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# Regime Switching in Volatilities and Correlation between Stock and Bond markets

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# Regime Switching in Volatilities and Correlation between Stock and Bond markets*

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**Abstract**

This paper studies the correlation and volatilities of the bond and stock markets in a regime-switching bivariate GARCH model. We extend the univariate Markov-Switching GARCH of Haas, Mittnik and Paolella (2004) into a bivariate Markov-switching GARCH model with Conditional Constant Correlation (CCC) specification within each regime, though the correlation may change across regimes. Our model allows separate state variable governing each of the three processes: bond volatility, stock volatility and bond-stock correlation. We find that a separate state variable for the correlation is needed while the two volatility processes could largely share a common state variable, especially for the 10-year bond paired with S&P500. The "low-to-high" switching in stock volatility is more likely to be associated with the "high-to-low" switching in correlation while the "low-to-high" switching in bond volatility is likely to be associated with the "low-to-high" switching in correlation. The bond-stock correlation is significantly lower when the stock market volatility is in the high regime, but higher when the bond volatility is in its high regime.

Keywords: Regime-Switching, GARCH, DCC, CCC, Bond-Stock Market Correlation

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# 1 Introduction

The correlation between bond and stock markets plays an important role in asset allocation as well as risk management. In tranquil time, investors would choose to invest more in equity markets to seek higher returns while they might "flee" to bond markets in turbulent market condition. So accurate modeling of the bond-stock correlation can provide investors with better diversification or hedging benefit. The most common econometric approaches in modeling correlations of multiple assets are the multivariate versions of the general autoregressive conditional heteroskedasticity (GARCH) type models of Engle (1982) and Bollerslev (1986)[1]. The GARCH type models have a fixed persistence level for the covariance process throughout the whole sample. But there is evidence that volatility is less auto-correlated and has larger response to a shock when the volatility is in a higher level. For example, Hamilton and Susmel (1994) distinguish a low-, moderate-, and high-volatility regime in weekly stock returns, with the high-volatility regime being associated with economic recessions. But very few papers have studied regime-switching in correlation. Because of the hedging relation between the bond and stock markets, bond-stock returns might have distinct relation in tranquil and turbulent market conditions. So allowing regime-switching in both volatility and correlation might provide better insight into the dynamic properties of the comovement of the stock and bond markets. Another motivation for using regime-switching GARCH model is that forecast errors are much more costly in high-correlation state than in low-correlation state for a risk averse investor, which is shown by Engle and Collacito (2006). Regime-switching would be better in capture extreme swings in correlation.

In this paper, we investigate the bond-stock correlation in a regime-switching bivariate GARCH model that has separate state variable for each of the three latent processes: bond volatility, stock volatility and bond-stock correlation. The model allows us to study the intertemporal and contemporary relation among the three state variables, such as the correlations in different market conditions characterized by the volatilities of the two markets. In the literature, even not limited to the bond-stock context, there is no study on these kinds of effects. In a study of international stock markets, Haas and Mittnik (2007) estimate a diagonal regime-switching GARCH model[2]. But in their specification, all the individual variance and covariance processes share the same

[1] For a survey of multivariate GARCH, see Bauwens, Laurent and Rombouts (2006).

[2] The actually sepcification in their paper is BEKK model of Engle and Kroner (1995), which guarantees positive-definetiveness of the covariance matrix.

latent Markov state variable. Even though it can nest a similar model to ours by increasing the number of states in their model, it is very difficult to recover the regime-switching parameters of the correlation from their model. Regime-switching in covariance could be caused by switching either in variance or in correlation. Our model is able to separate these two effects by allowing a separate latent state variable for each latent process.

Using the returns from a one-year bond, ten-year bond, and the S&P500 index, we study the bond-stock correlations for bonds with different maturities. Our main findings include: First, the contemporary state correlations indicate that a separate state variable for the bond-stock correlation is needed while the two volatility processes could largely share a common state variable, especially for the 10-year bond paired with S&P500. Second, the "low-to-high" switching in stock volatility is more likely to be associated with the "high-to-low" switching in correlation while the "low-to-high" switching in bond volatility is likely to be associated with the "low-to-high" switching in correlation. As a result, the expected bond-stock correlation conditional on stock's high volatility state is significant lower than that conditional on stock's low volatility state. But results are opposite for those conditional on bond's volatility states. Since shocks to the bond price must be mainly related shocks to the discount factor, which will move the bond and stock prices in the same direction, the bond-stock correlation will increase as a result. However, when there are shocks to the stock price, they should be mainly related to the cash flow news, which affect the bond and stock prices largely in the opposite direction[3]. As a result, the bond-stock correlation will decrease. Finally, we find that when the bond market is in its high volatility state and the stock market is in its low volatility state, the estimates of bond-stock correlation in both high and low correlation-states are non-negative. But when both the bond and stock markets are in high volatility state, the bond-stock correlation has the highest correlation estimate at its high correlation-state and almost lowest correlation estimate at its low correlation-state. This might be attributed to the relative impacts of shocks to the cash flow and shocks to the discount factor on the pricing of bonds and stocks at different stages of the business cycle. According to the findings of Boyd, Hu and Jagannathan (2005), stocks are dominated by the cash flow effect during recessions while respond mainly to the discount rate news during expansions. As a result, the bond-stock correlation should

[3]Cash flow news ultimately will also affect discount factor indirectly. A positve cash flow news will increase both the growth expectation and the discount factor. The increasing growth expecation and discount factor work in opposite way on the price of stock. When the former dominates the latter, the bond and stock move in opposite way.

be positive during expansions and negative during recessions, which is also consistent with the findings of Anderson, Bollerslev,Diebold, and Vega (2007). But we also find large swings in the bond-stock correlation between positive and negative values after 2003, which can not be explained by business cycles, and possibly can be driven by the time-varying equity premium.

The paper is laid out as follows: Section 2 presents a review of the recent literature on bond-stock correlation and regime-switching GARCH, while section 3 covers the econometric methodology employed in this paper. In section 4, the data and estimation method is explained while some comparative quantities are also introduced. Section 5 presents the empirical results and section 6 concludes and discusses areas for further research.

## 2 Literature Review

The main econometric model in this paper is a multivariate extension of univariate Markov-switching(MS) GARCH model of Haas, Mittnik and Paolella (2004). The basic idea of their approach is to assume there are several parallel GARCH processes, and the volatility is switching among these processes. Haas and Mittnik (2007) also propose a multivariate extension of the model. Their extension models the covariance matrix directly, which is governed by a single regime-switching state variable. In our simple bivariate context, we are able to assume both volatilities and correlation have their own state variables. This generalization allows us to answer more interesting questions about the bond-stock correlation. Pelletier (2006) proposes a regime-switching constant correlation model and claim to have better fit than Dynamic Conditional Correlation (DCC) model of Engle (2002). But he does not apply the model to study the bond-stock correlations. We are similar in the correlation part of the model but we further allow regime-switching in each volatility series.

Regime-switching modeling of volatility is relatively new in literature. As mentioned above, GARCH type models are the main approaches in modeling volatility of financial asset returns. However, it has been argued that the close-to-unity of the persistence parameter estimate in many GARCH models could be due to the regime-switching in volatility. For example, Diebold and Inoue (2001) analytically show that non-persistent series with stochastic regime switching can appear to have strong persistence, or long memory. In the volatility context, Mikosch and Starica (2004) show that deterministic shifts in the unconditional variance do indeed drive the estimate of persistence parameter toward unity. A natural step forward is to combine a GARCH type model with Markov-switching mechanism, which has been widely adopted in economics since Hamilton (1989). Cai (1994) and Hamilton and Susmel (1994) are among the first to combine an ARCH model with Markov-switching. The reason why they restrict their attention to ARCH structures within each regime rather than a GARCH-type structure is due to the problem of path-dependence of GARCH, which makes maximum likelihood estimation infeasible (to be detailed later). The first attempt to combine the GARCH with Markov-switching is Gray (1996), which is essentially an approximation. Haas, Mittnik and Paolella (2004) propose an approach where regime-switching is both on parameters and the latent process, and so it is able to avoid the path-dependence problem while keeping the GARCH structure. In this paper, We adopt their approach and make a multivariate extension for our study of bond-stock correlations.

On bond-stock correlations, the past literature is mainly using the GARCH-type models. Cappiello, Engle and Sheppard (2006) propose an extension of the DCC model to allow for different structure parameters for different correlation pairs. They estimate the model using weekly returns on the FTSE All-World equity indices for 21 countries and government bond indices for 13 countries, and find a significant asymmetric affect in the conditional correlation between stock and bond return, and that the correlations tend to decrease following an increase in stock volatility or a negative stock return. De Goeij and Marquering (2004) employ a similar approach to Cappiello, *et al*. (2006). They estimate a diagonal VECH extension of Glosten, Jagannathan and Runkle (1993) model using daily returns on a short-term bond, a long-term bond , and the returns on the S&P 500 and NASDAQ indexes. They find strong evidence of time-varying conditional covariance between stock and bond market returns. Their results indicate that not only variances, but also covariances respond asymmetrically to return shocks. But they do not study the implied correlation dynamics from the VECH framework. Gulko (2002) studies the change in correlation between stock and bond market around period of market crashes. He finds that stock and bond correlations change from weakly positive in normal time to strongly negative during stock market crashes, which means treasure bonds could act as a hedging vehicle against stock market crashes. Li (2002) studies the impacts of various macroeconomic factors on stock and bond correlations. He first uses daily data to construct a non-parametric estimate of the correlation for a given month, the so-called "realized correlation", then regresses the realized correlation on various macro factors. He finds that long-term expected inflation and real interest rate has the largest (positive) impact on the stock-bond correlation. Baur and Lucey (2006) study the daily stock-bond correlations of seven European countries. They first estimate the DCC model of Engle (2002) to obtain a time series of estimated conditional correlations between each stock and bond return series, and then regress these estimated conditional correlations on some factors to study the sources of variation in correlations. Although suffering from econometric problems by using estimated correlations in the second-stage regression, they find that the correlation for US markets are about 0.5 at the normal time and -0.4 at the 1997 crisis. Finally, the recent work of Anderson, Bollerslev,Diebold, and Vega (2007) find that during the expansion the stock-bond correlations are positive albeit small, whereas during the contraction they are negative and large[4], which is largely consistent with our results.

# 3 Econometric Methodology

## 3.1 Univariate Markov-switching GARCH (MS-GARCH)

To model the regime-switching behavior in the univariate volatility process, we start with the *standard* two-state Markov-switching GARCH(1,1) model:

$$\begin{aligned} R_t &= \mu_t + \epsilon_t \\ \epsilon_t &\sim N(0, V_t) \\ V_t &= \omega_{s(t)} + \beta_{s(t)} V_{t-1} + \alpha_{s(t)} \epsilon_{t-1}^2 \\ P(s(t) &= i | s(t-1) = j) = P_{ij} \quad \text{with } i = 0,1 \text{ and } j = 0,1 \end{aligned} \tag{1}$$

This model allows the volatility process to have different dynamics with different persistence and level parameters in different regimes. And the latent Markov state variable $s(t)$ determines which regime prevails at each point in time. Model (1) look apparently simple and intuitively straightforward. But because the recursive structure of GARCH, current conditional variance is decided by the complete history of the state variable. So the (conditional) likelihood of observation at time t, $f_t(R_t|\theta; R_{[t-1,0]}, s(0))$, needs to be computed from $f_t(R_t, s(t)..., s(1)|\theta; R_{[t-1,0]}, s(0))$

[4] They calculated the unconditional correlations separately for the expansion period from July 1998 through February 2001 and the contraction period from March 2001 through December 2002. And determination of contraction and expansion could be found in their paper.

through integration on $s(1)$ up to $s(t)$. So as the number of observations increases, the integration dimension increases, which makes Maximum likelihood estimation of this Markov-switching GARCH is essentially infeasible in practice. To avoid the path-dependence problem while maintaining the GARCH feature of modeling, we adopt an approach proposed by Haas Mittnik and Paolella (2004). The idea, for a two-state model, is to model two parallel volatility processes, each of which has the GARCH dynamics. The latent Markov state variable determines which *process* is selected for each time. Its trick is switching on both parameters and processes while the standard MS model only switches on parameters. Formally, it is as follows:

$$
\begin{aligned}
R_t &= \mu_t + \epsilon_t \\
\epsilon_t &\sim N(0, V_{s(t),t}) \\
V_{i,t} &= \omega_i + \beta_i V_{i,t-1} + \alpha_i \epsilon_{t-1}^2 \quad i = 0,1 \\
P(s(t) &= i|s(t-1) = j) = P_{ij} \quad \text{with } i = 0,1 \text{ and } j = 0,1
\end{aligned} \tag{2}
$$

The benefit of this approach is to be able to avoid the path-dependence problem so we are able to compute the likelihood function without integration over the whole path of the volatility process. As suggested in Haas, Mittnik and Paolella (2004), the interpretation of $\beta_i$ and $\alpha_i$ are still the persistence parameters in each regime. But the unconditional level in each regime is not only characterized by the three parameters of the GARCH formula. The transition probability $P_{ij}$, which determines how often the process is in a regime, also determines the expected volatility level of a regime. The formula for computing the unconditional level of volatility is in the Appendix. Another property of this model is that one of the regimes could be non-stationary, i.e. $\beta_i + \alpha_i >= 1$, but the whole system could still be stationary if the transition matrix together with all the persistence parameters satisfies a certain condition, which is also detailed in the Appendix.

### 3.2 The main model: Markov-switching CCC-GARCH(MSCCC)

A direct multivariate generalization of model (2) is illustrated in Haas and Mittnik (2007). They base the regime-switching on the BEKK model of Engle and Kroner (1995) model. But their setup is not well-suited for our research purpose. Since volatility may have different dynamics to correlation, we would like to have an explicit process for the correlation, as well as a corresponding state variable. So we adopt the Constant Conditional Correlation (CCC) model of Bollerslev (1990) as the base model, and allow both the individual volatility processes and correlation process to switch among different states. While using the DCC Model as the base model would be more general, we will explain later why we use CCC as base model.

To be specific, we model the covariance of the bond and stock markets as three components: stock market volatility, $V_t^s$, bond market volatility, $V_t^b$, and the correlation between two markets, $C_t$. Let $S_t^s$ be a two-state variable governing the regime-switching of $V_t^s$, and a two-state variable $S_t^b$ governing that of $V_t^b$. Then for each of the four combinations of $S_t^s$ and $S_t^b$ ( hereinafter called "combined states"), we further split it into two "sub-states" which correspond to high and low correlation regimes within each combined state. Finally, to represent the regime-switching of the whole covariance matrix, we define a general state variable $S_t$ with eight states to govern the regime-switching of the covariance matrix[5]. Table 1 provides detailed definition on $S_t$. This specification allows us to investigate how the regime-switching of two volatility processes responds to each other, and between what levels the correlations would switch within each of the four combined sates.

With the definition of $S_t$, we formally define our general bivariate Markov-Switching Constant Conditional Correlation model (MSCCC). Let $\mathbf{R}_t \equiv \begin{bmatrix} R_t^s & R_t^b \end{bmatrix}^T$ denote the returns of stock index and bond, either 1-year bond or 10-year bond, and $\boldsymbol{\mu}_t \equiv \begin{bmatrix} \mu_t^s & \mu_t^b \end{bmatrix}^T$ be conditional mean vector, $\boldsymbol{\epsilon}_t \equiv \begin{bmatrix} \epsilon_t^s & \epsilon_t^b \end{bmatrix}^T$ be the residual vector. Then the main model is as follows:

$$
\begin{aligned}
\mathbf{R}_t &= \boldsymbol{\mu}_t + \boldsymbol{\epsilon}_t \\
\boldsymbol{\epsilon}_t &\sim N(\mathbf{0}, \mathbf{V}_{s(t),t}) \\
\text{with } \mathbf{0} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \mathbf{V}_{s(t),t} = \begin{bmatrix} V_{s(t),t}^s & C_{s(t),t}\sqrt{V_{s(t),t}^s V_{s(t),t}^b} \\ C_{s(t),t}\sqrt{V_{s(t),t}^s V_{s(t),t}^b} & V_{s(t),t}^b \end{bmatrix} \\
V_{i,t}^k &= \omega_i^k + \beta_i^k V_{i,t-1}^k + \alpha_i^k \epsilon_{t-1}^{k2} \quad i = 1, 2...8 \text{ and } k = s(stock), b(bond) \\
C_{i,t} &= C_i \quad i = 1, , 2...8 \\
P(s(t) = i|s(t-1) = j) &= P_{ij} \quad \text{with } \ i \text{ and j} = 1, 2 \cdots 8
\end{aligned} \tag{3}
$$

[5] The GARCH parameters of stock volatility are the same in states $\{1; 2; 3; 4\}$ or $\{5; 6; 7; 8\}$ , the bond volatility has the same set of GARCH parameters in $\{1; 2; 5; 6\}$ or in $\{3; 4; 7; 8\}$ .And we do not restrict the low and high correlation level in those four combined states to be equal. So there are eight possible values for correlation.

Model (3) is different from Haas and Mittnik (2007) mainly in two ways. First, our base model is Constant Conditional Correlation model (CCC), which means that we explicitly model the correlation. This provides us direct observation of how correlation interacts with the volatilities in both markets. As mentioned earlier, Haas and Mittnik (2007) use BEKK as their base model. Although their approach is very flexible in that it allows any interaction of the squared terms and cross product, it is very difficult to recover the regime-switching parameters of the correlation from their model. Their model could only tell how the covariance switches among different regimes, instead of the correlation. The covariance could appear to have a jump because of either one or both of the volatility processes have a jump even when the correlation remains constant. Our setup enables us to disentangle these two effects neatly. Secondly, our setup allows us to do two-stage maximum likelihood estimation, which is useful in higher dimension problem or robustness for complex model estimation. The justifications for the two-stage estimation of the model are similar to those of CCC or DCC[6].

### 3.3 Discussion on model specification

A common concern about regime-switching models regards the number of regimes. By allowing each latent process to be governed by its own state variable, our joint state variable $S_t$ needs to have eight states to characterize the joint model. Of course, as mentioned before, our model is much more restricted than the general 8-state model without restrictions since the GARCH parameters in our model are only switching between two sets of values. But it is still natural to consider smaller models by shrinking the general one, which is a way of assessing the importance of having different state variable for different process. We could restrict $S_t$ in various ways, which would result in different restricted models. For example, the simplest restriction is to let the volatility processes of both markets and the correlation share the same two-regime state variable. Alternatively, we could allow two volatility processes share the same state variable, but the correlation has a separate state variable. The full set of specifications we considered in this paper are detailed in Table 2.

[6] We have not formally derived the stationarity conditions for the joint model. But since the correlation part of the model does not affect stationarity, the stationarity conditions for the joint model are similar to the conditions for the univariate model in Haas, Mittnik and Paolella (2004). Basically if both of the non-stationary states of the two volatility processes are transient and have high probability to switch to the stationary states, then the joint model is stationary.

On the choice of the "base model", a more general framework would be building the regime-switching upon the DCC, which allows for time-varying correlation within each correlation regime. But because DCC has a volatility-standardized term in its recursive formula, combining DCC with the regime-switching GARCH would hugely complicate the already complicated model structure. And as we will see in the result section below, three out of the four combined states are only visited in very low frequency. Adding 16 extra DCC parameters would not likely produce a great improvement. And we believe eight correlation states should capture well the variation in the bond-stock correlation. As results from Pelletier (2006) suggest, switching constant correlation is adequate to capture the dependence in the correlation.

# 4 Estimation and Comparative Statistics

## 4.1 Data and estimation

We use daily returns from both the stock and bond markets to estimate the model. Stock market returns are computed as daily returns on S&P500 index from WRDS. For the bond market, two series of daily returns are computed from bond yields on 10-year and 1-year US government bond, which are available from federal reserve bank in Chicago. The sample period is from January 5, 1986 to December 29, 2006. Table 3 presents the statistics summary of the two return series.

Before estimation of the volatility models, we filter each return series by an ARMA(p,q) model, with p and q chosen to have the best fit according to Bayesian Information Criterion (BIC). The optimal filtering orders are also presented in Table 3. Because bond returns are computed from bond yield, which itself is computed by some interpolations of "rolling bond" returns, there are some autocorrelations in the bond returns, which does not necessarily mean there are any arbitrage opportunities in the market.

After we estimate the conditional mean by an ARMA model, we can estimate the main Markov-Switching volatility model (3). The estimation are done in a two-step procedure. First, for each of the two time series, a univariate two-regime MS-GARCH model (2) is estimated. Then with $\varpi_k^i, \beta_k^i$ and $\alpha_k^i$ fixed at the estimates from the previous stage, the rest of the parameters of model (3) are estimated.

With 6 GARCH parameters for each volatility process, 8 correlations and $7 * 8 = 56$ transition probabilities, the general model has 76 free parameters. The two-step procedure could separate the problem in a lower dimension sub-problem, which could provide robustness in estimation. However there are still 64 free parameters in the second stage estimation, and the numerical standard errors of the estimates are difficult to compute. To get a more robust standard error estimate of the general transition matrix, we further fix those transition probabilities with second-stage estimate less than 0.0001 to be zero. This results in a estimation problem with only half of the parameters. The specific maximum likelihood estimation algorithm is standard. It combines a Markov-switching estimation algorithm with GARCH likelihood calculation. The details can be found in the Appendix.

## 4.2 Comparative statistics

Once we finish the estimation of the joint model, we analyze the intertemporal and contemporary relationships among the three state variables and three latent processes by the following comparative statistics. All of them are mainly derived from the general transition matrix of the main model, and their calculations are detailed in Appendix.

**Conditional transition probabilities**

As ways of analyzing the information contained in the transition matrix, we compute two types of "conditional transition probability". The first one is $t-$ *conditional transition probability*, which is conditional on time $t$ information. It is the probability of one state variable's regime-switching from $t$ to $t+1$ given the state of another state variable at time $t$. For example, $\Pr(S^b_{t+1} = high|S^b_t = low; S^s_t = high)$ is the transition probability of bond volatility switching from low to high regimes conditional on the current stock volatility state being in high regime. So this kind of conditional transition probabilities can reveal the forecasting benefit of joint modeling. If this transition probabilities are significant different from the marginal transition probabilities implied by the transition matrix, then by joint modelling, we have better forecasting power for the future state of the considered variable. Due to the focus of the paper, we only compute the $t-$conditional transition probabilities of the bond-stock correlation conditional on volatility states in the bond and stock markets.

The second one is $t+1-$*conditional transition probability*, which is conditional on information of up to time $t+1$. It is the probability of one state variable's regime-switching from $t$ to $t+1$ given that there is a regime-switching from $t$ to $t+1$ in another state variable. For example, $\Pr(S^b_{t+1} = high|S^s_{t+1} = high; \{S^s_t = low\ ;\ S^b_t = low\})$ is the transition probability of bond volatility switching from low to high regime conditional on stock volatility's switching from low to high regimes. These conditional transition probabilities are ways of measuring how two considered state variables jump together in their own directions. While the $t - conditionals$ are on intertemporal relationships, $t + 1 - conditionals$ are evaluating the contemporary relationships among the state variables.

#### Conditional expected correlations

Another type of statistics we are interested in is the expected correlation conditional on the contemporary volatility state in either market. For example, $E(C_t|S_t^s = low)$ is the expected correlation given a low volatility state in stock market. In the result section, we further compute the expected correlations conditional on regimes of both volatility state variables, such as $E(C_t|S_t^s = low; S_t^b = low)$. So these quantities can tell us the bond-stock correlation given all four "combined states", and such have important implications for portfolio diversification. Similar relation between correlation and volatility can not be measured in the traditional GARCH-type models.

As we can see, all above "conditional quantities" can be derived only when we allow separate state variable for each of the three latent processes. So we can not derive similar quantities from the model of Haas and Mittnik (2007). And the empirical results of the paper will be mainly focusing on these aspects of the model.

## 5 Empirical Results

In this section, we first present the estimation results on the univariate model. Second, the results on the joint model are demonstrated, mainly around the "conditional quantities" introduced previously. Third, we compare the filtered bond-stock correlation from our model with that estimated from the DCC model. Finally, we discuss the results on goodness of fit for various restricted model.

### 5.1 Results on univariate model estimations

The estimation results of the univariate MS-GARCH model (2) using S&P500 daily returns are presented in Table 4, while Table 5 and 6 show the results for 1-year bond and 10-year bond returns. The parameter estimates of the conditional mean are not shown, which will also applies to other tables of the results. The same model has been estimated by Mittnik and Paolella (2004) on foreign exchange returns. We find similar results on equity return data. First, there is a significant difference in the persistence levels of the GARCH processes of the two regimes. When the volatility process is in the low-volatility regime, the GARCH coefficient $\beta$ is larger, but far away from unity, which is usually the case when one fits a simple GARCH model to daily stock return data. The ARCH coefficient $\alpha$ is much smaller than that of the high-volatility regime, less than one-tenth of the latter[7]. So in tranquil periods, the volatility remains very stable, and the shock has very small effect on the conditional variance of the next day. But in volatile periods, the impact of today's shock has much larger effect on next day's conditional variance. The total persistence level of each process are measured by the sum of $\alpha$ and $\beta$. We could see in the table that in the low-volatility regime, the GARCH process has persistence level smaller than unit, which means it is stationary. But as found in Mittnik and Paolella (2004), the volatile regime has larger-than-unity persistence level, so the GARCH process of the high-volatility regime itself is not stationary. The stationarity of the whole system is determined jointly by the transition matrix and the two sets of persistence parameters in both regimes, which is summarized by the largest eigenvalue of the matrix $M$, which is explained in Appendix. We can see that all three univariate models are stationary with $eig(M)$ less than one. The stationary probability of low-volatility regime is higher than those found in exchange return, which is mostly less than 0.8. The expected level of volatility in high-volatility regime is about 4 times the expected level of low-volatility regime.

Table 5 and 6 report the estimation results of univariate MS-GARCH on the two bond return series. The difference between the two regimes is similar to that found in stock index return. Higher level of persistence, smaller $\beta$ estimate and larger $\alpha$ estimate are associated with high-volatility regime The persistence levels in the high-volatility regime of both bond series are larger than unity as well. For 10-year bond returns the difference in $\beta$ estimates is not significant. The difference in $\beta$ estimates for one-year bond is small as well, compared to that found in stock market. So for bond market, the volatility dynamics of the two regimes mainly distinguish from each other in their response to shocks. For one-year bond returns, the ratio of expected volatility in the high regime

[7] More precisely, the comparable quantity should be the standardized impact coefficient, which is the ARCH coefficient devided by the t-1 conditional variance of the regime. Considering the ratio of unconditional level of the high-volatility regime to that of low-volatility regime is much smaller than the ratio of their ARCH coefficients, the standardized impact of shock in high-volatility regime is still much higher.

to the low regime is much higher than that of the stock index return or ten-year bond return. Thus the one-year bond would experience much larger jump upon regime-switching than stock return, while 10-year bond behaves more similar to stock return.

### 5.2 Results on estimations of the joint model

We have studied the difference of the bond and stock markets in the univariate MS-GARCH model in last section. Now we discuss their relation in the joint model. The joint model allows us to see how the two state variables correspond to each other, and how the bond-stock correlation behaves within the 4 "combined states". First, we present the main estimation results of model (3) using two pairs of stock-bond returns in Table 7 and Table 8. The first two blocks in the tables are the regime-switching GARCH parameter estimates for the stock and bond series in the joint estimation. The third block reports the constant correlation levels in each of the eight regimes. The last block is the transition matrix estimate, together with the stationary probabilities of each regime. In both tables, the estimates of GARCH parameters are similar to those from the univariate estimations, but with smaller standard deviations due to efficiency increase by using more data. And here we focus on the results on the joint part of the model. Then in Table 9 to Table 12, some derived statistics are presented to better understand the intertemporal and contemporary relationships among the three latent processes. Finally, we have some discussions on the results of the bond-stock correlations in a simple discounted cash-flow model for the bond and stock prices.

#### 5.2.1 Stationary probabilities and correlations within the combined states

First, we can indirectly see how likely the two volatility processes switch together by comparing the stationary probabilities of the four "combined states". These can be obtained simply by summing up the stationary probabilities of $S_t$ states two by two. For the one-year bond pair, in about 76% of time both the stock and bond return are in low-volatility regime, and in about 3.6% of time both of them are in high-volatility regime. So there is about 20% of time that they are in different volatility regimes. For the 10-year bond pair, the proportions of time that bond and stock volatilities are in the same regime is similar to the one-year bond case. The main difference is that there is a much larger percentage, about 8%, of time when both bond and stock are in high-volatility regimes. This feature of ten-year bonds is a disadvantage for diversification. Large stationary probability of two volatilities in the same regime for both pairs indicates that two volatility state variables correspond to each other to a certain extent, which will be re-confirmed by the following results. We can also see the smoothed probabilities of each state across time in Figure 1 and 2[8].

Next, we compare the levels of correlation within the four "combined states". The two correlation levels in the two "sub-states" of all four "combined states" are all significantly different. These results indirectly suggest a separate state variable is needed for both stock-bond correlation processes. Taking into account of estimation errors, only in the "low-stock-high-bond" state are both of the two correlation estimates non-negative. And in the "high-stock-high-bond" state, the high and low correlation estimates are the highest and lowest respectively among the eight correlation estimates for the one-year bond paired with stock. Similar results are found for the 10-year paired with the stock. We will further discuss these findings in more details later. Finally, for the one-year bond pair, taking into account of standard errors, we could see there are mainly four correlation regimes: one extremely high at about 0.8, one negative at -0.4, and the other two in between are around 0.35 and 0.05. A similar state reduction for correlation could be done for ten-year bond pair, and we examine this formally in section 5.5.

#### 5.2.2 Conditional transition probabilities

**Results on the $t-$conditional transition probabilities** We study the intertemporal relations between correlation state variable and the volatility state variables in two market through the $t-$ *conditional transition probabilities* reported in Table 9. Comparing these numbers with the "unconditional transition probabilities" in Table 10 can reveal the forecasting benefit of allowing correlation to have separate state variable. First, we look at the transition probabilities of $C_t < 0$ to $C_{t+1} \geqslant 0$ given different volatility states in different markets. The correlation transition probability is higher when conditional on high volatility state in either bond or stock than conditional on low volatility state. And the difference is more significant for the one-year bond case. Comparing the information contained in different markets, the correlation transition probabilities conditional on low volatility states in two markets are not significant different for both pairs. However, correlation transition probability conditional on high volatility states in one-year bond is larger than that conditional on high volatility state in stock market. And for 10-year bond paired with stock market, conditioning on either market produces similar correlation transition probabilities. So it

[8] The "smoothed" state probabilities of each state across time in the whole sample is computed as in Kim (1994), which is recited in Appendix.

seems that only correlation between the 1-year bond and stock has very different conditional "low to high" transition probabilities for conditioning on different markets or different volatility states, and the high volatility state of 1-year bond is mostly likely to be followed by a jump up in correlation.

Next, we turn to the transition probabilities of $C_t \geqslant 0$ to $C_{t+1} < 0$ given various volatility states. The correlation transition probability conditional on bond's high volatility state is significant higher than that conditional on bond's low volatility state for both pairs. But the two volatility states in stock market has similar information on the correlation transition probability, which is different from above "high-to-low" case. Similar to the "high-to-low" case, there is no significant difference in conditioning on the low volatility state of either stock or bond market for both pairs, and conditioning on high volatility states in both bond markets has higher correlation transition probabilities than conditioning on high volatility state in stock market.

To summarize, high volatility states of bond markets, especially 1-year bond, imply higher t-conditional transition probabilities of bond-stock correlation for both "low-to-high" and "high-to-low" cases. And the differences between these numbers and those "unconditional transition probabilities" in Table 10 are statistically significant. But there is no significant difference in conditioning on today's high and low volatility states in stock market in terms of predicting regime-switching of the bond-stock correlation.

**Results on the $t+1-$conditional transition probabilities** The contemporary relations among the three state variables can be measured through the estimates of $t+1-$ *conditional transition probabilities* reported in Table 11. The first row of both panel is on how correlation switches conditional on a regime-switching in bond or stock volatility, which can again be compared with the unconditional ones in Table 10. For both bond pairs, when stock market volatility jumps from low to high, the transition probabilities of correlation switching from high to low is significantly higher than the unconditional ones while "high-to-low" correlation (conditional) transition probabilities are lower than the unconditional ones. However, conditioning on a "low-to-high" jump in bond volatility has the opposite results: the correlation (conditional) transition probability is higher, than the unconditional one for the "low-to-high" switching in correlation, and lower for "high-to-low" case. So the "low-to-high" switching in stock volatility is more likely to be associated with "high-to-low" switching in correlation while the "low-to-high" switching in bond volatility is likely to be associated with "low-to-high" switching in correlation. The second row of both panels evaluate how both of the volatility state variables switches conditioning on a regime-switching in bond-stock correlation, which can be compared with the "unconditional (or marginal) transition probabilities" in Table 4-6. The results are similar to those of first row since they inherit the main properties of the quantities in first row by definition.

The last row of both panels describes how likely the two volatility processes would jump in the same direction together. For ten-year bond, it tends to jump with stock in either direction with high probability. But for one-year bond, it has high probability to jump down with stock, but very low probability to jump up together. So the one-year bond does not always react the information from stock market. But if it does react, probably to extreme shock, it resolves the uncertainty together with stock market. Comparing these numbers to those in Table 4-6 also indicates that only the case of "low-to-high" conditioning in the 1-year bond pair does not provide significant more information than the marginal ones. So the two volatility state variables correspond to each other quite well, especially for the 10-year bond pair.

The contemporary relationship revealed in above results indicate that the correlation state variable has very different switching behavior from the volatility state variables while two volatility processes can largely share the same state variable, especially for the 10-year bond paired with S&P500.

### 5.2.3 Conditional expected correlations

An important feature of the general model is that it is able to capture the contemporary relations among the correlation and volatilities in two markets. We quantify these relations through the conditional expected correlations which are reported in Table 12.

First, we compare the expected correlations conditional on a single volatility state variable, which are presented in the last row and column of both panels. For both pairs, the expected correlation conditional on stock's high volatility state is significant lower than that conditional on stock's low volatility state, although the difference is significant only for the 1-year bond pair. But the results are opposite for those conditional on bond volatility states. Judging from the differences in conditioning on high and low volatility-states, it seems that stock market has larger contemporary impact on the bond-stock correlation than bond market. A related observation is that only the expected correlation between 1-year bond and stock conditional on the high volatility state in stock market has a negative point estimate. This reflects that during "flight to safety" period, investors transfer money from stock market more to bonds with shorter maturities than to bonds with longer maturities. This is again due to the fact that long-term bond is more like a stock.

Table 12 also presents results on the "joint-conditional" expected correlations. These results reveal the expected correlations in four "combined states", which is basically to aggregate the eight correlation estimates from Table 7 and Table 8 into four correlation levels. For both pairs, the "high-stock-low-bond" state has the lowest expected correlation while the "high-bond-low-stock" state has the highest one. And in the most volatile market condition characterized as the "high-stock-high-bond" state, the expected correlation is surprisingly low, even lower than the peaceful "low-stock-low-bond" state. In the high stock volatility state, the difference in expected correlations is not significant between conditioning on the high and low volatility states of the bond market. But the difference is significant when the stock market is in low volatility state. In contrast, the difference in the expected correlation is more significant between conditioning on stock's high and low volatility states when the bond market is in the high volatility state, especially for the 1-year bond case. So when the stock market is very volatile, the bond-stock correlation is mainly dominated by the stock market, usually with low correlation level. And the effect of the stock volatility on the bond-stock correlation becomes more prominent when the bond market is also in volatile condition.

### 5.2.4 Discussions on the bond-stock correlations

Above empirical findings can be explained in a simple discounted cash-flow model for both the bond and stock prices. The bond price is the expected value of the discounted factor while the stock price is co-driven by the future cash flow and discount factor[9]. So the high bond-volatility state is when the expected discount rate has high volatility. But for the stock volatility, it comes from either one or both of the two shocks, cash-flow news and discount factor news. The recent findings of Boyd, Hu and Jagannathan (2005) show that the stock price is dominated by the cash flow effect during recessions while it mainly responds to discount rate news during expansions. Our results are consistent with their findings. In the "high-bond-low-stock" state, the shock must come form discount factor which will make the bond and stock prices move in same direction irrespective

[9] Different from discount factor for bond, the discount factor for stock is sum of equity premium and the risk-free rate. But for ease of illustration, we assume they are the same.

of business cycle. This is confirmed by the result in Table 7. Only in the "high-bond-low-stock" state can we have positive point estimates for both the high and low correlation states[10]. So when we average them across the whole sample, we have highest unconditional correlation in the "high-bond-low-stock" state as in Table 12.

However, for the "high-stock-high-bond" state, we have two different situations since the source of the stock volatility is dependent on the business cycle. During expanssions, both the bond and stock prices are mainly driven by discount rate news. So we will have highest correlation during expansions; But during recessions, the stock price is mainly dominated by the cash flow effect, which works in the opposite direction for bond price as rising future cash flow will increase the discount rate and lower the bond price. As a result, we have lowest bond-stock correlation during recessions. This is confirmed by the regime correlation estimates in Table 7. And when we average the correlations across the whole sample, the correlation is close to zero in the "high-bond-high-stock" state as in Table 12. Results in the other two "combined states" can be explained similarly by the relative effect of shocks to discount factor and cash flow in the discounted cash-flow model. If we further differentiate the discount factors for the bond and stock prices, then the time-varying equity premium will also help to explain the time-varying bond-stock correlations. This is especially relevant for explaining the bond-stock correlation in sample after 2003 when it frequently swings between large positive values and large negative values, which can not be explained by the business cycle. A structural model is needed to measure the effect of time-varying equity premium on the bond-stock correlation, which is out of the scope of this paper.

### 5.3 Estimated correlations from the MSCCC model

Although we assume constant correlation within each regime, the correlation estimated from our model has quite rich dynamics. Figure 3 and 4 provides the view of the correlation dynamic for both pair over the past 20 years. Both correlations appear to jump to a negative level when there is a crisis in equity market. As mentioned previously, since after 2003 negative correlations happen very frequently, irrespective of having a stock market crisis or not. We believe it might be because that the time-varying equity premium is becoming more important for the bond-stock correlation after 2003. As a comparison, we plot the MSCCC correlations together with correlations estimated

[10] Since 10-year bond have stock-like characteristics, we mainly focus on 1-year bond in expaining our findings in the discounted cash flow model.

from DCC. As shown in Figure 5.1 and 6.1, the main difference between DCC's correlations and the MSCCC correlations is that the MSCCC correlations reveal more extreme, both positive and negative, correlation level than the DCC. And forecast error in such situation are most costly as pointed out by Engle and Collacito (2006). As another view for comparison, we fit the DCC correlations by an 4-order polynomials of MSCCC correlations in Figure 5.2 and 6.2. We could see DCC correlation usually has downward bias for day with high smoothed correlation, and upward bias for day with low smoothed correlation. Although our model is not developed for forecasting, it does point out potential way of improvement for the traditional GARCH type dynamic correlation models, especially for modeling bond-stock correlation.

### 5.4 Results on restricted models

Although all above analysis is based on the general model, we also estimate some restricted models specified in Table 4 to compare their goodness of fit[11]. Table 13 reports the goodness-of-fit results on all considered models. We rank all the model according to AIC and BIC. For one-year bond pair, the most general model MSCCC(2,2,8), is the best model according to AIC. But according to BIC, which penalize heavily extra parameters, MSCCC($\overline{2}$,$\overline{2}$,4) is the best model, in which two volatility process share the same Markov state variable but the correlation has a separate one. The same best models are identified under the two criterion for ten-year bond paired with S&P500. So generally according AIC, we do need three separate state variables for volatilities and correlation, which justifies our main model. According to BIC the most general model still rank second for the ten-year bond case, but very poorly for the one-year bond case. The MSCCC($\overline{2}$,$\overline{2}$,4) is always the best under BIC and the second under AIC. So while a separate state variable for correlation is very important, stock and bond volatilities could share the same state variable without too much loss in goodness of fit. This seems to reflect the information could be transmitted across two markets efficiently within a day. The good performance of MSCCC($\overline{2}$,$\overline{2}$,4) could also make it a good candidate as a multivariate regime-switching GARCH model in forecasting exercise since it is quite parsimonious, having only 28 parameters.

[11] We do not conduct formal tests for the number of regimes. As indicated in Hansen(1992) and McLachlam and Peel (2002), the standard likelihood ratio test is not valid for testing the number of regimes. And test on regime number still remains very difficult even in much simpler setups. A more recent study on regime number test is Cho and White(2007).

## 6 Concluding Remarks

To study the bond-stock correlation and its relation with volatilities in the two markets, we extend the univariate Markov-Switching GARCH of Haas Mittnik and Paolella (2004) into a bivariate MS-GARCH model with Conditional Constant Correlation (CCC) specification. Our specification allows a separate state variable governing each of the three processes: bond volatility, stock volatility and bond-stock correlation, which is different from other multivariate generalizations of their model. We estimate our model using two pairs of daily returns: S&P500 with one-year bond and S&P500 with ten-year bond, to study the difference in bond-stock correlation for different bond maturities.

From the univariate model estimation, we find both volatility processes switch between a stationary and a non-stationary state while the whole system is till stationary, similar to the findings in Haas Mittnik and Paolella (2004). By allowing different latent process to have separate Markov state variable, the results from the joint model estimation has the following main findings: First, we find that a separate state variable for the bond-stock correlation is needed while the two volatility processes could largely share a common state variable, especially for the 10-year bond paired with S&P500. Second, the "low-to-high" switching in stock volatility is more likely to be associated with the "high-to-low" switching in correlation while the "low-to-high" switching in bond volatility is likely to be associated with the "low-to-high" switching in correlation. As a result, we show that the expected bond-stock correlation conditional on stock's high volatility state is significant lower than that conditional on stock's low volatility state. But the results are opposite for those conditional on bond's volatility state. Finally, we find that when the bond market is in its high volatility state and the stock market is in its low volatility state, the estimates of bond-stock correlation in both high and low correlations states are non-negative. But when both bond and stock markets are in high volatility state, the bond-stock correlation has the highest correlation estimate at its high correlation-state and almost lowest correlation estimate at its low correlation-state. This might be attributed to the relative impacts of shocks to the cash flow and shocks to the discount factor on the pricing of bonds and stocks at different stages of business cycles. But we also find large swings in the bond-stock correlation between positive and negative values after 2003, which can not be explained by the business cycle, and possibly can be driven by the time-varying equity premium.

The proposed model assumes constant correlation within each state. The estimate correlations suggest that smaller number of correlation states is needed if we allow time-varying correlation within each state. So modeling the correlation as two-state regime-switching DCC may be adequate to capture both the structural break in the level of the correlation and the variation within each regime. Future work will consider how to effectively incorporate DCC in this regime-switching GARCH framework in a way that is parsimonious and numerically tractable. A better covariance forecasting model could also be developed in a similar approach. Another interesting research topic will be comparing the filtered states with the macro news announcement. For example, the state-7 should be related to the discount rate news in expansion while state-8 should be related to cash flow news in recession. Finally, our model might also be applied to study the contagion effect in international markets.

## Appendix: Details

### A1: details of maximum likelihood estimation

The general model could be estimated by maximum likelihood method. The algebra is standard, as in Hamilton (1994). Let $\Theta = \{\omega_i^k, \alpha_i^k, \beta_i^k, P_{ij,} C_i\}$ *with* $i = 1, 2...8$ and $k = s$ or $b$, be the parameter space of model (3). And $R_{[t,0]}$ is the whole history of return from 0 to $t$. Then the likelihood of the observed return pairs $\begin{bmatrix} R_t^s & R_t^b \end{bmatrix}^T$ at time $t$ can be written as:

$$
\begin{aligned}
& f_t(R_t^s, R_t^b|\Theta; R_{[t-1,0]}^s, R_{[t-1,0]}^b) \\
= & \sum_{i=1}^N f_t(R_t^s, R_t^b; S_t = i|\Theta; R_{[t-1,0]}^s, R_{[t-1,0]}^b) \\
= & \sum_{i=1}^N f_t(R_t^s, R_t^b|\Theta; S_t = i; R_{[t-1,0]}^s, R_{[t-1,0]}^b) * \Pr(S_t = i|\Theta; R_{[t-1,0]}^s, R_{[t-1,0]}^b) \\
= & \sum_{i=1}^N f_t(R_t^s, R_t^b|\Theta; S_t = i; R_{[t-1,0]}^s, R_{[t-1,0]}^b) * [\sum_{j=1}^N \Pr(S_t = i|\Theta; S_{t-1} = j) * \Pr(S_{t-1} = j|\Theta; R_{[t-1,0]}^s, R_{[t-1,0]}^b)] \\
= & \sum_{i=1}^N \eta_t(i) * [\sum_{j=1}^N P_{i,j} * \xi_{t-1|t-1}(j)]
\end{aligned}
$$

where $\eta_t(i) = f_t(R_t^s, R_t^b|\Theta; S_t = i; R_{[t-1,0]}^s, R_{[t-1,0]}^b)$, $P_{i,j} = \Pr(S_t = i|\Theta; S_{t-1} = j)$, and $\xi_{t-1|t-1}(j) = \Pr(S_{t-1} = j|\Theta; R_{[t-1,0]}^s, R_{[t-1,0]}^b)$.

Then $\eta_t(i)$ could be calculated as the usual GARCH likelihood function for the subset of the parameters $\Theta$ where $S_t = i.$, given the whole history of past returns. And $P_{i,j}$ is just the entries of the transition matrix, which is part of $\Theta$. To compute the remaining $\xi_{t-1|t-1}(j)$, let

$$\xi_{t|t-1}(j) = \Pr(S_t = j|\Theta; R_{[t-1,0]}^s, R_{[t-1,0]}^b)$$

Then $\xi_{t|t}(j)$ is computed as:

$$
\begin{aligned}
\xi_{t|t}(j) &= \Pr(S_t = j|\Theta; R^s_{[t,0]}, R^b_{[t,0]}) \\
&= \frac{\Pr(S_t = j; R^s_t, R^b_t|\Theta; R^s_{[t-1,0]}, R^b_{[t-1,0]})}{\Pr(R^s_t, R^b_t|\Theta; R^s_{[t-1,0]}, R^b_{[t-1,0]})} \\
&= \frac{\Pr(R^s_t, R^b_t|\Theta; S_t = j; R^s_{[t-1,0]}, R^b_{[t-1,0]}) * \Pr(S_t = j|\Theta; R^s_{[t-1,0]}, R^b_{[t-1,0]})}{\sum_{j=1}^{N} f_t(R^s_t, R^b_t|\Theta; S_t = j; R^s_{[t-1,0]}, R^b_{[t-1,0]}) * \Pr(S_t = j|\Theta; R^s_{[t-1,0]}, R^b_{[t-1,0]})} \\
&= \frac{\eta_t(j) * \xi_{t|t-1}(j)}{\sum_{i=1}^{N} \eta_t(j) * \xi_{t|t-1}(j)}
\end{aligned}
$$

So given $\eta_t(j)$ and $\xi_{t|t-1}(j)$, we could compute $\xi_{t|t}(j)$. Finally, the Markov property of the state variable results in the following relation between $\xi_{t|t-1}$ and $\xi_{t-1|t-1}$ :

$$\xi_{t|t-1}(j) = P * \xi_{t-1|t-1}$$

So the likelihood function could be computed by iterating $\xi_{t|t-1}$ and $\xi_{t-1|t-1}$ from t=1 to T. And the maximum likelihood estimator of $\widehat{\Theta}$ is simply as:

$$\widehat{\Theta} = \arg\max_{\Theta} \sum_{t=1}^{T} \log\{f_t(R^s_t, R^b_t|\Theta; R^s_{[t-1,0]}, R^b_{[t-1,0]})\}$$

In this paper, we numerically maximize this log-likelihood function. And standard error of parameter estimates are also obtained numerically.

With the estimate of the transition matrix, the "smoothed probability" $\xi_{t|T}$, of the states at t are computed as in Kim (1993):

$$\xi_{t|T} = \xi_{t|t} \odot \{\widehat{P}^T * [\xi_{t+1|T}(\div)\xi_{t+1|t}]\} \tag{4}$$

where sign $\odot$ denotes element-by-element multiplication, and $(\div)$ denotes element-by-element division. The smoothed probabilities could be computed by iterating on [4] backward from t=T to 1.

### A2: Stationarity condition and unconditional volatility

This part of appendix is reproduced from Haas Mittnik and Paolella (2004) Let $P_{ij}$ be the ith row and jth column element of the transition matrix, and $\alpha$ and $\beta$ are 2 by1 vectors of the ARCH and

GARCH parameters for two states. Then define

$$
\begin{aligned}
M &= \begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} \\
M_{i,j} &= P_{ij}(\beta + \alpha * e(i)^T), \quad i,j = 1,2
\end{aligned}
$$

where $e(i)$ is a vector with ith element equal to one and the rest being zero

Then model (2) is stationary if largest eigenvalue of matrix M, denoted as $eig(M)$ is less than one. If this stationarity condition holds, then the unconditional expectation of two volatility processes is given by

$$E(V) = [I_2, I_2](I_4 - M)^{-1}(\overline{P} \otimes \omega)$$

where $I_k$ is k by k identity matrix, $\overline{P}$ is the unconditional state probability vector to be computed in A3, $\omega$ is defined as in model (2). and $\otimes$ is the Kronecker product operator. The proof for above result could be found in Haas Mittnik and Paolella (2004).

## A3: Algebra for comparative analysis:

### Expected correlation conditional on volatility regime

To compute the expected correlation level conditional on volatility regime, we first need to compute the unconditional probability implied by the transition matrix. Let $N = 8$ be the number of state, $P$ be the N by N transition matrix, and $C$ is N by 1 vector of correlation. The unconditional state probability $\overline{P}$ can be computed as:

$$
\begin{aligned}
\overline{P} &= (A^T A)^{-1} A^T e_{N+1} \\
\text{where } \underset{(N+1)\times N}{A} &= \begin{bmatrix} I_N - P \\ 1^T \end{bmatrix}
\end{aligned}
$$

with $I_N$ being the N by N identity matrix, $e_{N+1}$ being the $(N+1)th$ column of $I_{N+1}$, and 1 being N by 1 vector of 1s.

With the unconditional state probability vector,and the transition matrix, we can calculate the expected correlation conditional volatility regime as follows, for the case of low volatility regime:

$$
\begin{aligned}
E(C_t|S^k &= low) = \sum_{\{i|S_t=i;S_t^k=low\}} C^i * \Pr(S_t^c = i|S_t^k = low) \\
&= \sum_{\{i|S_t=i;S_t^k=low\}} C^i * \frac{\Pr(S_t^c = i, S_t^k = low)}{\Pr(S_t^k = low)} \\
&= \sum_{\{i|S_t=i;S_t^k=low\}} C^i * \frac{\Pr(S_t^c = i, S_t^k = low)}{\sum_{\{i|S_t=i;S_t^k=low\}} \Pr(S_t^c = i, S_t^k = low)}
\end{aligned}
$$

where $\Pr(S_t^c = i, S_t^k = low)$ is one entry in the N by 1 unconditional state probability $\overline{P}$, and $k = s$ for conditioning on stock volatility regime and $k = b$ for conditioning on bonding volatility regime.

Similar calculations can be done for expected correlation conditioned on both of $S_t^s$ and $S_t^b$, such as $E(C_t|S^s = low; S^b = low)$.

$t - Conditional$ **and** $t+1-conditional$ **transition probabilities**

There are two types of conditional transition probabilities we consider. One is the usual transition probability conditional on observing today's realization on other state variable, such as $\Pr(S_{t+1}^c = low|S_t^c = high; S_t^s = low)$.We call this $t - conditional\ transition\ probability$. The second one is conditional on $t+1$ information. It is the transition probabilities of one process conditioning on there is an regime-switching in another process, such as $\Pr(S_{t+1}^b = low|S_{t+1}^s = high; \{S_t^s = low\ ; S_t^b = high\})$.We call this $t+1-conditional\ transition\ probability$. Both of these quantities can be derived from the general transition matrix.

First, we calculate the $t - conditional\ transition\ probability$ as follows:

$$
\begin{aligned}
\Pr(S_{t+1}^c &= low|S_t^c = high; S_t^s = low) \\
&= \Pr(S_{t+1} = J \mid S_t = I) \\
with\ \{\omega|S_{t+1}(\omega) &= J\} = \{\omega \mid S_{t+1}^c(\omega) = low\}; \\
\{\omega|S_t(\omega) &= I\} = \{\omega \mid S_t^c(\omega) = high; S_t^s(\omega) = low\ \}
\end{aligned}
$$

We can derive the $t+1-conditional\ transition\ probability$ in a similar way. For example, the probability of t+1's correlation would be in low regime conditioned on the stock volatility switching from low regime (at t) to high regime (at t+1), and t's correlation is in high regime:

$$
\begin{aligned}
\Pr(S^c_{t+1} &= low | S^s_{t+1} = high; \{S^s_t = low\ ;\ S^c_t = high\}) \\
&= \frac{\Pr(S^c_{t+1} = low\ ; S^s_{t+1} = high \mid S^s_t = low\ ;\ S^c_t = high)}{\Pr(S^s_{t+1} = high \mid S^s_t = low\ ;\ S^c_t = high)} \\
&= \frac{\Pr(S_{t+1} = J \mid S_t = I)}{\Pr(S_{t+1} = V \mid S_t = I)} \\
with\ \{\omega | S_{t+1}(\omega) &= J\} = \{\omega \mid S^c_{t+1}(\omega) = low\ ; S^s_{t+1}(\omega) = high\ \}; \\
\{\omega \mid S_t(\omega) &= I\} = \{\omega \mid S^s_t(\omega) = low\ ;\ S^c_t(\omega) = high\ \} \\
\{\omega \mid S_{t+1}(\omega) &= V\} = \{\omega \mid S^s_{t+1}(\omega) = high\ \}
\end{aligned}
$$

Finally, for both the $t-$ and $t+1-$*conditional transition probability*, we can use the stationary probabilities $\Pr(S_t = i)$ to compute:

$$
\begin{aligned}
\Pr(S_{t+1} &= J | S_t = I) \\
&= [\Pr(S_{t+1} = J | S_t = I_1) * \Pr(S_t = I_1) + \ldots.\Pr(S_{t+1} = J | S_t = I_I) * \Pr(S_t = I_I)] / \Pr(S_t = I) \\
&= \frac{\sum_{i \in I} \left\{ \sum_{j \in J} [\Pr(S_{t+1} = j | S_t = i)] * \Pr(S_t = i) \right\}}{\sum_{i \in I} \Pr(S_t = i)}
\end{aligned}
$$

where $\Pr(S_{t+1} = j \mid S_t = i)$ is the $j\ th$ row and $i\ th$ column of the transition matrix $P$. And $I, J, V$ are numbers from 1 to 8 So we could compute all the relevant quantities from the transition matrix in a similar way.

Note that by $S^c_{t+1} = low$, we mean those states with significantly negative correlation. So for both bond-stock pairs, this includes $S_t = 2, 6$ and 8, as could be seen in Table 9-10. Even though $\widetilde{C}_4<0$ for ten-year bond, it is not significant less than zero. So we still treat $S_t = 4$ to be the low state for ten-year bond. This would make easier the comparison of bonds with different maturities.

# 7 Tables and Figures:

**Table 1: State variable definitions for the general model**

For volatility state variables, 0 indicates low level regime, and 1 indicate regime with high level. For correlation state, the actual level of high or low state in different "general state" are different. So correlation state variable $S^c$ is the same as the general state variable $S$, both of which have eight states.

| $S$ (General State) | $S^s$ (stock vol) | $S^b$(bond vol) | $S^c$(Correlation) |
|---|---|---|---|
| 1 | 0 | 0 | high1 |
| 2 | 0 | 0 | low1 |
| 3 | 0 | 1 | high2 |
| 4 | 0 | 1 | low2 |
| 5 | 1 | 0 | high3 |
| 6 | 1 | 0 | low3 |
| 7 | 1 | 1 | high4 |
| 8 | 1 | 1 | low4 |

**Table 2: State specifications of all models**

This table specifies different model's restrictions on each of the three individual state variables among various states of the general state variable $S$. The general model is in the first row. The remaining rows present the more restricted models. The numbers in the same curly bracket represent in which states of $S$ individual state variable has the same value. So the collection of curl bracketed-subsets represents the finest $\sigma-$algebra of each state variable in different model specification.

| Model | Num. of Regimes | Restriction on $S^s$ | Restriction on $S^b$ | Restriction on $S^c$ |
|---|---|---|---|---|
| MSCCC(2,2,8) | 8 | $\{1,2,34\}\{5,6,7,8\}$ | $\{1,2,5,6\}\{3,4,7,8\}$ | $\{1\}\{2\}\{3\}\{4\}\{5\}\{6\}\{7\}\{8\}$ |
| MSCCC(2,2,4) | 4 | $\{1,2,3,4\}\{5,6,7,8\}$ | $\{1,2,5,6\}\{3,4,7,8\}$ | $\{1,2\}\{3,4\}\{5,6\}\{7,8\}$ |
| MSCCC($\overline{2}$,2,$\overline{2}$) | 4 | $\{1,2,3,4\}\{5,6,7,8\}$ | $\{1,2,5,6\}\{3,4,7,8\}$ | $\{1,2,3,4\}\{5,6,7,8\}$ |
| MSCCC(2,$\overline{2}$,$\overline{2}$) | 4 | $\{1,2,3,4\}\{5,6,7,8\}$ | $\{1,2,5,6\}\{3,4,7,8\}$ | $\{1,2,5,6\}\{3,4,7,8\}$ |
| MSCCC($\overline{2}$,$\overline{2}$,4) | 4 | $\{1,2\}\{7,8\}$ | $\{1,2\}\{7,8\}$ | $\{1\}\{2\}\{7\}\{8\}$ |
| MSCCC($\overline{2}$,$\overline{2}$,$\overline{2}$) | 2 | $\{1,2\}\{7,8\}$ | $\{1,2\}\{7,8\}$ | $\{1,2\}\{7,8\}$ |

**Table 3: Descriptive Statistics and Optimal Filtering Orders**

This table gives descriptive statistics as well as the optimal ARMA filtering orders (p* and q*) for the returns on the S&P 500 index and 1-year Treasury bond and the 10-year Treasury bond, for the period of January 2, 1986 to December 31 2006. All returns are daily returns in percentages.

| | 1-year bond | 10-year bond | S&P 500 |
|---|---|---|---|
| Mean | 0.021021 | 0.032231 | 0.042042 |
| Std. dev. | 0.054204 | 0.45427 | 1.0665 |
| Minimum | -0.31801 | -2.6762 | -20.467 |
| Maximum | 0.80969 | 4.8227 | 9.0994 |
| Skewness | 1.1069 | -0.042843 | -1.4327 |
| Kurtosis | 18.573 | 7.3358 | 33.622 |
| Jarque-Bera | 54052 | 4107 | 2.0667e+005 |
| p* | 1 | 0 | 0 |
| q* | 2 | 1 | 0 |

### Table 4: Parameters estimates for the MS-GARCH model: S&P500

This table reports the parameter estimates of the MS-GARCH model for S&P500 daily return series. The sample period is from January 5, 1986 to December 29, 2006. Standard deviation of the estimate is in parenthesis. $eig(M)$ is the stationarity statistics estimate. The estimated model is stationary if $eig(M) < 1$ The unconditional expectation of the each state's probability and volatility level in each regime are also reported.

$$\epsilon_t^s \sim N\left(0, \quad V^s_{S^s_{t,}t}\right), \quad S_t^s = 0 \text{ or } 1$$

$$\begin{cases} V^s_{0,t} = \underset{(0.0015)}{0.0026} + \underset{(0.0083)}{0.9528} * V^s_{0,t-1} + \underset{(0.0071)}{0.0321} * \epsilon^s_{t-1} \\ \\ V^s_{1,t} = \underset{(0.1523)}{0.2522} + \underset{(0.0811)}{0.7146} * V^s_{1,t-1} + \underset{(0.2249)}{0.6216} * \epsilon^s_{t-1} \end{cases}$$

$$\text{P}=\begin{bmatrix} & low & high \\ low & \underset{(0.0603)}{0.8745} & 0.9217 \\ high & 0.1255 & \underset{(0.0501)}{0.0783} \end{bmatrix}$$

loglikelihood=-6806.98; $eig(M)$ =0.9913

$P_0^\infty$ =0.8803; $E(V_0)$ =0.7781

$P_1^\infty$ =0.1197; $E(V_1)$ =3.1949

**Table 5: Parameters estimates for the MS-GARCH model: one-year bond**

This table reports the parameter estimates of the MS-GARCH model for 1-year bond daily return series. The sample period is from January 5,1986 to December 29,2006. Standard deviation of the estimate is in parenthesis. $eig(M)$ is the stationarity statistics estimate. The estimated model is stationary if $eig(M) < 1$ The unconditional expectation of the each state's probability and volatility level in each regime are also reported.

$$\epsilon_t^b \sim N\left(0,\ V^b_{S^b_{t},t}\right),\ S_t^b = 0 \text{ or } 1$$

$$\begin{cases} V^b_{0,t} = \underset{(0.0397)^{\dagger}}{0.1359^{\dagger}} + \underset{(0.0081)}{0.9546} * V^b_{0,t-1} + \underset{(0.0043)}{0.0219} * \epsilon^b_{t-1} \\ \\ V^b_{1,t} = \underset{(0.2127)^{\dagger}}{0.4251^{\dagger}} + \underset{(0.0167)}{0.9095} * V^b_{1,t-1} + \underset{(0.0936)}{0.3480} * \epsilon^b_{t-1} \end{cases}$$

$$\text{P=}\begin{bmatrix} & low & high \\ low & \underset{(0.0232)}{0.8671} & 0.8377 \\ high & 0.1329 & \underset{(0.0602)}{0.1627} \end{bmatrix}$$

loglikelihood=8693.43; $eig(M) = 0.9967$.

$P_0^{\infty}$ =0.8630; $E(V_0)$ =0.0033

$P_1^{\infty}$ =0.1370; $E(V_1)$ =0.0241

†: these numbers are multiplied by $10^4$.

**Table 6: Parameters estimates for the MS-GARCH model: 10-year bond**

This table reports the parameter estimates of the MS-GARCH model for10-year bond daily return series. the sample period is from January 5,1986 to December 29,2006. Standard deviation of the estimate is in parenthesis. $eig(M)$ is the stationarity statistics estimate. The estimated model is stationary if $eig(M) < 1$ The unconditional expectation of the each state's probability and volatility level in each regime are also reported.

$$\epsilon_t^b \sim N\left( 0, \quad V^b_{S^b_{t,},t} \right), \quad S_t^b = 0 \text{ or } 1$$

$$\begin{cases} V^b_{0,t} = \underset{(0.0007)}{0.0018} + \underset{(0.0104)}{0.9437} * V^b_{0,t-1} + \underset{(0.0045)}{0.0242} * \epsilon^b_{t-1} \\ \\ V^b_{1,t} = \underset{(0.0077)}{0.0123} + \underset{(0.0315)}{0.9202} * V^b_{1,t-1} + \underset{(0.0490)}{0.1314} * \epsilon^b_{t-1} \end{cases}$$

$$\text{P=}\begin{bmatrix} & low & high \\ low & \underset{(0.0539)}{0.7268} & 0.8851 \\ high & 0.2732 & \underset{(0.0531)}{0.1149} \end{bmatrix}$$

loglikelihood=-2958.43; $eig(M)$ =0.9810

$P_0^\infty = 0.7641$; $E(V_0) = 0.1216$

$P_1^\infty = 0.2359$; $E(V_1)$ =0.4990

### Table 7: Parameters estimates for the MSCCC(8) model: S&P500 and 1-year bond

This table reports the parameter estimates of the MSCCC(2,2,8) model for S&P500 and 1-year bond returns. The sample period is from January 5,1986 to December 29,2006. Standard deviation of the estimate is in parenthesis. The unconditional expectation of the each state's probability is also reported.

$$
\begin{cases}
V^s_{0,t} = & \underset{(0.0011)}{0.0033} & + & \underset{(0.0065)}{0.9548} & * V^s_{0,t-1} + & \underset{(0.0045)}{0.0302} & * \epsilon^s_{0,t-1} \\
V^s_{1,t} = & \underset{(0.1405)}{0.4179} & + & \underset{(0.0775)}{0.6629} & * V^s_{1,t-1} + & \underset{(0.1395)}{0.5349} & * \epsilon^s_{1,t-1}
\end{cases}
$$

$$
\begin{cases}
V^b_{0,t} = & \underset{(0.0375)^{\dagger}}{0.1406^{\dagger}} & + & \underset{(0.0073)}{0.9524} & * V^b_{0,t-1} + & \underset{(0.0038)}{0.0233} & * \epsilon^b_{0,t-1} \\
V^b_{1,t} = & \underset{(0.1579)^{\dagger}}{0.3177^{\dagger}} & + & \underset{(0.0129)}{0.9306} & * V^b_{1,t-1} + & \underset{(0.0629)}{0.2551} & * \epsilon^b_{1,t-1}
\end{cases}
$$

| | $(S^s = 0, S^b = 0)$ | $(S^s = 0, S^b = 1)$ | $(S^s = 1, S^b = 0)$ | $(S^s = 1, S^b = 1)$ |
|---|---|---|---|---|
| *high corr* | 0.2839 (0.0271) | 0.4323 (0.0539) | 0.0569 (0.1005) | 0.8264 (0.0831) |
| *low corr* | −0.3530 (0.0637) | 0.0795 (0.1401) | −0.2444 (0.1307) | −0.4856 (0.1392) |

| | (low,low,1) | (low,low,2) | (low,high,3) | (low,high,4) | (high,low,5) | (high,low,6) | (high,high,7) | (high,high,8) |
|---|---|---|---|---|---|---|---|---|
| **(low,low,1)** | 0.8222 (0.0215) | 0.0093 (0.0028) | 0.6161 (0.0858) | | 0.6366 (0.1006) | | 0.8285 (0.0035) | |
| **(low,low,2)** | | 0.7918 (0.0507) | | 0.7058 (0.2130) | | 0.7493 (0.0752) | | 0.8175 (0.0477) |
| **(low,high,3)** | 0.1338 (0.0147) | | | | 0.3100 (0.1329) | | | |
| **(low,high,4)** | | 0.0243 (0.0255) | | 0.0113 (0.1135) | | 0.0836 (0.0974) | 0.1269 (0.0667) | 0.1825 (0.0477) |
| **(high,low,5)** | 0.0440 (0.0173) | | 0.2247 (0.0844) | 0.1174 (0.1071) | 0.0534 (0.0478) | | | |
| **(high,low,6)** | | 0.1180 (0.0373) | | 0.1655 (0.1847) | | | | |
| **(high,high,7)** | | | 0.1592 (0.0215) | | | | | |
| **(high,high,8)** | | 0.0566 (0.0225) | | | | 0.1671 (0.0831) | 0.0446 (0.0485) | |
| **unconditional** | 0.53044 | 0.24959 | 0.084875 | 0.015237 | 0.044823 | 0.038193 | 0.013515 | 0.023326 |

Log Likelihood: 2132.1 AIC:-4112.2 BIC:-3613.2

## Table 8: Parameters estimates for the MSCCC(8) model: S&P and 10-year bond

This table reports the parameter estimates of the MSCCC(2,2,8) model for S&P500 and 10-year bond returns. The sample period is from January 5,1986 to December 29,2006. Standard deviation of the estimate is in parenthesis. The unconditional expectation of the each state's probability is also reported.

$$
\begin{cases}
\begin{array}{lllllll}
V^s_{1,t} = & 0.0028 & + & 0.9597 & * V^s_{1,t-1} + & 0.0263 & * \epsilon^s_{1,t-1} \\
 & (0.0010) & & (0.0061) & & (0.0041) & \\
V^s_{2,t} = & 0.2151 & + & 0.7423 & * V^s_{2,t-1} + & 0.5420 & * \epsilon^s_{2,t-1} \\
 & (0.0958) & & (0.0619) & & (0.1183) &
\end{array}
\end{cases}
$$

$$
\begin{cases}
\begin{array}{lllllll}
V^b_{1,t} = & 0.0020 & + & 0.9441 & * V^b_{1,t-1} + & 0.0228 & * \epsilon^b_{1,t-1} \\
 & (0.0007) & & (0.0101) & & (0.0040) & \\
V^b_{2,t} = & 0.0125 & + & 0.9259 & * V^b_{2,t-1} + & 0.1140 & * \epsilon^b_{2,t-1} \\
 & (0.0071) & & (0.0271) & & (0.0370) &
\end{array}
\end{cases}
$$

$$
\left[
\begin{array}{lcccc}
 & (S^s = 0, S^b = 0) & (S^s = 0, S^b = 1) & (S^s = 1, S^b = 0) & (S^s = 1, S^b = 1) \\
\multirow{2}{*}{$high\ corr$} & 0.4334 & 0.5501 & 0.2978 & 0.6931 \\
 & (0.0326) & (0.0554) & (0.1484) & (0.0691) \\
\multirow{2}{*}{$low\ corr$} & \text{-0.2810} & \text{-0.0641} & \text{-0.5121} & \text{-0.5062} \\
 & (0.0399) & (0.1188) & (0.1247) & (0.0965)
\end{array}
\right]
$$

| | (low,low,1) | (low,low,2) | (low,high,3) | (low,high,4) | (high,low,5) | (high,low,6) | (high,high,7) | (high,high,8) |
|---|---|---|---|---|---|---|---|---|
| **(low,low,1)** | 0.7206 (0.0208) | | 0.7117 (0.0660) | | 0.5618 (0.2162) | | 0.9779 (0.0231) | |
| **(low,low,2)** | 0.0014 (0.0033) | 0.7547 (0.0365) | 0.0052 (0.0134) | 0.6249 (0.1780) | | | | 0.7829 (0.2015) |
| **(low,high,3)** | 0.1948 (0.0208) | 0.0151 (0.0048) | 0.0286 (0.0380) | | 0.3842 (0.2158) | | | |
| **(low,high,4)** | | 0.0976 (0.0397) | | 0.1351 (0.2499) | | 0.3673 (0.1508) | | 0.2171 (0.2015) |
| **(high,low,5)** | | | 0.2400 (0.0590) | | | | | |
| **(high,low,6)** | 0.0027 (0.0029) | | | 0.0431 (0.0499) | | 0.6327 (0.1508) | 0.0221 (0.0231) | |
| **(high,high,7)** | 0.0805 (0.0151) | | 0.0146 (0.0380) | | 0.0540 (0.0760) | | | |
| **(high,high,8)** | | 0.1326 (0.0335) | | 0.1969 (0.1638) | | | | |
| **unconditional** | 0.49021 | 0.23586 | 0.11267 | 0.041214 | 0.027035 | 0.011066 | 0.042557 | 0.039381 |

Log Likelihood: -9329.29 AIC:18811 BIC:19310

**Table 9: $t-$Conditional Transition Probabilities for Correlation**

This table reports the transition probabilities of correlation state variable conditioned on observing the volatility state variable. We denote stock-vol, bond-vol and their correlation by S, B and C respectively. So $P(C_{+,-}|S_0) \equiv P(C_{t+1} < 0|C_t \geqslant 0; S_t^s = 0)$, and similar meaning for other notations. Note that by $C < 0$, we means those states with *significantly* negative correlation. So for both bond-stock pairs, this includes $S_t = 2, 6$ and 8. Standard deviations of the estimation are in parenthesis. All these are derived from the general transition matrix in Table 9-10.

| 1-year bond and S&P500 | | | | |
|---|---|---|---|---|
| $C_- \Rightarrow C_+$ | P(C$_{-,+}$\|$S_0$) | P(C$_{-,+}$\|$S_1$) | P(C$_{-,+}$\|$B_0$) | P(C$_{-,+}$\|$B_1$) |
| | 0.0336 | 0.1211 | 0.0403 | 0.1825 |
| | (0.0219) | (0.0547) | (0.0201) | (0.0477) |
| $C_+ \Rightarrow C_-$ | P(C$_{+,-}$\|$S_0$) | P(C$_{+,-}$\|$S_1$) | P(C$_{+,-}$\|$B_0$) | P(C$_{+,-}$\|$B_1$) |
| | 0.0242 | 0.0103 | 0 | 0.1394 |
| | (0.0366) | (0.0108) | (-†) | (0.0798) |

| 10-year bond and S&P500 | | | | |
|---|---|---|---|---|
| $C_- \Rightarrow C_+$ | P(C$_{-,+}$\|$S_0$) | P(C$_{-,+}$\|$S_1$) | P(C$_{-,+}$\|$B_0$) | P(C$_{-,+}$\|$B_1$) |
| | 0.1128 | 0.2500 | 0.1242 | 0.2171 |
| | (0.0391) | (0.1346) | (0.0400) | (0.1784) |
| $C_+ \Rightarrow C_-$ | P(C$_{+,-}$\|$S_0$) | P(C$_{+,-}$\|$S_1$) | P(C$_{+,-}$\|$B_0$) | P(C$_{+,-}$\|$B_1$) |
| | 0.0594 | 0.0135 | 0.0039 | 0.1892 |
| | (0.0474) | (0.0124) | (0.0032) | (0.0937) |

†No standard deviations due to the zero constraints on the transition matrix, same as reported in Table 9-10.

**Table 10: Correlation transition matrix**

This table reports the correlation transition matrix estimates. All these are derived from the general transition matrix in Table 9-10. Note that by $C < 0$, we means those states with significantly negative correlation. So for both bond-stock pairs, this includes $S_t = 2, 6$ and 8. Even though $\widetilde{C}_4<0$ for ten-year bond, it is not significant less than zero. So we still treat $S_t = 4$ to be one of state in $\{C \geqslant 0\}$ for ten-year bond. This would make easier the comparison of bonds with different maturities.

| 1-year bond and S&P500 | | |
|---|---|---|
| | $C \geqslant 0$ | $C < 0$ |
| $C \geqslant 0$ | 0.9770<br>(0.0359) | 0.0509 |
| $C < 0$ | 0.0230 | 0.9491<br>(0.0174) |

| 10-year bond and S&P500 | | |
|---|---|---|
| | $C \geqslant 0$ | $C < 0$ |
| $C \geqslant 0$ | 0.9451<br>(0.0447) | 0.1370 |
| $C < 0$ | 0.0549 | 0.8630<br>(0.0407) |

**Table 11: $t+1-$Conditional Transition Probabilities**

This table reports the transition probabilities of one state variable conditioned on there is a switching in other state variables. We denote stock-vol, bond-vol and their correlation by S, B and C respectively. So $P(C_{+,-}|S_{0,1}) \equiv P(C_{t+1} < 0|C_t > 0; S_t^s = 0; S_{t+1}^s = 1)$, and similar meaning for other notations. Note that by $C < 0$, we means those states with significantly negative correlation. So for both bond-stock pairs, this includes $S_t = 2, 6$ and 8. Standard deviations of the estimation are in parenthesis.

| 1-year bond and S&P500 | | | | |
|---|---|---|---|---|
| $S/B \Rightarrow C$ | P($C_{+,-}|S_{0,1}$) | P($C_{+,-}|B_{0,1}$) | P($C_{-,+}|S_{0,1}$) | P($C_{-,+}|B_{0,1}$) |
| | 0.1632 | 0 | 0 | 0.3110 |
| | (0.0987 ) | (-)† | (-) | (0.1996 ) |
| $C \Rightarrow S/B$ | P($S_{0,1}|C_{+,-}$) | P($B_{0,1}|C_{+,-}$) | P($S_{0,1}|C_{-,+}$) | P($B_{0,1}|C_{-,+}$) |
| | 0.7163 | NA‡ | 0 | 0.7999 |
| | (0.2194) | (NA) | (-) | (0.1639) |
| $S \Leftrightarrow B$ | P($B_{0,1}|S_{0,1}$) | P($B_{1,0}|S_{1,0}$) | P($S_{0,1}|B_{0,1}$) | P($S_{1,0}|B_{1,0}$) |
| | 0.2112 | 0.8352 | 0.1551 | 1 |
| | (0.0929) | (0.0481) | (0.0779) | (-) |
| **10-year bond and S&P500** | | | | |
| $S/B \Rightarrow C$ | P($C_{+,-}|S_{0,1}$) | P($C_{+,-}|B_{0,1}$) | P($C_{-,+}|S_{0,1}$) | P($C_{-,+}|B_{0,1}$) |
| | 0.1416 | 0 | 0 | 0.4951 |
| | (0.0996) | (-)† | (-) | (0.1284) |
| $C \Rightarrow S/B$ | P($S_{0,1}|C_{+,-}$) | P($B_{0,1}|C_{+,-}$) | P($S_{0,1}|C_{-,+}$) | P($B_{0,1}|C_{-,+}$) |
| | 0.2936 | 0 | 0 | 1 |
| | (0.1395 ) | (-) | (-) | (-) |
| $S \Leftrightarrow B$ | P($B_{0,1}|S_{0,1}$) | P($B_{1,0}|S_{1,0}$) | P($S_{0,1}|B_{0,1}$) | P($S_{1,0}|B_{1,0}$) |
| | 0.9813 | 0.8945 | 0.3668 | 0.9872 |
| | (0.0156) | (0.0974) | (0.0516) | (0.0096) |

† No standard deviations due to the zero constraints on the transition matrix as in Table 9-10.

‡The conditioning event is an empty set. So there is no estimate for such conditional probability

**Table 12: Expected correlation conditioned on volatility regime**

This table reports the expected bond-stock correlation conditioned on contemporary volatility states in bond and stock markets. The upper panel is for 1-year bond and the lower panel for 10-year bond. For both panels, the last row and last column are the expected correlation conditioned on only one volatility state variable. Standard deviations of the estimation are in parenthesis.

| 1-year bond and S&P | | | |
|---|---|---|---|
| | $S^s = 0$ | $S^s = 1$ | $E(C_t \vert S^b)$ |
| $S^b = 0$ | 0.0801<br>(0.0642) | -0.0817<br>(0.1580) | 0.0645<br>(0.0582) |
| $S^b = 1$ | 0.3786<br>(0.0524) | -0.0043<br>(0.1464) | 0.2756<br>(0.0711) |
| $E(C_t \vert S^s)$ | 0.1140<br>(0.0596) | -0.0579<br>(0.1191) | |

| 10-year bond and S&P | | | |
|---|---|---|---|
| | $S^s = 0$ | $S^s = 1$ | $E(C_t \vert S^b)$ |
| $S^b = 0$ | 0.2013<br>(0.0694) | 0.0626<br>(0.1602) | 0.1944<br>(0.0703) |
| $S^b = 1$ | 0.3856<br>(0.0904) | 0.1167<br>(0.1427) | 0.2922<br>(0.0934) |
| $E(C_t \vert S^s)$ | 0.2336<br>(0.0673) | 0.0995<br>(0.1229) | |

**Table 13: Likelihood-based Goodness-of-fit**

This table shows the likelihood-based goodness-of-fit for models fitted to two pairs of stock and bond return series. The specifications for each model notation in the first column are detailed in Table 2. "Likelihood" is the value of the maximum log-likelihood value, AIC is the Akaike information criterion (1973). And BIC is the Bayesian Information criterion of Schwarz. For both criteria, the ranking of each model is show in parenthesis. Boldface entries indicate the best model for each criterion.

| **S&P500 and 1-year bond** | | | | | |
|---|---|---|---|---|---|
| Model | Num. of Regimes | Parameter number | Likelihood | AIC(rank) | BIC(rank) |
| MSCCC(2,2,8) | 8 | 76 | 2132.1 | **-4112.2(1)** | -3613.2(6) |
| MSCCC(2,2,4) | 4 | 28 | 1972 | -3888(3) | -3704.1(4) |
| MSCCC($\overline{2}$,2,$\overline{2}$) | 4 | 28 | 1964.55 | -3877.1(5) | -3706.4(3) |
| MSCCC(2,$\overline{2}$,$\overline{2}$) | 4 | 26 | 1952.82 | -3853.6(6) | -3682.9(5) |
| MSCCC($\overline{2}$,$\overline{2}$,4) | 4 | 28 | 2083.6 | -4111.3(2) | **-3927.4(1)** |
| MSCCC($\overline{2}$,$\overline{2}$,$\overline{2}$) | 2 | 16 | 1956.4 | -3880.7(4) | -3775.7(2) |
| **S&P 500 and 10-year bond** | | | | | |
| Model | Num. of Regimes | Parameter number | Likelihood | AIC(rank) | BIC(rank) |
| MSCCC(2,2,8) | 8 | 76 | -9329.29 | **18811(1)** | 19310(2) |
| MSCCC(2,2,4) | 4 | 28 | -9579.99 | 19216(3) | 19400(4) |
| MSCCC($\overline{2}$,2,$\overline{2}$) | 4 | 26 | -9645.61 | 19343(6) | 19514(6) |
| MSCCC(2,$\overline{2}$,$\overline{2}$) | 4 | 26 | -9627.6 | 19307(5) | 19478(5) |
| MSCCC($\overline{2}$,$\overline{2}$,4) | 4 | 28 | -9402.46 | 18861(2) | **19045(1)** |
| MSCCC($\overline{2}$,$\overline{2}$,$\overline{2}$) | 2 | 16 | -9610.36 | 19253(4) | 19358(3) |

**Figure 1: state probabilities of correlation: 1-year bond and S&P500**

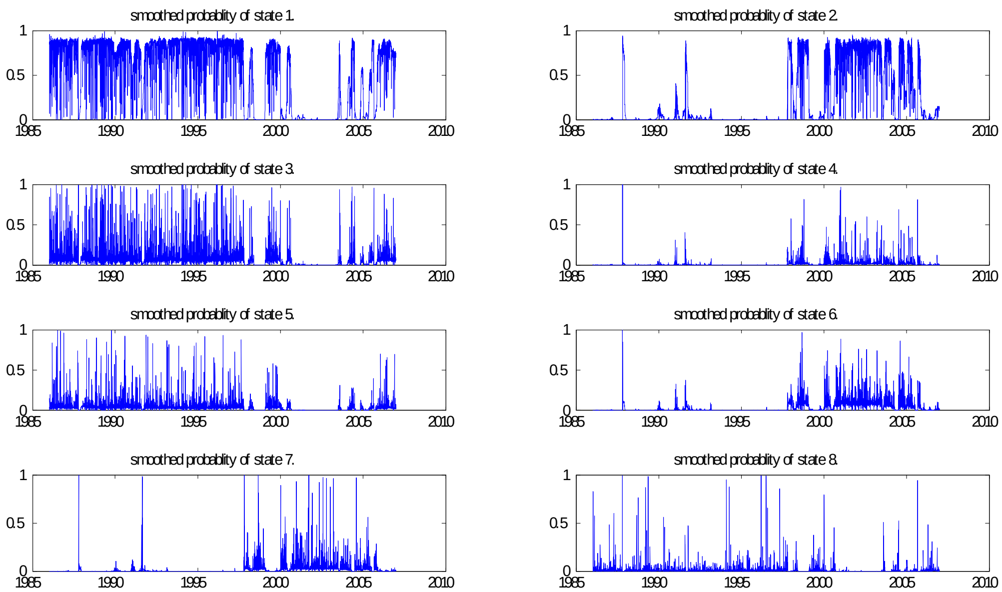



**Figure 2: state probabilities of correlation: 10-year bond and S&P500**

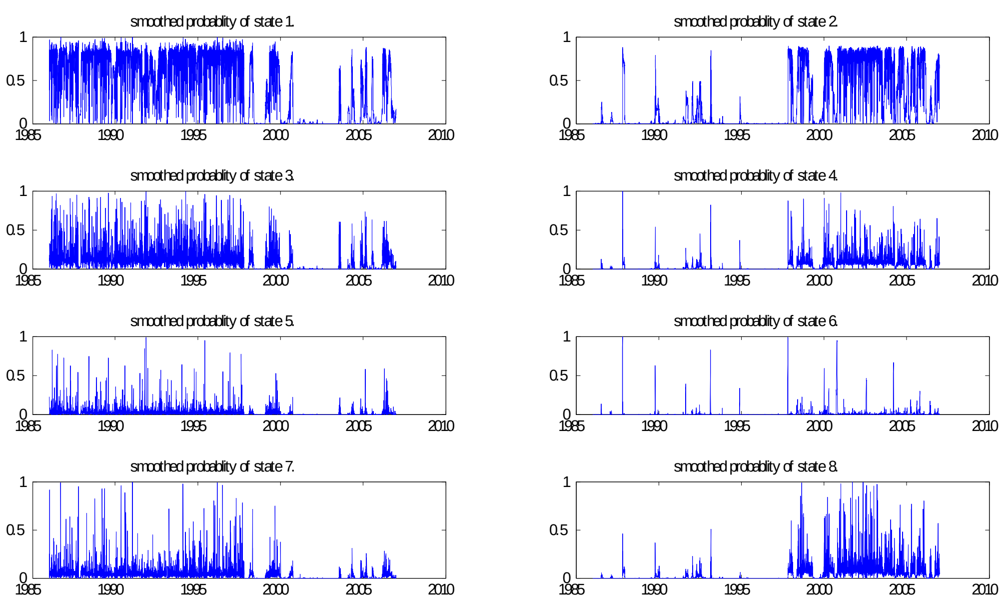


**Figure 3: smoothed correlation: 1-year bond and S&P500**

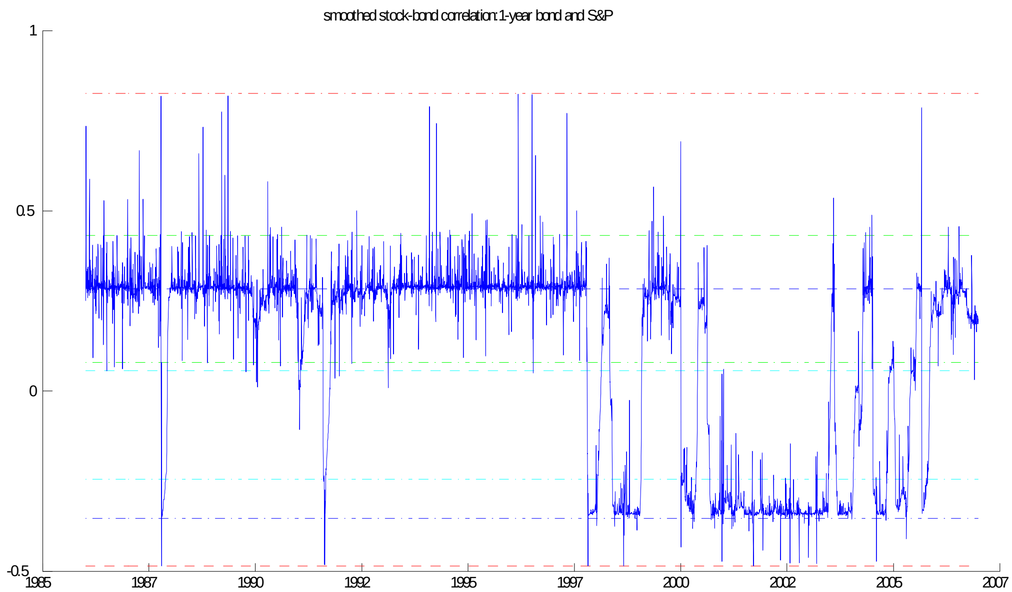



**Figure 4: smoothed correlation: 10-year bond and S&P**

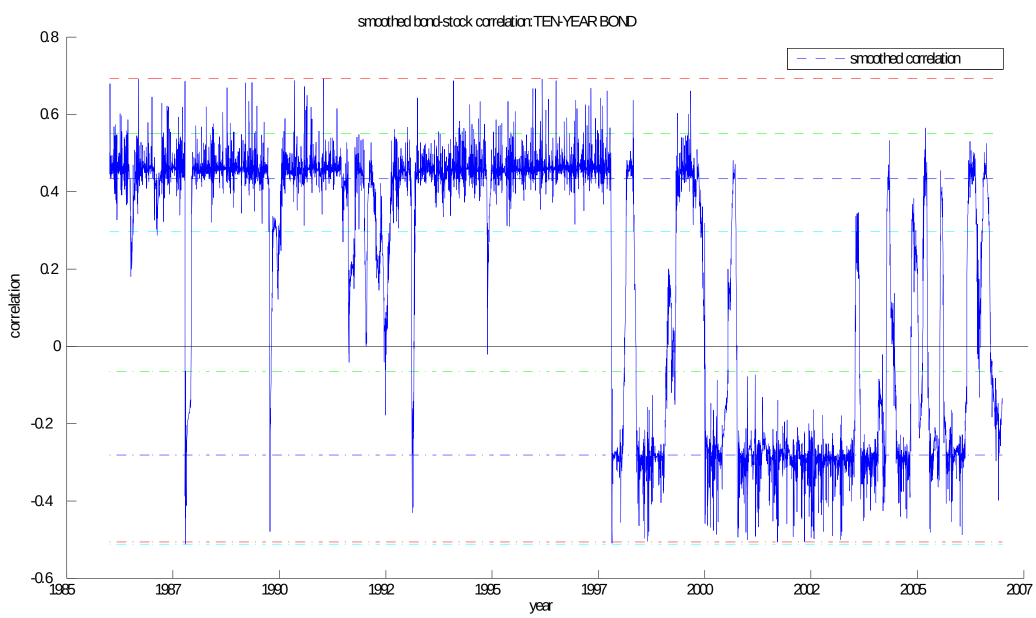


**Figure 5.1: MSCCC VS DCC—1-year bond and S&P(1)**

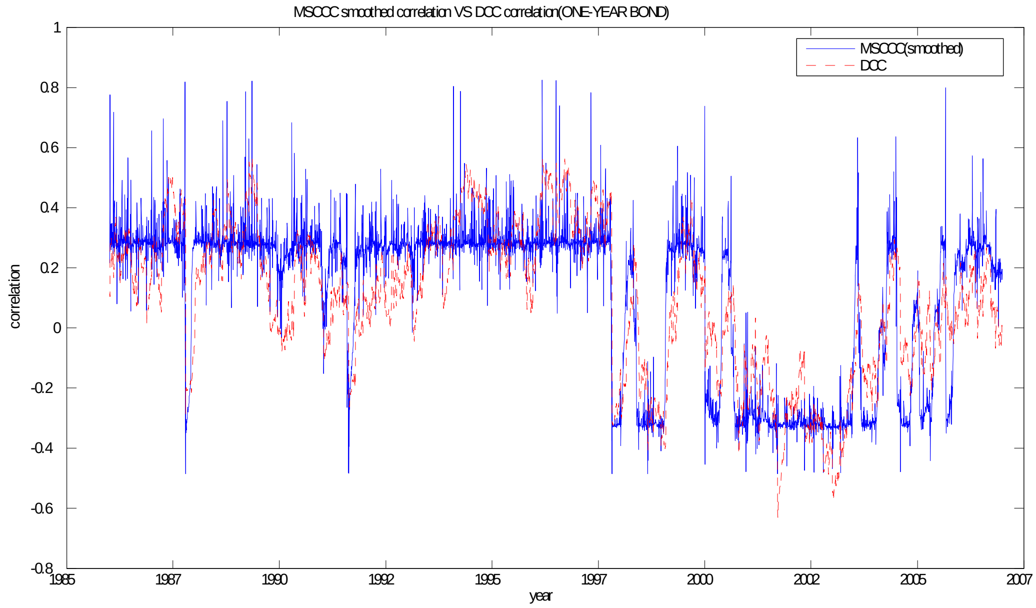



**Figure 5.2: MSCCC VS DCC—1-year bond and S&P(2)**

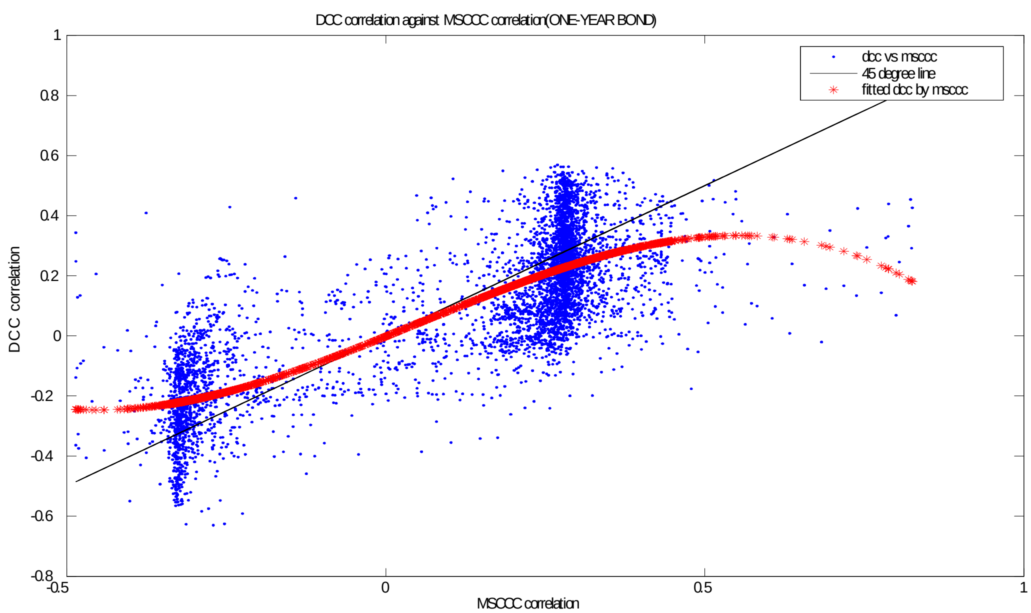


Figure 6.1: MSCCC VS DCC—10-year bond and S&P(1)

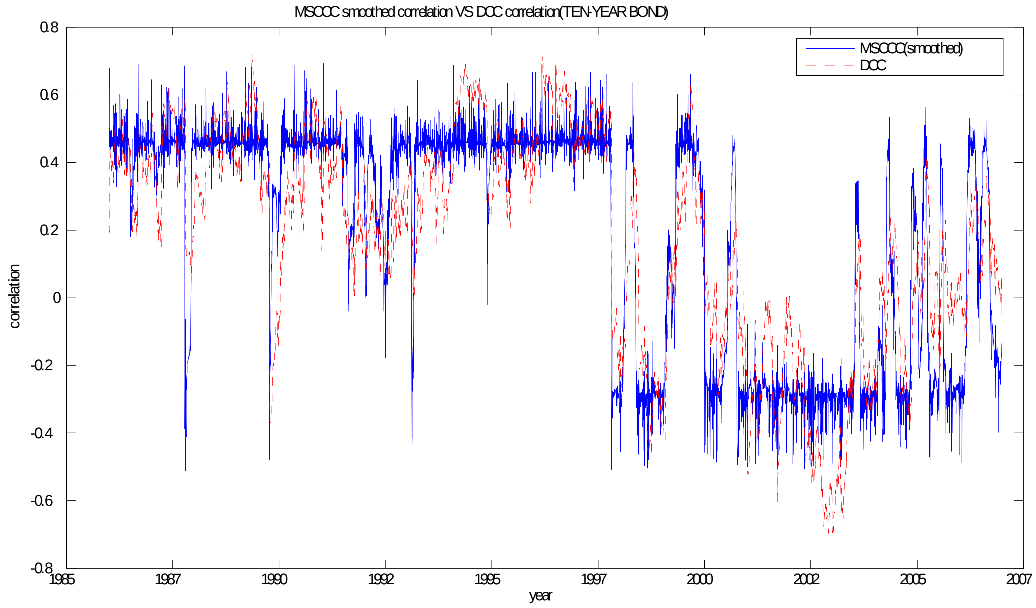
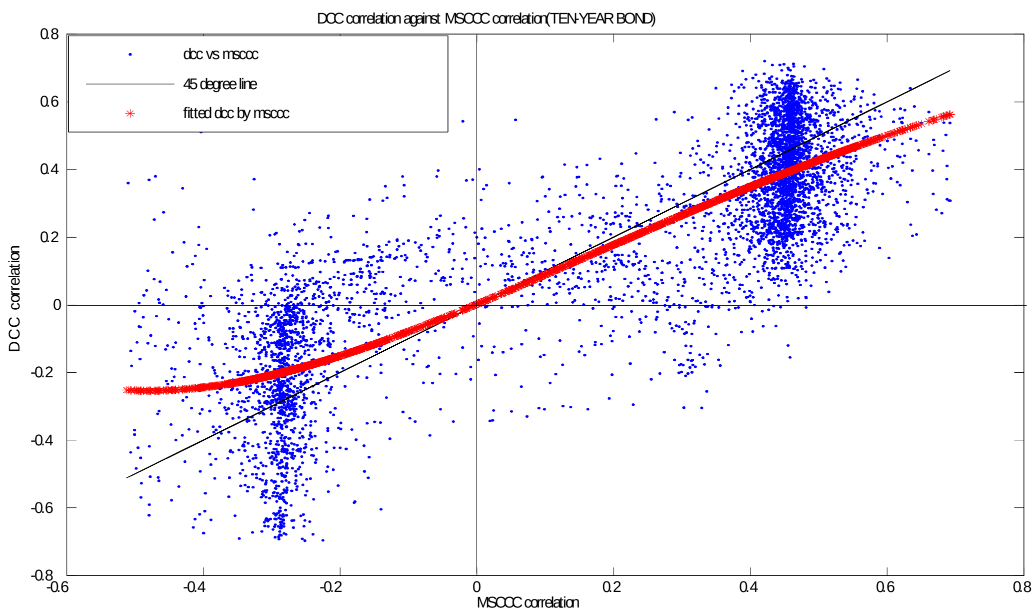



Figure 6.2: MSCCC VS DCC—10-year bond and S&P(2)



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