Understanding Portfolio Efficiency with Conditioning Information

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Understanding Portfolio Efficiency with Conditioning Information*

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Abstract

Contrary to the classic framework of passive strategies, if investors exploit return predictability through active strategies then there is a tension between the mean-variance frontiers that drive empirical work and the mean-variance preferences that are used in finance theory. We show that standard preferences choose portfolios on a frontier that has not been studied in the literature, develop new betas and Sharpe ratios to construct portfolio efficiency tests, and highlight some concerns with current empirical work. An empirical application to active strategies on stock portfolios sorted by size and book-to-market confirms the relevance of our theoretical results.

Keywords: Beta-pricing, Dynamic portfolio strategies, Jensen's alpha, Mean-variance frontiers, Sharpe ratios.

JEL: C12, G11, G12

1   Introduction

The mean-variance analysis developed by Markowitz (1952) is regarded as the cornerstone of modern investment theory and continues to be widely used in economics and finance. Its applications cover key issues such as portfolio choice, asset pricing tests and performance evaluation. On the other hand, the predictability of asset returns is an empirical fact, therefore investors are better off designing active portfolio strategies based on their conditioning information. See e.g. Cochrane (2001) for a summary of the empirical evidence on mean predictability.

The seminal paper of Hansen and Richard (1987) developed the theoretical framework to study mean-variance efficiency of active strategies. They analyzed a frontier based on conditional moments and a frontier based on their unconditional counterparts, named the conditional and unconditional return frontiers (CRF and URF) respectively. They showed that active returns on the URF are also on the CRF, but not viceversa. Their main goal was to clarify the tension between the conditional implications of asset pricing theory and the use of unconditional moments in empirical work, as they can be easily estimated with sample averages. For instance, the CAPM implies that the market portfolio should be on the CRF, but not necessarily on the URF. However, many papers have used the URF to guide portfolio choice. See e.g. Ferson and Siegel (2001) for the case of mean predictability and Ferson and Siegel (2007) for a portfolio efficiency test on the URF. Other papers such as Brandt and Santa-Clara (2006) and Bansal, Dahlquist, and Harvey (2004) approximate portfolios on this frontier through passive strategies of managed portfolios.

Understanding portfolio efficiency requires understanding the connection between mean-variance frontiers and preferences. The main contribution of this paper is the analysis of this connection and its implications for portfolio efficiency tests. Such a connection is trivial in the Markowitz set-up without conditioning information, but it becomes much more complex when we take into account that investors design active strategies. In fact, different mean-variance preferences may disagree on the relevant test of portfolio efficiency. We show that standard preferences choose portfolios on a frontier that has not been studied in the literature, develop new betas and Sharpe ratios to construct portfolio efficiency tests, and highlight some concerns with current empirical work.

In the classic Markowitz set-up based on passive strategies, any family of mean-variance preferences can be chosen to explore the whole efficient mean-variance frontier. For instance,

the passive returns p that maximize $E\left(p\right) - \left(b/2\right)E\left(p^2\right)$ for each real number b lie on the mean-variance frontier, and each of those optimal returns also maximizes $E\left(p\right) - \left(\theta/2\right)Var\left(p\right)$ for the corresponding real number[1] θ. As a result, the specific family of mean-variance preferences does not matter and all of them agree on the relevant test of portfolio efficiency, which relies on beta-pricing: A passive return p_β different from the minimum variance one is on the frontier if and only if, for some real number E_β called the zero-beta return, we find $E\left(p\right) - E_\beta = \beta\left[E\left(p_\beta\right) - E_\beta\right]$ for every passive return p, where $\beta = Cov\left(p, p_\beta\right)/Var\left(p_\beta\right)$. Gibbons, Ross, and Shanken (1989) is a well known reference on testing portfolio efficiency in the Markowitz set-up.

From now on, we will take into account that investors design active strategies given an information set G_1 instead. Ferson and Siegel (2001) show that the URF can be rationalized by quadratic utility $E\left(p|G_1\right) - \left(b/2\right)E\left(p^2|G_1\right)$, but not by e.g. CARA utility plus normality of returns conditional on a signal, which implies preferences $E\left(p|G_1\right) - \left(\theta/2\right)Var\left(p|G_1\right)$. This fact represents a tension between the empirical work based on the URF and the theoretical literature that uses mean-variance preferences. For instance, Dybvig and Ross (1985) use mean-variance preferences of the second type to study the complexity of performance evaluation of an informed manager by an uninformed agent. Areas such as market microstructure and rational expectations equilibria often rely on those preferences too,[2] see Brunnermeier (2001) for a survey of asset pricing theory under asymmetric information or Easly and O'Hara (2004) as a recent reference.

For this reason, we characterise the subset of the CRF where $E\left(p|G_1\right) - \left(\theta/2\right)Var\left(p|G_1\right)$ choose portfolios. This subset can be represented by a frontier based on unconditional moments that has not been studied in the literature. We will refer to it as the residual frontier (RRF) because it focuses on $E\left[Var\left(p|G_1\right)\right]$ instead of the total variance $Var\left(p\right)$ as the URF does. Similarly, portfolio efficiency tests from the perspective of those standard preferences should rely on a new type of beta, which we call the residual beta.

We show that the URF and RRF are different subsets of the CRF (see Figure 1 as an illustration), apart from a few cases that are far from plausible. Therefore, we find an important novelty with respect to the classic Markowitz set-up in the sense that there is not a single relevant beta anymore. We can think of conditional, unconditional, and residual betas associated with the

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[1] Actually both parameters are linked to risk aversion and hence assumed to be strictly positive.

[2] They are also used in continuous time asset allocation. See Basak and Chabakauri (2008) and the references therein.

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CRF, URF, and RRF respectively. As Figure 1 illustrates, the conditional beta can be used to test if a portfolio is on the CRF. This test is compatible with general mean-variance preferences, not only the two types commented above, and hence it is not as powerful as possible if we think of particular preferences. In that sense, a residual beta can be used to test the location of a portfolio on the RRF or equivalently the subset of the CRF defined by standard mean-variance preferences in finance theory. On the other hand, empirical work relies often on unconditional betas and hence it is actually testing if a portfolio lies on the URF, which has a weak justification in terms of preferences.
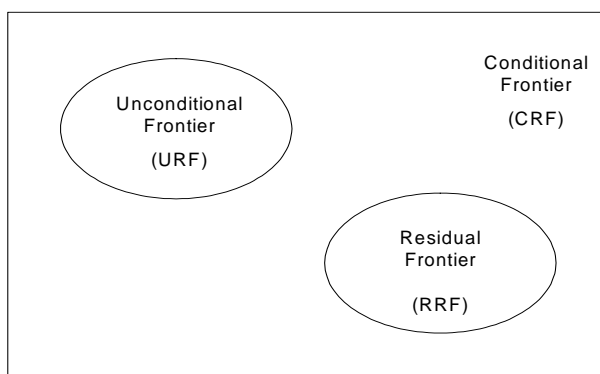

Figure 1: The Unconditional and Residual Frontiers represented as different subsets of the Conditional Frontier.

We also analyze the presence of a safe asset and find additional concerns with the URF. In the classic Markowitz set-up, there is a linear frontier, a unique risk-return trade off defined by the Sharpe ratio on the frontier, and a unique zero-beta return in beta-pricing given by the safe asset itself. We show that in general the URF does not satisfy any of these properties and hence the use of unconditional Sharpe ratios and Jensen's alphas in empirical work may be misleading because of a null hypothesis that is not well defined. On the other hand, the RRF satisfies the classic properties, which simplifies the construction of efficiency tests. In fact, we can use both unconditional and residual Sharpe ratios to test the location of a portfolio on the RRF.

Finally, a standard empirical application of active strategies confirms the quantitative relevance of the theoretical results outlined above. Importantly, we do not need to rely on too many assets, neither too many predictors or a complex model of conditional moments. We consider six portfolios of US stocks sorted by size and book-to-market in the spirit of Fama and French

(1993). The information set is constructed with three standard predictors of stock returns (the dividend yield, the term spread, and the default spread), and the connection between predictors and conditional moments follows simple linear predictive regressions.

The corresponding URF and RRF have quite different properties. Their plots show that, from the perspective of one frontier, the other may look as inefficient as passive strategies. The unconditional Sharpe ratio on the URF changes considerably and can be lower than the single value 0.94 on the RRF, or even lower than the ratios from passive strategies. For instance, as we move from an expected return of 6% to 10% on the URF, its unconditional Sharpe ratio changes from 0.38 to 0.97, while it changes from 0.73 to 0.92 on the passive frontier. In terms of the unconditional Jensen's alpha, as we move from an expected return of 6% to 10% on the URF, the alphas change from an average of 9% to 2% across the original six portfolios. Hence the alphas with respect to the URF can be very high compared to the average risk premium of 10%, and higher than they are on the RRF, where the average alpha is 2%.

The rest of the paper is organized as follows. We introduce the general theoretical set-up of active strategies and a simple binomial example in Section 2, while Section 3 reviews the CRF and URF. Next, we study the link between mean-variance preferences and frontiers in Section 4, where we develop the RRF and its asset pricing implications, and discuss its connection to the URF. We analyze the presence of a riskless asset, and the use of Sharpe ratios and Jensen's alpha, in Section 5. Finally, we develop an empirical application in Section 6, and present our conclusions in Section 7. Proofs are gathered in the appendix.

2   Active Portfolio Strategies

Let us introduce active strategies with a simple binomial example. We start from a 2×1 vector $\mathbf{x}$ that represents annual gross returns, or unit cost investments, on two risky assets. We consider an investor that observes a signal at date 1 that may take two values, revealing one of two possible expected return vectors and two possible covariance matrices. That is, we can understand the investor's information set as containing a binomial random variable z that can take the following two values: If $z = 1$ then the relevant mean and variances are given by the real numbers

$$E\left(\mathbf{x}|z=1\right) = \begin{pmatrix} 1.20 \\ 1.15 \end{pmatrix}, \qquad Var\left(\mathbf{x}|z=1\right) = \begin{pmatrix} 0.0204 & 0 \\ 0 & 0.0006 \end{pmatrix},$$

while if $z = 2$ then those real numbers become

$$E\left(\mathbf{x}|z=2\right) = \begin{pmatrix} 1.1 \\ 1.02 \end{pmatrix}, \qquad Var\left(\mathbf{x}|z=2\right) = \begin{pmatrix} 0.036 & 0.0024 \\ 0.0024 & 0.012 \end{pmatrix}.$$

Those numbers try to capture a simple situation where the investor chooses an optimal portfolio of stocks and hedge funds, which are represented by the first and second entry of $\mathbf{x}$ respectively. The signal realization $z = 1$ represents a "bull market", i.e. high return and low risk, where the hedge funds show low correlation with stocks and a high ratio of mean over standard deviation. The signal realization $z = 2$ represents a "bear market", i.e. low return and high risk, where the hedge funds show higher correlation with stocks and perform poorly. We assign a probability of 0.8 to a "bull market", and hence the unconditional moments are

$$E\left(\mathbf{x}\right) = \begin{pmatrix} 1.18 \\ 1.12 \end{pmatrix}, \qquad Var\left(\mathbf{x}\right) = \begin{pmatrix} 0.0251 & 0.0026 \\ 0.0026 & 0.012 \end{pmatrix}.$$

There are three important dates in this economy: $\mathbf{0}$, $\mathbf{1}$ and $\mathbf{2}$. We identify $\mathbf{0}$ as the decision date, $\mathbf{1}$ as the trading date and $\mathbf{2}$ as the payoff date. Investors design portfolio strategies at $\mathbf{0}$ that may depend on the information that they will observe at $\mathbf{1}$, when trading takes place, and they finally receive payoffs at $\mathbf{2}$. The key feature of a binomial set-up is that any portfolio strategy can be easily represented by two real 2×1 vectors $\mathbf{w}\left(z=1\right)$ and $\mathbf{w}\left(z=2\right)$ and, as a result, the payoff space is indeed finite dimensional from the point of view of date $\mathbf{0}$. We say that a portfolio strategy is passive if the two vectors $\mathbf{w}\left(z=1\right)$ and $\mathbf{w}\left(z=2\right)$ are equal; otherwise, we say that it is active.

We will illustrate the next sections with this simple example for pedagogical reasons. However, our results will not depend on that structure and will cover also situations where the payoff space at date $\mathbf{2}$ is infinite dimensional from the perspective of date $\mathbf{0}$. Even though investors only have access to a finite set of primitive asset payoffs, say N risky assets whose random payoffs $\mathbf{x} = (x_1, \ldots, x_N)'$ are defined on an underlying probability space, they can use any piece of information known at date $\mathbf{1}$ in designing their investment strategies.

Let G_1 denote the investors' information at date $\mathbf{1}$, typically containing signals observed at $\mathbf{1}$ that are informative about future asset payoffs. We denote the set of all random variables that are measurable with respect to G_1 by I_1. The first two conditional moments of the primitive payoffs and their conditional costs are denoted by

$$E\left(\mathbf{x}|G_1\right), \qquad E\left(\mathbf{x}\mathbf{x}'|G_1\right), \qquad C\left(\mathbf{x}|G_1\right), \tag{1}$$

respectively, all of which belong to I_1.

Investors can condition their portfolios weights on the information they know they will have at the time of trading, which is given by G_1. Consequently, they can construct active portfolio strategies with weights $\mathbf{w}_1 \in I_1$ that will deliver payoffs $p = \mathbf{x}'\mathbf{w}_1$, and we will refer to the corresponding payoff space as the active payoff space P_a. Although we allow asset prices $C(\mathbf{x}|G_1)$ to depend on the values of the signals, there are two important examples of payoffs whose costs are non-random: gross returns, which are payoffs with unit prices, and arbitrage portfolios, or zero-cost payoffs.[3]   They will be relevant in the analysis that follows and hence we will define the corresponding two subsets of P_a,

$$\mathcal{R}_a = \{p \in P_a : C(p|G_1) = 1\},$$

$$\mathcal{A}_a = \{p \in P_a : C(p|G_1) = 0\}.$$

If an investor is endowed with some positive wealth, which we can normalize to 1 without loss of generality, then she will only be interested in portfolio strategies that cost 1 at date 1 for every possible value of the signals in G_1. Therefore, the mean-variance frontiers that we will study are constructed from returns in $\mathcal{R}_a$.

3   Mean-Variance Frontiers

This section briefly reviews the two frontiers that Hansen and Richard (1987) studied.

3.1   The Conditional Return Frontier

The Conditional Return Mean-Variance Frontier (CRF) is defined as the active returns with minimum conditional variance $Var(p|G_1)$ for a given profile of conditional expected returns $E(p|G_1)$. That is, the set of active returns that solve the optimization problem

$$\min_{p \in \mathcal{R}_a} E(p^2|G_1) \qquad s.t. \qquad E(p|G_1) = \nu_1 \in I_1, \tag{2}$$

which can be represented as

$$p_C(\nu_1) = R_a^* + \omega_1(\nu_1) A_a^+, \qquad \omega_1(\nu_1) = \frac{\nu_1 - E(R_a^*|G_1)}{E(A_a^+|G_1)}, \tag{3}$$

where R_a^* is the return with minimum conditional second moment, i.e. the solution to problem

$$\min_{p \in \mathcal{R}_a} E(p^2|G_1),$$

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[3]We assume that the vector $C(\mathbf{x}|G_1)$ has at least one entry different from 0 a.s. so that it is possible to construct returns.

and A_a^+ represents the conditional mean of arbitrage portfolios with an uncentred second moment in the sense that it is the unique arbitrage portfolio that satisfies

$$E\left(A_a^+ p|G_1\right) = E\left(p|G_1\right), \qquad \forall p \in \mathcal{A}_a.$$

The following lemma provides an alternative representation of the CRF. It is the active counterpart of the representation that Chamberlain and Rothschild (1983) developed for the passive mean-variance frontier. Its proof in the Appendix shows the relationship[4] between (R_a^*, A_a^+) and (R_a^{**}, A_a^{++}).

Lemma 1 *The CRF returns (3) can also be represented as*

$$p_C\left(\nu_1\right) = R_a^{**} + w_1\left(\nu_1\right) A_a^{++}, \qquad w_1\left(\nu_1\right) = \frac{\nu_1 - E\left(R_a^{**}|G_1\right)}{E\left(A_a^{++}|G_1\right)}, \tag{4}$$

where R_a^{**} is the return with minimum conditional variance, i.e. the solution to problem

$$\min_{p \in \mathcal{R}_a} Var\left(p|G_1\right),$$

and A_a^{++} represents the conditional mean of arbitrage portfolios with a centred second moment in the sense that it is the unique arbitrage portfolio that satisfies

$$E\left(p|G_1\right) = Cov(A_a^{++}, p|G_1), \quad \forall p \in \mathcal{A}_a.$$

The CRF is a hyperbola on the $[Var^{1/2}\left(p|G_1\right), E\left(p|G_1\right)]$ space for a particular value of the conditioning variables in G_1. Figure 2 shows the CRF at both values of z for the binomial example described in Section 2, where each return on the CRF is associated with two pairs of volatility and expected return, $[Var^{1/2}\left(p|z=1\right), E\left(p|z=1\right)]$ and $[Var^{1/2}\left(p|z=2\right), E\left(p|z=2\right)]$. For instance, R_a^{**} is located at the minimum volatility of each hyperbola. Two CRF points are shown on Figure 2 instead. They coincide in $[Var^{1/2}\left(p|z=1\right), E\left(p|z=1\right)]$, but take different values in terms of $[Var^{1/2}\left(p|z=2\right), E\left(p|z=2\right)]$. One point is labelled as URF and the other as RRF for reasons that will be explained later.

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[4]We can relate these important returns and arbitrage portfolios to the concept of mean and cost representing portfolios following Hansen and Richard (1987) and Chamberlain and Rothschild (1983).
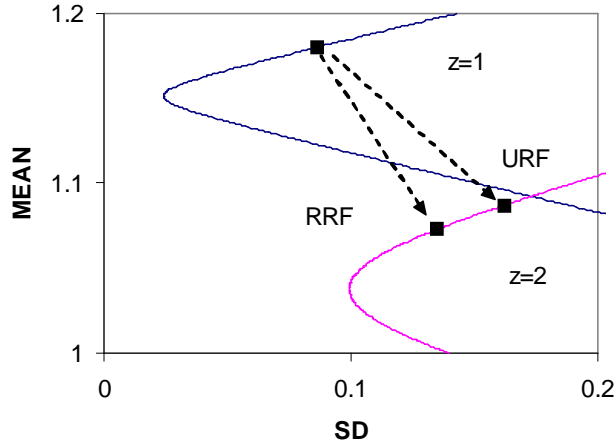
Figure 2: The Conditional Frontier on space $[Var^{1/2}(p|z), E(p|z)]$ at both signal values $z = 1$ and $z = 2$. Two points on the frontier are shown, one belongs to the Unconditional Frontier (URF) and the other to the Residual Frontier (RRF).

Portfolio efficiency tests are often based on the beta-pricing implications of the mean-variance frontier. The beta-pricing of the CRF is based on the conditional beta of a return $p \in \mathcal{R}_a$ with respect to a return $p_\beta \in \mathcal{R}_a$

$$\beta_1 = \frac{Cov(p, p_\beta|G_1)}{Var(p_\beta|G_1)},$$

and can be expressed as follows: A return $p_\beta \in \mathcal{R}_a$ different from R_a^{**} is on the CRF if and only if

$$E(p|G_1) - E_1 = \beta_1 [E(p_\beta|G_1) - E_1], \qquad \forall p \in \mathcal{R}_a,$$

for some $E_1 \in I_1$. That random variable is interpreted as the conditional mean of the corresponding zero-beta return.[5]

Note two difficulties in the implementation of the previous beta-pricing equation. First, it is fully based on conditional moments. Second, given a return p_β, we should check that equation for every $p \in \mathcal{R}_a$, which may be a daunting task.

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[5] Each return p_1 on the CRF, apart from R_a^{**}, has a unique zero-beta counterpart p_2 on the CRF, i.e. a return such that $Cov(p_1, p_2|G_1) = 0$. Moreover, such a return has a similar geometry to the case of passive strategies in the sense that the return p_1 is the solution to

$$\max_{p \in \mathcal{R}_a} \frac{E(p|G_1) - E(p_2|G_1)}{Var^{1/2}(p|G_1)},$$

which represents the tangency from $E(p_2|G_1)$ to the CRF on the $[Var^{1/2}(p|G_1), E(p|G_1)]$ space for a particular value of the conditioning variables in G_1.

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3.2 The Unconditional Return Frontier

Most empirical work tries to avoid the use of conditional moments as required in the previous conditional beta-pricing equation. See e.g. chapter 8 in Cochrane (2001). Conditional moments require the choice of which are the relevant variables in G_1 and the specific equations that define (1), and both decisions are subject to misspecification. This is not the case with unconditional moments, which can be easily estimated by sample averages. In addition, in many practical situations the observer of the agents' decisions only has access to an information set that is much coarser than the agents' information set. The performance evaluation of a portfolio manager is a typical example of the use of unconditional moments by an outside evaluator who may not have access to the proprietary strategies followed by the manager.

If we define the unconditional beta of a return $p \in \mathcal{R}_a$ with respect to a return $p_\beta \in \mathcal{R}_a$ as

$$\beta_U = \frac{Cov\left(p, p_\beta\right)}{Var\left(p_\beta\right)}$$

then the usual beta-pricing equation in empirical work is actually

$$E\left(p\right) - E_\beta = \beta_U \left[E\left(p_\beta\right) - E_\beta\right], \qquad \forall p \in \mathcal{R}_a,$$

for some $E_\beta \in \mathbb{R}$. This equation is similar to the classic set-up of passive strategies as commented in the Introduction, since it is fully based on unconditional moments. Nevertheless, there is an important difference because the relevant return space is every $p \in \mathcal{R}_a$, not only passive returns.

Hansen and Richard (1987) show that a return $p_\beta \in \mathcal{R}_a$ different from the minimum (unconditional) variance one satisfies the unconditional beta-pricing above if and only if p_β lies on the Unconditional Return Mean-Variance Frontier (URF). This frontier is defined as the active returns with minimum unconditional variance $Var\left(p\right)$ for each level of unconditional expected return $E\left(p\right)$. Hence, the URF will be given by the set of active returns that solve the problem

$$\min_{p \in \mathcal{R}_a} E\left(p^2\right) \qquad s.t. \qquad E\left(p\right) = \nu \in \mathbb{R}, \tag{5}$$

which can be represented as

$$p_U\left(\nu\right) = R_a^* + \omega_U\left(\nu\right) A_a^+, \qquad \omega_U\left(\nu\right) = \frac{\nu - E\left(R_a^*\right)}{E\left(A_a^+\right)}. \tag{6}$$

The URF is a hyperbola on the $\left[Var^{1/2}\left(p\right), E\left(p\right)\right]$ space, like the two curves in Figure 2 were on a different space. The URF should not be confused with the classic Markowitz frontier, which is constructed with passive strategies. The passive frontier would be located to the right of the URF on the $\left[Var^{1/2}\left(p\right), E\left(p\right)\right]$ space.

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4   Mean-Variance Preferences

This section studies portfolio efficiency from the perspective of specific mean-variance preferences and their link to specific frontiers. In a set-up of conditioning information, general mean-variance preferences are represented by $U\left(E\left(p|G_1\right), Var^{1/2}\left(p|G_1\right)\right)$ for some function $U\left(\cdot\right)$ that may depend on G_1 too and satisfies some standard properties: strictly increasing in the first argument, strictly decreasing in the second argument, and strictly concave in both.[6]   These preferences choose portfolios on the CRF: If we plot the corresponding indifference curves on Figure 2 then a tangency with the "bull" and "bear" hyperbolas defines the optimal profile ν_1 for a given $U\left(\cdot\right)$, which satisfies $\nu_1 \geq E\left(R_a^{**}|G_1\right)$.

4.1   Preferences Underlying the Conditional and Unconditional Frontiers

It will be convenient to map mean-variance preferences, or equivalently the CRF problem (2), into simple criteria based on possibly random risk-return trade-offs.

Proposition 1 *The following mean-variance criteria choose portfolios on the CRF:*

1. The optimal portfolio of problem

$$\max_{p \in \mathcal{R}_a} E\left(p|G_1\right) - \frac{\theta_1}{2} Var\left(p|G_1\right),$$

given some strictly positive $\theta_1 \in I_1$ is equal to (4) if we choose a profile $\nu_1 \in I_1$ such that

$$w_1\left(\nu_1\right) = \frac{1}{\theta_1}.$$

2. The optimal portfolio of problem

$$\max_{p \in \mathcal{R}_a} E\left(p|G_1\right) - \frac{b_1}{2} E\left(p^2|G_1\right),$$

given some strictly positive $b_1 \in I_1$ is equal to (3) if we choose a profile $\nu_1 \in I_1$ such that

$$\omega_1\left(\nu_1\right) = \frac{1}{b_1}.$$

Ferson and Siegel (2001) show that the optimal portfolio of an agent with quadratic utility $E\left[p - (b/2)\,p^2|G_1\right]$ for some strictly positive $b \in \mathbb{R}$ is actually a return on the URF. In our set-up, Proposition 1 shows that the solution of problem

$$\max_{p \in \mathcal{R}_a} E\left(p|G_1\right) - \frac{b}{2} E\left(p^2|G_1\right) \tag{7}$$

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[6]The justification of mean-variance preferences was linked to elliptical distributions by Chamberlain (1983) and Owen and Rabinovitch (1983) in the context of passive strategies and unconditional moments.

is equal to the solution of problem (5) if we choose a $\nu \in \mathbb{R}$ such that

$$\omega_U (\nu) = \frac{1}{b},$$

that is, the active return

$$R_a^* + \frac{1}{b} A_a^+.$$

This link of the URF to specific preferences clarifies that while returns on the URF always lie on the CRF, the converse is not generally true. Similarly, the actual difference between problems (5) and (2) is the mean constraint, not the criterion to minimize. The first problem only constrains $E(p)$ while the latter constrains the whole profile $E(p|G_1)$. A CRF portfolio will also be located on the URF if and only if we choose the conditional mean profile as

$$\nu_1 = \omega_U (\nu) E\left(A_a^+|G_1\right) + E\left(R_a^*|G_1\right),$$

which we can interpreted as the optimal mean profile given $E(p) = \nu$, so that

$$\omega_1 (\nu_1) = \omega_U (\nu).$$

In the context of Figure 2, the URF focuses on particular pairs $(E(p|z = 1), E(p|z = 2))$ in the sense that, given a value of $E(p|z = 1)$, there is only one value of $E(p|z = 2)$ on the URF. However, all possible pairs are on the CRF.

We can rationalize the URF by means of quadratic utility but such preferences are not used in finance theory. They are not a plausible description of investor behavior because they have decreasing marginal utility and increasing absolute risk aversion.

4.2 Standard Preferences and the Residual Return Frontier

When conditioning information is taken into account, probably the most common mean-variance preferences in finance theory are given by

$$\max_{p \in \mathcal{R}_a} E(p|G_1) - \frac{\theta}{2} Var(p|G_1). \tag{8}$$

for some strictly positive $\theta \in \mathbb{R}$. This criterion is often justified by CARA utility $E\left[-\exp(-\theta p)|G_1\right]$ plus conditional normality of p, but none of the following results require CARA utility and/or normality.

Given Proposition 1, we can easily characterize the specific subset of the CRF where the choices of this criterion are located.

Corollary 1 *The optimal portfolio that solves problem (8) is*

$$R_a^{**} + \frac{1}{\theta} A_a^{++}.$$

Now we will study the mean-variance frontier where only these returns are located because this is the key feature to develop an efficiency test. We can decompose the unconditional variance of a portfolio payoff as

$$Var\left(p\right) = E\left[Var\left(p|G_1\right)\right] + Var\left[E\left(p|G_1\right)\right],$$

and we will refer to the first component as the residual variance because it can be interpreted as follows. If we consider the residual of a portfolio payoff as $\tilde{p} = p - E\left(p|G_1\right)$ then we find

$$Var\left(\tilde{p}\right) = E\left(\tilde{p}^2\right) = E\left[Var\left(p|G_1\right)\right].$$

We define the Residual Return Mean-Variance Frontier (RRF) as the active returns that minimize the residual variance $E\left[Var\left(p|G_1\right)\right]$ for a given target of expected return $E\left(p\right)$. Thus, the RRF will be given by the set of active returns that solve the problem

$$\min_{p \in \mathcal{R}_a} E\left[Var\left(p|G_1\right)\right] \qquad s.t. \qquad E\left(p\right) = \nu \in \mathbb{R}. \tag{9}$$

The next proposition shows the link between this non-standard frontier and standard mean-variance preferences.

Proposition 2 *The optimal portfolio that solves problem (9) can be represented as*

$$p_R\left(\nu\right) = R_a^{**} + \omega_R\left(\nu\right) A_a^{++}, \qquad \omega_R\left(\nu\right) = \frac{\nu - E\left(R_a^{**}\right)}{E\left(A_a^{++}\right)}, \tag{10}$$

and it is also the solution of the portfolio problem (8) if we choose a $\nu \in \mathbb{R}$ such that

$$\omega_R\left(\nu\right) = \frac{1}{\theta}.$$

The RRF is a hyperbola on the $\left[E^{1/2}\left[Var\left(p|G_1\right)\right], E\left(p\right)\right]$ space and the minimum residual variance is given by R_a^{**}. Figure 3 illustrates the RRF for the binomial example described in Section 2.
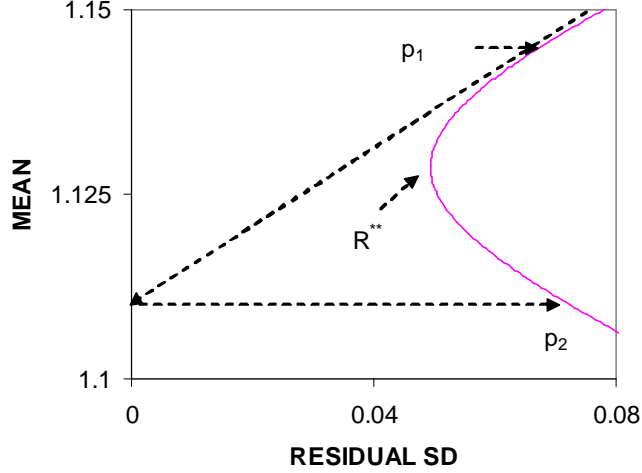
Figure 3: The Residual Frontier on space $[E^{1/2}[Var(p|G_1)], E(p)]$. The location of R_a^{**} is shown, jointly with the geometry of the zero-residual beta return p_2 associated to a return p_1 on the frontier.

On the other hand, R_a^* is located on the CRF, but not on the RRF. The actual difference between problems (9) and (2) is the mean constraint, not the criterion to minimize. Given Lemma 1 and Proposition 2, we can easily characterize the connection between the RRF and the CRF.

Corollary 2 *The returns on the RRF are also on the CRF. A CRF portfolio will also be located on the RRF if and only if we choose the conditional mean profile as*

$$\nu_1 = \omega_R(\nu) E\left(A_a^{++}|G_1\right) + E\left(R_a^{**}|G_1\right),$$

so that

$$w_1(\nu_1) = \omega_R(\nu).$$

In the binomial example of Figure 2, given a value of $E(p|z=1)$, there is only one value of $E(p|z=2)$ on the RRF. Note also that a standard mean-variance agent chooses $\omega_R(\nu) \geq 0$ or equivalently $\nu \geq E(R_a^{**})$, which defines the efficient part of the RRF. Corollary 2 shows that $\nu \geq E(R_a^{**})$ implies $\nu_1 \geq E(R_a^{**}|G_1)$, therefore returns on the efficient part of the RRF are also located on the efficient part of the CRF. However, this is not necessarily the case with efficient returns in the URF sense.

Our remaining task is the particular beta-pricing associated to the RRF. Let us define the residual beta of a return $p \in \mathcal{R}_a$ with respect to a return $p_\beta \in \mathcal{R}_a$ as

$$\beta_R = \frac{E[Cov(p, p_\beta|G_1)]}{E[Var(p_\beta|G_1)]}.$$

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Each return p_R on the RRF, apart from R_a^{**}, has a unique zero-beta counterpart on the RRF.[7] Specifically, two returns on the RRF, say $p_1 = R_a^{**} + \omega_1 A_a^{++}$ and $p_2 = R_a^{**} + \omega_2 A_a^{++}$, satisfy

$$E\left[Cov\left(p_1, p_2 | G_1\right)\right] = 0$$

if and only if (ω_1, ω_2) are related by $E\left[Var\left(R_a^{**} | G_1\right)\right] + \omega_1 \omega_2 E\left(A_a^{++}\right) = 0$. Moreover, the return p_1 is the solution to

$$\max_{p \in \mathcal{R}_a} \frac{E\left(p\right) - E\left(p_2\right)}{E^{1/2}\left[Var\left(p | G_1\right)\right]},$$

which represents the tangency from $E\left(p_2\right)$ to the RRF on the $\left[E^{1/2}\left[Var\left(p | G_1\right)\right], E\left(p\right)\right]$ space. See Figure 3 for a particular p_1.

The following proposition can be interpreted as the RRF counterpart of the results in Roll (1977) for passive strategies and Hansen and Richard (1987) for the URF.

Proposition 3 *A return $p_\beta \in \mathcal{R}_a$ different from R_a^{**} is on the RRF if and only if*

$$E\left(p\right) - E_\beta = \beta_R\left[E\left(p_\beta\right) - E_\beta\right], \qquad \forall p \in \mathcal{R}_a,$$

for some $E_\beta \in \mathbb{R}$.

This proposition defines the null hypothesis of a portfolio efficiency test that is relevant for the standard mean-variance preferences defined by (8). This equation is standard in the sense of using unconditional means $E\left(p\right)$ and $E\left(p_\beta\right)$, which can be estimated by historical averages, but it is not in the sense of using a new type of beta β_R. As in the case of conditional and unconditional betas, the beta-pricing equation should be satisfied for every $p \in \mathcal{R}_a$ since we are considering active returns.

Contrary to the classic Markowitz set-up, where all mean-variance preferences agree on a single beta, we find three different betas to test portfolio efficiency. A conditional beta tests if a particular portfolio is on the CRF or equivalently if it may be optimal for general mean-variance preferences. Therefore we think of the whole square in Figure 1 or any random θ_1 or b_1 in Proposition 1. However, we can increase the test power if we have specific preferences in mind and hence specific subsets of the CRF. In particular, we have studied the RRF and URF, represented by subsets in Figure 1, or any constant $\theta_1 = \theta$ (RRF) or $b_1 = b$ (URF) in Proposition 1. The RRF and URF have specific associated betas, β_R and β_U respectively.

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[7]The return R_a^{**} cannot have a zero-beta counterpart because

$$Cov\left(p, R_a^{**} | G_1\right) = Var\left(R_a^{**} | G_1\right), \qquad \forall p \in \mathcal{R}_a.$$

This difference in betas is relevant because the next section shows that, apart from a few implausible cases, the RRF and URF are different objects. In general a portfolio that is on the RRF cannot be on the URF, therefore residual and unconditional beta-pricing are incompatible. Testing portfolio efficiency from the URF perspective, which is the usual context of empirical work, is actually relevant for quadratic utility but not for more standard mean-variance preferences.

4.3   Relationship between the Residual and Unconditional Frontiers

Both the RRF and the URF are subsets of the CRF, but we still need to study if those subsets share any element. In the very special case of all expected returns being equal then $A_a^+ = 0$ and the CRF collapses to the singleton $R_a^* = R_a^{**}$. Therefore the RRF and URF collapse to that point too. In the binomial example of Figure 2, we would find a single mean-volatility pair in a "bull market" and another one in a "bear market". This situation can be associated with the equilibrium of an economy with a risk-neutral agent or simply a single payoff in $\mathbf{x}$.

In general the RRF and URF cannot share a point.[8]   Each frontier gives the best return for different criteria $Var(p)$ and $E[Var(p|G_1)]$ and hence the other frontier will be located to the right on the corresponding space as Figure 4 shows.
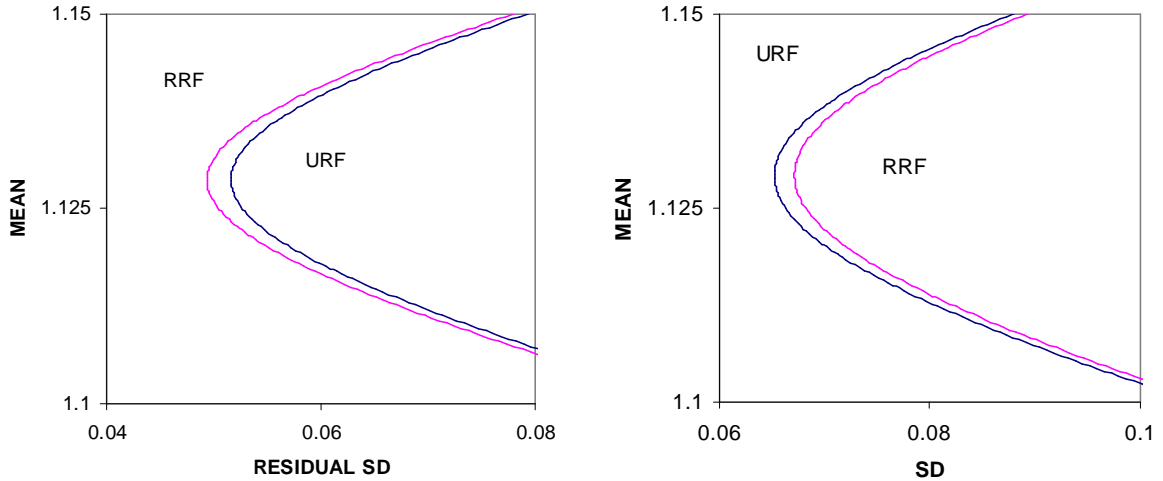

Figure 4: The Residual and Unconditional Frontiers (RRF and URF) on spaces
$[E^{1/2}[Var(p|G_1)], E(p)]$ and $[Var^{1/2}(p), E(p)]$.

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[8]For instance, both R_a^{**} and R_a^* belong to the CRF. However, R_a^{**} belongs to the RRF but not to the URF, while R_a^* is located on the URF but not on the RRF.

The next proposition characterizes the special case where the RRF and URF are tangent when $A_a^+ \neq 0$.

Proposition 4 *Let us assume $A_a^+ \neq 0$. The RRF and the URF are tangent if and only if there are two real numbers (a, b) such that*

$$E\left(R_a^*|G_1\right) = a + bE\left(A_a^+|G_1\right),$$

in which case the shared element has a constant conditional mean.

Unconditional two-fund spanning holds in both frontiers, in the sense that we can use an unconditional combination of two elements on one frontier to replicate the remaining elements of that frontier. Hence the RRF and URF will be equal if the previous condition is satisfied at two points at the same time, which translates into the following result.

Corollary 3 *Let us assume $A_a^+ \neq 0$. The RRF and URF are equal if and only if*

$$E\left(R_a^*|G_1\right) \in \mathbb{R}, \qquad E\left(A_a^+|G_1\right) \in \mathbb{R},$$

in which case every element on those frontiers has a constant conditional mean.

In this special case, returns on the RRF and URF have constant conditional means and hence unconditional and residual variances (and betas) are equal. The hyperbolas that represent the CRF on the $[Var^{1/2}\left(p|G_1\right), E\left(p|G_1\right)]$ space for a each value of the conditioning variables in G_1, e.g. the "bull" and "bear" hyperbolas in Figure 2, share the same location of the minimum and the same asymptotes. This situation is far from plausible and does not depend exclusively on asset payoffs having constant conditional means because the conditional covariance matrices also matter in the construction of active strategies.

5   Frontiers with a Safe Asset

The existence of a safe asset may simplify the study of portfolio efficiency with a single optimal risk-return trade-off defined by an optimal Sharpe ratio. We will describe its implications for the different frontiers that we studied so far, with a special emphasis on Sharpe ratios and Jensen's alphas given their relevance in portfolio efficiency tests.

In our binomial example of "bull" and "bear" markets, we introduce a conditionally riskless asset with a return R_0 defined by the real numbers

$$R_0\left(z = 1\right) = 1.07, \qquad R_0\left(z = 2\right) = 1.02,$$

which imply the unconditional moments

$$E\left(R_0\right) = 1.06, \qquad Var\left(R_0\right) = 0.0004.$$

Regarding our general theoretical framework, imagine that investors have access to a set of assets that includes not only the original risky asset payoffs[9] in $\mathbf{x}$, but also the safe payoff $x_0 = 1$, with cost $C(x_0|G_1) \in I_1$. In this context, $Q_a \supset P_a$ will denote the corresponding enlarged payoff space constructed by active strategies on both $\mathbf{x}$ and x_0, and we will denote by $S_a \supset \mathcal{R}_a$ the subset of returns in Q_a. In this set-up, the conditionally safe return is given by

$$R_0 = \frac{1}{C(x_0|G_1)} \in I_1,$$

and sometimes we will work with the excess return of a gross return $p \in S_a$, which is defined as

$$e = p - R_0.$$

5.1 The Conditional Frontier

The elements of the CRF solve the same problem as (2), except that p is allowed to belong to the enlarged set S_a. Therefore, equation (3) also defines the CRF after the introduction of a safe payoff if we translate R_a^* and A_a^+ to the new space Q_a. We denote the counterpart of A_a^+ by B_a^+ and we can show that R_a^* becomes $R_0(1 - B_a^+)$, therefore the CRF on S_a is

$$p_C(\nu_1) = R_0 + (\omega_1(\nu_1) - R_0) B_a^+, \qquad \omega_1(\nu_1) - R_0 = \frac{\nu_1 - R_0}{E(B_a^+|G_1)}.$$

We also have an equivalent representation along the lines of equation (4). The minimum conditional variance return is trivially R_0 and there is a well defined arbitrage portfolio B_a^{++} that plays the same role as A_a^{++} in the new space of arbitrage portfolios, Using those objects, we can express the CRF as

$$p_C(\nu_1) = R_0 + w_1(\nu_1) B_a^{++}, \qquad w_1(\nu_1) = \frac{\nu_1 - R_0}{E(B_a^{++}|G_1)}.$$

The elements of the CRF lie along two straight lines on the $[Var^{1/2}(p|G_1), E(p|G_1)]$ space for each possible value of the signals in G_1, and those two lines intersect on the vertical axis at R_0. Figure 5 illustrates these features with our binomial example. We can also see that there is a tangency between the CRF on S_a and the CRF on $\mathcal{R}_a$. In general there is a conditional mean profile ν_1 such that the weight of the CRF on the conditionally safe payoff x_0 will be identically 0 for every possible signal realization, which implies that it will be equal to the CRF on $\mathcal{R}_a$ at that point.

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[9]So far we assumed implicitely the smallest eigenvalue of $Var(\mathbf{x}|G_1)$ is uniformly bounded away from 0 a.s., which implies that none of the primitive assets is either conditionally riskless or redundant.

Lemma 2 *If $E\left(R_a^{**}|G_1\right) \neq R_0$ then there is a tangency portfolio between the CRF with and without a safe asset given by*

$$R_a^{**} + \left[\frac{Var\left(R_a^{**}|G_1\right)}{E\left(R_a^{**}|G_1\right) - R_0}\right] A_a^{++} = R_a^* + \left[\frac{E\left(R_a^{*2}|G_1\right) - R_0 E\left(R_a^*|G_1\right)}{E\left(R_a^*|G_1\right) + R_0 E\left(A_a^+|G_1\right) - R_0}\right] A_a^+. \tag{11}$$

The risky component of the elements of the CRF on $\mathcal{S}_a$ is conditionally proportional to this element of the CRF on $\mathcal{R}_a$.
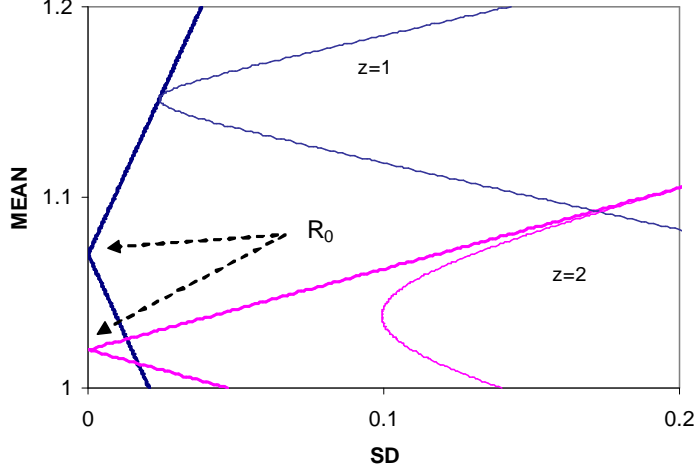

Figure 5: Conditional Frontier with and without a safe asset on space $\left[Var^{1/2}\left(p|z\right), E\left(p|z\right)\right]$ at both signal values $z = 1$ and $z = 2$. The location of the safe return R_0 is shown.

There is a single optimal risk-return trade-off on the $\left[Var^{1/2}\left(p|G_1\right), E\left(p|G_1\right)\right]$ space for each signal value in the sense that conditional Sharpe ratios of risky returns $p \in \mathcal{S}_a$ defined as

$$SR_1 = \frac{E\left(e|G_1\right)}{Var^{1/2}\left(e|G_1\right)}$$

reach their maximum value for risky returns on the efficient side of the CRF. Those excess returns are given by $w_1\left(\nu_1\right) B_a^{++}$ and hence

$$\max_{p \in \mathcal{S}_a} SR_1^2 = E\left(B_a^{++}|G_1\right),$$

a result that we can use it to test portfolio efficiency. In our binomial example, this random variable takes two values given by the slopes of the straight lines in Figure 5.

Regarding conditional beta-pricing, now there is a unique zero-beta return given by the safe asset itself. In particular, a return $p_\beta \in \mathcal{S}_a$ different from R_0 is on the CRF if and only if

$$E\left(p|G_1\right) - R_0 = \beta_1\left[E\left(p_\beta|G_1\right) - R_0\right], \qquad \forall p \in \mathcal{S}_a.$$

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That pricing equation can also be expressed in terms of excess returns and we can define the conditional version of Jensen's alpha as

$$\alpha_1 = E\left(e|G_1\right) - \frac{Cov\left(e, e_\beta|G_1\right)}{Var\left(e_\beta|G_1\right)} E\left(e_\beta|G_1\right),$$

which is equal to zero for every return on the CRF.

5.2 The Residual Frontier

Equation (10) also defines the RRF on $\mathcal{S}_a$ if we translate R_a^{**} and A_a^{++} to the new space,

$$p_R\left(\nu\right) = R_0 + \omega_R\left(\nu\right) B_a^{++}, \qquad \omega_R\left(\nu\right) = \frac{\nu - E\left(R_0\right)}{E\left(B_a^{++}\right)}.$$

The RRF becomes two straight lines on the $[E^{1/2}\left[Var\left(p|G_1\right)\right], E\left(p\right)]$ space with zero residual variance minimum at $\nu = E\left(R_0\right)$, as Figure 6 shows. However, contrary to conventional wisdom on mean-variance frontiers with a safe asset, in general there is no tangency portfolio as we can see in Figure 6. The risky component of the elements of the RRF on $\mathcal{S}_a$ is conditionally proportional to the tangency portfolio of the CRF on $\mathcal{R}_a$ (11), which does not belong to the RRF on $\mathcal{R}_a$ in general because its weight on A_a^{++} is random.[10]
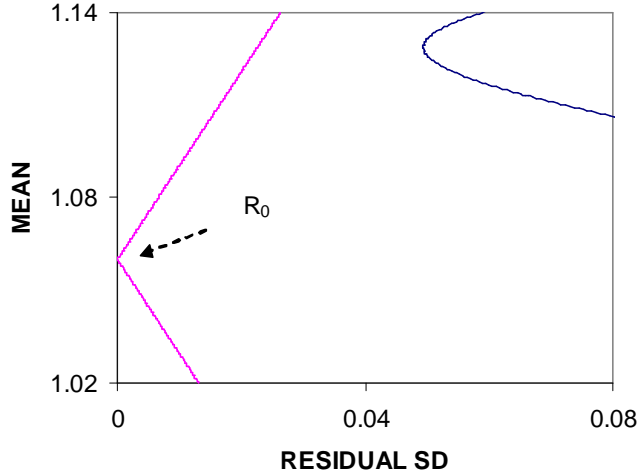

Figure 6: The Residual Frontier with and without a safe asset on space $[E^{1/2}\left[Var\left(p|G_1\right)\right], E\left(p\right)]$. The location of the safe return R_0 is shown.

[10]We can follow the proof of Corollary 3 to show that there is a tangency if and only if

$$\frac{E\left(R_a^{**}|G_1\right) - R_0}{Var\left(R_a^{**}|G_1\right)} \in \mathbb{R},$$

in which case we can span the RRF on $\mathcal{S}_a$ by means of passive strategies in the safe asset and the tangency portfolio.

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Regarding Sharpe ratios and Jensen's alpha, we find standard properties on the RRF if we use their residual versions. There is a single optimal risk-return trade-off on the $[E^{1/2}[Var(p|G_1)], E(p)]$ space in the sense that residual Sharpe ratios of risky returns $p \in \mathcal{S}_a$ defined as

$$SR_R = \frac{E(e)}{E^{1/2}[Var(e|G_1)]}$$

reach their maximum value for risky returns on the efficient side of the RRF. Those excess returns are given by $\omega_R(\nu) B_a^{++}$ and hence

$$\max_{p \in \mathcal{S}_a} SR_R^2 = E\left(B_a^{++}\right).$$

We can rely on SR_R to construct a simple test of portfolio efficiency. This Sharpe ratio has not been used in empirical work so far even though it is the relevant one for preferences like (8). We also find the following relationship between SR_R on the RRF and SR_1 on the CRF

$$\max_{p \in \mathcal{S}_a} SR_R^2 = E\left[\max_{p \in \mathcal{S}_a} SR_1^2\right].$$

Residual beta-pricing has a unique zero-beta return given by the safe asset itself, as the following corollary states.

Corollary 4 *A return $p_\beta \in \mathcal{S}_a$ different from R_0 is on the RRF if and only if*

$$E(p) - E(R_0) = \beta_R [E(p_\beta) - E(R_0)], \qquad \forall p \in \mathcal{S}_a.$$

That pricing equation can also be expressed in terms of excess returns and we can define the residual version of Jensen's alpha as

$$\alpha_R = E(e) - \frac{E[Cov(e, e_\beta|G_1)]}{E[Var(e_\beta|G_1)]} E(e_\beta),$$

which is equal to zero for every return on the RRF.

5.3   The Unconditional Frontier

The elements of the URF solve the same problem as (5), except that p is allowed to belong to the enlarged set $\mathcal{S}_a$. The solution (6) is still valid with the proper translation of R_a^* and A_a^+ to the new space Q_a

$$p_U(\nu) = R_0 + (\omega_U(\nu) - R_0) B_a^+, \qquad \omega_U(\nu) = \frac{\nu - E[R_0(1 - B_a^+)]}{E(B_a^+)}.$$

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Peñaranda and Sentana (2008) point out two facts about the URF that contradict conventional wisdom on mean-variance frontiers with a safe asset. First, as Figure 6 illustrated with the RRF, in general there is no tangency portfolio. The risky component of the elements of the URF on $\mathcal{S}_a$ is again conditionally proportional to an element on the original CRF on $\mathcal{R}_a$ (11) that does not belong to the URF on $\mathcal{R}_a$ in general because its weight on A_a^+ is random.[11]

Second, there is an important difference with respect to the RRF. The elements of the URF do not lie along two straight lines on the $[Var^{1/2}(p), E(p)]$ space if R_0 is random. In fact, Hansen and Richard (1987) show that R_0 is not on the URF in that case. Figure 7 compares the RRF and the URF in this context.
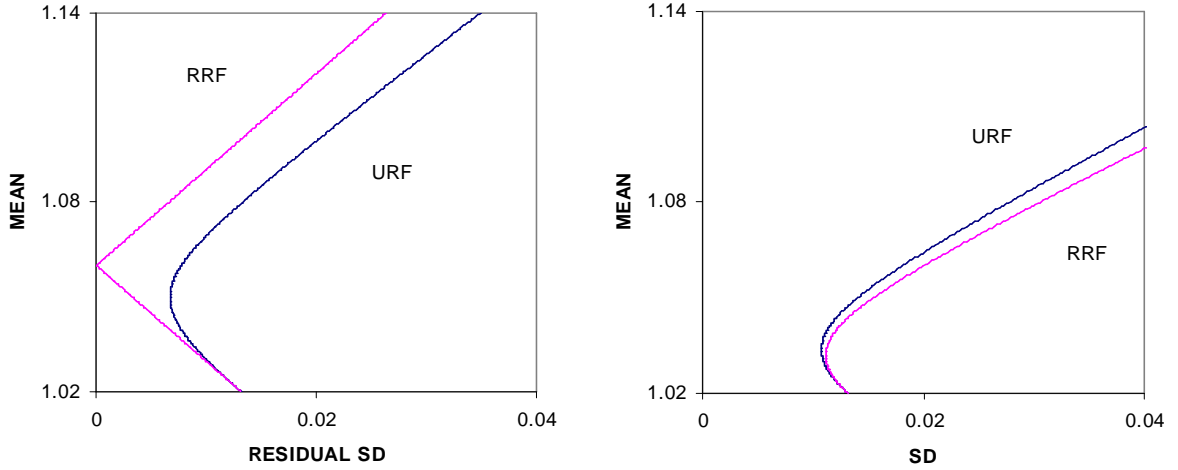

Figure 7: The Residual and Unconditional Frontiers (RRF and URF) with a safe asset on spaces $[E^{1/2}[Var(p|G_1)], E(p)]$ and $[Var^{1/2}(p), E(p)]$.

Now let us consider Sharpe ratios and Jensen's alpha. The unconditional Sharpe ratio of a risky return $p \in \mathcal{S}_a$, defined as

$$SR_U = \frac{E(e)}{Var^{1/2}(e)},$$

is not unique on the URF, where excess returns are given by $(\omega_U(\nu) - R_0) B_a^+$, and hence SR_U is not useful to test if a portfolio is on the URF. Moreover, the unconditional beta-pricing equation

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[11] We can follow the proof of Corollary 3 to show that there is a tangency if and only if

$$\frac{E(R_a^{*2}|G_1) - R_0 E(R_a^*|G_1)}{E(R_a^*|G_1) + R_0 E(A_a^+|G_1) - R_0} \in \mathbb{R}.$$

Nevertheless, we cannot represent the URF on $\mathcal{S}_a$ with passive strategies in the safe asset and the tangency portfolio unless $R_0 \in \mathbb{R}$.

is the same as if there was not a safe asset in the sense that the zero-beta return will depend on the chosen element of the URF. An unconditionally risky safe asset is not the proper zero-beta return in unconditional beta-pricing. Therefore, we can define the unconditional version of Jensen's alpha as

$$\alpha_U = E(e) - \frac{Cov(e, e_\beta)}{Var(e_\beta)} E(e_\beta),$$

but it is not equal to zero for returns on the URF, and its value also depends on the particular e_β.

Empirical work tends to use unconditional Sharpe ratios and Jensen's alpha because they do not require a model of conditional moments as commented before. Unfortunately, they simply do not work as conventional wisdom suggests when testing the location of a return on the URF. The null hypothesis is not well defined for both Sharpe ratios (not unique on the URF) and Jensen's alpha (not zero on the URF).

On the other hand, the RRF has better properties in terms of those measures because the excess returns e_β on the RRF are simply $\omega_R(\nu) B_a^{++}$. There is a unique unconditional Sharpe ratio on the RRF equal to $E(B_a^{++})/Var^{1/2}(B_a^{++})$ even though it is not a straight line on the $[Var^{1/2}(p), E(p)]$ space. As a result, we can use both unconditional and residual Sharpe ratios to test the location of a portfolio on the RRF. In addition, unconditional alphas on the RRF are not zero but they do not depend on e_β, only on the particular e that we are pricing.

There are more properties to comment such as SR_R and α_R on the URF, but we leave them for the empirical application. We finally emphasize that all these different properties of the URF and RRF are compatible with both having the same maximum SR_1 and zero α_1 at every signal value because they belong to the CRF.

5.4 A Safe Asset with Constant Return

The empirical work on efficiency tests has focused on properties that are compatible with a nonrandom R_0. This may be due to the fact that Treasury-Bills' historical variance is much lower than e.g. stocks' historical variance. Nevertheless, the empirical application will show that the relevant properties are the ones that we have described above not the ones derived from a constant return.

In any case, before turning to the empirical application, we will describe the frontiers when $R_0 \in \mathbb{R}$ or equivalently the safe asset is unconditionally riskless because $C(x_0|G_1)$ is nonrandom.

In our binomial example we simply define a constant return equal to the mean of the previous safe asset, $R_0 = 1.06$.

The CRF would have a single intersection on the vertical axis of Figure 5, and Figure 6 would be similar in the sense that there is no tangency portfolio on the RRF irrespective of whether R_0 is random or not. There is not a tangency portfolio on the URF either, but it will indeed consist of two straight lines that intersect on the vertical axis at R_0. Figure 8 illustrates this case for our binomial example. A constant R_0 is a necessary but not sufficient condition for the URF and RRF to be equal. Note that the translation of Corollary 3 to the safe asset context shows that the URF and RRF on $\mathcal{S}_a$ are equal if and only if both R_0 and $E\left(B_a^+|G_1\right)$ are constant.
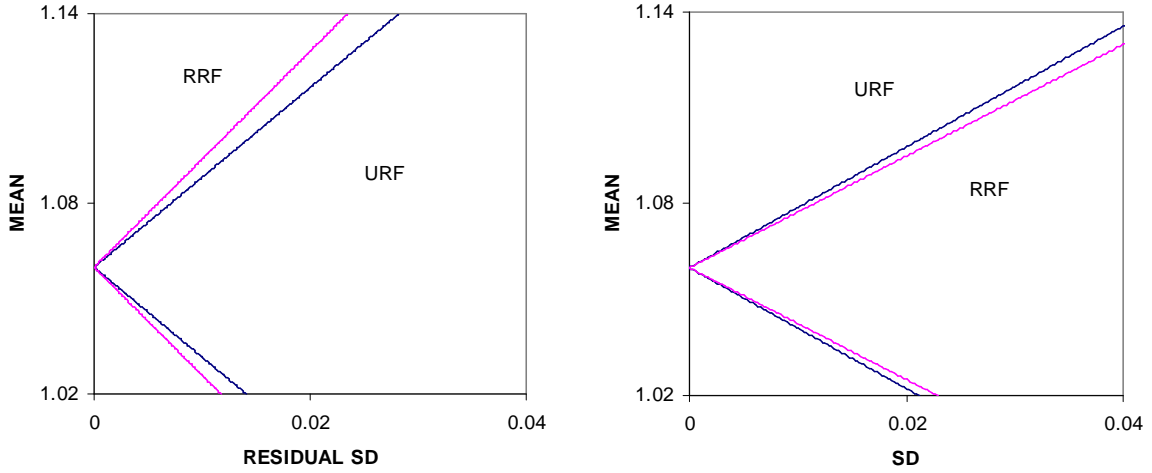

Figure 8: The Residual and Unconditional Frontiers (RRF and URF) with a constant return on spaces $\left[E^{1/2}\left[Var\left(p|G_1\right)\right], E\left(p\right)\right]$ and $\left[Var^{1/2}\left(p\right), E\left(p\right)\right]$.

The properties of the RRF are similar to the general case of a random R_0, but the properties of the URF change drastically. Now excess returns on the URF are unconditionally proportional to B_a^+. There is a single optimal SR_U for risky returns on the efficient side of the URF given by $E\left(B_a^+\right)/\left(1 - E\left(B_a^+\right)\right)$, and we can rely on SR_U to test portfolio efficiency. Therefore, in the case of an unconditionally riskless asset, we can find a relationship between SR_U on the URF and SR_1 on the CRF

$$\frac{1}{1 + \max_{p \in \mathcal{S}_a} SR_U^2} = E\left[\frac{1}{1 + \max_{p \in \mathcal{S}_a} SR_1^2}\right],$$

which was already developed by Jagannathan (1996). Here we emphasize that such a relationship, where both sides are equal to $1 - E\left(B_a^+\right)$, does not hold in the realistic context of a nonconstant return. Regarding Jensen's alpha, there is a unique zero-beta return on the URF, the safe asset itself. Now a return $p_\beta \in \mathcal{S}_a$ different from R_0 is on the URF if and only if its α_U is zero.

6   Empirical Application

Our results have been illustrated so far with a simple binomial example for pedagogical reasons. Now, by means of a standard empirical application, we will confirm that the quantitative relevance of our results does not depend on the binomial feature neither on the chosen numbers. Moreover, we do not need to rely on too many assets, neither too many predictors or a complex model of conditional moments. We will think of our sample estimates as population parameters, or simply calibrated parameters, since our goal is not inference but a plausible illustration of the right null hypothesis in efficiency tests.

The vector $\mathbf{x}$ is constructed with six gross returns whose nonrandom $C\left(\mathbf{x}|G_1\right)$ is equal to a vector of ones from six portfolios of US stocks sorted by size (small and large) and book-to-market (high, medium and low). We will refer to those six portfolios as SL, SM, SH, BL, BM and BH. We use data from 1953 to 2007 from Ken French's web page; see Fama and French (1993) for further details.

The main difference with our binomial example is that we will use continuous signals and hence, from the perspective of date 0, the payoff space at date 2 will be infinite dimensional. Similarly, the counterpart of Figure 2 would contain infinite hyperbolas since there is a different one for each signal value. The new information set is constructed with three standard predictors of stock returns: the dividend yield, the term spread, and the default spread. See e.g. Ferson and Siegel (2007) and the references therein.

We model conditional moments following the standard approach in e.g. Ferson and Siegel (2001), linear predictive regressions where $E\left(\mathbf{x}|G_1\right)$ is linear in the three predictors and $Var\left(\mathbf{x}|G_1\right)$ is constant. In what follows, we will use overlapping annual gross returns from our monthly data. We think of a one-year investment horizon to have enough return predictability and hence relevant active strategies. The R^2 of the six predictive regressions range from 9 to 14%, which are typical values with annual data.

Figure 9 shows the RRF and URF in this set-up,[12] jointly with the passive frontier (dashed line). The latter frontier is defined like the URF in (5) but constrained to passive strategies and hence it represents the classic Markowitz frontier. Interestingly, from the point of view of unconditional variance, the RRF looks as suboptimal as the passive frontier.
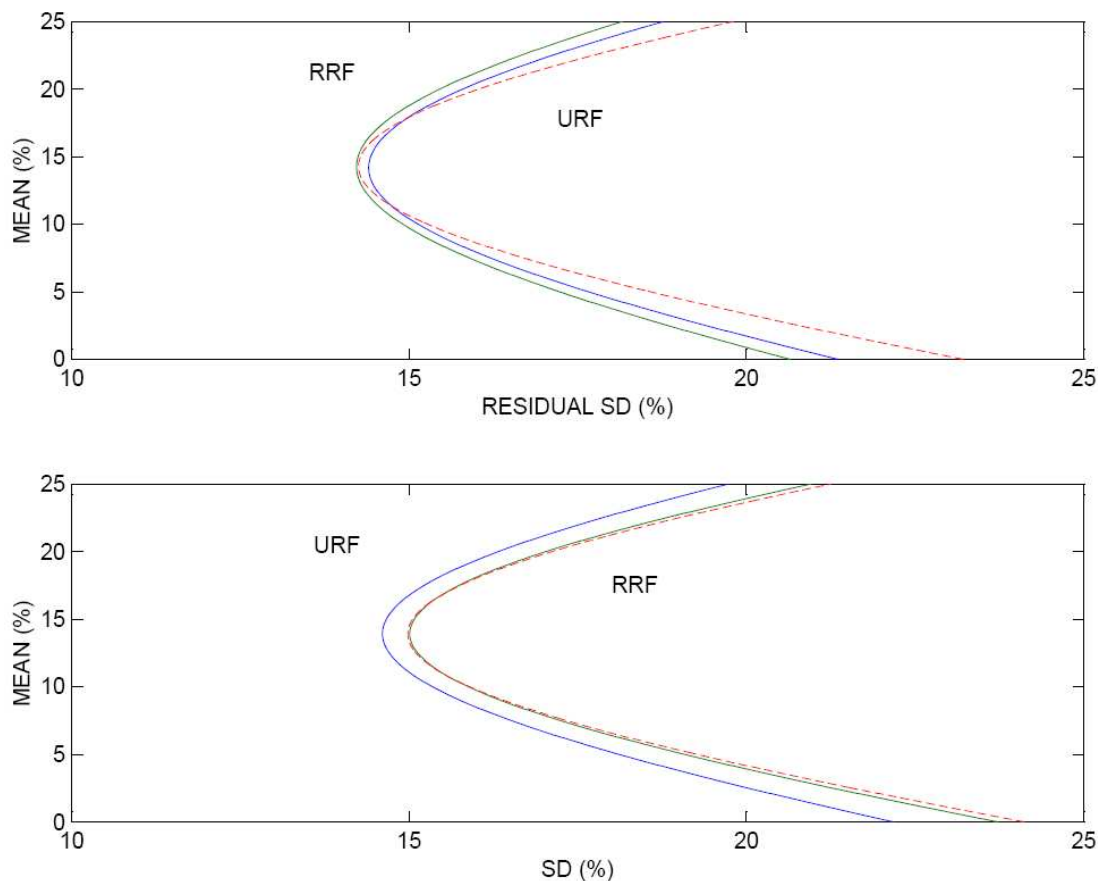

Figure 9: Empirical Residual and Unconditional Frontiers (RRF and URF) on spaces $[E^{1/2}[Var(p|G_1)], E(p)]$ and $[Var^{1/2}(p), E(p)]$. The dashed line represents the passive frontier.

This is also the case when we add the safe asset to our empirical application, see Figure 10. The safe return has a sample mean of 5.2%, with a volatility of 2.8%. Compared to the six risky returns of our primitive portfolios, whose average mean is 15.3% and average volatility is 20.1%, the safe return volatility is very low. Nevertheless, it really makes a difference with respect to

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[12]Corollary 3 showed that the RRF and URF are equal if and only if $E(R_a^*|G_1)$ and $E(A_a^+|G_1)$ are constant. In this data set, $E(R_a^*|G_1)$ has mean 0.672 and standard deviation 0.202, while $E(A_a^+|G_1)$ has mean 0.410 and standard deviation 0.183.

a zero volatility as we will see. The shape of the URF is far from the conventional property of straight lines that intersect the y-axis, while the RRF has this property on its own space.
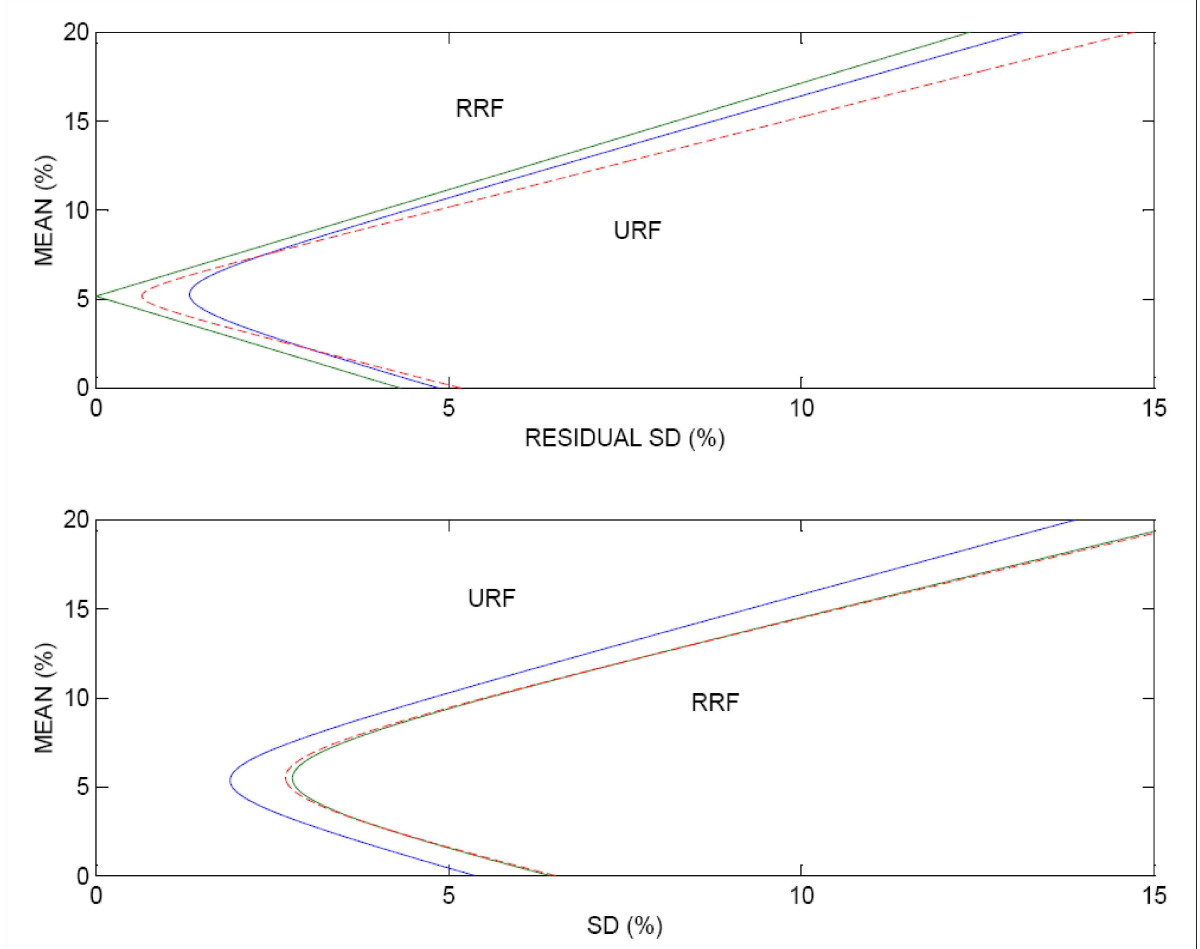

Figure 10: Empirical Residual and Unconditional Frontiers (RRF and URF) with a safe asset on spaces $[E^{1/2}\left[Var\left(p|G_1\right)\right], E\left(p\right)]$ and $[Var^{1/2}\left(p\right), E\left(p\right)]$. The dashed line represents the passive frontier.

Now let us turn to the computation of Sharpe ratios and Jensen's alpha. Table 1 shows that the SR_U on the URF changes considerably, and can be lower than the value 0.94 on the RRF, which is unique even though the RRF is not a straight line on the $[Var^{1/2}\left(p\right), E\left(p\right)]$ space. For instance, the return on the URF with mean 6% has an unconditional Sharpe ratio of 0.38, while the return with mean 10% has a ratio of 0.97. In addition, passive strategies can give a higher Sharpe ratios than the URF, and these ratios are different to the RRF even though the passive frontier and the RRF look similar on the $[Var^{1/2}\left(p\right), E\left(p\right)]$ space. Specifically, the SR_U on the passive frontier is 0.73, 0.92, and 0.93 for return means 6, 10, and 20% respectively. Regarding

the SR_R, it takes values 0.80, 1.00, and 1.01 on the passive frontier for the same mean returns. The RRF provides the unique optimal value of SR_R.

Table 1: Empirical Residual and Unconditional Sharpe ratios on the Residual and Unconditional Frontiers (RRF and URF).

The URF ratios depend on the particular return mean, which is shown in parenthesis.

	SR_R	SR_U
RRF	1.197	0.942
URF(6%)	0.565	0.380
URF(10%)	1.096	0.970
URF(20%)	1.128	1.053

Table 2 shows the unconditional and residual alphas of the six primitive portfolios with respect to returns on the RRF and URF. We could apply the beta-pricing equations to any active return but we focus on the primitive risky assets to simplify the analysis.

Table 2: Empirical Residual and Unconditional Jensen's alphas on the Residual and Unconditional Frontiers (RRF and URF).

The alphas are computed for each of the six primitive portfolios, whose risk premia are 7.3, 12.5, 14.8, 7.0, 8.4, and 11.0% respectively. The URF alphas depend on the particular return mean, which is shown in parenthesis.

	α_R (%)						α_U (%)					
	SL	SM	SH	BL	BM	BH	SL	SM	SH	BL	BM	BH
RRF	0	0	0	0	0	0	1.7	2.2	2.8	1.8	1.7	2.5
URF(6%)	6.2	10.1	11.8	5.0	6.0	8.3	7.1	11.4	13.3	5.8	6.9	9.5
URF(10%)	0.7	1.3	1.3	0.0	0.3	0.5	2.1	2.6	3.0	0.9	1.0	1.8
URF(20%)	-0.0	0.3	0.2	-0.3	-0.1	-0.2	0.5	0.5	0.5	0.0	0.0	0.2

The URF shows a nonzero α_U that can be very high, even higher than on the RRF (where alpha is independent of the particular point on the frontier), but decreases with the mean of the particular portfolio on the URF. In particular, moving from an expected return of 6% to 10% on the URF changes the average alpha across the six portfolios from 9 to 2%, while the average risk premia on these portfolios is 10% and the average α_U for the RRF is 2%. Regarding the

α_R, the URF shows again different values for a given asset depending on the particular point on the frontier, while α_R on the RRF is zero for every active return with respect to any point on the frontier.

Obviously, the "mispricing" in terms of α_U on the URF is only due to using the safe asset as the wrong zero-beta, but this is not taken into account in the usual null hypothesis of portfolio efficiency tests. The conventional wisdom is actually based on properties that hold with an unrealistic constant return. To compare both set-ups, Tables 3 and 4 replicate Tables 1 and 2 under the assumption of a constant return of 5.2%. Now all the frontiers in Figure 10 would turn into straight lines that touch the y-axis at R_0 in both spaces.

Table 3: Empirical Sharpe ratios on the Residual and Unconditional Frontiers (RRF and URF) with an artificially constant return.

	SR_R	SR_U
RRF	1.195	0.929
URF	1.121	1.057

The properties of the URF change drastically with a constant return, while the RRF is not so sensitive to the presence of such an asset. The URF has a unique SR_U, which is higher than the unique value on the RRF; and a unique SR_R, which is lower than the RRF. For comparison, the passive frontier has a unique SR_U equal to 0.94, slightly better than the RRF, while its unique SR_R is equal to 1.01. Finally, Table 4 shows a zero α_U and a unique nonzero α_R (for each asset that is priced) on the URF.

Table 4: Empirical Residual and Unconditional Jensen's alphas on the Residual and Unconditional Frontiers (RRF and URF) with an artificially constant return. The alphas are computed for each of the six primitive portfolios, whose risk premia are 7.3, 12.5, 14.8, 7.0, 8.4, and 11.0% respectively.

	α_R (%)						α_U (%)					
	SL	SM	SH	BL	BM	BH	SL	SM	SH	BL	BM	BH
RRF	0	0	0	0	0	0	1.5	2.2	2.8	1.9	1.8	2.6
URF	-0.1	0.2	0.1	-0.3	-0.1	-0.3	0	0	0	0	0	0

7 Conclusion

The trivial connection between mean-variance frontiers and preferences in the classic Markowitz set-up becomes much more complex when investors design active strategies. This paper studies the tension between the frontiers that drive empirical work and the preferences that are used in finance theory, and its implications for efficiency tests. Our results can be considered as both a guideline for future empirical work and an accurate interpretation of the current evidence.

We distinguish two subsets of the conditional frontier (CRF), the unconditional and the residual frontiers (URF and RRF), as illustrated in Figure 1. The URF is the main focus of empirical work but has a weak connection to mean-variance preferences. The RRF has not been studied in the literature but it is where standard mean-variance preferences choose portfolios. We show that both subsets of the CRF do not share any active return unless returns satisfy a few implausible cases.

In the classic Markowitz set-up there is a single beta and Sharpe ratio to use in efficiency tests, but return predictability implies that different mean-variance preferences may disagree on the relevant test of portfolio efficiency. An unconditional beta or Sharpe ratio are not the relevant tests for standard mean-variance preferences, whose specific tests are based on their residual counterparts, while the power of a conditional test may be too low since all mean-variance preferences are compatible with it.

When there is a safe asset, we also highlight that conventional features with passive strategies such as a unique Sharpe ratio and a zero alpha on the frontier do not necessarily hold on the URF. Hence the use of unconditional Sharpe ratios and Jensen's alphas in empirical work may be misleading. On the other hand, the RRF has the classic properties that simplify the construction of efficiency tests.

A standard empirical application to US stocks confirms the quantitative relevance of our theoretical concerns. The corresponding URF and RRF have quite different properties, and their plots show that, from the perspective of one frontier, the other may look as inefficient as passive strategies. Moreover, the unconditional Sharpe ratio on the URF changes considerably and can be lower than its unique value on the RRF. The unconditional Jensen's alphas can be very high on the URF, even higher than on the RRF.

Finally, we did not study the translation of beta-pricing results to stochastic discount factors because the duality between portfolio and stochastic discount factor frontiers is developed in

Peñaranda and Sentana (2007).[13]   An interesting avenue of further research is the application of the RRF framework to the analysis of factor mimicking portfolios, as Ferson, Siegel, and Xu (2006) have computed such portfolios in the URF set-up. The analysis of other families of mean-variance preferences whose corresponding subsets of the CRF have interesting properties (like the RRF and URF), and more general preferences that include higher order moments, are additional topics of further research.

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[13]In addition, Basu and Stremme (2006) relate the pricing of portfolios with general cost to Sharpe ratios on frontiers with general cost.

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Appendix

Proofs

Lemma 1:

The connection between (R_a^*, A_a^+) and (R_a^{**}, A_a^{++}) is given by

$$R_a^{**} = R_a^* + E(R_a^*|G_1) A_a^{++}, \qquad A_a^{++} = \frac{1}{1 - E(A_a^+|G_1)} A_a^+.$$

Note that $R_a^{**} - R_a^*$ and A_a^{++} are conditionally proportional to A_a^+. The factor of proportionality is well defined because the safe asset does not belong to $\mathcal{A}_a$. Equivalently, we can write

$$R_a^* = R_a^{**} - E(R_a^{**}|G_1) A_a^+, \qquad A_a^+ = \frac{1}{1 + E(A_a^{++}|G_1)} A_a^{++}.$$

Finally, the lemma is a straightforward implication of the relationships above,

$$R_a^{**} + w_1(\nu_1) A_a^{++} = \left[R_a^* + \frac{E(R_a^*|G_1)}{1 - E(A_a^+|G_1)} A_a^+ \right] + w_1(\nu_1) \left[\frac{1}{1 - E(A_a^+|G_1)} A_a^+ \right]$$

$$= R_a^* + \left[\frac{E(R_a^*|G_1) + w_1(\nu_1)}{1 - E(A_a^+|G_1)} \right] A_a^+ = R_a^* + \omega_1(\nu_1) A_a^+,$$

which shows the link between $w_1(\nu_1)$ in (4) and $\omega_1(\nu_1)$ in (3). $\qquad\square$

Proposition 1:

This proof and others below are based on orthogonal projections. We assume that the diagonal elements of $E(\mathbf{xx'}|G_1)$ are uniformly bounded with probability one (a.s.), so that all the elements of $\mathbf{x}$ belong to L^2, which is the collection of all random variables defined on the underlying probability space with bounded (unconditional) second moments.

The first point of the proposition is based on the decomposition of any $p \in \mathcal{R}_a$ along the lines of Chamberlain and Rothschild (1983) framework for passive strategies. We will use the covariance projection based on $Cov(x, y|G_1)$ as the inner product between random variables x and y, and $Var^{1/2}(x|G_1)$ as its corresponding norm. A priori, this may not be a proper norm in the sense that $Var(x|G_1) = 0$ implies $p = E(p|G_1)$ but not necessarily $p = 0$. However, if there is not a safe asset, i.e. there is no p such that $p = E(p|G_1) \neq 0$, then $Var(x|G_1) = 0$ implies $p = 0$. Moreover, even if there was a safe asset, this inner product would define a proper norm in $\mathcal{A}_a$ if there are no arbitrage opportunities because a safe asset cannot belong to $\mathcal{A}_a$.

Therefore we can decompose $p - R_a^{**} \in \mathcal{A}_a$ into two components that belong to $\mathcal{A}_a$

$$p - R_a^{**} = \lambda_1 A_a^{++} + u, \qquad \lambda_1 = \frac{Cov(A_a^{++}, p - R_a^{**}|G_1)}{Var(A_a^{++}|G_1)} = \frac{E(p - R_a^{**}|G_1)}{E(A_a^{++}|G_1)},$$

where the former component is its conditional covariance projection onto the conditional span of A_a^{++}, and the latter one is the corresponding residual. The residual satisfies $E(u|G_1) = 0$ because $Cov(A_a^{++}, u|G_1) = 0$.

We can characterize any $p \in \mathcal{R}_a$ by its corresponding pair (λ_1, u). The optimal portfolio maximizes

$$\max_{(\lambda_1, u)} E(p|G_1) - \frac{\theta_1}{2} Var(p|G_1),$$

a criterion that we can express now as

$$\left[E(R_a^{**}|G_1) + \lambda_1 E(A_a^{++}|G_1) \right] - \frac{\theta_1}{2} \left[Var(R_a^{**}|G_1) + \lambda_1^2 Var(A_a^{++}|G_1) + Var(u|G_1) \right],$$

because $Cov(R_a^{**}, p|G_1) = 0$ for every $p \in \mathcal{A}_a$. As a result, the optimal portfolio chooses a u such that $Var(u|G_1) = 0$, and hence $u = E(u|G_1) = 0$, and a λ_1 such that

$$\max_{\lambda_1} \left(\lambda_1 - \frac{\theta_1}{2} \lambda_1^2 \right) E(A_a^{++}|G_1)$$

since $E(A_a^{++}|G_1) = Var(A_a^{++}|G_1)$. The optimal λ_1 is θ_1^{-1} and the optimal return can be expressed as

$$R_a^{**} + \frac{1}{\theta_1} A_a^{++},$$

which completes the proof of the first part of the proposition.

The second point of the proposition is based on a different decomposition developed by Hansen and Richard (1987). They introduced a conditional analogue to a standard Hilbert space based on the mean square inner product $E(xy|G_1)$, and the associated mean square norm $E^{1/2}(x^2|G_1)$. Their decomposition is

$$p = R_a^* + \eta_1 A_a^+ + u, \qquad \eta_1 = \frac{E(A_a^+(p - R_a^*)|G_1)}{E(A_a^{+2}|G_1)} = \frac{E(p - R_a^*|G_1)}{E(A_a^+|G_1)},$$

where u satisfies $E(u|G_1) = 0$ because $E(A_a^+ u|G_1) = 0$. The optimal portfolio maximizes

$$\left[E(R_a^*|G_1) + \eta_1 E(A_a^+|G_1) \right] - \frac{b_1}{2} \left[E(R_a^{*2}|G_1) + \eta_1^2 E(A_a^{+2}|G_1) + E(u^2|G_1) \right]$$

and hence the optimal u is such that $E(u^2|G_1) = 0$, or equivalently $u = 0$, and the optimal η_1 is b_1^{-1} given that $E(A_a^{+2}|G_1) = E(A_a^+|G_1)$. We can express the optimal return as

$$R_a^* + \frac{1}{b_1} A_a^+,$$

which completes the proof the second part of the proposition. $\qquad \square$

Proposition 2:

This proof relies on the residual inner product $E\left[Cov(x,y|G_1)\right]$ between random variables x and y and its corresponding norm as $E^{1/2}\left[Var(x|G_1)\right]$. For any $p \in \mathcal{R}_a$, we can decompose $p - R_a^{**} \in \mathcal{A}_a$ into two components that belong to $\mathcal{A}_a$

$$p - R_a^{**} = \lambda A_a^{++} + u, \qquad \lambda = \frac{E\left[Cov\left(A_a^{++}, p - R_a^{**}|G_1\right)\right]}{E\left[Var\left(A_a^{++}|G_1\right)\right]} = \frac{E\left(p - R_a^{**}\right)}{E\left(A_a^{++}\right)},$$

where the former component is its residual projection onto the unconditional span of A_a^{++}, and the latter component is simply the residual of this projection. Hence $E(u) = 0$ because u satisfies $E\left[Cov\left(A_a^{++}, u|G_1\right)\right] = 0$.

We can characterize any $p \in \mathcal{R}_a$ by its corresponding pair (λ, u) with moments

$$E(p) = E\left(R_a^{**}\right) + \lambda E\left(A_a^{++}\right),$$

$$E\left[Var(p|G_1)\right] = E\left[Var\left(R_a^{**}|G_1\right)\right] + \lambda^2 E\left[Var\left(A_a^{++}|G_1\right)\right] + E\left[Var(u|G_1)\right].$$

Therefore we can choose λ to satisfy the mean constraint of problem (9)

$$\lambda = \frac{\nu - E\left(R_a^{**}\right)}{E\left(A_a^{++}\right)} = \omega_R(\nu),$$

and simultaneously we can minimise the criterion of problem (9) by choosing a u such that $E\left[Var(u|G_1)\right] = 0$. Equivalently, the optimal residual is $u = E(u|G_1) = 0$ because a safe asset cannot belong to $\mathcal{A}_a$ under lack of arbitrage opportunities.

In sum, we found that returns

$$R_a^{**} + \omega_R(\nu) A_a^{++}$$

are optimal in the RRF sense. Finally, the first part of Proposition 1 implies that choosing $\omega_R(\nu) = \theta^{-1}$ we solve problem (8). $\qquad\square$

Proposition 3:

This proof follows the argument that Hansen and Richard (1987) develop for the CRF. To prove necessity, let us start with two returns on the RRF different from R_a^{**}, say $p_1 = R_a^{**} + \omega_1 A_a^{++}$ and $p_2 = R_a^{**} + \omega_2 A_a^{++}$, such that $E\left[Cov(p_1, p_2|G_1)\right] = 0$. That is, one return is the zero-beta counterpart of the other.

The proof of Proposition 2 shows that any return $p \in \mathcal{R}_a$ can be expressed as

$$p = p_R + u$$

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where p_R is on the RRF, and u satisfies $E(u) = 0$ and has zero residual covariance with any return on the RRF. Moreover, by unconditional two-fund spanning on the RRF, we can write

$$p_R = \lambda p_1 + (1 - \lambda) p_2$$

for some $\lambda \in \mathbb{R}$, and hence

$$E(p) = E(p_R) = E(p_2) + \lambda [E(p_1) - E(p_2)].$$

In addition,

$$E[Cov(p, p_1|G_1)] = E[Cov(p_R, p_1|G_1)] = \lambda E[Var(p_1|G_1)],$$

or equivalently

$$\lambda = \frac{E[Cov(p, p_1|G_1)]}{E[Var(p_1|G_1)]}.$$

Therefore, we obtain the desired result by choosing $p_\beta = p_1$, $E_\beta = E(p_2)$, and $\beta_R = \lambda$.

Now we turn to prove sufficiency. Let us assume there is some $p_\beta \in \mathcal{R}_a$ that satisfies

$$E(p) - E_\beta = \beta_R [E(p_\beta) - E_\beta], \qquad \forall p \in \mathcal{R}_a.$$

We can again decompose such a return as

$$p_\beta = p_R + u$$

where p_R is on the RRF, and u satisfies $E(u) = 0$ and has zero residual covariance with any return on the RRF. Obviously, p_R itself should satisfy the residual beta-pricing

$$E(p_R) - E_\beta = \frac{E[Cov(p_R, p_\beta|G_1)]}{E[Var(p_\beta|G_1)]} [E(p_\beta) - E_\beta],$$

which implies

$$\frac{E[Cov(p_R, p_\beta|G_1)]}{E[Var(p_\beta|G_1)]} = 1$$

because $E(p_\beta) = E(p_R)$.

The previous decomposition of p_β also implies

$$E[Var(p_\beta|G_1)] = E[Var(p_R|G_1)] + E[Var(u|G_1)],$$

and putting both implications together, u must satisfy $E[Var(u|G_1)] = 0$, or equivalently $u = E(u|G_1)$. This translates into $u = 0$ because a safe asset cannot belong to $\mathcal{A}_a$ under lack of arbitrage opportunities. $\qquad \square$

Proposition 4:

If we start from the RRF and use the relationships developed in the proof of Lemma 1,

$$R_a^{**} + \omega_R(\nu) A_a^{++} = \left[R_a^* + \frac{E(R_a^*|G_1)}{1 - E(A_a^+|G_1)} A_a^+ \right] + \omega_R(\nu) \left[\frac{1}{1 - E(A_a^+|G_1)} A_a^+ \right]$$

$$= R_a^* + \left[\frac{E(R_a^*|G_1) + \omega_R(\nu)}{1 - E(A_a^+|G_1)} \right] A_a^+,$$

then we can see that the RRF has a random weight on A_a^+. However, the URF returns $R_a^* + \omega_U(\nu) A_a^+$ require a nonrandom weight on A_a^+, and hence both frontiers share a point if and only if

$$\frac{E(R_a^*|G_1) + \omega_R(\nu)}{1 - E(A_a^+|G_1)} = \omega_U(\nu).$$

This expression can be rewritten as

$$E(R_a^*|G_1) = a + bE(A_a^+|G_1),$$

where $a = \omega_U(\nu) - \omega_R(\nu)$ and $b = -\omega_U(\nu)$. In fact, a is equal to the constant conditional mean of the shared element. $\square$

Lemma 2:

The arbitrage portfolio B_a^{++} belongs to the conditional span of A_a^{++} and $R_a^{**} - R_0$, which are orthogonal under the covariance inner product,

$$B_a^{++} = A_a^{++} + \gamma_1(R_a^{**} - R_0), \qquad \gamma_1 = \frac{E(R_a^{**}|G_1) - R_0}{Var(R_a^{**}|G_1)}.$$

Using that relationship, the CRF on $\mathcal{S}_a$ can be expressed as

$$R_0 + w_1(\nu_1) B_a^{++} = R_0 + w_1(\nu_1) \left[A_a^{++} + \gamma_1(R_a^{**} - R_0) \right]$$

$$= \left[1 - w_1(\nu_1)\gamma_1 \right] R_0 + w_1(\nu_1) \left[\gamma_1 R_a^{**} + A_a^{++} \right],$$

which separates the riskless and risky components of the CRF. We have $\gamma_1 \neq 0$ under the assumption $E(R_a^{**}|G_1) \neq R_0$, and hence we can choose $w_1(\nu_1) = \gamma_1^{-1}$ to skip the riskless component,

$$R_a^{**} + \frac{1}{\gamma_1} A_a^{++} = R_a^{**} + \left[\frac{Var(R_a^{**}|G_1)}{E(R_a^{**}|G_1) - R_0} \right] A_a^{++}.$$

If we had $\gamma_1 = 0$ then the cost of the risky component of the CRF on $\mathcal{S}_a$, given by $w_1(\nu_1) \left[\gamma_1 R_a^{**} + A_a^{++} \right] = w_1(\nu_1) A_a^{++}$, would be zero at every ν_1. In that case, we cannot find a common point with the CRF on $\mathcal{R}_a$ because its cost must be 1 by definition.

We can use a similar argument to obtain the second expression in (11). The arbitrage portfolio B_a^+ belongs to the conditional span of A_a^+ and $(R_a^* - R_0) + R_0 A_a^+$, which are orthogonal under the inner product defined by uncentred second moments,

$$B_a^+ = A_a^+ + \delta_1 \left[(R_a^* - R_0) + R_0 A_a^+ \right], \qquad \delta_1 = \frac{E\left[(R_a^* - R_0) + R_0 A_a^+ | G_1 \right]}{E\left[\left((R_a^* - R_0) + R_0 A_a^+ \right)^2 | G_1 \right]}.$$

Therefore the CRF on $\mathcal{S}_a$ can be expressed as

$$R_0 + (\omega_1(\nu_1) - R_0) B_a^+ = R_0 + (\omega_1(\nu_1) - R_0) \left[A_a^+ + \delta_1 \left[(R_a^* - R_0) + R_0 A_a^+ \right] \right]$$

$$= \left[1 - (\omega_1(\nu_1) - R_0) \delta_1 \right] R_0 + (\omega_1(\nu_1) - R_0) \left[\delta_1 R_a^* + (1 + \delta_1 R_0) A_a^+ \right],$$

and choosing $\omega_1(\nu_1) - R_0 = \delta_1^{-1}$ we obtain

$$R_a^* + \left[\frac{1 + \delta_1 R_0}{\delta_1} \right] A_a^+ = R_a^* + \left[\frac{E\left(R_a^{*2} | G_1 \right) - R_0 E\left(R_a^* | G_1 \right)}{E\left(R_a^* | G_1 \right) + R_0 E\left(A_a^+ | G_1 \right) - R_0} \right] A_a^+,$$

which completes the proof. $\qquad\square$