m = m_1, m_2, \dots \uparrow \infty,$$

converging to ξ in the norm of \mathcal{L}_1 . Since μ is a unitary measure, the norm-convergent sequence $(\tilde{\xi}_m)$ also converges in measure and, selecting a subsequence if necessary (henceforth *s.s.i.n.*) without changing the notation, converges a.e. to the same limit.

Next, a sequence (ξ_n) from \mathcal{L}_1 is called *weakly precompact* if it contains a subsequence converging weakly to some limit in \mathcal{L}_1 . A set Ξ in \mathcal{L}_1 is called *weakly*

sequentially compact if every infinite sequence from Ξ is weakly precompact. A necessary and sufficient condition for this property is that for $\xi \in \Xi$ the norms $\int |\xi| d\mu$ are uniformly bounded and the indefinite integrals are uniformly continuous, i.e. that for every $\epsilon > 0$ there is a $\delta > 0$ such that, for every $\xi \in \Xi$, the conditions $H \in \mathcal{H}$ and $\mu(H) < \delta$ imply $\int_H |\xi| d\mu < \epsilon$. It is sufficient if there is a $\xi_0 \in \mathcal{L}_1$ such that, for each $\xi \in \Xi$, $|\xi| \leq |\xi_0|$ a.e., and a fortiori if all $|\xi|$ are uniformly bounded (by a constant). It is also sufficient if, for some $\epsilon > 0$,

$$\sup \left[\int |\xi(s)|^{1+\epsilon} d\mu(s) : \xi \in \Xi \right] < \infty,$$

see [Mey1] II 17–23, [DuS] Ch.IV, also [F1] S 3.

We return to the Standard Model. Given a sequence (c_n) from \mathcal{C} , we denote by (U_n) the corresponding utility sequence defined by $U_n(s) = u[c_n(s); s]$ – see (3.6). The sequence is called maximising if $\varphi(c_n) = \int U_n d\mu \rightarrow \varphi^*$ with $\varphi^* = \sup\{\varphi(c) : c \in \mathcal{C}\}$ finite. We have the

EXISTENCE LEMMA.

Let φ^* be finite. Suppose that there is a maximising sequence (c_n) from \mathcal{C} such that (U_n) is weakly precompact in $\mathcal{L}_1 = \mathcal{L}_1(\mathcal{H}, \mu)$. Then there is a $c_* \in \mathcal{C}$ such that $\varphi(c_*) = \varphi^*$.

PROOF. It may be assumed s.s.i.n. that (U_n) converges weakly to some U_* in \mathcal{L}_1 .

Consequently there is a sequence (\tilde{U}_m) of convex combinations

$$\tilde{U}_m = \sum_{j=1}^{J_m} \alpha_{jm} U_{m+j}, \quad \sum_{j=1}^{J_m} \alpha_{jm} = 1, \quad m = m_1, m_2, \dots \uparrow \infty, \quad \dots (9.1)$$

converging to U_* in the norm of \mathcal{L}_1 , and we have

$$\int U_*(s) d\mu = \lim_{m \rightarrow \infty} \int \tilde{U}_m(s) d\mu = \lim_{n \rightarrow \infty} \int U_n(s) d\mu = \varphi^* \quad \dots (9.2)$$

by the definition of weak and norm convergence and the fact that (U_n) is maximising. Since μ is a unitary measure the sequence \tilde{U}_m also converges in measure and s.s.i.n. converges a.e. to the same limit. Using the non-negative constants α_{jm} we define the convex combinations of consumption plans

$$\tilde{c}_m = \sum_{j=1}^{J_m} \alpha_{jm} c_{m+j}, \quad m = m_1, m_2, \dots, \quad (9.3)$$

and $\tilde{c}_m \in \mathcal{C}$ since this set is convex. By the concavity of u we have

$$u[\tilde{c}_m(s); s] \geq \sum_{j=1}^{J_m} \alpha_{jm} u[c_{m+j}(s); s] = \tilde{U}_m(s) \quad \text{a.e.} \quad (9.4)$$

Now define a function c_* by

$$c_*(s) = \lim_{m \rightarrow \infty} \inf \tilde{c}_m(s) \quad (9.5)$$

and note that $c_* \in \mathcal{C}$ by (B)(iv) of the Feasibility Lemma in Chapter 4. Using the continuity and monotonicity of u with respect to its first argument and the a.e. convergence of the \tilde{U}_m we have

$$u[c_*(s); s] = \lim_{m \rightarrow \infty} \inf u[\tilde{c}_m(s); s] \geq \lim_{m \rightarrow \infty} \inf \tilde{U}_m(s) = U_*(s) \quad \text{a.e.}, \quad (9.6)$$

and integrating on both sides yields

$$\int u[c_*(s); s] d\mu \geq \int U_*(s) d\mu = \varphi^*. \quad (9.7)$$

Since $c_* \in \mathcal{C}$ and φ^* is the supremum of the utility integral on \mathcal{C} we have equality in (7) and c_* is optimal. ||

As in the case of the saving model, it is immediate that an optimum exists if $|u|$ is bounded on $[0, \infty] \times \mathcal{S}$, since then φ^* is finite and all U -plans are uniformly bounded, so that every sequence is weakly precompact. For unbounded utility functions, there are essential differences between the cases of u bounded above and u bounded below, or equivalently $u \leq 0$ and $u \geq 0$.

CRITERION FOR NEGATIVE UTILITY.

In case $u \leq 0$, an optimum exists if

$$\text{there is a plan } c_+ \in \mathcal{C} \text{ such that } \varphi(hc_+) > -\infty \text{ for each } h \in (0, 1] \quad (9.8)$$

PROOF. This is the same, up to changes of notation, as for the corresponding proposition in the saving model, see [F1] Assumption (iii), taking into account the convexity of \mathcal{C} established by B(iii) of the Feasibility Lemma.

APPLICATION Suppose

$$0 \leq (1-b)u(c; \omega, t) \leq c^{1-b} e^{\zeta(\omega, t)} \quad \text{for } 0 \leq c \leq \infty \quad \text{a.e.} \quad (9.9)$$

where ζ is a progressive process and $b > 1$. Assuming for brevity that $f \geq 0$ always, hence $\mathfrak{D}(k; t) \geq 0$ for any plan, the feasibility condition (4.9) shows that a sufficient condition for a process $c \geq 0$ to define a plan is that

$$\int_0^{\infty} c(t) dt \leq K_0 \quad \text{a.s.} \quad (9.10)$$

Consequently a sure (i.e. non-random) plan may be defined by setting

$$c(t) = \lambda K_0 e^{-\lambda t}, \quad t \in \mathcal{T} \text{ a.s.}, \quad \lambda > 0 \quad (9.11)$$

Then

$$(1-b)\varphi(c) \leq (\lambda K_0)^{1-b} \int_0^{\infty} e^{(b-1)\lambda t} \cdot E\{e^{\zeta(t)} q(t)\} dt, \quad (9.12)$$

and the convergence of this integral for some (small) $\lambda > 0$ is sufficient for existence of an optimum.

In particular, consider the SNC model defined in Chapter 2 with $b > 1$, and suppose that e^W is m -integrable; then we may set $c = \underline{c}$, $(1-b)u = c^{1-b}$, $q = e^W$ as in (2.36) and $\zeta = 0$. For c constructed as in (11) we have

$$(1-b)\varphi = (\lambda K_0)^{1-b} \int_0^{\infty} e^{(b-1)\lambda t} \cdot E e^{W(t)} dt, \quad (9.13)$$

and convergence of this integral implies both existence and m -integrability of e^W . In the BNC model, we have $w(t) = a_w t + \sigma_w B_w(t)$ where B_w is a Brownian motion — cf. Chapter 2, fn. 6 — so that

$$E e^{W(t)} = e^{[a_w + \frac{1}{2}\sigma_w]t}$$

and $a_w + \frac{1}{2}\sigma_w < 0$ is a sufficient condition for existence.¹

¹ An attempt to find a criterion analogous to (9.13) by choosing a sure $\bar{c} = \underline{c}e^x$ (instead of a sure \underline{c}) does not work, because in general $c = \underline{c} = \bar{c}e^x$ will not then satisfy (10) unless x is uniformly bounded below, which in interesting cases is not so. However, if one drops the assumption $\psi(0) = 0$ and introduces a small but sure exogenous income $\psi(0; t) \geq \delta \cdot \exp\{-\lambda' t\}$, where δ and λ' are positive constants, it becomes feasible to set $\bar{c}(t) = \delta \cdot \exp\{-\lambda' t\}$. Then, setting $(1-b)u(c) = (\underline{c}e^x)^{1-b}$,

CRITERION FOR POSITIVE UTILITY.

Suppose $u \geq 0$. If there is an $\epsilon > 0$ such that

$$\sup \left\{ \int U_c^{1+\epsilon} d\mu : c \in \mathcal{C} \right\} < \infty, \quad (9.14)$$

then $\varphi^* < \infty$ and an optimum exists.

PROOF. The finiteness of $\varphi^* = \sup \int U_c d\mu$ follows from the fact that $[\int U_c^r d\mu]^{1/r} \uparrow$ with $r > 0$. Also, (14) implies that the set $\{U_c : c \in \mathcal{C}\}$ is weakly sequentially compact in $\mathcal{L}_1(\mathcal{H}, \mu)$. The Existence Lemma does the rest.

APPLICATION. Consider the SNC model with $0 < b < 1$ and e^w m -integrable, and as in (2.36) set $c = \underline{c}$, $(1-b)u = c^{1-b}$, $q = e^w$, $d\mu = e^w dt dP$. Define $M(r, c) = (\int c^r d\mu)^{1/r}$, $M^*(r) = \sup\{M(r, c) : c \in \mathcal{C}\}$ and $r^+ = \sup\{r > 0 : M^*(r) < \infty\}$. It follows easily from the above Criterion that $\varphi^* < \infty$ and an optimum exists if $1-b < r^+$, whereas $\varphi^* = \infty$ if $1-b > r^+$; only in case $1-b = r^+$ is it an open question whether one can have $\varphi^* < \infty$ without existence.

These remarks apply in particular to the BNC model, with $w(t) = a_w t + \sigma_w B_w(t)$. Then the condition that $e^{w(t)}$ is m -integrable, i.e. that $a_w + \frac{1}{2}\sigma_w^2 < 0$, is necessary if φ^* is to be finite for any $b \in (0, 1)$. To see this, choose $\underline{c} = c$ as in (11) and evaluate $(1-b)\varphi$ as in (13); if $a_w + \frac{1}{2}\sigma_w^2 > 0$ then $(1-b)\varphi = \infty$ for small λ , and if $a_w + \frac{1}{2}\sigma_w^2 = 0$ then $(1-b)\varphi = (\lambda K_0)^{1-b} / \lambda(1-b) \rightarrow \infty$ as $\lambda \rightarrow 0$. This argument shows that $a_w + \frac{1}{2}\sigma_w^2 < 0$ is necessary for existence if $b < 1$, whereas for $b > 1$ this condition is sufficient.²

$c = \underline{c}$, $q = \exp(v)$ as in (2.37) and $\zeta = 0$, one finds that the convergence of an expression like (13) but with δ, λ', v in place of $\lambda K_0, \lambda, w$ implies both the existence of an optimum and the integrability of $\exp(v)$. With the stronger assumption $\psi(0) \geq \delta > 0$ a.e., the term in λ' drops out, and in the BNC case with $v(t) = a_v t + \sigma_v B_v(t)$ one obtains $a_v + \frac{1}{2}\sigma_v^2 < 0$ as a sufficient condition for existence. Concerning the case of an exogenous income, see also Chapter 3, fn 4 and [F4].

² Pursuing this analogy, it may be shown as in the preceding footnote that, if the value $\psi(0) = 0$ is replaced by a positive constant, the condition $a_v + \frac{1}{2}\sigma_v^2 < 0$ is necessary for existence when $b < 1$.

DIRECT TEST FOR POSITIVE UTILITY.

One can give a general condition which often allows existence in the Standard Model to be inferred directly from the data of the problem. The argument is an extension of that for the saving model given in [F1]. Suppose u satisfies (9) but with $0 < b < 1$. Note that $\partial(k;t) \leq f'(0;t) = \partial(0;t)$ for all $k > 0$ — see (3.4) — and write $\mathcal{D}_0(T) = \int_0^T \partial(0;t)dt$. We have $\mathcal{D}(k;T) \leq \mathcal{D}_0(T)$ for all plans — see (4.7a), and (4.9) shows that a plan must satisfy

$$\int_0^m c(t)e^{-\mathcal{D}_0(t)}dt \leq K_0 \quad \text{a.s.} \quad \dots(9.15)$$

Then, for $c \in \mathcal{C}$ and $H \in \mathcal{H}$ we have, using Hölder's inequality,

$$\begin{aligned} 0 \leq (1-b) \int_H U_c d\mu &\leq \int_H \left\{ c^{1-b} e^{\zeta} q \right\} d\mu \\ &\leq \left\{ \int_H c e^{-\mathcal{D}_0} d\mu \right\}^{1-b} \times \left\{ \int_H [e^{(1-b)\mathcal{D}_0 + \zeta} q]^{1/b} d\mu \right\}^b \end{aligned} \quad \dots(9.16)$$

where $\mathcal{D}_0 = \mathcal{D}_0(T)$. In view of (15), the integral of $ce^{-\mathcal{D}_0}$ does not exceed K_0 ; thus, if the last integral in (16) converges with $H = \mathcal{H}$, i.e. if

$$\int_0^m E[e^{(1-b)\mathcal{D}_0(t) + \zeta(t)} q(t)]^{1/b} dt < \infty, \quad \dots(9.17)$$

it follows that the norms $\int |U_c| d\mu$ are uniformly bounded and the indefinite integrals $\int_H |U_c| d\mu$ uniformly μ -continuous. But then every sequence from $\{U_c : c \in \mathcal{C}\}$ is weakly precompact in $\mathcal{L}_1(\mathcal{H}, \mu)$, and the existence of an optimum follows from the Existence Lemma.

APPLICATION. In the BNC model with $b < 1$ we may set $\mathcal{D}_0(t) = t\psi'(0)$, $\zeta = 0$, $c = \underline{c}$, $(1-b)u = c^{1-b}$ and $q = e^w$, provided that $\alpha_w + \frac{1}{2}\sigma_w^2 < 0$. The criterion (17) reduces to $(1-b)\psi'(0) + \alpha_w + \frac{1}{2}\sigma_w^2/b < 0$, which implies the preceding inequality and so is a sufficient condition for existence.

REMARK. The sufficient conditions for existence in the BNC model given above, while simple, are not the best possible. It follows from the results of [F5-6] that, for $b > 1$, an optimum exists if

$$\begin{aligned} & \text{either (i) } a_w + \frac{1}{2}\sigma_w^2 < 0 \\ & \text{or (ii) } a_v + \frac{1}{2}\sigma_v^2 < 0 \text{ and } (1-b)\psi'(0) + a_w + \frac{1}{2}\sigma_w^2 < 0; \end{aligned} \quad \dots (9.18)$$

while, for $0 < b < 1$, an optimum exists if

$$\begin{aligned} & \text{either (i) } (1-b)\psi'(0) + a_w + \frac{1}{2}\sigma_w^2 < 0 \\ & \text{or (ii) } a_v + \frac{1}{2}\sigma_v^2 < 0 \text{ and } a_w + \frac{1}{2}\sigma_w^2 < 0. \end{aligned} \quad \dots (9.19)$$

Sufficient conditions for logarithmic utility are obtained by setting $b = 1$ in (18)

or (19). Work in progress suggests that these conditions are also necessary.

As in [F1], the results for positive and negative utilities can be combined if μ is unbounded both above and below; we omit the details of the argument but state the result as part (v) of the following theorem.

EXISTENCE THEOREM.

An optimum exists in the following cases:

- (i) if $|u|$ is bounded;
- (ii) if $u \leq 0$ and (9.8) holds;
- (iii) if $u \geq 0$, $\varphi^* < \infty$, and if this remains true when u is replaced by $u^{1+\epsilon}$ with some $\epsilon > 0$;
- (iv) if $0 \leq (1-b)u(c;s) \leq c^{1-b}e^{\zeta(s)}$ for $c \geq 0$ a.e., $0 < b < 1$, where ζ is a progressive process and (9.17) holds;
- (v) if u is unbounded in both directions and satisfies (possibly with different choices of scale and origin) first (9.8), and secondly either (9.14) or both (9.9) and (9.17), with u replaced by $u^+ = u \cdot \mathcal{I}[(c,s): u(c;s) \geq 0]$ in (9.9) or (9.14).

UNIQUENESS

In general, an optimal plan is not unique. However, if (c^*, k^*) and (c, k) are distinct optimal plans, i.e. $c^* \neq c$ on a non-null progressive set and $\varphi(c^*) = \varphi(c) = \varphi^*$, then Property (iii) of Chapter 4 shows that the convex combination of consumption plans $\tilde{c}^\alpha = \alpha c + (1-\alpha)c^*$ is also feasible, as is the convex combination of capital plans $k^\alpha = \alpha k + (1-\alpha)k^*$ with corresponding consumption plan $c^\alpha = f(k^\alpha) - \dot{k}^\alpha$. Then $c^\alpha \geq \tilde{c}^\alpha$ – see (4.11) – together with the concavity of φ implies

$$\varphi(c^\alpha) \geq \varphi(\tilde{c}^\alpha) \geq \alpha\varphi(c) + (1-\alpha)\varphi(c^*) = \varphi^*, \quad (9.20)$$

so that c^α and \tilde{c}^α are both optimal. If now it is assumed that $f'(\cdot; \omega, t)$ is (strictly) decreasing for (ω, t) in some progressive set of positive measure, then on that set (4.11) is replaced by $c^\alpha > \tilde{c}^\alpha$, hence $\varphi(c^\alpha) > \varphi(\tilde{c}^\alpha)$, a contradiction which shows that in this case an optimum must be unique. Similarly, if $u'(\cdot; \omega, t)$ is (strictly) decreasing on some progressive set of positive measure, the second inequality in (20) becomes sharp, which again implies uniqueness.

(A) *Survey and Example.*

Our formulation of the growth model is unusual in that the generalised Lipschitz condition (3.2) is imposed for $k \geq 0$, whereas most authors adopt an 'Inada' condition at $k = 0$, say in the form

$$f'(0; \omega, t) = \infty, \quad (\omega, t) \in \mathcal{S}. \quad (10.1)$$

The main reason for considering the Inada condition is that it allows certain production functions such as Cobb-Douglas with nice properties of homogeneity. The Inada condition has also on occasion been justified on other grounds of convenience in theoretical work, but it is in some respects very inconvenient, in particular because of the occurrence of multiple solutions of (1.1) at points with zero capital. In addition, the average returns process $\mathfrak{D}(k; T) = \int_0^T \mathfrak{D}[k(t); t] dt$ may diverge to infinity along capital paths $k(t)$ which reach zero at finite times, thus destroying the equivalence between the o.d.e. (1.1) and the integral equations (4.8) or (3.12). Also, the possibility must be considered that paths of $k^*(t)$ reach zero at finite times with $R(T) = \int_0^T f'[k^*(t); t] dt$ diverging to infinity, which would invalidate formulae expressed in discounted units. The purpose of this chapter is to explore the implications of these problems for the theory presented in the preceding chapters.

It is instructive to begin with an example. Consider the Cobb-Douglas production function, which in the notation of Chapter 2 takes the form (up to a scaling constant)

$$\Psi(k, L) = k^{1-\lambda} L^{-\lambda} \quad \text{or} \quad \psi(k) = k^{1-\lambda} \quad (10.2)$$

where $0 < \lambda < 1$ — cf. (2.2) and (2.6). Then, for a plan $\underline{k} = k$, $\underline{c} = c$ in standardised units with $k > 0$ for all (ω, t) , (2.38) yields

$$f[k(\omega, t); \omega, t] = k(\omega, t)^{1-\lambda} e^{-\lambda x(\omega, t)} \quad (10.3)$$

where $\exp\{-\lambda x(\omega, t)\}$ is required to be a s. locally integrable so as to satisfy (3.2) for $k > 0$. Thus, omitting ω , (1.1) with (4.1) has the form

$$\dot{k}(t) = \begin{cases} k(t)^{1-\lambda} e^{-\lambda x(t)} - c(t) & k(t) \geq 0, \\ -c(t) & k(t) < 0. \end{cases} \quad \dots (10.4a)$$

$$\dots (10.4b)$$

Let (c_n, k_n) be a sequence of plans with $c_n > 0$ and $k_n > 0$ for all (ω, t) . Let ν be a finite stopping time, and suppose that there are processes c_ω and k_ω such that, for each ω , $c_n \uparrow c_\omega$ for $t < \nu$ and $k_n \downarrow k_\omega$ for $t \leq \nu$, with $k_\omega(t) > 0$ for $t < \nu$ and $k_\omega(\nu) = 0$. Further, for $t \geq \nu$ let (c_n, k_n) be such that $c_n(t) = \theta[k_n(t); t]$ with some constant $\theta \in [0, 1]$; then, on dropping the subscript n , (4a) yields

$$\dot{k}(t) = (1-\theta)k(t)^{1-\lambda} e^{-\lambda x(t)}, \quad t \geq \nu, \quad \dots (10.5)$$

and the 'initial' value $k(\nu) = k_n(\nu) > 0$ determines a unique forward solution

$$k(T) = \left[k(\nu)^\lambda + \lambda(1-\theta) \int_\nu^T e^{-\lambda x(t)} dt \right]^{1/\lambda}, \quad T \geq \nu, \quad (10.6)$$

where

$$c(T) = \theta e^{-\lambda x(T)} \left[k(\nu)^\lambda + \lambda(1-\theta) \int_\nu^T e^{-\lambda x(t)} dt \right]^{(1-\lambda)/\lambda}, \quad T \geq \nu. \quad \dots (10.7)$$

Now let $n \rightarrow \infty$. For $T \leq \nu$, it is easily seen, as in the proof of Property (iv) in Chapter 4(B), that k_ω is the unique solution of (4) through $(K_0, 0)$ corresponding to c_ω . For $T \geq \nu$, let $c_\omega(T)$ and $k_\omega(T)$ be defined by the limits of (7) and (6) with $c = c_n$, $k = k_n$, so that here $c_n \downarrow c_\omega$ and $k_n \downarrow k_\omega$. Clearly k_ω is a solution of (4) corresponding to c_ω satisfying $k(\nu) = 0$; thus both processes are non-negative on \mathcal{T} . Nevertheless (c_ω, k_ω) cannot be regarded as a 'proper' plan if $\theta > 0$, because k_ω 'bounces' back to positive values after reaching zero at ν , whereas the natural interpretation of the conditions $k(\nu) = 0$ and $f(0; t) = 0$ for $t \geq \nu$ is that production cannot get started again without positive capital. Of course, there is another forward solution of (4) with initial condition $k(\nu) = 0$ corresponding to c_ω , namely

$$k(T) = - \int_\nu^T c_\omega(t) dt, \quad T \geq \nu, \quad \dots (10.8)$$

which is not obtainable as a limit of the c_n ; this solution is not feasible unless $c_\omega = 0$ for a.a. $T \geq \nu$, but it has a natural interpretation as the cumulative amount which it

would be necessary to borrow (if this were possible) in order to sustain the consumption c_{ω} .

In the special case where $c_{\omega} = 0$ for $T \geq \nu$, corresponding to $\theta = 0$, the pair (c_{ω}, k_{ω}) does define a proper, feasible plan, albeit one which represents starvation. To be explicit, note that $k = k_{\omega} = 0$ for $T \geq \nu$ in (4) implies $c = 0$, apart from null sets. On the other hand, if c is specified by setting $c = c_{\omega}$ for $T < \nu$ and $c = 0$ for $T \geq \nu$, then (4) has two solutions for $T > \nu$, namely the 'proper' but 'special' solution $k_{\omega} = 0$ and the 'improper' but 'general' solution $k_{\omega} > 0$ defined by (6) with $k(\nu) = 0$ and $\theta = 0$; note that proper convex combinations of these two solutions are not solutions.

Although in the preceding calculations we have chosen c_n and k_n so that $c_n \uparrow c_{\omega}$ for $t < \nu$, $c_n \downarrow c_{\omega}$ for $t \geq \nu$ and $k_n \downarrow k_{\omega}$ for all t , it is also possible to represent (c_{ω}, k_{ω}) as the limit of a sequence of plans $(\tilde{c}_n, \tilde{k}_n)$ with $\tilde{c}_n \uparrow c_{\omega}$, $\tilde{k}_n \downarrow k_{\omega}$ for all t . For example, it suffices to take $\tilde{c}_n = c_{\omega}(1-1/n)$ and let \tilde{k}_n be the corresponding solution of (1.1) with $\tilde{k}_n(0) = K_0$. The use of such monotone sequences simplifies the general theory.

We return to the general discussion of the Standard Model but postpone formal definitions. Let Π denote the set of proper plans (c, k) , i.e. those for which k stays at zero after first arrival, Π^+ the subset of plans with positive capital, i.e. those with $k(t) > 0$ on \mathcal{S} a.s., and $\overset{\rightarrow}{\Pi}$ the set of extended plans, i.e. pairs of processes (c, k) which can be represented as limits $c_n \uparrow c$, $k_n \downarrow k$ with $(c_n, k_n) \in \Pi$; elements of $\overset{\rightarrow}{\Pi} \setminus \Pi$ are improper plans.

Now, the existence theory of Chapter 9 depends on the closure of the set of (proper) c -plans under passage to the limit inferior, at least in the case of maximising sequences, whereas the Cobb-Douglas example shows that, with $f'(0) = \infty$, the limit of a sequence (c_n, k_n) from Π may be improper. Thus the procedure of Chapters 4 and 9, suitably modified, will deliver at most an optimum in $\overset{\rightarrow}{\Pi}$. The first main problem will therefore be to establish the form in which the properties of feasible sets

in Chapter 4 carry over to $\vec{\Pi}$; once this is done, the main existence results in Chapter 9 will carry over to $\vec{\Pi}$

The second main problem is then to characterise an optimum in $\vec{\Pi}$. It is easily seen that the Conditions for Optimality (3.28–33) remain sufficient for an optimum in $\vec{\Pi}$, which under these conditions must be in Π and even in Π^+ , while the conditions (3.29–34) remain necessary for an optimum *relative to* Π^+ . As regards necessity for an optimum in $\vec{\Pi}$, we shall not review all the discussion of earlier chapters, but rather outline an argument to show that (under slight assumptions) an optimum (c^*, k^*) in $\vec{\Pi}$ is in Π^+ . At first sight this may seem obvious, and it is indeed easily shown, as in Chapter 6, that $u'(0; t) = \infty$ implies $c^* > 0$ a.e., and then in view of $f(0; t) = 0$ it follows that, a.s., $k^*(t) > 0$ except perhaps on a Lebesgue null set of \mathcal{T} (which may vary with ω). The problem is to show that $k^*(t) > 0$ for *all* t . After all, it might seem attractive to run out of capital from time to time and then take off again along an improper solution of (1.1).

Of course, these difficulties do not arise in cases where it is possible to construct a plan in Π^+ which satisfies a set of sufficient conditions. In particular, in the BNC model with Cobb-Douglas production, an optimal consumption function with positive consumption can be constructed, for suitable values of the parameters, as the solution of a boundary value problem along lines similar to [F5–6]. (However the boundary conditions for optimal consumption as $\kappa \rightarrow 0$ are quite different and give rise to interesting new technical problems; I hope to present this theory elsewhere.)

(B) *Definitions concerning Plans* – cf Chapters 3(A) and 4(A).

We turn now to a more systematic review of parts of the theory for the Standard Model, but set out details only where changes of substance are needed. In this Chapter it will be assumed for simplicity that, for each $\kappa > 0$, $f'(\kappa; \omega, t) \geq 0$, $f(\kappa; \omega, t) > 0$ for

each $(\omega, t) \in \mathcal{S}$, and

$$\int_0^T f(k; \omega, t) dt < \infty \quad \text{for each } (\omega, T) \in \mathcal{S}; \quad (10.9)$$

also, of course, $f'(0; \omega, t) = \infty$, $f(0; \omega, t) = 0$, while (3.2) remains in force for $k > 0$.

The discussion in Chapter 4(A) of the existence and uniqueness of pathwise solutions of (1.1) with (4.1) then needs some changes. For fixed ω , let a locally integrable $c(t) \geq 0$ be given, and refer to the conditions set out in the paragraph following (4.1). Given $\bar{k} \geq 0$, $\bar{t} \geq 0$ and a rectangle

$$Q = [k_1, k_2] \times [\bar{t}_1, \bar{t}_2], \quad k_1 < k < k_2,$$

the existence of a local forward solution through (\bar{k}, \bar{t}) now results from

$$0 \leq f(k; t) \leq f(k_2; t), \quad (k, t) \in Q,$$

together with (8) and the local integrability of c . If $\bar{k} > 0$, one can choose $k_1 > 0$, and then the local uniqueness follows from

$$|k'' - k'| f'(k_2; t) \leq |f(k''; t) - f(k'; t)| \leq |k'' - k'| f'(k_1; t),$$

(k', t) and (k'', t) in Q ,

and the validity of (3.2) for $k > 0$. For $\bar{k} < 0$, existence and uniqueness are trivial. For $\bar{k} = 0$ uniqueness generally fails, but it is known that there is a maximal and a minimal solution through (\bar{k}, \bar{t}) .

To check that no solution of (1.1) can explode to $+\infty$ in finite time, suppose that $k(t)$ passes through (\bar{k}, \bar{t}) with $\bar{k} > 0$ and note that, since $f \geq 0$, $f' \geq 0$ and $c \geq 0$, $k(t)$ is bounded above for $t \geq \bar{t}$ by the solution $k^+(t)$ of $\dot{k} = f(k; t)$ through (\bar{k}, \bar{t}) ; but $k^+(t)$ is non-decreasing in t and stays finite because

$$\dot{k}^+(t) = f[k^+(t); t] \leq f[\bar{k}; t] + (k^+(t) - \bar{k})f'(\bar{k}; t),$$

while $f(\bar{k}; t)$ and $f'(\bar{k}; t)$ are integrable on finite intervals of \mathcal{S} . On the other hand, no solution can explode to $-\infty$ because $f - c \geq -c$ and c is integrable on finite intervals. Thus all solutions can be continued forward on \mathcal{S} , though perhaps not uniquely. In particular, for given $c(t)$ it makes sense to speak of *the* maximal (or *the* minimal) solution on \mathcal{S} through $(K_0, 0)$.

Returning to Chapter 3(A), we now amend the definition of plans as follows.

Let c be a progressive, non-negative and a.s. locally integrable process and let k be a progressive, non-negative process with $k(0) = K_0$ such that, a.s., $k(\omega, \cdot)$ is a solution of (1.1) on \mathcal{T} corresponding to c . Let a stopping time $\bar{\nu}$ be defined by

$$\bar{\nu}(\omega) = \inf\{t \in \mathcal{T} : k(\omega, t) = 0\} \quad \dots (10.10)$$

if this exists and $\bar{\nu}(\omega) = \infty$ otherwise. Since $K_0 > 0$, the solution $k(\omega, \cdot)$ is unique on some interval $[0, \bar{\nu}(\omega))$ of positive length

Given such $c, k, \bar{\nu}$, we say that (c, k) is a *proper plan*, or simply a *plan*, if $k(\omega, t) = 0$ for $t \geq \bar{\nu}(\omega)$ a.s.; (clearly, this implies that $c(\omega, t) = 0$ for a.a. $t \geq \bar{\nu}(\omega)$, a.s.). In case $\bar{\nu} = \infty$ a.s., we have a *plan with positive capital*. A sequence of plans (c_n, k_n) of plans is called *monotone* if $c_n \uparrow$ a.e., hence $k_n \downarrow$ on \mathcal{T} a.s.; then, making changes on null sets if necessary, it may be assumed that $c_n \uparrow$ and $k_n \downarrow$ for all (ω, t) . A stationary sequence is monotone. Given c, k , and $\bar{\nu}$ as in the preceding paragraph, we say that (c, k) is an *extended plan* if there is a monotone sequence (c_n, k_n) of plans such that $c_n \uparrow c$ and $k_n \downarrow k$ for all (ω, t) . An extended plan is called *improper* if

$$P\{\omega : k(\omega, t) > 0 \text{ for some } t > \bar{\nu}(\omega)\} > 0 \quad \dots (10.11)$$

The sets of proper plans, plans with positive capital, and extended plans are denoted by Π, Π^+ and $\vec{\Pi}$. Extending the notation of Chapter 3(A) in an obvious way, we also write

$$\vec{\mathcal{E}} = \{c : \exists k, (c, k) \in \vec{\Pi}\}, \quad \vec{\mathcal{K}} = \{k : \exists c, (c, k) \in \vec{\Pi}\} \quad \dots (10.12)$$

etc., and assume that the functional φ (or Φ) is well defined on $\vec{\mathcal{E}}$ (or $\vec{\mathcal{K}}$ or $\vec{\Pi}$) and obeys a finite supremum condition there. It follows by monotone convergence that the supremum φ^* of φ on \mathcal{E} is also the supremum on $\vec{\mathcal{E}}$. The definition of an extended plan contains some redundancy: it is enough to say that (c, k) is a pair of progressive processes such that $c_n \uparrow c$ and $k_n \downarrow k$ for some sequence from Π . That c and k are then progressive and non-negative is obvious. To show that c is locally

integrable and k is a solution of (1.1) corresponding to c , it is enough to verify that, for fixed ω , we have

$$\begin{aligned} \int_0^T c(t)dt &= \lim_n \int_0^T c_n(t)dt = \lim_n \left\{ \int_0^T f[k_n(t);t]dt - k_n(T) + K_0 \right\} \\ &= \int_0^T f[k(t);t]dt - k(T) + K_0 < \infty, \quad T \in \mathcal{T}. \end{aligned} \quad (10.13)$$

The first equality results from monotone convergence. The second follows directly from (1.1). The third equality (and the finiteness) are due to the continuity and monotonicity of f and dominated convergence; explicitly,

$$0 \leq k(t) \leq k_1(t) \quad \text{implies} \quad 0 \leq f[k(t);t] \leq f[k_1(t);t],$$

and $f[k_1(t);t]$ is integrable on $[0, T]$ because k_1 is a plan, so that we may pass to the limit under the integral sign and then use $k_n \downarrow k$, $f(k_n) \downarrow f(k)$.¹

(C) *Properties of the Set of Extended Plans* — cf. Chapters 4(B) and 9.

The extension to $\vec{\Pi}$ of the properties of Π established in Chapter 4(B) is not quite straightforward, though the proofs are simplified by the assumption that $f \uparrow$ in k .

First, if (c, k) is improper and $\mathcal{D}(k; \omega, T)$ is the average returns process defined by (4.7a), the stopping time $t = \inf\{T: \mathcal{D}(k, T) = \infty\}$ may take finite values with positive probability, and then for $T \geq t$ the integral equation (4.8) is no longer equivalent to (1.1) with $k(0) = K_0$, and in particular says nothing about the sign of $k(T)$; this calls for some changes in the later argument. Note that $t \geq \bar{\nu}$.

We now set out the new version of Properties (i)–(iv), with proofs where needed.

PROPERTY (ī). If $c \equiv 0$, there exists k such that $(c, k) \in \Pi \subset \vec{\Pi}$.

PROPERTY (īī). If $(c_1, k_1) \in \vec{\Pi}$ and $0 \leq c_2 \leq c_1$ a.e., c_2 progressive, there exists k_2 such that $(c_2, k_2) \in \vec{\Pi}$; moreover $0 \leq k_1(t) \leq k_2(t)$ on \mathcal{T} a.s.

¹ It can be shown that (c, k) is an improper plan iff c is progressive, non-negative, locally integrable, k is the maximal solution of (1.1) on \mathcal{T} with $k(0) = K_0$, and k is non-negative and such that (11) holds.

PROOF. Note first that the corresponding assertions for Π^+ and Π are obvious under present assumptions, because with $f \uparrow$ in κ it is clear that $0 \leq c_2 \leq c_1$ implies $k_2 \geq k_1$. Now let (c_1, k_1) be improper, with $0 \leq c_{1n} \uparrow c_1$, $k_{1n} \downarrow k_1$, $(c_{1n}, k_{1n}) \in \Pi$, and let $0 \leq c_2 \leq c_1$ with c_2 progressive. Define $c_{2n} = (c_2/c_1)c_{1n}$ if $c_1 > 0$ and $c_{2n} = 0$ otherwise; then $0 \leq c_{2n} \leq c_{1n}$, so that by the preceding remark we have $(c_{2n}, k_{2n}) \in \Pi$ with unique k_{2n} such that $k_{2n} \geq k_{1n} \geq k_1$. Also $0 \leq c_{2n} \uparrow c_2 \leq c_1$, so that $k_{2n} \downarrow$ some limit function $k_2 \geq k_1 \geq 0$. It then follows from the definition of an extended plan that $(c_2, k_2) \in \vec{\Pi}$. ||

PROPERTY (iii). Let (c^*, k^*) and $(c, k) = (c^* + \delta c, k^* + \delta k)$ be elements of $\vec{\Pi}$, and for $0 < \alpha < 1$ let $\tilde{c}^\alpha = \alpha c + (1-\alpha)c^*$, $k^\alpha = \alpha k + (1-\alpha)k^*$. For each α ,

(a) there exists \tilde{k}^α such that $(\tilde{c}^\alpha, \tilde{k}^\alpha) \in \vec{\Pi}$, i.e. \mathcal{E}^α is convex;

(b) if $(c, k) \in \Pi^+$, or if $(c, k) \in \Pi$ and $k > 0$ on $\{k \neq k^*\}$, there exists c^α such that $(c^\alpha, k^\alpha) \in \Pi^+$.

PROOF. By definition, there exist (c_n^*, k_n^*) and (c_n, k_n) in Π such that $c_n^* \uparrow c^*$, $c_n \uparrow c$, $k_n^* \downarrow k^*$, $k_n \downarrow k$. Since f is continuous and non-decreasing in κ , it follows from (1.1) that $\tilde{k}_n^* \downarrow \tilde{k}^*$, $\tilde{k}_n \downarrow \tilde{k}$.

(a) Let $\tilde{c}_n^\alpha = \alpha c_n + (1-\alpha)c_n^*$. Since the proof of Property (iii) in Chapter 4 remains valid for proper plans, there is some \tilde{k}_n^α such that $(\tilde{c}_n^\alpha, \tilde{k}_n^\alpha) \in \Pi$. When $n \rightarrow \infty$, $\tilde{c}_n^\alpha \uparrow \tilde{c}^\alpha$, so $\tilde{k}_n^\alpha \downarrow$ some limit $\tilde{k}^\alpha \geq 0$, and it follows as before that $(\tilde{c}^\alpha, \tilde{k}^\alpha) \in \vec{\Pi}$.

(b) For brevity we consider only the case $(c, k) \in \Pi^+$. Let $k_n^\alpha = \alpha k_n + (1-\alpha)k_n^*$. Using Property (iii) again and abridging the notation, we have $(c_n^\alpha, k_n^\alpha) \in \Pi$ with

$c_n^\alpha = f(k_n^\alpha) - \tilde{k}_n^\alpha$. Now let $c^\alpha = f(k^\alpha) - \tilde{k}^\alpha$. Since $k_n^\alpha \downarrow k^\alpha$ and $\tilde{k}_n^\alpha \downarrow \tilde{k}^\alpha$, it is clear that $c_n^\alpha \rightarrow c^\alpha$, but the last convergence need not be monotonic; (if it were, the present argument could be used to prove the convexity of \mathcal{E}^α , which is not claimed). We

accordingly consider the processes $\bar{c}_n^\alpha = \inf_{m \geq n} c_m^\alpha$; since $0 \leq \bar{c}_n^\alpha \leq c_n^\alpha$, it follows from Property (ii) in Chapter 4 that $(\bar{c}_n^\alpha, \bar{k}_n^\alpha) \in \Pi$ for some $\bar{k}_n^\alpha \geq k_n^\alpha$. When $n \rightarrow \infty$, $\bar{c}_n^\alpha \uparrow c^\alpha$, so $\bar{k}_n^\alpha \downarrow$ some limit $\bar{k}^\alpha \geq k^\alpha$; it then follows as before that $(c^\alpha, \bar{k}^\alpha) \in \vec{\Pi}$, so that \bar{k}^α is

a solution of (1.1) with $c = c^\alpha$ and $k(0) = K_0$. Now k^α is a solution of this equation, and k^α is positive because k is positive; but then k^α is the unique solution, implying $\bar{k}^\alpha = k^\alpha$ and $(c^\alpha, k^\alpha) \in \Pi^+$. ||

This argument also proves

PROPERTY (iiv). Let $(c_n, k_n) \in \vec{\Pi}$, $n = 1, 2, \dots$, and define $c_* = \lim_n \inf_{m \geq n} c_m$ as in (4.12); there exists k_* such that $(c_*, k_*) \in \vec{\Pi}$.

Once these properties are proved, the Existence Lemma in Chapter 9 applies, except that it is now asserted that there is a $c_* \in \mathcal{E}^*$ (rather than in \mathcal{E}) such that $\varphi(c_*) = \varphi^*$. Only part (a) of Property (iii) is needed in this argument. Similarly the Criteria for Negative and for Positive Utility assert the existence of an optimum in $\vec{\Pi}$ rather than in Π . The Applications immediately following the statements of these Criteria remain valid (subject to a minor change of proof in the negative case to avoid reference to eq (4.9)); however the proof of the Direct Test for Positive Utility fails.

(D) *Conditions for Optimality* — cf. Chapters 3(B) and 5–6.

Let (c^*, k^*) be a distinguished element of $\vec{\Pi}$ and let $\bar{\nu}$ be the first arrival time of k^* at zero — cf. (10). Further, let R be the process defined as in (3.7) and define a stopping time σ by

$$\sigma(\omega) = \inf\{T \in \mathcal{T} : R(\omega, T) = \infty\} \quad \dots(10.14)$$

if such a number exists and $\sigma(\omega) = \infty$ otherwise; obviously $\sigma \geq \bar{\nu}$ a.s. The definitions (3.8–10) of J , G and y yield $J(T) = 0$, $G(T) = G(\sigma)$ and $y(T) = \infty$ for $T > \sigma$, so that formulae involving these processes and their derivatives are in general either trivial or invalid for $T > \sigma$. The Conditions for Optimality (3.28–33), even if meaningful, cannot be fulfilled if $P\{\sigma < \infty\} > 0$ since $\chi_n \uparrow \infty$ a.s. implies $P\{\chi_n > \sigma\} > 0$ for large n , hence $Ey(\chi_n) = \infty$, contrary to the definition of a local martingale. Clearly our approach to characterising an optimum will not work for $\vec{\Pi}$ unless R is finite on \mathcal{T} , a.s.; but since R represents the marginal log-return to

capital along the star plan, and $\exp(-R)$ the discount process, it is to be expected on economic grounds that this condition will be satisfied by an optimal plan.

Consider first the sufficiency of the Conditions for Optimality (3.28-33). By assumption, y is a local martingale, so c^* is positive a.e., R is finite on \mathcal{T} a.s., the meaning of the Conditions presents no problems and the proof of sufficiency stands (However, if $P\{\bar{v} < \infty\} > 0$, the Conditions cannot be fulfilled with $\chi_n = \nu_i$ as in (3.36), because then $i \uparrow K_0$ implies $\nu_i \uparrow \bar{v}$ a.s.)

Turning to necessity, let (c^*, k^*) be optimal in $\bar{\Pi}$. We wish to show that, under slight assumptions to be stated, (c^*, k^*) is actually in Π^+ , i.e. that $\bar{v} = \infty$ a.s. This will imply that the Conditions (3.29-34) are necessary for optimality. We proceed somewhat informally in order to avoid undue repetition of earlier arguments. Starting with Chapter 5, we assume as usual that (5.6) is satisfied. By virtue of Property (iii) the formula (5.1) for $D\varphi(c^*, \delta c)$ has meaning for $(c, k) = (c^* + \delta c, k^* + \delta k) \in \bar{\Pi}$, while formula (5.2) for $D\Phi(k^*, \delta k)$ has meaning if $(c, k) \in \Pi^+$ or if $k > 0$ on $\{\delta k \neq 0\}$. The formulae (5.3-5) and (5.7) obtained by differentiating under the integral sign then remain valid under assumption (5.6), (with the reservation that the versions involving g, j, G, J etc. require R to be finite). It then follows, as in Chapter 6, that $c^*(\omega, t) > 0$ a.e., but unlike the situation in Chapter 6 we can at this stage only infer that the absolutely continuous function $k^*(\omega, \cdot)$ is positive for *almost* all $t \in \mathcal{T}$, hence is positive on a sequence of intervals of positive length, all of them open except the first which has the form $[0, \bar{v}(\omega))$.

We next sketch a proof that optimality requires that $R(\omega, T) < \infty$ on \mathcal{T} , a.s. Suppose on the contrary that $P\{\sigma < \infty\} > 0$. For $t < \sigma$ and we may choose a variation $\delta c < 0$, say $\delta c = -\epsilon c^*$, $c = (1-\epsilon)c^*$ with $0 < \epsilon < 1$; this is feasible since $c^* > 0$ a.e. Now, abridging the notation, we have

$$\delta \dot{k}(t) = f[k(t); t] - f[k^*(t); t] + \epsilon c^*(t),$$

and clearly $\delta k(t) > 0$ for $t \leq \sigma$. By concavity of f ,

$$f[k(t);t] - f[k^*(t);t] \geq \delta k(t) \cdot f'[k(t);t]$$

hence $\dot{k}(t) > \delta k(t) \cdot f'[k(t);t]$. Bearing in mind that σ is a predictable time, we choose a time τ such that $0 < \tau < \sigma$, hence $\delta k(\tau) > 0$ a.s., and obtain

$$\delta k(\sigma)/\delta k(\tau) > \exp\left\{\int_{\tau}^{\sigma} f'[k(t);t]dt\right\}$$

As $\epsilon \downarrow 0$, $k \downarrow k^*$, $f'[k(t);t] \uparrow f'[k^*(t);t] = r(t)$, leading to

$$\lim_{\epsilon \downarrow 0} \delta k(\sigma)/\delta k(\tau) \geq \exp\{R(\sigma) - R(\tau)\} = \infty \text{ if } \sigma < \infty.$$

In other words, an arbitrarily small proportional cut in consumption (which may even be restricted to a short interval preceding σ) permits the accumulation of an arbitrarily large sum of additional capital $\delta k(\sigma)$ at σ . Taking into account (5.6), it is found as in the discussion leading to (5.7) that the loss of welfare before σ is finite and tends to zero with ϵ . On the other hand, $\delta k(\sigma)$ may be consumed after σ , and for ϵ small enough the additional consumption can be chosen so that the increment of welfare after σ exceeds the loss before σ , contrary to optimality.

Turning to Chapter 6, the proof of the supermartingale property of y depends essentially only on finiteness of R and on variations of the consumption plan, and so the property remains valid. We assume (6.18), so that we may choose c^* right continuous, and then k^* is also right continuous. Now suppose that $\bar{\nu}(\omega) < \infty$ for $\omega \in A$ with $PA > 0$, so that $k^*(\bar{\nu}) = 0$ for $\omega \in A$ while $k^*(t) > 0$ on some (random) right neighbourhood of $\bar{\nu}$, hence also $f[k^*(t);t] > 0$. By right continuity, we have $\dot{k}(t) > 0$ on some (possibly smaller) right neighbourhood of $\bar{\nu}$. Since $c^*(t) = f[k^*(t);t] - \dot{k}^*(t) \geq 0$ and $0 \leq f[k^*(t);t] \rightarrow f[k^*(\bar{\nu});\bar{\nu}] = 0$ as $t \downarrow \bar{\nu}$, it follows that $c^*(t) < f[k^*(t);t]$ and $c^*(t) \rightarrow c^*(\bar{\nu}) = 0$. But then $y(\bar{\nu}) = \infty$ for $\omega \in A$, contrary to the supermartingale property. This contradiction shows that $\bar{\nu} = \infty$ a.s., so that $(c^*, k^*) \in \Pi^+$.

The main results of this Chapter may be summed up, somewhat imprecisely, in the following

OPTIMALITY THEOREM FOR THE STANDARD MODEL WITH INADA CONDITION.

Suppose that, for each (ω, t) , $f'(0; \omega, t) = \infty$ and $f(0; \omega, t) = 0$, also $f'(k; \omega, t) \geq 0$ and $f(k; \omega, t) \geq 0$ for each $k > 0$; further that, for each ω and each $k > 0$, $f'(k; \omega, \cdot)$ and $f(k; \omega, \cdot)$ are integrable on finite intervals of \mathcal{T} .

The Feasibility Lemma remains valid, apart from the assertions involving (4 7–9), if plans are replaced by extended plans and Properties (i)–(iv) by (ī)–(iv̄). The Existence Lemma and the Existence Criteria for Negative and Positive Utilities then remain valid for extended plans.

The Conditions for Optimality (3 28–33) are sufficient for an optimum in the set $\vec{\Pi}$ of extended plans. Under the additional assumptions (5.6) and (6 18), an optimal extended plan is proper and is a plan with positive capital; therefore (3 29–34) are necessary conditions for optimality in $\vec{\Pi}$.

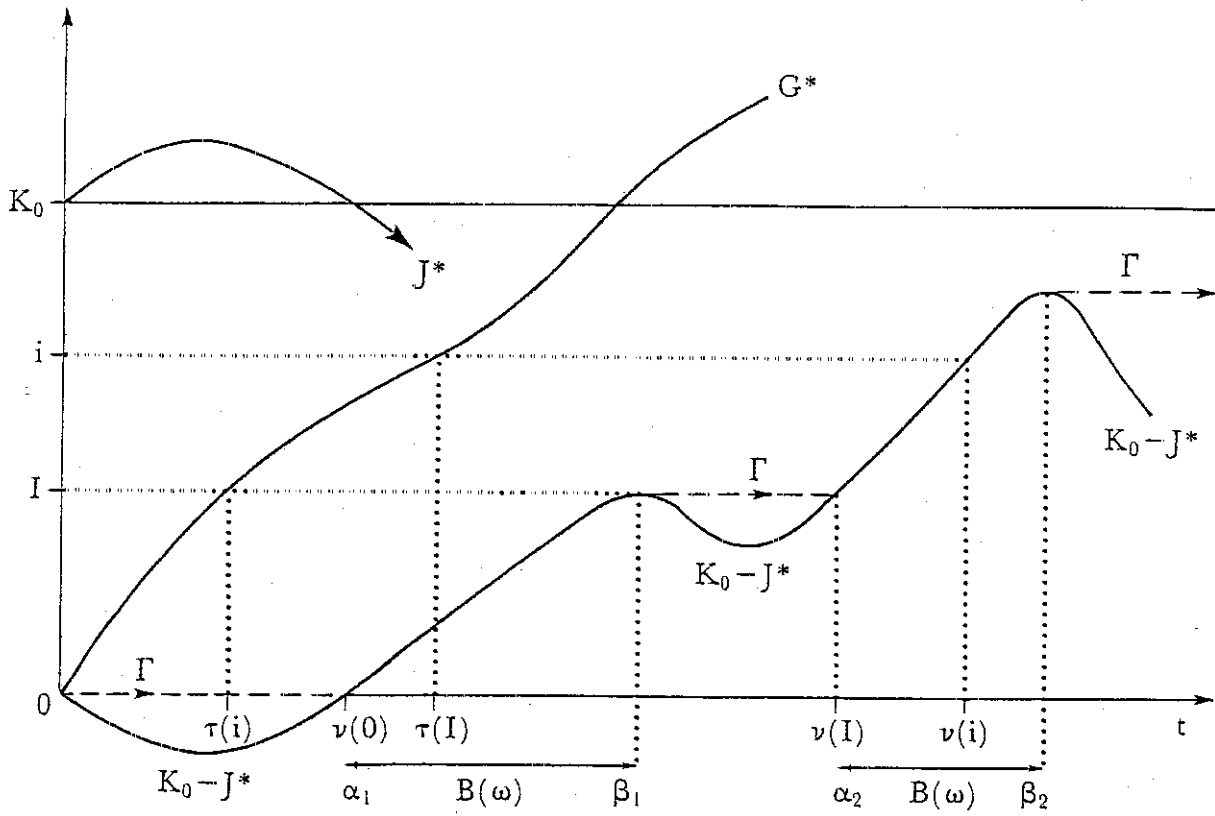


Figure 1

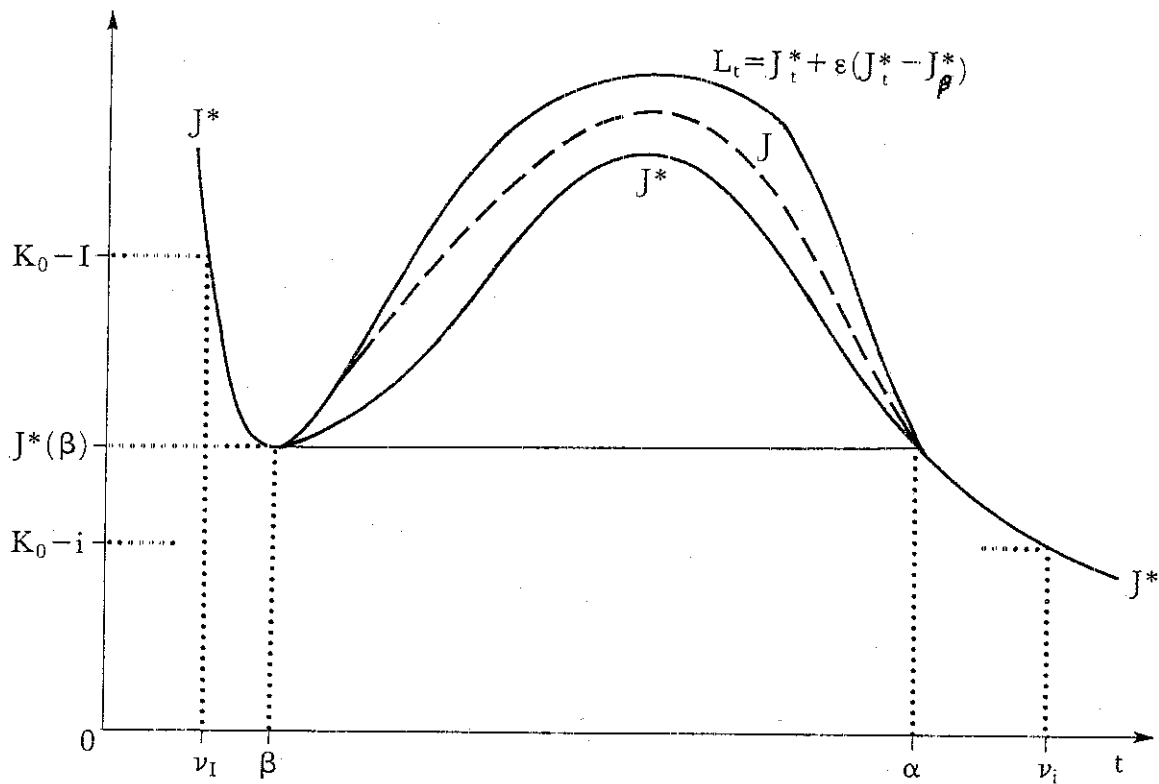


Figure 2

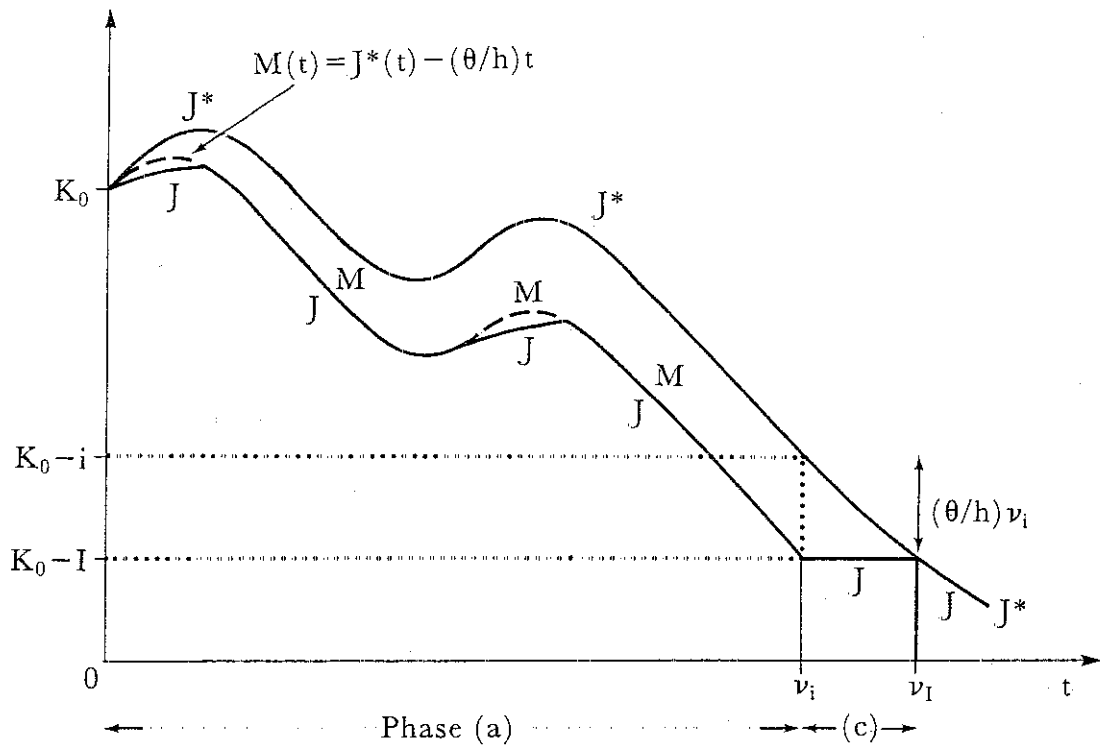


Figure 3: Case with $\nu_i < h, \Gamma = \nu_I, I = i + (\theta/h)\nu_i$

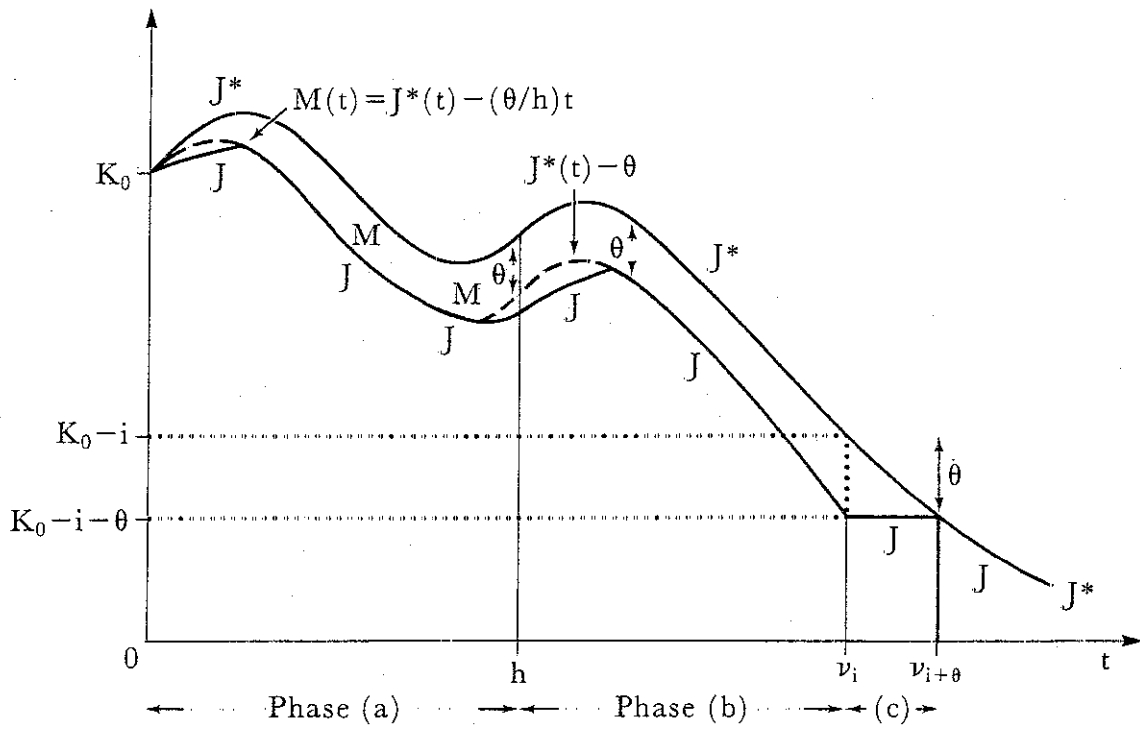


Figure 3: Case with $h < \nu_i, \Gamma = \nu_i + \theta$

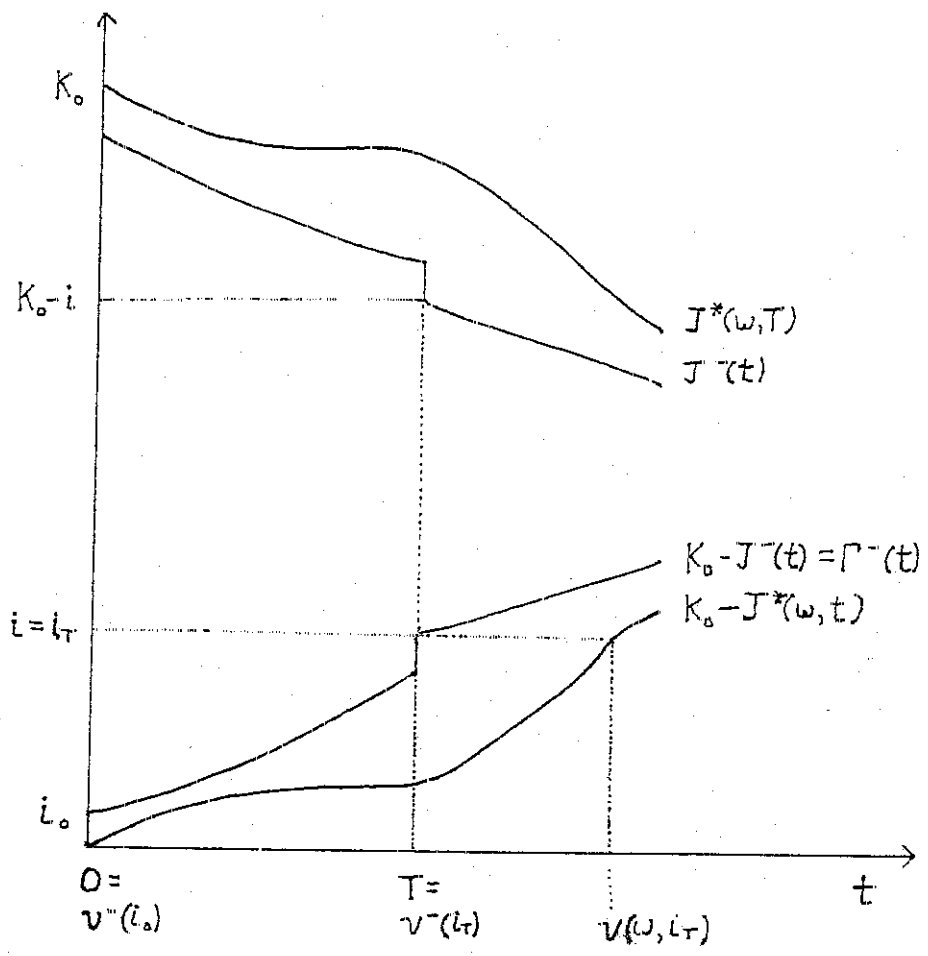


Figure 4

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