A Preferred-Habitat Model of Term Premia, Exchange Rates and Monetary Policy Spillovers

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Abstract

We propose an integrated preferred-habitat model of bond and currency markets across two countries. Prices are determined by arbitrageurs trading with investors with preferences for specific assets. Risk premia vary over time in response to shocks to short rates and to bond and currency demand. This variation generates empirically documented violations of Expectations Hypothesis and Uncovered Interest Parity. Large-scale asset purchases in one country cause that country's currency to depreciate, bond yields in that country to drop, and yields in the other country to drop by a smaller amount. A short-rate cut in one country has the same qualitative effects, although our estimated model reveals that the spillovers to the other country's term structure are significantly smaller.

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1 Introduction

This paper proposes an integrated preferred-habitat model of bond and currency markets. Our model features two countries and three types of investors: bond investors, specialized in specific maturity segments of the domestic or foreign bond market; currency investors; and risk-averse global rate arbitrageurs with a limited amount of capital. Because these global rate arbitrageurs operate on both on the domestic and foreign bond market, and in currency markets, bond and currency risk premia are linked in equilibrium. Crucially, changes in demand and supply of bonds or currency must be absorbed in equilibrium by global rate arbitrageurs, with resulting—and joint—changes in risk premia, expected returns, long term yields and exchange rates.

Our model provides new and important insights on the international transmission of conventional and unconventional monetary policy. It also offers a potential resolution to several long-standing puzzles in the finance literature, such as the Uncovered Interest Parity (UIP) puzzle or deviations from the Expectation Hypothesis (EH). Under UIP, domestic and foreign bonds are perfect substitutes, and the expected rate of depreciation of the nominal exchange rate offsets the difference between domestic and foreign nominal yields. Under the EH, bonds of various maturities are perfect substitutes and the shape of the yield curve reflects expectations about future short rates.

Consider the standard international macro model with perfect capital mobility and floating exchange rates. In that model, up to constant risk premia, both UIP and the EH hold. This has powerful implications for the transmission of monetary policy, both along the yield curve, and across countries. First, the yield curve in each country only depends on expectations of the local policy rate, which is controlled by local monetary authorities. This immediately implies that nonconventional policies, such as Quantitative Easing (QE), whereby the central bank purchases long-dated bonds while keeping short rates unchanged, have no effect on the yield curve. Second, this also implies that each country's yield curve is fully insulated from other countries' monetary policy. This insulation obtains because, according to UIP, the expected rate of depreciation of the exchange rate provides all the necessary adjustment. This result is nothing more than a slightly broader statement of the well-known Friedman-Obstfeld-Taylor Trilemma: with flexible exchange rates and perfect capital mobility, a floating exchange rate provides local monetary policy autonomy, not just in setting policy rates, but also in shaping the local yield curve.

Four broad empirical observations cast doubts on the validity of this standard model. First, a large empirical literature documents strong and systematic patterns in the structure of currency returns, in violation of UIP (see Fama (1984) and the subsequent literature): high interest rate countries typically earn high expected returns on short term deposits, an indication that currency risk premia are time-varying. These deviations from UIP form the basis for currency carry trade (CCT) strategies that borrow in currencies with low short interest rates and invest in currencies with high short interest rates.

Second, a similarly large empirical literature documents strong and systematic deviations from the EH. Two seminal papers in this literature, Fama and Bliss (1987) and Campbell and Shiller (1991), establish that the slope of the term structure has predictive power for excess bond returns and for future change in yields, an indication that bond risk premia are time-varying. These deviations form the basis for bond carry trade (BCT) strategies that borrow in maturities with a low interest rate and invest in maturities with a higher interest rate.

Third, while the empirical literature on currency and bond returns largely followed parallel but separate tracks, recent papers establish that the foreign exchange and bond risk premia are deeply connected. For instance, Chernov and Creal (2020) as well as Lloyd and Marin (2020) find that yield curve slope differentials matter for the predictability of the currency carry trade (CCT)—investment strategies that borrow in low interest rate currencies and invest in high ones—while Lustig, Stathopoulos, and Verdelhan (2019) find that the profitability of the currency carry trade declines when the trade is carried out with long-term bonds rather than short term ones. This last result indicate that bond and currency risk premia tend to offset each other as the maturity of the bond instruments increases.

Lastly, since the 2008 Global Financial Crisis, monetary authorities around the world have experimented with various forms of 'Unconventional Monetary Policies' (UMP) including but not limited to Quantitative Easing (QE), Forward Guidance, yield curve control or negative interest rates. A growing body of evidence, surveyed in Bhattarai and Neely (2018) suggests that central banks' asset purchases announcements had a significant impact not only on domestic yields, but also on exchange rates and foreign yields (see also Neely (2015) and Bauer and Neely (2014)).

The challenge is to build a tractable asset pricing framework that is consistent with these four broad facts. As Lustig, Stathopoulos, and Verdelhan (2019) observe, leading representative no-arbitrage models of international finance typically have a hard time reproducing these empirical patterns. For instance, these authors observe that no-arbitrage models cannot replicate both the strong evidence of deviations from UIP at the short end of the maturity structure, and its absence when using longer term instruments, since both arise from the same pricing equation. Similarly, Engel (2016) observes that standard representative agent models cannot explain simultaneously the UIP puzzle—which through the lens of these models implies that high interest rate currencies are more risky–and the fact that high interest rate currencies tend to be stronger than implied by future interest-rate differentials under UIP—which through the lens of these models suggests that the high interest rate currencies are less risky.

Our paper develops such a framework. It builds on the recent and promising line of research that recognizes the importance of financial intermediaries and of the limits to arbitrage across partially segmented financial markets. At the theoretical level, this relaxes the hypothetical representative-agent's arbitrage condition and focuses instead on the risk-return tradeoff of the relevant global investors. Gabaix and Maggiori (2015) present a stylized model of currency markets along those lines, reviving an important older literature on portfolio balance models (Kouri, 1982). These models naturally generate deviations from UIP as arbitrageurs need to be compensated for their currency exposure. Similarly, Vayanos and Vila (2021) present a preferred-habitat model of segmentation along the yield curve in a closed economy. That model naturally generate deviations from the EH as arbitrageurs need to be compensated for their bond exposure. Our model proposes an integrated analysis of global rate markets which delivers sharp predictions on the co-movements between bond and currency risk premia. The model is particularly useful to investigate how 'local shocks' to the supply of or demand for specific maturities can propagate along the domestic and foreign term structure.

At the institutional level, market segmentation seems a very plausible assumption: the marginal investor in currency markets is much more likely to be a specialized investor such as a large macro global hedge fund, the trading desk of a multinational corporation, a sovereign wealth fund, or the fixed-income desk of a global broker-dealer, rather than the representative household trying to diversify the risks to the marginal utility of its consumption stream.

In each country, a monetary authority sets short term policy rate exogenously. Further, local investors are situated along the domestic and foreign term structure. These investors are specialized in a given currency and maturity segment. In addition, there are specialized investors in the currency market. These investors are price elastic and their demand for bonds and currency constitute another source of exogenous variation. Lastly 'global rates market' risk averse arbitrageurs can invest limited capital in all fixed-income instruments, foreign and domestic. Because these global arbitrageurs operate both on the term structure in each country, and in currency markets, term premia and currency risk premia are linked in equilibria.

Our framework allows us to answer a number of specific questions. First, we can characterize the time series behavior of term premia and currency risk premia, given the underlying policy and demand shocks. Our model recovers deviations from UIP and also very naturally the Lustig, Stathopoulos, and Verdelhan (2019) term structure of currency risk premia. In our model, as the maturity of the bond increases, the short term excess return decreases to zero. The reason is precisely that long term bond and currency risk premia are linked: as arbitrageurs become more exposed to domestic policy shocks, domestic long term bonds and foreign currency are equally undesirable: their premia increase by similar amounts, which account for the decline in the term structure of currency risk premia.

Second, our framework allows us to explore how shocks to the policy rate in one country transmit to the domestic term structure, the currency, and the foreign term structure. We now provide the core intuition for our results. Consider first the case of a decrease in the domestic policy rate and the impact on the domestic yield curve. This makes domestic long term bonds more desirable, increasing the price of domestic long term bonds. This leads price-elastic domestic bond investors to retrench. In equilibrium, global arbitrageurs must increase their holdings of domestic long term bonds. This requires a higher expected return, hence the yield on foreign bonds does not decline all the way to the level implied by the EH: the required rent that accrues to global arbitrageurs attenuates the transmission of monetary policy along the domestic yield curve, compared to the standard case. Consider now the impact on the exchange rate. The lower domestic policy rate makes foreign currency more desirable, appreciating the foreign currency. This leads price-elastic currency traders to retrench. In equilibrium, global arbitrageurs must increase their foreign currency holdings. This requires a higher expected currency return, hence the foreign currency does not appreciate all the way to the level implied by UIP. Finally, consider the impact on the foreign yield curve. A larger exposure to foreign currency makes global rate arbitrageurs more exposed to the risk of a decline in foreign interest rates (and the associated depreciation of the foreign currency). Foreign long term bonds provide a natural hedge since their price increases when the foreign short rate declines. Hence, in response to a decline in the *domestic* policy rate, global rate arbitrageurs will increase their demand for foreign long term bonds. This will decrease the yield on foreign bonds and flatten the foreign yield curve. Hence, the transmission of conventional monetary policy to the domestic economy is weakened, and spills over to the foreign yield curve, even when exchange rates are flexible: the required rents that accrue to global rate arbitrageurs connect domestic, foreign and currency markets. To the extent that long rates matter for economic activity, as in Ray (2019), the Friedman-Obstfeld-Taylor Trilemma fails.

Our framework also allows us to investigate how non-conventional policies such as Quantitative Easing, Forward Guidance or Foreign Exchange intervention transmit, both domestically and abroad. Consider first the case of a purchase of domestic long term bonds by the domestic central bank. This increase in demand leads to an increase in price of those bonds and decline in their yield. Global arbitrageurs respond by reducing their demand for these long term bonds. This reduction in their holdings of domestic long term bonds make them less exposed to the risk of a rise in the domestic interest rate. Therefore, they become more willing to hold assets exposed to that risk. Foreign currency and foreign long term bonds are two such assets. Hence the model predicts that a domestic asset purchase will depreciate the domestic currency and lower foreign yields—flattening the foreign yield curve.

We illustrate the above mechanisms analytically in the case where the short rates in each country are the only risk factors and are mutually independent. We complement our analytical results with a quantitative exercise, where we allow for three additional risk factors corresponding to bond and currency demand, and for correlation between short rates. We estimate the model parameters using second moments of yields and exchange rates in the US and the Eurozone. As a test for the estimated model, we compute the coefficients of common return predictability regressions for bonds and currencies. These coefficients are untargeted in our calibration. The model replicates key properties of these coefficients, with the fit being somewhat better for bonds than for currencies. We next use the model to evaluate the effects of conventional and non-conventional monetary policies. We find that conventional and non-conventional policies are comparable in terms of their effects on the exchange rate. Non-conventional policies, however, have sizeable international spillover effects on the term structure, while such effects are small for conventional policies.

Greenwood, Hanson, Stein, and Sunderam (2019) develop independently a model similar to ours, with arbitrageurs trading bonds and currency across two countries. They find, as we do, that bond and currency carry trades are profitable, and that an increase in bond demand in one country causes the currency of that country to depreciate and bond prices in both countries to rise. They also introduce segmented arbitrage, e.g., some arbitrageurs can only trade bonds in one country, and some can trade only currency. Their model is set up in discrete time and assumes only a short and a long bond. By contrast, ours is set up in continuous time and derives the entire term structure of interest rates in each country. This allows us to compare the predictability of bond and currency movements across different horizons, and to perform a quantitative exercise.

Our paper connects four strands of literature. First, there is an abundant empirical literature on bond and currency 'puzzles.' (Add citations...) Second, a more recent empirical literature emphasizes the role of quantities in asset pricing. (Koijen and Yogo (2019), ...)

Third, from a modeling perspective, we build on recent models of market segmentation in currency markets and bond markets. (Kouri (1982), Gabaix and Maggiori (2015), Vayanos and Vila (2021)). Itskhoki and Mukhin (2017) present such a model where financial arbitrageurs also need to absorb liquidity demand arising from noise traders, as in Jeanne and Rose (2002). These liquidity demand shocks translate, in equilibrium, into 'UIP shocks', i.e. deviations from the UIP condition. Quantitatively, Itskhoki and Mukhin (2017) conclude that these UIP shocks account for more than 90% of the fluctuations in the nominal and real exchange rate, but very little of the fluctuations in output, thus potentially explaining the well-known disconnect between exchange rate movements and traditional macroeconomic fundamentals such as monetary policy, output growth, or external imbalances (see Meese and Rogoff (1983) and the literature on the 'exchange rate disconnect puzzle').

Fourth, our paper explores how both conventional and unconventional monetary policy transmit, both domestically and internationally. Ray (2019) embeds such a segmented asset market structure into a New Keynesian model and explores how non-conventional policies, such as QE or forward guidance can be deployed effectively. References on international transmission of monetary policy (Gali Monacelli, Corsetti, Itskhokin-Mukhin)

2 Model

Time is continuous and goes from zero to infinity. There are two countries, Home (H) and Foreign (F). We define the exchange rate as the units of home currency that one unit of foreign currency can buy, and denote it by e_t at time t. An increase in e_t corresponds to a home currency depreciation.

In each country $j = H, F$, a continuum of zero-coupon government bonds can be traded. The bonds' maturities lie in the interval $(0, T)$, where T can be finite or infinite. The country-j bond with maturity τ at time t pays off one unit of country j's currency at time $t+\tau$. We denote by $P_{jt}^{(\tau)}$ the time-t price of that bond, expressed in units of country j's currency, and by $y_{jt}^{(\tau)}$ the bond's yield. The yield is the spot rate for maturity τ, and is related to the price through

$$y_{jt}^{(\tau)} = -\frac{\log\left(P_{jt}^{(\tau)}\right)}{\tau}. \tag{2.1}$$

The country-j and time-t short rate i_{jt} is the limit of the yield $y_{jt}^{(\tau)}$ when τ goes to zero. We take i_{jt} as exogenous, and describe its dynamics later in this section (Equation 2.9). An exogenous i_{jt} can be interpreted as the result of actions that the central bank in country j takes when targeting the short nominal rate by elastically supplying liquidity.

There are three types of agents: arbitrageurs, bond investors and currency traders. Arbitrageurs are competitive and maximize a mean-variance objective over instantaneous changes in wealth. We express their wealth in units of the home currency, thus assuming that the home currency is the riskless asset for them. We allow arbitrage to be global or segmented. When arbitrage is global, arbitrageurs can invest in the currencies and bonds of both countries. When instead arbitrage is segmented, arbitrageurs can invest in the currency of the home country (the riskless asset), and in a single additional asset class: foreign currency for some arbitrageurs, home bonds for others, and foreign bonds for the remainder. We assume that the arbitrageurs investing in foreign bonds have a zero net position in foreign-currency instruments: they hedge their bond position with an equally sized position in the foreign short rate. Segmented arbitrage is a useful benchmark, as the interactions between bond and currency markets that global arbitrage generates are not present.

In the case of global arbitrage, we denote by W_t the arbitrageurs' time-t wealth, by W_{Ht} and W_{Ft} their net position in home and foreign-currency instruments, respectively, and by $X_{Ht}^{(\tau)} d\tau$ and $X_{Ft}^{(\tau)} d\tau$ their position in the home and foreign bonds with maturities in $[\tau, \tau + d\tau]$, respectively, all expressed in units of the home currency. The position of arbitrageurs in the bonds with maturities in $[\tau, \tau + d\tau]$ is of order $d\tau$ in equilibrium because preferred-habitat demand for those bonds is assumed to be of the same order.

The arbitrageurs' budget constraint is

$$W_{t+dt} = \left(W_{Ht} - \int_0^T X_{Ht}^{(\tau)} d\tau \right) (1 + i_{Ht} dt) + \int_0^T X_{Ht}^{(\tau)} \frac{P_{H,t+dt}^{(\tau-dt)}}{P_{Ht}^{(\tau)}} d\tau$$
$$+ \left(W_{Ft} - \int_0^T X_{Ft}^{(\tau)} d\tau \right) (1 + i_{Ft} dt) \frac{e_{t+dt}}{e_t} + \int_0^T X_{Ft}^{(\tau)} \frac{P_{F,t+dt}^{(\tau-dt)} e_{t+dt}}{P_{Ft}^{(\tau)} e_t} d\tau. \tag{2.2}$$

The first term in the right-hand side of (2.2) corresponds to a position in the home short rate, the second term to a position in home bonds, the third term to a position in the foreign short rate, and the fourth term to a position in foreign bonds. In the third term, $W_{Ft} - \int_0^T X_{Ft}^{(\tau)} d\tau$ units of the home currency are converted at time t to units of the foreign currency by dividing by e_t. They earn the foreign short rate between time t and $t+dt$, and are converted back at time $t+dt$ to units of the home currency by multiplying by e_{t+dt}. In the fourth term, $X_{Ft}^{(\tau)}$ units of the home currency are converted at time t to units of the foreign currency by dividing by e_t, and then to units of the foreign bond with maturity τ by dividing by $P_{Ft}^{(\tau)}$, the price of the bond in foreign currency. They are converted back at time $t+dt$ to units of the home currency by multiplying by $P_{F,t+dt}^{(\tau-dt)}e_{t+dt}$.

Subtracting $W_t = W_{Ht} + W_{Ft}$ from both sides of (2.2) and rearranging, we find

$$\begin{aligned} dW_t =& W_t i_{Ht}dt + W_{Ft}\left(\frac{de_t}{e_t} + (i_{Ft} - i_{Ht})dt\right) \\ &+ \int_0^T X_{Ht}^{(\tau)}\left(\frac{dP_{Ht}^{(\tau)}}{P_{Ht}^{(\tau)}} - i_{Ht}dt\right)d\tau + \int_0^T X_{Ft}^{(\tau)}\left(\frac{d(P_{Ft}^{(\tau)}e_t)}{P_{Ft}^{(\tau)}e_t} - \frac{de_t}{e_t} - i_{Ft}dt\right)d\tau. \end{aligned} \tag{2.3}$$

If arbitrageurs invest all their wealth in the home short rate, then the instantaneous change dW_t in their wealth is $W_t i_{Ht}dt$, the first term in the right-hand side of (2.3). Relative to that case, arbitrageurs can earn an additional return from investing in three sets of assets: foreign currency, home bonds, and foreign bonds. The returns from these investments correspond to the second, third and fourth term, respectively, in the right-hand side of (2.3).

The optimization problem of a global arbitrageur is

$$\max_{W_{Ft},\{X_{jt}^{(\tau)}\}_{\tau\in(0,T),j=H,F}} \left[\mathbb{E}_t(dW_t) - \frac{a}{2}\mathbb{V}\text{ar}_t(dW_t)\right], \tag{2.4}$$

where $a \geq 0$ is a coefficient that characterizes the trade-off between mean and variance. The coefficient a can capture innate risk aversion or, in reduced form, constraints such as Value at Risk. By possibly redefining a, we assume that global arbitrageurs are in measure one. Arbitrageurs with the objective (2.4) can be interpreted as overlapping generations living over infinitesimal periods.

In the case of segmented arbitrage, the budget constraint of any given arbitrageur is derived from (2.3) by setting two of the terms to zero. For an arbitrageur who can invest only in foreign currency, the third and fourth terms are zero ($X_{Ht}^{(\tau)} = X_{Ft}^{(\tau)} = 0$); for an arbitrageur who can invest only in home bonds, the second and fourth terms are zero ($W_{Ft} = X_{Ft}^{(\tau)} = 0$); and for an arbitrageur who can invest only in foreign bonds, with a zero net position in foreign-currency instruments, the second and third terms are zero ($W_{Ft} = X_{Ht}^{(\tau)} = 0$). The optimization problem is derived from (2.4) by restricting the choice variables accordingly. We denote by a_e, a_H and a_F, respectively, the risk-aversion coefficient of an arbitrageur who can invest in foreign currency, home bonds and foreign bonds. By possibly redefining (a_e, a_H, a_F), we assume that each type of arbitrageur is in measure one.

Bond investors have preferences ("habitats") for specific countries and maturities. For example, pension funds in the home country prefer long-maturity home bonds because these match their pension liabilities, which are long term and denominated in home currency. At the other end of the maturity spectrum, home money-market funds are required by their mandates to hold short-maturity home bonds. For tractability, we assume that preferences take an extreme form, where investors demand only the bond closest to their preferred characteristics. That is, investors with preferences for country j and maturity τ at time t hold a position $Z_{jt}^{(\tau)}$ in the country-j bond with maturity τ and hold no other bond. We assume that maturity preferences cover the interval $(0, T)$, and investors with preferences for country j and maturities in $[\tau, \tau + d\tau]$ are in measure $d\tau$. We express the position $Z_{jt}^{(\tau)}$ in units of the home currency, and assume that it is affine and decreasing in the logarithm of the bond price:

$$Z_{jt}^{(\tau)} = -\alpha_j(\tau) \log\left(P_{jt}^{(\tau)}\right) - \beta_{jt}^{(\tau)}. \tag{2.5}$$

The slope coefficient $\alpha_j(\tau) \geq 0$ is constant over time but can depend on country j and maturity τ. The intercept coefficient $\beta_{jt}^{(\tau)}$ can depend on t, τ and j. For simplicity, we refer to $\alpha_j(\tau)$ and $\beta_{jt}^{(\tau)}$ as demand slope and demand intercept, respectively. The actual intercept is $-\beta_{jt}^{(\tau)}$.

The demand intercept $\beta_{jt}^{(\tau)}$ takes the form

$$\beta_{jt}^{(\tau)} = \zeta_j(\tau) + \theta_j(\tau)\beta_{jt}, \tag{2.6}$$

where $(\zeta_j(\tau), \theta_j(\tau))$ are constant over time but can depend on country j and maturity τ, and β_{jt} is independent of τ but can depend on country j and time t. We refer to β_{jt} as a demand risk factor, and describe its dynamics later in this section (Equation 2.9). **?** provide an optimizing foundation for the demand specification (2.5)-(2.6) in a setting where investors form overlapping generations consuming at the end of their life, are infinitely risk-averse, and can invest in bonds and in a private opportunity with exogenous return.

We assume that currency traders generate a downward-sloping demand for foreign currency as a function of the exchange rate e_t. These agents can be interpreted as exporters and importers, or as central banks intervening on currency markets. For example, when e_t is low, the central bank in the home country may want to increase its holdings of foreign currency, perhaps to stabilize the currency. Similarly, when e_t is low, the flow demand for foreign currency arising from exporters and importers may increase, as in Gabaix and Maggiori (2015), and this may push up the stock demand for foreign currency. For tractability, we assume that the stock demand of currency traders, expressed in units of the home currency, is affine and decreasing in the logarithm of the exchange rate:

$$Z_{et} = -\alpha_e \log(e_t) - (\zeta_{et} + \theta_e \gamma_t), \tag{2.7}$$

where $\alpha_e \geq 0$ is a slope coefficient, ζ_{et} is a deterministic term, θ_e is a constant, and γ_t is a demand risk factor. We describe the dynamics of γ_t and motivate the deterministic term ζ_{et} later in this section.

The demand (2.7) for foreign currency is expressed in the spot market. We allow for additional currency demand in the forward market. Indeed, according to BIS (2019), spot transactions accounted for only one-third of total trading volume in the currency market over recent years, with forward and swap transactions accounting for most of the remainder. We assume that the demand of currency traders, expressed in units of the home currency, for the foreign-currency forward contract with maturity τ is

$$Z_{et}^{(\tau)} = -(\zeta_e(\tau) + \theta_e(\tau)\gamma_t), \tag{2.8}$$

where $(\zeta_e(\tau), \theta_e(\tau))$ are functions of τ.

Under Covered Interest Parity (CIP), the demand $Z_{et}^{(\tau)}$ for the foreign-currency forward contract with maturity τ is equivalent to the combination of (i) a demand $Z_{et}^{(\tau)}$ for foreign currency in the spot market, (ii) a demand $Z_{et}^{(\tau)}$ for the foreign bond with maturity τ, and (iii) a demand $-Z_{et}^{(\tau)}$ for the home bond with maturity τ. Hence, the equilibrium with the forward market is equivalent to one without it but with the demands (i)-(iii) added to (2.5) and (2.7). We use that equivalence to study the effects of currency demand in the forward market. CIP holds only under global arbitrage since it is only then that a common set of agents can trade all the instruments involved in CIP arbitrage. Accordingly, we allow for currency demand in the forward market only under global arbitrage.

The 5×1 vector $q_t \equiv (i_{Ht}, i_{Ft}, \beta_{Ht}, \beta_{Ft}, \gamma_t)^\top$ follows the process

$$dq_t = -\Gamma(q_t - \bar{q})dt + \Sigma dB_t, \tag{2.9}$$

where $\bar{q}$ is a constant 5×1 vector, (Γ, Σ) are constant 5×5 matrices, B_t is a 5×1 vector $(B_{iHt}, B_{iFt}, B_{\beta Ht}, B_{\beta Ft}, B_{\gamma t})^\top$ of independent Brownian motions, and $\top$ denotes transpose. Equation (2.9) nests the case where the factors $(i_{Ht}, i_{Ft}, \beta_{Ht}, \beta_{Ft}, \gamma_t)$ are mutually independent, and the case where they are correlated. Independence arises when the matrices (Γ, Σ) are diagonal. When instead Σ is non-diagonal, shocks to the factors are correlated, and when Γ is non-diagonal, the drift (instantaneous expected change) of each factor depends on all other factors. We assume that the eigenvalues of Γ have negative real parts so that q_t is stationary. Equation (2.9) implies that the long-run mean of a stationary q_t is $\bar{q}$. We set the long-run means of the demand factors to zero ($\bar{q}_3 = \bar{q}_4 = \bar{q}_5 = 0$). This is without loss of generality since we can redefine $\{\zeta_j(\tau)\}_{j=H,F}$ and ζ_{et} to include a non-zero long-run mean. We set the supply of each bond and of foreign currency to zero by redefining demand to be net of supply.

Key to the tractability of our model is that all demand functions are expressed in terms of the same numeraire, which is the riskless asset for arbitrageurs. The numeraire can be the currency of one of the two countries, and we take it to be the home currency. One limiting feature of this assumption is that the home currency must be the riskless asset for all arbitrageurs, even foreign ones. Our assumption also precludes that exchange-rate movements holding foreign bond yields constant affect foreign bond demand in home currency terms.

Our model can be given both a nominal and a real interpretation. Our presentation so far focuses on the nominal interpretation: bonds pay in currency units, the exchange rate is the price of one currency relative to the other, preferences of arbitrageurs concern their nominal wealth, preferences of bond investors concern their nominal consumption, and the demand of currency traders is a function of the nominal exchange rate. A difficulty with the nominal interpretation is that the demand of currency traders such as exporters and importers is better viewed as a function of the real rather than the nominal exchange rate. To put it differently, while it is reasonable for the real exchange rate to be stationary, we want to allow for a non-stationary nominal exchange rate. To make the nominal interpretation compatible with a real currency demand, we can replace the nominal exchange rate e_t in (5.1) by the real exchange rate. This amounts to keeping e_t inside the logarithm and adding $\alpha_e(\log(p_{Ft}) - \log(p_{Ht}))$ to ζ_{et}, where p_{jt} is the price level in country $j = H, F$. Hence, under the nominal interpretation, we can take ζ_{et} to be $\alpha_e(\log(p_{Ft}) - \log(p_{Ht}))$. This interpretation is valid as long as we ignore inflation risk, i.e. as long as we treat $\log(p_{Ft}) - \log(p_{Ht})$ as a deterministic process. More generally, the term ζ_{et} captures all deterministic forces that lead to a non-stationary nominal exchange rate.

An alternative interpretation of our model is real: bonds pay in units of goods with a real price $P_{jt}^{(\tau)}$, the exchange rate e_t is the real exchange rate defined as the price of goods in one country relative to the other, preferences of arbitrageurs concern their real wealth, preferences of bond investors concern their real consumption, and the demand of currency traders depends on the real exchange rate. Under the real interpretation, we can take ζ_{et} to be a constant, ζ_e.

In what follows, we present the nominal interpretation of the model in the special case where the inflation rate is constant in each country: $\zeta_{et} = \zeta_e + \alpha_e(\pi_F - \pi_H)t$, where π_j is the constant inflation rate in country j and ζ_e is a constant.

3 Segmented Arbitrage

In this section we study the case of segmented arbitrage, where foreign currency, home bonds, and foreign bonds are traded by three disjoint sets of arbitrageurs. For simplicity, we assume that the home and foreign short rates (i_{Ht}, i_{Ft}) are independent, that demand for bonds and foreign currency does not vary stochastically and hence the demand factors $(\beta_{Ht}, \beta_{Ft}, \gamma_t)$ are equal to their mean of zero in steady state, that one-off shocks to the demand factors do not affect the short rates or other demand factors, and that all currency demand is expressed in the spot market. This amounts to taking the matrices (Γ, Σ) in (2.9) to be diagonal and to setting $\Sigma_{3,3} = \Sigma_{4,4} = \Sigma_{5,5} = \zeta_e(\tau) = \theta_e(\tau) = 0$. Setting $(\Gamma_{1,1}, \Gamma_{2,2}, \bar{q}_1, \bar{q}_2, \Sigma_{1,1}, \Sigma_{2,2}) \equiv (\kappa_{iH}, \kappa_{iF}, \bar{i}_H, \bar{i}_F, \sigma_{iH}, \sigma_{iF})$, we can write the dynamics of the country-j short rate as

$$di_{jt} = \kappa_{ij}(\bar{i}_j - i_{jt})dt + \sigma_{ij} dB_{ijt}. \tag{3.1}$$

3.1 Equilibrium

We conjecture that the equilibrium exchange rate is a log-affine function of the home short rate, the foreign short rate and a linear time trend, and that equilibrium bond yields in country $j = H, F$ are affine functions of that country's short rate. That is, there exist three scalars $(\{A_{ije}\}_{j=H,F}, C_e)$ and four functions $\{A_{ij}(\tau), C_j(\tau)\}_{j=H,F}$ that depend only on τ, such that

$$\log e_t = -\left[A_{iHe} i_{Ht} - A_{iFe} i_{Ft} + C_e + (\pi_F - \pi_H)t\right], \tag{3.2}$$

$$\log P_{jt}^{(\tau)} = -\left[A_{ij}(\tau) i_{jt} + C_j(\tau)\right]. \tag{3.3}$$

When arbitrage is segmented, the exchange rate, the yields of home bonds, and the yields of foreign bonds are determined independently, and they reflect the risk aversion of the corresponding arbitrageurs. Our conjectured solution (3.2)-(3.3) implies that the real exchange rate $(e_t p_{Ft})/p_{Ht} = e_t \exp((\pi_F - \pi_H)t)(p_{F0}/p_{H0})$ and bond prices $P_{jt}^{(\tau)}$ are stationary while the nominal exchange rate exhibits a trend $\exp((\pi_H - \pi_F)t)$.

3.1.1 Exchange Rate

We determine the exchange rate by deriving the arbitrageurs' first-order condition and combining it with market clearing. Applying Ito's Lemma to (3.2), and using the dynamics (3.1) of i_{jt}, we find that the instantaneous return on foreign currency is

$$\frac{de_t}{e_t} = \mu_{et}dt - A_{iHe}\sigma_{iH}dB_{iHt} + A_{iFe}\sigma_{iF}dB_{iFt}, \tag{3.4}$$

where

$$\mu_{et} \equiv -A_{iHe}\kappa_{iH}(\bar{i}_H - i_{Ht}) + A_{iFe}\kappa_{iF}(\bar{i}_F - i_{Ft}) - (\pi_F - \pi_H) + \frac{1}{2}A_{iHe}^2\sigma_{iH}^2 + \frac{1}{2}A_{iFe}^2\sigma_{iF}^2 \tag{3.5}$$

is the expected return. Substituting the return (3.4) into the budget constraint of the subset of arbitrageurs who can invest in foreign currency (and whose budget constraint is derived from (2.3) by setting $X_{Ht}^{(\tau)} = X_{Ft}^{(\tau)} = 0$), we find

$$dW_t = \left[W_t i_{Ht} + W_{Ft}\left(\mu_{et} + i_{Ft} - i_{Ht}\right)\right]dt - W_{Ft}\left(A_{iHe}\sigma_{iH}dB_{iHt} - A_{iFe}\sigma_{iF}dB_{iFt}\right).$$

The optimization problem of these arbitrageurs is

$$\max_{W_{Ft}} \left[W_{Ft}\left(\mu_{et} + i_{Ft} - i_{Ht}\right) - \frac{a_e}{2}W_{Ft}^2\left(A_{iHe}^2\sigma_{iH}^2 + A_{iFe}^2\sigma_{iF}^2\right)\right],$$

and their first-order condition is

$$\mu_{et} + i_{Ft} - i_{Ht} = a_e W_{Ft}\left(A_{iHe}^2\sigma_{iH}^2 + A_{iFe}^2\sigma_{iF}^2\right). \tag{3.6}$$

Equation (3.6) describes the arbitrageurs' risk-return trade-off when investing in the *currency carry trade* (CCT). We term CCT the trade of borrowing short-term in the home country, exchanging the borrowed amount in the foreign currency, investing it short-term in the foreign country, and exchanging it back in the home currency.[1] The CCT's return is $\frac{de_t}{e_t} + (i_{Ft} - i_{Ht})dt$, equal to the return on foreign currency plus that on the foreign-home short-rate differential.

If arbitrageurs invest an extra unit of home currency in the CCT, then their expected return increases by the CCT's expected return $\mu_{et}+i_{Ft}-i_{Ht}$. This is the left-hand side of (3.6). The right-hand side is the increase in the the arbitrageurs' portfolio risk, times their risk-aversion coefficient a_e. The increase in portfolio risk is equal to the variance of the CCT's return, times the arbitrageurs' wealth W_{Ft} invested in foreign currency.

We next combine the arbitrageurs' first-order condition (3.6) with market clearing in foreign currency. Market clearing requires that the time-t positions of arbitrageurs and currency traders sum to zero:

$$W_{Ft} + Z_{et} = 0. \tag{3.7}$$

Using (3.7), we can write (3.6) as

$$\begin{aligned}
\mu_{et} + i_{Ft} - i_{Ht} &= -a_e Z_{et} \left(A_{iHe}^2 \sigma_{iH}^2 + A_{iFe}^2 \sigma_{iF}^2 \right) \\
&= a_e \left[\alpha_e \log(e_t) + \zeta_e + \alpha_e(\pi_F - \pi_H)t \right] \left(A_{iHe}^2 \sigma_{iH}^2 + A_{iFe}^2 \sigma_{iF}^2 \right) \\
&= a_e \left[\zeta_e - \alpha_e \left(A_{iHe} i_{Ht} - A_{iFe} i_{Ft} + C_e \right) \right] \left(A_{iHe}^2 \sigma_{iH}^2 + A_{iFe}^2 \sigma_{iF}^2 \right),
\end{aligned} \tag{3.8}$$

where the second step follows from (2.7) and $\gamma_t = 0$, and the third step follows from (3.2). Substituting μ_{et} from (3.5) into (3.8), we can write the latter equation as

$$\begin{aligned}
&- A_{iHe}\kappa_{iH}(\bar{i}_H - i_{Ht}) + A_{iFe}\kappa_{iF}(\bar{i}_F - i_{Ft}) - (\pi_F - \pi_H) + \frac{1}{2} A_{iHe}^2 \sigma_{iH}^2 + \frac{1}{2} A_{iFe}^2 \sigma_{iF}^2 + i_{Ft} - i_{Ht} \\
&= a_e \left[\zeta_e - \alpha_e \left(A_{iHe} i_{Ht} - A_{iFe} i_{Ft} + C_e \right) \right] \left(A_{iHe}^2 \sigma_{iH}^2 + A_{iFe}^2 \sigma_{iF}^2 \right).
\end{aligned} \tag{3.9}$$

Equation (3.9) is affine in (i_{Ht}, i_{Ft}). Identifying the linear terms in (i_{Ht}, i_{Ft}) and the constant terms yields three equations for the three scalars $(\{A_{ije}\}_{j=H,F}, C_e)$.

Proposition 3.1. *When arbitrage is segmented, the exchange rate e_t is given by (3.2), with $(\{A_{ije}\}_{j=H,F}, C_e)$ equal to the unique solution of the system*

$$\kappa_{ij} A_{ije} - 1 = -a_e \alpha_e A_{ije} \left(\sigma_{iH}^2 A_{iHe}^2 + \sigma_{iF}^2 A_{iFe}^2 \right), \tag{3.10}$$

$$\begin{aligned}
&- \kappa_{iH} \bar{i}_H A_{iHe} + \kappa_{iF} \bar{i}_F A_{iFe} - (\pi_F - \pi_H) + \frac{1}{2} \sigma_{iH}^2 A_{iHe}^2 + \frac{1}{2} \sigma_{iF}^2 A_{iFe}^2 \\
&= a_e \left(\zeta_e - \alpha_e C_e \right) \left(\sigma_{iH}^2 A_{iHe}^2 + \sigma_{iF}^2 A_{iFe}^2 \right).
\end{aligned} \tag{3.11}$$

[1] For simplicity, we deviate from market terminology, according to which the CCT borrows in the currency with the low interest rate.

In the special case where arbitrageurs are risk neutral ($a_e = 0$), (3.6) implies that Uncovered Interest Parity (UIP) holds: $\mu_{et} = i_{Ht} - i_{Ft}$. In addition, for the solution to be of the form conjectured in (3.2), Proposition 3.1 requires that the unconditional mean of the two countries' real interest rates, $\bar{i}_j - \pi_j$, be equated, up to a convexity adjustment term equal to $\frac{\sigma_{iH}^2}{2\kappa_{iH}^2} + \frac{\sigma_{iF}^2}{2\kappa_{iF}^2}$:

$$\bar{i}_F - \pi_F + \frac{\sigma_{iH}^2}{2\kappa_{iH}^2} + \frac{\sigma_{iF}^2}{2\kappa_{iF}^2} = \bar{i}_H - \pi_H, \tag{3.12}$$

This is quite intuitive: if these unconditional real interest rates were different and arbitrageurs were risk neutral, then the real exchange rate would appreciate or depreciate forever, violating the conjectured stationarity in (3.2). From (3.10), the sensitivity of the nominal exchange rate to short rate shocks is $A_{ije}^{UIP} = 1/\kappa_{ij}$. When arbitrageurs are risk neutral, the response of the exchange rate to the short rate only depends on the persistence of the short rate process. The more persistent the process is (a lower κ_{ij}), the larger is the nominal exchange rate response.

When arbitrageurs are risk-averse, UIP does not hold, even in the limit when risk-aversion goes to (but is not equal to) zero. In that case, the real exchange rate remains stationary as conjectured in (3.2), regardless of the unconditional mean of the two countries' real interest rates. The reason is that any permanent difference in real interest rates is absorbed in equilibrium by a an adjustment in currency risk premia. The currency of the country with permanently higher real interest rate is permanently stronger. This reduces the demand from currency traders, and requires an offsetting adjustment in risk premia, but no trend appreciation of the currency. In the limit $a_e \to 0$, the position of arbitrageurs in the CCT becomes arbitrarily large.

The following corollary summarizes these results.

Corollary 3.1. *Suppose that arbitrage is segmented.*

- *When currency arbitrageurs are risk-neutral ($a_e = 0$), UIP holds: the expected return on foreign currency is $\mu_{et}^{UIP} \equiv i_{Ht} - i_{Ft}$. The sensitivity of the exchange rate to short-rate shocks is $A_{ije}^{UIP} \equiv \frac{1}{\kappa_{ij}}$. Stationarity of the real exchange rate requires that (3.12) holds*

- *When the risk aversion of currency arbitrageurs goes to zero ($a_e \to 0$), the expected return on foreign currency does not converge to μ_{et}^{UIP}, but the sensitivity of the exchange rate to short-rate shocks converges to A_{ije}^{UIP}. The real exchange rate is stationary and satisfies (3.2), even if (3.12) is not satisfied.*

3.1.2 Bond Yields

The determination of bond yields parallels that of the exchange rate. Applying Ito's Lemma to (3.3) for $j = H$, using the dynamics (3.1) of i_{jt} for $j = H$, and noting that $t + \tau$ stays constant when taking the derivative, we find that the time-t instantaneous return on the home bond with maturity τ is

$$\frac{dP_{Ht}^{(\tau)}}{P_{Ht}^{(\tau)}} = \mu_{Ht}^{(\tau)} dt - A_{iH}(\tau)\sigma_{iH} dB_{iHt}, \tag{3.13}$$

where

$$\mu_{Ht}^{(\tau)} \equiv A'_{iH}(\tau) i_{Ht} + C'_H(\tau) - A_{iH}(\tau)\kappa_{iH}(\bar{i}_H - i_{Ht}) + \frac{1}{2} A_{iH}(\tau)^2 \sigma_{iH}^2 \tag{3.14}$$

is the expected return. Likewise, (3.1) and (3.3) for $j = F$, combined with (3.2), imply that the time-t instantaneous return on the foreign bond with maturity τ, expressed in home-currency terms, minus the instantaneous return on foreign currency, is

$$\frac{d(P_{Ft}^{(\tau)} e_t)}{P_{Ft}^{(\tau)} e_t} - \frac{de_t}{e_t} = \mu_{Ft}^{(\tau)} dt - A_{iF}(\tau)\sigma_{iF} dB_{iFt}, \tag{3.15}$$

where

$$\mu_{Ft}^{(\tau)} \equiv A'_{iF}(\tau) i_{Ft} + C'_F(\tau) - A_{iF}(\tau)\kappa_{iF}(\bar{i}_F - i_{Ft}) + \frac{1}{2} A_{iF}(\tau) \left(A_{iF}(\tau) - 2A_{iFe}\right) \sigma_{iF}^2 \tag{3.16}$$

and A_{iFe} is solved for in Proposition 3.1. We next substitute the return (3.13) into the budget constraint of the subset of arbitrageurs who can invest in home bonds (and whose budget constraint is derived from (2.3) by setting $W_{Ft} = X_{Ft}^{(\tau)} = 0$). We do the same for (3.15) and the subset of arbitrageurs who can invest in foreign bonds and have a zero net exposure in foreign-currency instruments (and whose budget constraint is derived from (2.3) by setting $W_{Ft} = X_{Ht}^{(\tau)} = 0$). For the arbitrageurs investing in the bonds of country $j = H, F$, we find

$$dW_t = \left[W_t i_{Ht} + \int_0^T X_{jt}^{(\tau)} \left(\mu_{jt}^{(\tau)} - i_{jt} \right) d\tau \right] dt - \int_0^T X_{jt}^{(\tau)} A_{ij}(\tau)\sigma_{ij} dB_{ijt}.$$

The optimization problem of these arbitrageurs is

$$\max_{\{X_{jt}^{(\tau)}\}_{\tau \in (0,T)}} \left[\int_0^T X_{jt}^{(\tau)} \left(\mu_{jt}^{(\tau)} - i_{jt} \right) d\tau - \frac{a_j}{2} \left(\int_0^T X_{jt}^{(\tau)} A_{ij}(\tau) d\tau \right)^2 \sigma_{ij}^2 \right],$$

and their first-order condition, which follows from point-wise differentiation, is

$$\mu_{jt}^{(\tau)} - i_{jt} = a_j A_{ij}(\tau) \left(\int_0^T X_{jt}^{(\tau)} A_{ij}(\tau) d\tau \right) \sigma_{ij}^2. \tag{3.17}$$

Equation (3.17) describes the arbitrageurs' risk-return trade-off when investing in the *bond carry trade* (BCT) in country j. We term BCT in country j the trade of borrowing short-term in that country and investing the borrowed amount in that country's bonds.[2] The return on the BCT in the home country and for maturity τ is $\frac{dP_{Ht}^{(\tau)}}{P_{Ht}^{(\tau)}} - i_{Ht}dt$, equal to the return on the home bond with maturity τ minus that on the home short rate. The return on the BCT in the foreign country, expressed in home-currency terms, is $\frac{d(P_{Ft}^{(\tau)}e_t)}{P_{Ft}^{(\tau)}e_t} - \frac{de_t}{e_t} - i_{Ft}dt$. This is equal to the return on the foreign bond with maturity τ, expressed in home-currency terms, minus that on foreign currency, minus that on the foreign short rate.

If arbitrageurs invest an extra unit of home currency in the BCT for country j and maturity τ, then their expected return increases by the BCT's expected return $\mu_{jt}^{(\tau)} - i_{jt}$. This is the left-hand side of (3.17). The right-hand side is the increase in the arbitrageurs' portfolio risk, times their risk-aversion coefficient a_j. The increase in portfolio risk is equal to the covariance between the return on the BCT in country j and for maturity τ, and the return on the BCT portfolio of arbitrageurs in country j and across all maturities. Since these returns depend only on the country j short rate i_{jt}, their covariance is the product of their sensitivities to i_{jt} times the instantaneous variance σ_{ij}^2 of i_{jt}. Equations (3.13) and (3.15) imply that the return sensitivities to i_{jt} are $-A_{ij}(\tau)$ and $-\int_0^T X_{jt}^{(\tau)} A_{ij}(\tau)$, respectively.

We next combine the arbitrageurs' first-order condition (3.17) with market clearing for country j bonds. Market clearing requires that the time-t positions of arbitrageurs and bond investors sum to zero:

$$X_{jt}^{(\tau)} + Z_{jt}^{(\tau)} = 0. \tag{3.18}$$

[2] For simplicity, we deviate from market terminology, according to which the BCT borrows at maturities with a low interest rate.

Using (3.18), we can write (3.17) as

$$\mu_{jt}^{(\tau)} - i_{jt} = -a_j A_{ij}(\tau) \left(\int_0^T Z_{jt}^{(\tau)} A_{ij}(\tau) d\tau \right) \sigma_{ij}^2$$

$$= a_j A_{ij}(\tau) \left(\int_0^T \left[\alpha_j(\tau) \log\left(P_{jt}^{(\tau)}\right) + \zeta_j(\tau) \right] A_{ij}(\tau) d\tau \right) \sigma_{ij}^2$$

$$= a_j A_{ij}(\tau) \left(\int_0^T \left[\zeta_j(\tau) - \alpha_j(\tau) \left(A_{ij}(\tau) i_{jt} + C_j(\tau) \right) \right] A_{ij}(\tau) d\tau \right) \sigma_{ij}^2 \tag{3.19}$$

where the second step follows from (2.5) and $\beta_{jt} = 0$, and the third step follows from (3.3). Substituting $\mu_{Ht}^{(\tau)}$ from (3.14) into (3.19) for $j = H$, we find an equation affine in i_{Ht}. Identifying the linear terms in i_{Ht} and the constant terms yields two ordinary differential equations (ODEs) for the two functions $(A_{iH}(\tau), C_{rH}(\tau))$. Repeating this process for the foreign bond, yields two ODEs for $(A_{iF}(\tau), C_{rF}(\tau))$. These ODEs are linear, with the complication that the linear coefficients depend on integrals involving these functions.

Proposition 3.2. *When arbitrage is segmented, bond prices $P_{jt}^{(\tau)}$ in country $j = H, F$ are given by (3.3), with $(A_{ij}(\tau), C_{rj}(\tau))$ equal to the unique solution of the system*

$$A'_{ij}(\tau) + \kappa_{ij} A_{ij}(\tau) - 1 = -a_j \sigma_{ij}^2 A_{ij}(\tau) \int_0^T \alpha_j(\tau) A_{ij}(\tau)^2 d\tau, \tag{3.20}$$

$$C'_j(\tau) - \kappa_{ij} \bar{i}_j A_{ij}(\tau) + \frac{1}{2} \sigma_{ij}^2 A_{ij}(\tau) \left(A_{ij}(\tau) - 2 A_{iFe} 1_{\{j=F\}} \right)$$

$$= a_j \sigma_{ij}^2 A_{ij}(\tau) \int_0^T \left[\zeta_j(\tau) - \alpha_j(\tau) C_j(\tau) \right] A_{ij}(\tau) d\tau, \tag{3.21}$$

with the initial conditions $A_{ij}(0) = C_j(0) = 0$.

In the special case where arbitrageurs are risk-neutral, the Expectations Hypothesis (EH) holds.

Corollary 3.2. *When arbitrage is segmented and bond arbitrageurs in country j are risk-neutral ($a_j = 0$), the EH holds in country j. The expected return on country-j bonds is $\mu_{jt}^{(\tau)EH} \equiv i_{jt}$, and the sensitivity of these bonds to shocks to the country-j short rate is $A_{ij}^{EH}(\tau) \equiv \frac{1-e^{-\kappa_{ij}\tau}}{\kappa_{ij}}$. The same results hold when the risk aversion of bond arbitrageurs in country j goes to zero ($a_j \to 0$).*

3.2 Short-Rate Shocks, Carry Trades and Risk Premia

We next determine how bond yields and the exchange rate respond to short-rate shocks, and what the implications are for the profitability of carry trades and risk premia.

3.2.1 Bonds

Proposition 3.3. *Suppose that arbitrage is segmented. Following a drop in the short rate in country j, bond yields drop in that country ($A_{ij}(\tau) > 0$) and do not change in the other country. When additionally bond arbitrageurs in country j are risk-averse ($a_j > 0$) and the demand of bond investors in that country is price-elastic ($\alpha_j(\tau) > 0$ in a positive-measure set of $(0, T)$):*

- *Bond yields do not drop all the way to the value implied by the EH: $A_{ij}(\tau) < A_{ij}^{EH}(\tau)$.*
- *The expected return of the BCT rises: $\frac{\partial\left(\mu_{jt}^{(\tau)} - i_{jt}\right)}{\partial i_{jt}} < 0$.*

When the short rate in country j drops, bond prices in that country rise (and bond yields drop) because of a standard discounting effect. Prices do not rise all the way to the value implied by the EH, however. Indeed, if prices remain the same as before the shock, then the drop in the short rate renders the BCT in country j more profitable, raising its expected return $\mu_{jt}^{(\tau)} - i_{jt}$. Hence, bond arbitrageurs in country j seek to invest in the BCT, increasing their bond holdings $X_{jt}^{(\tau)}$. This puts upward pressure on bond prices $P_{jt}^{(\tau)}$. When the demand by bond investors in country j is price-elastic, their holdings $Z_{jt}^{(\tau)}$ decreases as bond prices rise and that of bond arbitrageurs $X_{jt}^{(\tau)}$ increases in equilibrium. But according to (3.17), bond arbitrageurs need to be compensated for their larger bond position with a higher risk premium. Hence, as in **?** for the case of a closed economy, the BCT's expected return $\mu_{jt}^{(\tau)} - i_{jt}$ remains higher than before the shock. Bond prices adjust all the way to their EH value when bond arbitrageurs in country j are risk neutral, since they do not require such compensation. They also adjust to their EH value when the demand by bond investors in country j is price-elastic, because arbitrageurs' activity causes prices to rise until there is no change in $X_{jt}^{(\tau)}$.

Proposition 3.3 implies that the slope of the term structure in country j predicts positively the BCT's future return in that country. Indeed, slope and future return vary over time only because of the country j short rate i_{jt}, and are both high when i_{jt} is low. A positive relationship between the slope of the term structure and the BCT's future return is documented in Fama and Bliss (1987, FB), but is inconsistent with the EH according to which the BCT's expected return should be zero. Campbell and Shiller (1991, CS) document a related violation of the EH: the slope of the term structure in country j predicts negatively changes in future long rates in that country.

3.2.2 Foreign Currency

Proposition 3.4. *Suppose that arbitrage is segmented. Following a drop in the home short rate or a rise in the foreign short rate, the foreign currency appreciates ($A_{iHe} > 0$, $A_{iFe} > 0$). When additionally currency arbitrageurs are risk-averse ($a_e > 0$) and the demand of currency traders is price-elastic ($\alpha_e > 0$),*

- *The foreign currency does not appreciate all the way to the level implied by UIP: $A_{iHe} < A_{iHe}^{UIP}$, $A_{iFe} < A_{iFe}^{UIP}$.*
- *The expected return of the CCT rises: $\frac{\partial(\mu_{et}+i_{Ft}-i_{Ht})}{\partial i_{Ht}} < 0$ and $\frac{\partial(\mu_{et}+i_{Ft}-i_{Ht})}{\partial i_{Ft}} > 0$.*

When the home short rate drops or the foreign short rate rises, the foreign currency appreciates. These movements are in the direction implied by UIP. The foreign currency does not appreciate all the way to the value implied by UIP, however. Indeed, if the exchange rate remains the same as before the shock, then the drop in i_{Ht} or rise in i_{Ft} render the CCT more profitable, raising its expected return $\mu_{et} + i_{Ft} - i_{Ht}$. Hence, currency arbitrageurs seek to increase their holdings W_{Ft} of the foreign currency. When the demand by currency traders is price-elastic, both the exchange rate e_t and arbitrageurs' foreign-currency holdings W_{Ft} increase in equilibrium. Risk-averse arbitrageurs, however, do not trade all the way to the point where e_t reaches its UIP value. Instead, in a spirit similar to Gabaix and Maggiori (2015), the CCT's expected return $\mu_{et}+i_{Ft}-i_{Ht}$ remains higher than before the shock to compensate arbitrageurs for the risk generated by their larger foreign-currency position. The exchange rate adjusts all the way to its UIP value when currency arbitrageurs are risk-neutral or when the demand by currency traders is price-inelastic.

Proposition 3.4 implies that the difference between the foreign and the home short rate predicts positively the CCT's future return. This is consistent with the evidence in Bilson (1981) and Fama (1984), who document that following an increase in the foreign-minus-home short-rate differential, the expected return on the foreign currency typically increases. Moreover, even in samples where it decreases, it does so less than implied by UIP. Hence, the CCT becomes more profitable.

3.3 Demand Shocks

We next determine how bond yields and the exchange rate respond to changes in the demand for bonds and foreign currency. Since we assume no demand risk in this section, we take the demand changes to be unanticipated and one-off. Demand changes by bond investors in country j correspond to shocks to the demand factor β_{jt}. Demand changes by currency traders correspond to shocks to the demand factor γ_t. Following the shocks, the demand factors revert deterministically to their mean of zero. The effects of unanticipated and one-off shocks are the limit of those under anticipated and recurring shocks (Section 5) when the shocks' variance goes to zero.

Without loss of generality, we take θ_e to be positive, which means that an increase in γ_e corresponds to a drop in demand for foreign currency. We take $\theta_j(\tau)$ to be positive for all τ, which means that an increase in β_{jt} corresponds to a drop in demand for the bonds of country j.

Proposition 3.5. *Suppose that arbitrage is segmented, $\theta_e > 0$ and $\theta_j(\tau) > 0$ for all τ.*

- *An unanticipated one-off drop in investor demand for the bonds of country j (increase in β_{jt}) raises bond yields in country j if bond arbitrageurs in that country are risk-averse ($a_j > 0$). It has no effect on bond yields in the other country and on the exchange rate.*
- *An unanticipated one-off drop in currency traders' demand for foreign currency (increase in γ_e) causes the foreign currency to depreciate if currency traders are risk-averse ($a_e > 0$). It has no effect on bond yields.*

When arbitrage is segmented, changes to the demand for an asset class—foreign currency, home bonds, foreign bonds—affect that asset class only. When, for example, the demand for bonds in country j drops, these bonds become cheaper and their yields increase, while foreign currency and bonds in the other country are unaffected.

3.4 International Transmission and the Trilemma with Segmented Arbitrage

We next summarize the main implications of the model with segmented arbitrage for the domestic and international transmission of monetary policy. Consider a conventional monetary policy easing at home, such as a drop in the home short rate i_{Ht}. That drop propagates along the home term structure, although less than implied by EH (Proposition 3.3). Moreover, the home currency depreciates, although less than implied by UIP (Proposition 3.4). Propagation is imperfect (compared to EH and UIP) because bond and foreign-currency arbitrageurs must be compensated for the change in their portfolio holdings. The drop in the home short rate does not affect the foreign term structure (Proposition 3.3), and hence has no effect on foreign monetary conditions. In that sense, the model with segmented arbitrage features *full insulation*.

Consider next a quantitative easing at home, where the Central Bank unexpectedly increases its holdings of home bonds of some maturities $\tau > 0$. Through the lens of the model, this corresponds to an increase in the demand for home bonds, i.e. $\beta_{jt} < 0$. This policy decreases home bond yields (Proposition 3.5). It does not effect the foreign term structure, and hence has no effect on foreign monetary conditions. Once again, the model with segmented arbitrage features *full insulation*.

To understand why insulation arises, it is useful to frame the discussion in terms of the classic Friedman-Obstfeld-Taylor open-economy Trilemma. According to the Trilemma, a country that wants to maintain domestic monetary autonomy must either let its currency float, or impose capital controls. From that perspective, our finding that foreign monetary policy is insulated from home monetary policy may appear unsurprising at first glance. After all, we are assuming that the exchange rate is floating and that there are restrictions on capital flows since home-bond arbitrageurs cannot hold foreign bonds and vice-versa. According to the Trilemma, each one of these assumptions in isolation would be sufficient to ensure monetary policy insulation. As the next section will demonstrate, however, this is not the case in our framework. When arbitrageurs are global, they transmit monetary impulses from one country's term structure to the other, even when exchange rates are floating. In other words, while floating exchange rates keep short rates insulated, insulation of the term structure arises entirely from the assumption that the home and foreign bond markets are segmented.

In the model with segmented arbitrage, foreign-currency arbitrageurs can invest only in the home and the foreign short rate, which are pinned down, respectively, by the home and foreign central bank. Hence, unanticipated shocks to the demand for home bonds affect home bond yields but not the exchange rate (Proposition 3.5). One relevant implication is that unanticipated QE has no effect on the exchange rate. Hence in the segmented model, conventional monetary policy and QE transmit differently to the domestic economy: in the case of conventional policy, a monetary easing lowers bond yields and depreciates the currency, while in the case of unanticipated QE, a monetary easing lowers bond yields but leaves the exchange rate unchanged. This result no longer holds in Section 5, where shocks to bond demand affect both the term structure and the exchange rate.

4 Global Arbitrage

The remainder of the paper studies the case of global arbitrage. In this section we maintain the other assumptions of Section 3, i.e., independent short rates, no stochastic variation in the demand factors, one-off shocks to the demand factors that do not affect the short rates or other demand factors, and currency demand only in the spot market. We relax these assumptions in Section 5.

4.1 Equilibrium

We conjecture that the equilibrium exchange rate takes the same form (3.2) as in Section 3. In contrast to Section 3, we allow bond yields in each country $j = H, F$ to also depend on the other country's short rate because of potential spillovers, which we show occur in equilibrium. Thus, we replace (3.3) by

$$\log P_{jt}^{(\tau)} = -\left[A_{ijj}(\tau)i_{jt} + A_{ijj'}(\tau)i_{j't} + C_j(\tau)\right] \tag{4.1}$$

for $j' \neq j$ and six functions $(\{A_{ijj'}(\tau)\}_{j,j'=H,F}, \{C_j(\tau)\}_{j=H,F})$ that depend only on τ.

Proceeding as in Section 3, we find that the first-order condition of global arbitrageurs is

$$\mu_{et} + i_{Ft} - i_{Ht} = A_{iHe}\lambda_{iHt} - A_{iFe}\lambda_{iFt}, \tag{4.2}$$

$$\mu_{jt}^{(\tau)} - i_{jt} = A_{ijj}(\tau)\lambda_{ijt} + A_{ijj'}(\tau)\lambda_{ij't}, \tag{4.3}$$

where $j, j' = H, F$, $j \neq j'$ and

$$\lambda_{ijt} \equiv a\sigma_{ij}^2 \left(W_{Ft}A_{ije}(-1)^{1_{\{j=F\}}} + \sum_{j'=H,F} \int_0^T X_{j't}^{(\tau)} A_{ij'j}(\tau)d\tau \right). \tag{4.4}$$

The left-hand side of (4.2) and (4.3) is the increase in the arbitrageurs' expected return if they invest one unit of home currency in the CCT and in the country j BCT, respectively. The right-hand side is the increase in the arbitrageurs' portfolio risk, times their risk-aversion coefficient a. Portfolio risk increases by the covariance between the corresponding trade (CCT or country j BCT) and the arbitrageurs' portfolio. To compute the covariance, we multiply the sensitivity of the trade's return to the short rate in country j, times the sensitivity λ_{ijt} of the arbitrageurs' portfolio return to the same factor, times the factor's variance σ_{ij}^2. We then sum over $j = H, F$. In the terminology of no-arbitrage models, the sensitivity λ_{ijt} is the price of the risk factor i_{jt}. The key difference between (4.2) and (4.3), and their counterparts (3.6) and (3.17) is that the same factor prices λ_{ijt} apply to all trades (CCT, home BCT, foreign BCT). It is through the equalization of factor prices that global arbitrage connects bond and currency markets. Using market clearing to substitute $(W_{Ft}, \{X_{jt}^{(\tau)}\}_{j=H,F}$ in (4.4), and proceeding as in Section 3, we characterize the exchange rate and bond prices by a system of scalar equations and ODEs.

Proposition 4.1. *When arbitrage is global, the exchange rate e_t is given by (3.2) and bond prices $P_{jt}^{(\tau)}$ in country $j = H, F$ are given by (4.1), with $(\{A_{ije}\}_{j=H,F}, C_e)$ solving*

$$\kappa_{ij} A_{ije} - 1 = a\sigma_{ij}^2 \bar{\lambda}_{ijj} A_{ije} - a\sigma_{ij'}^2 \bar{\lambda}_{ijj'} A_{ij'e}, \tag{4.5}$$

$$- \kappa_{iH} \bar{i}_H A_{iHe} + \kappa_{iF} \bar{i}_F A_{iFe} - (\pi_F - \pi_H) + \frac{1}{2}\sigma_{iH}^2 A_{iHe}^2 + \frac{1}{2}\sigma_{iF}^2 A_{iFe}^2 \tag{4.6}$$
$$= a\sigma_{iH}^2 \bar{\lambda}_{iHC} A_{iHe} - a\sigma_{iF}^2 \bar{\lambda}_{iFC} A_{iFe},$$

and $(A_{ijj}(\tau), A_{ijj'}(\tau), C_j(\tau))$ solving

$$A'_{ijj}(\tau) + \kappa_{ij} A_{ijj}(\tau) - 1 = a\sigma_{ij}^2 \bar{\lambda}_{ijj} A_{ijj}(\tau) + a\sigma_{ij'}^2 \bar{\lambda}_{ijj'} A_{ijj'}(\tau), \tag{4.7}$$

$$A'_{ijj'}(\tau) + \kappa_{rj'} A_{ijj'}(\tau) = a\sigma_{ij}^2 \bar{\lambda}_{rj'j} A_{ijj}(\tau) + a\sigma_{ij'}^2 \bar{\lambda}_{rj'j'} A_{ijj'}(\tau), \tag{4.8}$$

$$C'_j(\tau) - \kappa_{ij} \bar{i}_j A_{ijj}(\tau) - \kappa_{rj'} \bar{i}_{j'} A_{ijj'}(\tau) + \frac{1}{2}\sigma_{ij}^2 A_{ijj}(\tau) \left(A_{ijj}(\tau) - 2A_{iFe} 1_{\{j=F\}}\right)$$
$$+ \frac{1}{2}\sigma_{ij'}^2 A_{ijj'}(\tau) \left(A_{ijj'}(\tau) + 2A_{iHe} 1_{\{j=F\}}\right) = a\sigma_{ij}^2 \bar{\lambda}_{ijC} A_{ijj}(\tau) + a\sigma_{ij'}^2 \bar{\lambda}_{ij'C} A_{ijj'}(\tau), \tag{4.9}$$

with the initial conditions $A_{ijj}(0) = A_{ijj'}(0) = C_j(0) = 0$, where $j' \neq j$ and

$$\bar{\lambda}_{ijj} \equiv -\left(\sum_{k=H,F} \int_0^T \alpha_k(\tau) A_{ikj}(\tau)^2 d\tau + \alpha_e A_{ije}^2\right), \tag{4.10}$$

$$\bar{\lambda}_{ijj'} \equiv -\left(\sum_{k=H,F} \int_0^T \alpha_k(\tau) A_{ikj}(\tau) A_{ikj'}(\tau) d\tau - \alpha_e A_{ije} A_{ij'e}\right), \tag{4.11}$$

$$\bar{\lambda}_{ijC} \equiv \sum_{k=H,F} \int_0^T \left(\zeta_k(\tau) - \alpha_k(\tau) C_k(\tau)\right) A_{ikj}(\tau) d\tau + \left(\zeta_e - \alpha_e C_e\right) A_{ije} (-1)^{1_{\{j=F\}}}. \tag{4.12}$$

Equations (4.7) and (4.8) form a system of two linear ODEs in $(A_{ijj}(\tau), A_{ijj'}(\tau))$, with the complication that the coefficients of $(A_{ijj}(\tau), A_{ijj'}(\tau))$ depend on integrals involving these functions, on integrals involving the functions obtained by inverting j and $j' \neq j$, and on (A_{iHe}, A_{iFe}). We solve the system taking $\bar{\lambda}_{ijj}$, $\bar{\lambda}_{ijj'} = \bar{\lambda}_{ij'j}$ and $\bar{\lambda}_{ij'j'}$ as given. We do the same for the system obtained by inverting j and j', and for the linear scalar system (4.5) in (A_{iHe}, A_{iFe}). We then substitute back into the definitions of $\bar{\lambda}_{ijj}$, $\bar{\lambda}_{ijj'} = \bar{\lambda}_{ij'j}$ and $\bar{\lambda}_{ij'j'}$ to derive a non-linear system of three equations in these three unknowns. The properties that we show in the remainder of this section hold for any solution of this system.

In the special case where arbitrageurs are risk-neutral and the parameters $(\psi_e, \bar{i}_F - \bar{i}_H)$ satisfy (3.12), UIP and EH hold. When instead $(\psi_e, \bar{i}_F - \bar{i}_H)$ are unrestricted and arbitrageurs are risk-averse, UIP and EH do not hold, even in the limit when risk-aversion goes to zero. Recall that in that limit, UIP fails but EH holds under segmented arbitrage. Under global arbitrage, failure of UIP causes failure of EH because the risk premia in the currency market, which do not converge to zero, spill over to the bond market.

Corollary 4.1. *When arbitrage is global, the results in Corollaries 3.1 and 3.2 continue to hold. The only exception is that when arbitrageur risk aversion goes to zero ($a \to 0$) and (3.12) does not hold, the expected return on country-j bonds does not converge to $\mu_{jt}^{(\tau)EH}$.*

4.2 Short-Rate Shocks, Carry Trades and Risk Premia

Proposition 4.2. *Suppose that arbitrage is global.*

- *The effects of short-rate shocks on the exchange rate and on the CCT's expected return have the same properties as in Proposition 3.4.*
- *The effects of shocks to the country-j short rate i_{jt} on bond yields in country j and on the BCT's expected return have the same properties as in Proposition 3.3, except that the price-elasticity condition can hold for currency traders or bond investors ($\alpha_e > 0$ or $\alpha_j(\tau) > 0$).*
- *When arbitrageurs are risk-averse ($a > 0$) and the demand by currency traders is price-elastic ($\alpha_e > 0$), a drop in i_{jt} causes bond yields in country $j' \neq j$ to drop ($A_{j'j}(\tau) > 0$) and the BCT's expected return to drop ($\frac{\partial\left(\mu_{j't}^{(\tau)} - i_{j't}\right)}{\partial i_{jt}} > 0$).*
- *The effect of i_{jt} on bond yields is smaller in country j' than in country j ($A_{jj}(\tau) > A_{j'j}(\tau)$).*

The response of the exchange rate to short-rate shocks is similar under global and segmented arbitrage: the exchange rate moves in the direction implied by UIP, and there is under-reaction when arbitrageurs are risk-averse ($a > 0$) and the demand by currency traders is price-elastic

($\alpha_e > 0$). Global and segmented arbitrage differ in how bond yields respond to shocks. Under segmented arbitrage, a shock to the short rate i_{jt} in country j affects bond yields in that country only. By contrast, under global arbitrage, and provided that $a\alpha_e > 0$, the shock affects bond yields in both countries, even though the short rate $i_{j't}$ in country $j' \neq j$ does not change. When i_{jt} drops, bond yields in both countries drop.

Since short-rate shocks are transmitted across countries, monetary policy in one country has a direct effect on the other country's interest rates. When the central bank in country j lowers the short rate i_{jt}, interest rates for longer maturities in country j' drop. This is so even though the central bank in country j' leaves the short rate $i_{j't}$ unchanged.

Short-rate shocks are transmitted across countries because global arbitrageurs engage in the CCT and use the bond market to hedge. Recall that under both segmented and global arbitrage, a drop in the home short rate i_{Ht} raises the profitability of the CCT, making it more attractive to arbitrageurs. When the demand by currency traders is price-elastic, the arbitrageurs' equilibrium investment in the CCT increases. Because arbitrageurs hold more foreign-currency instruments (higher W_{Ft}), they become more exposed to the risk that the foreign short rate i_{Ft} drops and the foreign currency depreciates. Global arbitrageurs hedge that risk by buying foreign bonds because their price rises when i_{Ft} drops. The arbitrageurs' activity pushes the prices of foreign bonds up and their yields down.

An additional consequence of hedging by global arbitrageurs is greater under-reaction of home bonds to the home short rate. When i_{Ht} drops, arbitrageurs invest more in the CCT, and hence become more exposed to a rise in i_{Ht}. Investing in home bonds, whose prices drop when i_{Ht} rises, adds to that risk. Hence, global arbitrageurs are less eager than segmented arbitrageurs to buy home bonds following a drop in i_{Ht}, and the expected return of the home BCT increases more than under segmented arbitrage. In particular, when the demand by home bond investors is price-inelastic (and that by currency traders is elastic), a drop in i_{Ht} raises the home BCT's expected return under global arbitrage but leaves it unaffected under segmented arbitrage.

We next turn to variants of the CCT studied in the empirical literature. We show that these trades can be viewed as combinations of the BCT and the (basic) CCT, and that Proposition 4.2 can shed light on empirical findings concerning these trades.

One variant is a hybrid CCT in which the trading horizon is short but the trading instruments are long-term. Borrowing in the home country and investing in the foreign country is done with the respective τ-year bonds, and the positions are held for a short horizon dt. The return of the hybrid CCT in home-currency units is

$$\frac{d(P_{Ft}^{(\tau)}e_t)}{P_{Ft}^{(\tau)}e_t} - \frac{dP_{Ht}^{(\tau)}}{P_{Ht}^{(\tau)}} = \left(\frac{de_t}{e_t} + (i_{Ft} - i_{Ht})dt\right) + \left(\frac{d(P_{Ft}^{(\tau)}e_t)}{P_{Ft}^{(\tau)}e_t} - \frac{de_t}{e_t} - i_{Ft}dt\right) - \left(\frac{dP_{Ht}^{(\tau)}}{P_{Ht}^{(\tau)}} - i_{Ht}dt\right). \tag{4.13}$$

Hence, the hybrid CCT can be viewed as a combination of (i) the basic CCT, (ii) a long position in the foreign BCT, and (iii) a short position in the home BCT.

A second variant is a long-horizon CCT, in which borrowing in the home country and investing in the foreign country is done with the respective τ-year bonds, and the positions are held until the bonds' maturity. The return of the long-horizon CCT in home-currency units and log terms is

$$\log\left(\frac{e_{t+\tau}}{P_{Ft}^{(\tau)}e_t}\right) - \log\left(\frac{1}{P_{Ht}^{(\tau)}}\right) = \int_t^{t+\tau}\left(\log\left(\frac{e_{s+ds}}{e_s}\right) + i_{Fs}ds - i_{Hs}ds\right) + \left(\tau y_{Ft}^{(\tau)} - \int_t^{t+\tau} i_{Fs}ds\right) - \left(\tau y_{Ht}^{(\tau)} - \int_t^{t+\tau} i_{Hs}ds\right), \tag{4.14}$$

where the equality follows from (2.1). Hence, the long-horizon CCT can be viewed as the combination of (i) a sequence of basic CCTs, (ii) a long position in a long-horizon foreign BCT, and (iii) a short position in a long-horizon home BCT. The long-horizon BCT in country j involves buying bonds in country j and financing that position by borrowing short-term and rolling over.

Proposition 4.3. *Suppose that arbitrage is global, arbitrageurs are risk-averse ($a > 0$), and the demand by currency traders or by bond investors is price-elastic ($\alpha_e > 0$ or $\alpha_j(\tau) > 0$).*

- *The hybrid CCT's and the long-horizon CCT's expected returns rise following a drop in the home short rate i_{Ht} or a rise in the foreign short rate i_{Ft}, provided that the maturity τ of the bonds involved in these trades lies in an interval $(0, \tau^*)$. The threshold τ^* is infinite when countries are symmetric.*
- *The sensitivity of the hybrid CCT's expected return to (i_{Ht}, i_{Ft}) is smaller than for the basic CCT. The sensitivity of the long-horizon CCT's expected return to (i_{Ht}, i_{Ft}) is smaller than for the corresponding sequence of basic CCTs.*
- *The sensitivity of the hybrid CCT's and the long-horizon CCT's expected returns to (i_{Ht}, i_{Ft}) goes to zero when the maturity τ of the bonds involved in these trades goes to infinity. The expected return of the hybrid CCT also goes to zero.*

Short-rate shocks move the expected returns of the hybrid CCT and the long-horizon CCT in the same direction as for the basic CCT, except possibly when the maturity τ of the bonds involved in these trades is very long. The effects of short-rate shocks on the hybrid CCT and the long-horizon CCT are smaller than for the corresponding basic CCTs because the shocks' effects through the BCTs work in the opposite direction. Consider, for example, a drop in the home short rate. Proposition 4.2 implies that the expected return of the basic CCT increases, but so does the expected return of the home BCT, which enters as a short position in the hybrid CCT and the long-horizon CCT.

When the maturity τ of the bonds involved in the hybrid CCT and the long-horizon CCT is very long, the effects of short-rate shocks through the BCTs offset almost fully those through the basic CCTs. As a consequence, short-rate shocks have almost no effect on the expected return of the hybrid CCT and the long-horizon CCT. These results are consistent with Lustig, Stathopoulos, and Verdelhan (2019), who document that short rates lose their predictive power for the return of the hybrid CCT, while they predict strongly the return of the basic CCT. They are also consistent with Chinn and Meredith (2004), who document that UIP cannot be rejected over long horizons.

Short rate shocks lose their predictive power for the hybrid and the long-horizon CCT because the risk of these trades arises from long-horizon exchange-rate movements, which are unrelated to current short-rate shocks. Indeed, an arbitrageur entering in the long-horizon CCT at time t receives a fixed amount of foreign currency and pays a fixed amount of home currency at time $t + \tau$. Mean-reverting short-rate shocks do not affect the risk borne by the arbitrageur when τ is large. The same is true for the hybrid CCT because that trade is identical to the long-horizon CCT except that it is unwound at time $t + dt$.

Under segmented arbitrage, the hybrid and the long-horizon CCT cannot be performed by any agent in the model as they require trading bonds and foreign currency simultaneously. Yet, we can compute these trades' expected returns, and show the second result in Proposition 4.2. The first and third result do not hold, however, because the effects of short-rate shocks on the BCTs and the basic CCT are driven by the risk aversion of different arbitrageurs, and are hence disconnected. In particular, the expected returns of the hybrid CCT and the long-horizon CCT may not approach zero when the maturity τ of the bonds involved in these trades is very long.

4.3 Demand Shocks

Under global arbitrage, shocks to the demand for an asset class—foreign currency, home bonds, foreign bonds—affect all three asset classes. This is in contrast to segmented arbitrage, where only the asset class for which demand changes is affected (Proposition 3.5).

Proposition 4.4. *Suppose that arbitrage is global, arbitrageurs are risk-averse ($a > 0$), the functions $(a_H(\tau), \alpha_F(\tau))$ are non-increasing, and the function $\theta_j(\tau)$ is positive. A drop in investor demand for the bonds of country j (increase in β_{jt}):*

- *Raises bond yields in country j.*
- *Raises bond yields in country $j' \neq j$ when the demand by currency traders is price-elastic ($\alpha_e > 0$).*
- *Causes the foreign currency to depreciate if $j = H$, and to appreciate if $j = F$.*

A drop in investor demand for home bonds depresses their prices, as in Proposition 3.5. Additionally, prices for foreign bonds drop and the foreign currency depreciates. The latter (cross) effects are driven by hedging of global arbitrageurs. Indeed, arbitrageurs accommodate the drop in demand for home bonds by holding more such bonds. Hence, they become more exposed to a rise in the home short rate i_{Ht} and less willing to hold assets that lose value when i_{Ht} rises. Foreign currency is such an asset, and hence it depreciates. Foreign bonds is another such asset (Proposition 4.2 shows that a rise in i_{Ht} drives foreign bond prices down when the demand by currency traders is price-elastic), and hence their prices drop. A drop in demand for foreign bonds has symmetric effects.

Proposition 4.5. *Suppose that arbitrage is global, arbitrageurs are risk-averse ($a > 0$), the functions $(a_H(\tau), \alpha_F(\tau))$ are non-increasing, and $\theta_e > 0$. A drop in currency traders' demand for foreign currency (increase in γ_t):*

- *Causes the foreign currency to depreciate.*
- *Raises bond yields in the home country.*
- *Lowers bond yields in the foreign country.*

A drop in currency traders' demand for foreign currency causes it to depreciate, as in Proposition 3.5. Additionally, hedging by global arbitrageurs causes home bond prices to drop and foreign bond prices to rise. Indeed, arbitrageurs accommodate the drop in demand for foreign currency by holding more of it. Hence, they become more exposed to a rise in the home short rate i_{Ht} and to a decline in the foreign short rate i_{Ft}. This makes them less willing to hold home bonds, which lose value when i_{Ht} rises, and more willing to hold foreign bonds, which gain value when i_{Ft} drops.

4.4 International Transmission and the Trilemma with Global Arbitrage

We next summarize the main implications of the model with global arbitrage for the domestic and international transmission of monetary policy. Consider a conventional monetary policy easing at home, such as a drop in the home short rate i_{Ht}. That drop propagates imperfectly along the home term structure and depreciates the home currency (Proposition 4.2). These effects are as in the case of segmented arbitrage. Unlike in that case, yields on foreign bonds decrease, even though the foreign short rate remains unchanged. Hence, foreign monetary conditions are affected by domestic monetary conditions. In that sense, the model with global arbitrage and floating exchange rates features *imperfect insulation*.

Consider next a quantitative easing at home, where the Central Bank increases its holdings of domestic bonds of some maturities $\tau > 0$. Through the lens of the model, this corresponds to an increase in the demand for domestic bonds, i.e. $\beta_{jt} < 0$. This policy decreases home bond yields (Proposition 4.4). This effect is as in the case of segmented arbitrage. Unlike that case, yields on foreign bonds decrease and the home currency depreciates. Hence, foreign monetary conditions are affected by domestic monetary conditions. Once again, the model with global arbitrage features *imperfect insulation*. For both types of policies, monetary conditions co-move positively: easing at home eases abroad and vice versa.

To understand why insulation fails, we can go back to our Trilemma analysis. According to the Trilemma, a country without restrictions on capital mobility should be able to maintain domestic monetary autonomy—interpreted as controlling the yield curve—by letting the exchange rate float. This is no longer the case under global arbitrage. The reason is that global rate arbitrageurs rebalance their entire portfolio in response to shocks. When global arbitrageurs are risk-averse, portfolio rebalancing requires adjustments in expected returns. In turn, this triggers movements in bond prices and the exchange rate.

For example, a lower home short rate induces global arbitrageurs to increase their holdings of domestic bonds (BCT) and of foreign currency (CCT). It also induces them to increase their holdings of foreign long term bonds (BCT), to hedge their larger holdings of foreign currency. This pushes down bond yields everywhere and depreciates the home currency.

The global arbitrage model implies additionally that sterilized foreign exchange interventions affect not only the exchange rate but also the home and foreign yield curves. A sterilized foreign exchange intervention designed to support the home currency can be interpreted as a drop in the demand for foreign currency (an increase in γ_t), while holding the short rate unchanged. This depreciates the foreign currency while tightening domestic monetary conditions and easing foreign monetary conditions (Proposition 4.5).

Insulation of monetary policy is restored if global investors are risk-neutral. In that case, expected returns satisfy both EH and UIP. Under EH, all bonds in a given country have the same instantaneous expected return, equal to that country's short rate. Under UIP, the foreign currency has instantaneous expected return equal to the difference between the home and the foreign short rate. Hence, the exchange rate adjusts so that bonds of all maturities in both countries have the same expected return: insulation is restored.

5 Global Arbitrage and Demand Risk

We now turn to the most general version of the model, allowing for stochastic demand by bond investors and currency traders. There are five risk factors: the home and foreign short rates (i_{Ht}, i_{Ft}), the demand factors for home and foreign bonds (β_{Ht}, β_{Ft}), and the demand factor for currency γ_t. The vector of state variables $q_t = (i_{Ht}, i_{Ft}, \beta_{Ht}, \beta_{Ft}, \gamma_t)^\top$ satisfies (2.9). We allow for a general correlation structure between the five factors (non-diagonal matrices Γ and Σ), and for currency demand in both the spot and the forward market, with appropriate substitutions.

5.1 Equilibrium

We conjecture and verify that the equilibrium exchange rate and bond yields are log-affine functions of q_t. That is, there exist six scalars $(\{A_{ije},\ A_{\beta je}\}_{j=H,F}, A_{\gamma e}, C_e)$ and twelve functions $(\{A_{ijj'}(\tau), A_{\beta jj'}(\tau)\}_{j,j'=H,F}, \{A_{\gamma j}(\tau)\}_{j=H,F}, \{C_j(\tau)\}_{j=H,F})$ that depend only on τ, such that

$$\log e_t = -\left[A_e^\top q_t + C_e + (\pi_F - \pi_H)t\right], \tag{5.1}$$

$$\log P_{jt}^{(\tau)} = -\left[A_j(\tau)^\top q_t + C_j(\tau)\right], \tag{5.2}$$

where $A_e \equiv (A_{iHe}, -A_{iFe}, A_{\beta He}, -A_{\beta Fe}, A_{\gamma e})^\top$ and $A_j(\tau) \equiv (A_{ijH}(\tau), A_{ijF}(\tau), A_{\beta jH}(\tau), A_{\beta jF}(\tau), A_{\gamma j}(\tau))^\top$.

Proceeding as in Sections 3 and 4, the first-order condition of the optimization problem of global arbitrageurs is

$$\mu_{et} + i_{Ft} - i_{Ht} = A_e^\top \lambda_t, \tag{5.3}$$

$$\mu_{jt}^{(\tau)} - i_{jt} = A_j(\tau)^\top \lambda_t, \tag{5.4}$$

where $j = H, F$, $\mu_{et} = \mathbb{E}_t(de_t/e_t)$ and $\mu_{jt}^{(\tau)} = \mathbb{E}_t(dP_{jt}^{(\tau)}/P_{jt}^{(\tau)})$, $\lambda_t \equiv (\lambda_{iHt}, \lambda_{iFt}, \lambda_{\beta Ht}, \lambda_{\beta Ft}, \lambda_{\gamma t})^\top$ and

$$\lambda_t \equiv a\Sigma\Sigma^\top \left(W_{Ft} A_e + \sum_{j=H,F} \int_0^T X_{jt}^{(\tau)} A_j(\tau) d\tau \right). \tag{5.5}$$

The expected return of the CCT in (5.3), and of the country j BCT in (5.4), are computed by multiplying the sensitivity of each trade's return to each risk factor times the factor's price, and summing over factors. We denote by $(\mathcal{E}_{iH}, \mathcal{E}_{iF}, \mathcal{E}_{\beta H}, \mathcal{E}_{\beta F}, \mathcal{E}_{\gamma})$ the five 5×1 vectors that correspond to the five consecutive columns of the 5×5 identity matrix. Using market clearing to substitute $(W_{Ft}, \{X_{jt}^{(\tau)}\}_{j=H,F})$ in (5.5), and proceeding as in Sections 3 and 4, we characterize the exchange rate and bond prices by a system of scalar equations and ODEs in the following proposition.

Proposition 5.1. *When arbitrage is global and demand for currency and bonds is stochastic according to (2.9), the exchange rate e_t is given by (5.1) and bond prices $P_{jt}^{(\tau)}$ in country $j = H, F$ are given by (5.2), with (A_e, C_e) solving*

$$MA_e - \mathcal{E}_{iH} + \mathcal{E}_{iF} = 0, \tag{5.6}$$

$$-A_e^\top \Gamma \bar{q} - (\pi_F - \pi_H) + \frac{1}{2} A_e^\top \Sigma\Sigma^\top A_e = A_e^\top \lambda_C, \tag{5.7}$$

and $(A_j(\tau), C_j(\tau))$ solving

$$A_j'(\tau) + MA_j(\tau) - \mathcal{E}_{ij} = 0, \tag{5.8}$$

$$C_j'(\tau) - A_j(\tau)^\top \Gamma \bar{q} + \frac{1}{2} A_j(\tau)^\top \Sigma\Sigma^\top \left(A_j(\tau) + 2A_e 1_{\{j=F\}} \right) = A_j(\tau)^\top \lambda_C, \tag{5.9}$$

with the initial conditions $A_j(0) = C_j(0) = 0$*, and*

$$M \equiv \Gamma^\top - a\left[\sum_{j=H,F}\int_0^T \left(\theta_j(\tau)\mathcal{E}_{\beta j} + \theta_e(\tau)\mathcal{E}_\gamma(-1)^{1_{\{j=H\}}} - \alpha_j(\tau)A_j(\tau)\right)A_j(\tau)^\top d\tau \right.$$

$$\left. + \left(\theta_e\mathcal{E}_\gamma + \int_0^T \theta_e(\tau)\mathcal{E}_\gamma d\tau - \alpha_e A_e\right)A_e^\top\right]\Sigma\Sigma^\top, \tag{5.10}$$

$$\lambda_C \equiv a\Sigma\Sigma^\top\left[\sum_{j=H,F}\int_0^T \left(\zeta_j(\tau) + \zeta_e(\tau)(-1)^{1_{\{j=H\}}} - \alpha_j(\tau)C_j(\tau)\right)A_j(\tau)d\tau \right.$$

$$\left. + \left(\zeta_e + \int_0^T \zeta_e(\tau)d\tau - \alpha_e C_e\right)A_e\right]. \tag{5.11}$$

Equation (5.8) is a linear ODE system in the 5×1 vector $A_j(\tau)$. We solve it taking the 5×5 matrix M as given, and do the same for the linear scalar system (5.6) in A_e. We then substitute $(\{A_j(\tau)\}_{j=H,F}, A_e)$ in (5.10) and derive M as a solution to a non-linear scalar system. Because the non-linear system is high-dimensional, it can no longer be solved analytically and must instead be solved numerically, as described in Appendix B.

5.2 Estimation and Data

We next lay out explicitly the model parameters required to solve the model numerically, and describe our estimation strategy. First, we parametrize the functions $\{\alpha_j(\tau)\}_{j=H,F}$ that describe the slope of preferred-habitat demand as function of maturity, and $\{\theta_j(\tau)\}_{j=H,F}$ that describe how shocks to the demand factors affect the demand intercept as function of maturity. The analytical results in the previous sections place only weak restrictions on these functions, but solving the model numerically requires a more explicit characterization. We assume the exponential specification:

$$\alpha_j(\tau) \equiv \alpha_{j0}\exp(-\alpha_{j1}\tau), \tag{5.12}$$

$$\theta_j(\tau) \equiv \theta_{j0}\tau\exp(-\theta_{j1}\tau), \tag{5.13}$$

for positive scalars $(\alpha_{j0}, \alpha_{j1}, \theta_{j0}, \theta_{j1})$. The exponential specification simplifies the estimation of the model, while also being sufficiently flexible. The function $\theta_j(\tau)$ is positive and hump-shaped with a peak at maturity $1/\theta_{j1}$. Thus, shifts to the demand factor β_{jt} shift the demand for bonds of all maturities in the same direction, with the effects being more pronounced at a specific maturity. The function $\alpha_j(\tau)\tau$, which describes the demand slope when demand is expressed as function of yield rather than price, has the same functional form as $\theta_j(\tau)$, with a peak at $1/\alpha_{j1}$. When $\alpha_{j1} = \theta_{j1}$, the term structure in the absence of arbitrageurs is flat, and shocks to β_{jt} generate parallel shifts. We set the maximum maturity T to infinity.

Next, we impose some structure on the dynamics matrix Γ and correlation matrix Σ in (2.9). First, we consider as our baseline a simple diagonal Γ matrix. Second, we allow the innovations to the short rates (i_{Ht}, i_{Ft}) to be correlated: $\Sigma_{i_H,i_F} \neq 0$. This allows to capture the dynamics in the short rates which are observed in the data. Third, since the data does not offer as tight guidance on the demand factors $(\beta_{Ht}, \beta_{Ft}, \gamma_t)$, which are not observable, we assume that they follow mutually independent processes, also independent from the short rate processes. These restrictions simplify the estimation of the model and the interpretation of the results, while allowing us to capture some key features of the data. Allowing for correlation between the demand factors, and between these factors and the short rates, could improve the fit even further and is an important extension of our research. With the imposed restrictions, Γ and Σ take the following form:

$$\Gamma = \begin{bmatrix} \Gamma_{i_H} & 0 & 0 & 0 & 0 \\ 0 & \Gamma_{i_F} & 0 & 0 & 0 \\ 0 & 0 & \Gamma_{\beta_H} & 0 & 0 \\ 0 & 0 & 0 & \Gamma_{\beta_F} & 0 \\ 0 & 0 & 0 & 0 & \Gamma_{\gamma_e} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_{i_H} & \Sigma_{i_H,i_F} & 0 & 0 & 0 \\ \Sigma_{i_H,i_F} & \Sigma_{i_F} & 0 & 0 & 0 \\ 0 & 0 & \Sigma_{\beta_H} & 0 & 0 \\ 0 & 0 & 0 & \Sigma_{\beta_F} & 0 \\ 0 & 0 & 0 & 0 & \Sigma_{\gamma_e} \end{bmatrix} \tag{5.14}$$

Finally, we do not estimate the long-run mean $\bar{q}$ of the vector of state variables q_t, the intercepts $(\{\zeta_j(\tau)\}_{j=H,F}, \zeta_e)$, and the inflation differential $\pi_F - \pi_H$. These parameters concern long-run averages rather than responses to shocks. We estimate our model using second moments of yields (implied by responses to shocks), and use it to determine other second moments and responses to shocks.

The above assumptions leave us with 22 parameters to estimate: eight bond demand parameters $(\{\alpha_{j0}, \alpha_{j1}\}_{j=H,F}, \{\theta_{j0}, \theta_{j1}\}_{j=H,F})$, two currency demand parameters (α_e, θ_e), five elements of Γ, six elements of Σ, and the arbitrageurs' risk-aversion coefficient a. Our estimation approach does not identify four of these moments: the three volatility parameters $(\{\Sigma_{\beta,j}\}_{j=H,F}, \Sigma_{\gamma_e})$ of the demand shocks, because they affect second moments only through their products with $(\{\theta_j(\tau)\}_{j=H,F}, \theta_e)$, and the risk-aversion coefficient a because it affects second moments only through its products with $(\{\alpha_{j0}(\tau), \theta_{j0}(\tau)\}_{j=H,F}, \alpha_e, \theta_e)$. The intuition in the case of a is that volatility of yields can be large if demand shocks are modest and arbitrageurs highly risk-averse, or if shocks are large and arbitrageur risk aversion is low. We bring in additional information later in this section to identify a. Finally, we impose $\alpha_{H1} = \theta_{H1} = \alpha_{F1} = \theta_{F1}$, an additional three restrictions. As noted above, when $\alpha_{j1} = \theta_{j1}$, the term structure in the absence of arbitrageurs is flat and bond demand shocks generate parallel shifts.

We estimate the 15 (=22-4-3) remaining parameters via Generalized Method of Moments, by targeting a large set of unconditional second moments of yields and exchange rates as well as bond turnover by maturity. We take the home country to be the United States and the foreign country to be the Eurozone, where we use data on German bunds for the foreign yield curve. We focus on these two countries mainly for data reasons: we require the availability of a long history of zero-coupon yield curve data and bond trading volume data by maturity. We use monthly yield data starting in 06/1986, for which long-term zero coupon yields are consistently available for US Treasuries and German bunds (for maturities up to 20 years). Our zero-coupon yield data for the US is from Gurkaynak, Sack, and Wright (2007), and the German yields are from the Bundesbank.[3] As in previous sections, the units of time t and maturity τ are years, so consecutive months are separated by a time equal to 1/12.

A first set of target moments concern the one-year yields. We include them to obtain information on the dynamics of the short rates. These moments are: the standard deviation of one-year yields $y_{jt}^{(1)}$ and of their annual change $\Delta y_{jt}^{(1)} \equiv y_{j,t+1}^{(1)} - y_{jt}^{(1)}$, and the correlation between the home and foreign annual change in one-year yields;

A second set of moments concern the exchange rate. We include them to obtain information on the dynamics of the currency demand factor γ_t. These moments are: the standard deviation of the annual (log) exchange rate change $\Delta \log e_t \equiv \log e_{t+1} - \log e_t$; the correlation between $\Delta \log e_t$ and the two-year change in the exchange rate $\Delta^2 \log e_t \equiv \log e_{t+2} - \log e_t$; and the correlation between the change in one-year yields $\Delta y_{Ht}^{(1)} - \Delta y_{Ft}^{(1)}$ and $\Delta \log e_t$.

A third set of moments concern yields across all maturities up to twenty years. We include them to obtain information on the dynamics of the demand factors (β_{Ht}, β_{Ft}) and how movements in the demand factors are transmitted to yields. We include the standard deviation of yields $y_{jt}^{(\tau)}$ and of their annual change $\Delta y_{jt}^{(\tau)} \equiv y_{j,t+1}^{(\tau)} - y_{jt}^{(\tau)}$; and the correlation between the annual changes $\Delta y_{jt}^{(1)}$ in one-year yields and $\Delta y_{jt}^{(\tau)}$ in all other yields.

[3]https://www.bundesbank.de/en/statistics/money-and-capital-markets/interest-rates-and-yields/term-structure-of-interest-rates

A final set of moments concern trading volume. We include them to obtain information on the functions $\{(\alpha_j(\tau), \theta_j(\tau)\}_{j=H,F}$ that describe the demand of preferred-habitat investors. We include the trading volume for short-term bonds (with maturities between 0 and 3 years for the US), relative to the total US bond trading volume.

Overall, we have $9 + 6 \times \mathcal{N}_T$ target moments where $\mathcal{N}_T$ refers to the number of maturities. We observe maturities up to twenty years (in annual increments), so there are $\mathcal{N}_T = 20$ maturities and 129 (=$9 + 6 \times 20$) target moments. We refer to the 9 moments that do not depend on maturity as scalar. Appendix Section D describes in more detail our data sources and moment calculations.

Collecting the 15 parameters into a vector $\boldsymbol{\rho}$, we estimate the model by choosing $\hat{\boldsymbol{\rho}}$ to minimize the weighted sum of square residuals:

$$L(\boldsymbol{\rho}) = \sum_{n=1}^{N} w_n(\hat{m}_n - m_n(\boldsymbol{\rho}))^2, \tag{5.15}$$

where $\{\hat{m}_n\}_n$ represents the moments from the data, and $\{m_n(\boldsymbol{\rho})\}_n$ the model-implied counterparts as a function of the calibration parameters. The terms w_n represent the weights placed on each target moment. We set the weight to one for scalar moments, and to $1/\mathcal{N}_T$ for moments that are a function of maturity.

5.3 Model Fit

5.3.1 Target Moments and Estimated Parameters

Table 1 compares the nine scalar moments in the data and in the model. Figure 1 does the same for the six moments that depend on maturity. In the figure, the red circles are the moments in the data and the blue solid lines are the model-implied counterparts. Both the table and the figure report moments in terms of standard deviations ($\sigma(x) = \sqrt{\mathbb{V}\text{ar}(x)}$) and correlations ($\rho(x,y) = \frac{\mathbb{C}\text{ov}(x,y)}{\sqrt{\mathbb{V}\text{ar}(x)\mathbb{V}\text{ar}(y)}}$), instead of the targeted variances and covariances. The model does reasonably well in fitting the large set of moments, both across maturities and across countries.

Table 2 reports the estimated parameters. The estimated bond demand slope and intercept coefficients α_{j0} and θ_{j0} are relatively similar in the US and the Eurozone. The innovations to the short rates are positively correlated in the US and the Eurozone ($\Sigma_{i_H,i_F} = 0.404$). The currency demand elasticity ($\alpha_e = 0.082$) is quite comparable to the elasticity of US bond demand to yields,

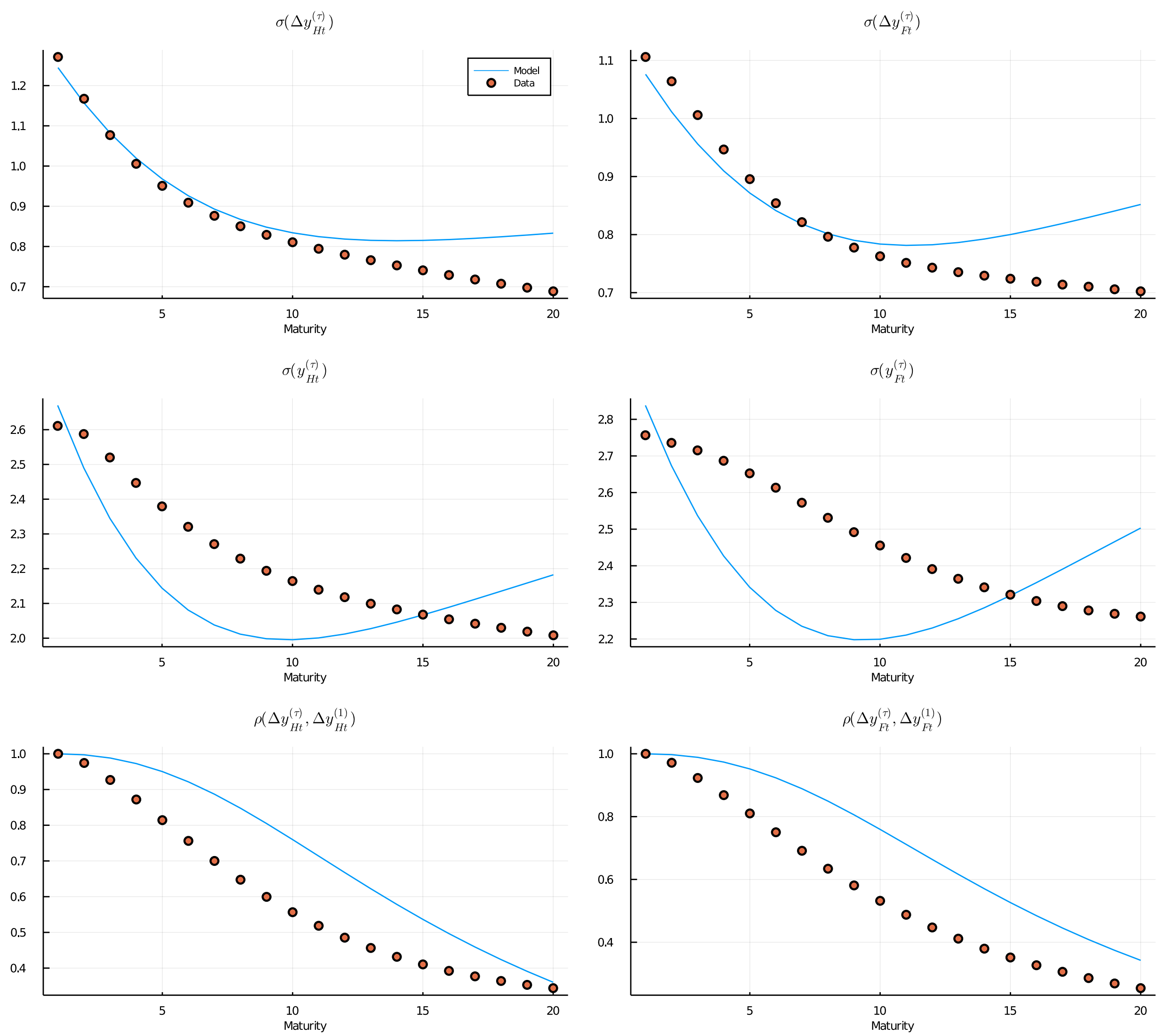

Figure 1: Maturity-Dependent Moments in the Data and the Model

Note: The red circles represent the moments in the data, and the solid lines represent their model-implied counterparts. Bond maturity is on the x-axis.

Target	Data	Model
$\sigma(y_{Ht}^{(1)})$	2.611	2.67
$\sigma(y_{Ft}^{(1)})$	2.757	2.838
$\sigma(\Delta y_{Ht}^{(1)})$	1.271	1.245
$\sigma(\Delta y_{Ft}^{(1)})$	1.106	1.076
$\rho(\Delta y_{Ht}^{(1)}, \Delta y_{Ft}^{(1)})$	0.345	0.335
$\sigma(\Delta \log e_t)$	10.27	10.27
$\rho(\Delta \log e_t, \Delta^2 \log e_t)$	0.656	0.615
$\rho(\Delta(y_{Ht}^{(1)} - y_{Ft}^{(1)}), \Delta \log e_t)$	-0.114	-0.126
$Vol_H(\tau \leq 3)$	0.361	0.427

Table 1: Scalar Moments in the Data and the Model

Notes: The table reports the scalar moments targeted in the estimation and their estimated counterpart. The table reports the standard deviation $\sigma(x) = \sqrt{\mathbb{V}\text{ar}(x)}$ and the correlation $\rho(x, y) = \frac{\mathbb{C}\text{ov}(x,y)}{\sqrt{\mathbb{V}\text{ar}(x)\mathbb{V}\text{ar}(y)}}$ instead of the targeted variance $\mathbb{V}\text{ar}(x)$ and covariance $\mathbb{C}\text{ov}(x, y)$.

Table 2: Estimation Results

Parameter	Value
$a\alpha_{H0}$	0.013
$a\Sigma_{\beta_H}\theta_{H0}$	0.356
$a\alpha_{F0}$	0.024
$a\Sigma_{\beta_F}\theta_{F0}$	0.419
$a\alpha_e$	0.082
$a\Sigma_\gamma\theta_e$	1.537
$a\delta$	0.343
Γ_{i_H}	0.115
Γ_{i_F}	0.075
Γ_{β_H}	0.074
Γ_{β_F}	0.056
Γ_{β_e}	0.672
Σ_{i_H}	1.386
Σ_{i_F}	1.101
Σ_{i_H,i_F}	0.404

Note: The table reports the GMM estimates of the model according to (5.15). We set $\delta \equiv \alpha_{1j} = \theta_{1j}$.

averaged across maturities ($\int_0^\infty \tau\alpha_H(\tau)d\tau = \frac{\alpha_{H0}}{\delta^2} = 0.110$). Finally, the estimates indicate that the currency factor is less persistent ($\Gamma_{\gamma_e} = 0.672$) than the short rates and the demand factors.

5.3.2 Return Predictability Regressions

We next examine the implications of our estimated model for the predictability of bond and currency returns. We do so by computing common regressions run in the asset pricing literature, and comparing the empirical coefficients in our US/Eurozone sample to the coefficients implied by our model. The regression coefficients are not targeted moments in our estimation. Hence, comparing the empirical coefficients to the model-implied ones is akin to an "out-of-sample" exercise. The calculations of the model-implied regression coefficients are in Appendix C.

Figure 2 reports empirical and model-implied coefficients for the Fama and Bliss (1987, FB) (top row) and Campbell and Shiller (1991, CS) (bottom row) regressions for the US (left column) and the Eurozone (right column). Under the Expectation Hypothesis, the FB coefficient should be zero and the CS coefficient should be one, as indicated by the dashed lines. The empirical coefficients, indicated by the red circles and the two-standard-error confidence intervals around them, are consistent with the findings of FB and CS. The EH is rejected and the deviations from EH are increasing with maturity.

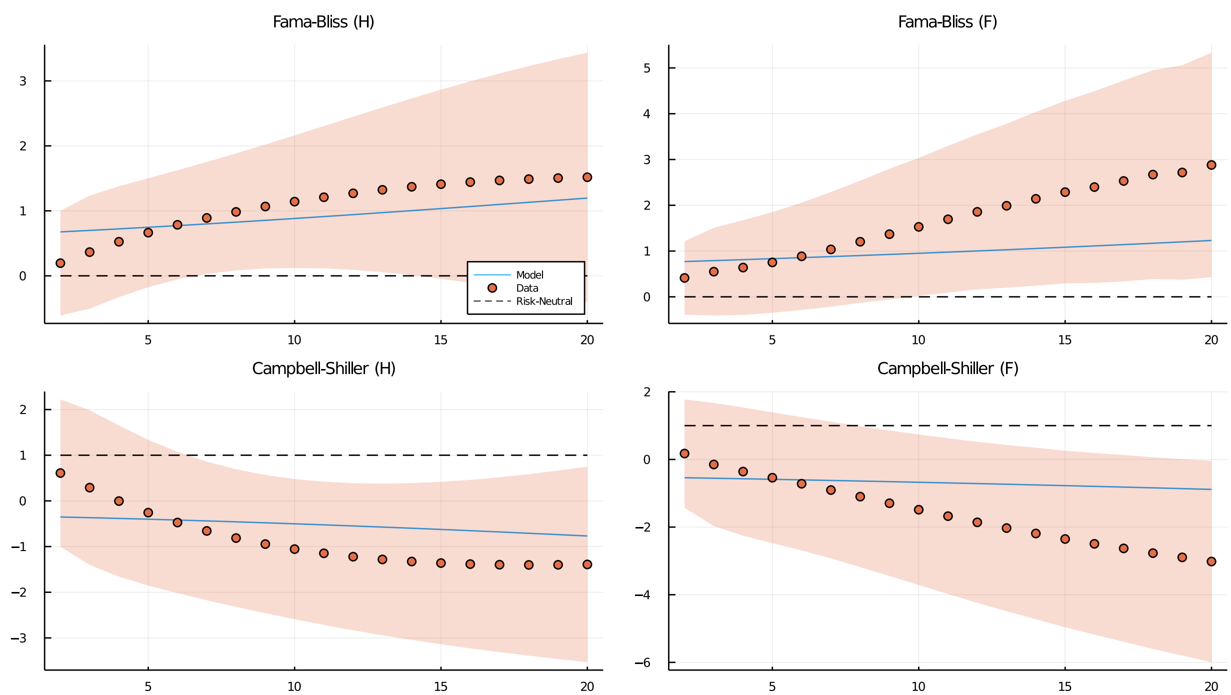

Figure 2: Term Structure Regression Coefficients

The model-implied coefficients in Figure 2 are indicated by the blue lines. The model broadly reproduces the empirical patterns. The FB coefficients are positive, increasing in maturity, and near or above one for long maturities. The CS coefficients are below one, decreasing in maturity, and negative for long maturities. The main discrepancy between the model and the data is that the deviations from the EH are not large enough for long maturities.

The model generates a positive FB coefficient through two mechanisms working in the same direction. The first mechanism is the under-reaction (relative to the EH) of bond yields to short-rate shocks. This mechanism is introduced by short-rate risk and is explained in Sections 3 and 4. The second mechanism is the over-reaction (relative to the EH, under which there is no reaction) of yields to demand shocks. This mechanism is introduced by demand risk. Suppose that demand by preferred-habitat investors in country j drops. Bond prices in that country then drop so that arbitrageurs are induced to buy the bonds. As a consequence, the term structure is steeply upward sloping and bonds offer large expected returns, generating a positive FB coefficient.

In addition to raising the FB coefficient, demand risk renders it increasing with maturity. Indeed, since bonds of longer maturities are riskier, their expected returns are impacted more heavily by demand shocks. The slope of the term structure is also impacted more heavily by demand shocks when it is calculated based on longer maturities, but the effect is not increasing as rapidly with maturity as with expected returns. This is because the effect on yields incorporates how demand shocks affect future expected returns, and that effect is weaker than for current expected returns because demand shocks mean-revert.

Figure 3 reports empirical and model-implied coefficients for various types of UIP regressions. The top left panel concerns the hybrid UIP regression of Lustig, Stathopoulos, and Verdelhan (2019, LSV), in which the return of the hybrid CCT constructed using bonds with maturity τ is regressed on the foreign-minus-home short-rate differential. This regression nests as a special case, for small τ, the standard UIP regression of Bilson (1981) and Fama (1984). Under the UIP, the LSV coefficient should be zero. The empirical coefficients are positive and statistically significant for short maturities, consistent with Bilson (1981) and Fama (1984). They decline with maturity and become statistically insignificant for long maturities. This is consistent with LSV, although LSV's coefficients, computed over multiple currency pairs rather than over only dollar/euro, are closer to zero. The model-implied coefficients are positive and decline with maturity.

The top right panel in Figure 3 concerns the long-horizon UIP regression of Chinn and Meredith (2004, CM), in which the realized rate of foreign currency depreciation over horizon τ is regressed on the foreign-minus-home τ-year yield differential. Under the UIP, the CM coefficient should be one. The empirical coefficient is not statistically different from zero at short maturities, although confidence intervals are large because we use only one currency pair. As horizon increases, the regression coefficient converges to one, consistent with CM (and UIP). While the model-implied coefficients converge to one as maturity increases, they do so slowly. The model thus generates significant departures from UIP even at relatively long horizons.

The bottom two graphs concern regressions run in Chernov and Creal (2020) and Lloyd and Marin (2020), whereby the realized rate of foreign currency depreciation over horizon τ is regressed on the foreign-minus-home τ-year yield differential (level – same regressor as in CM), and the foreign-minus-home slope differential (slope). Under UIP, the level coefficient should be one and the slope coefficient should be zero. As with the CM regression, the coefficients using only one currency pair are imprecisely estimated, but the point estimates are consistent with the literature. In particular, the slope coefficient is positive, meaning that for a given yield differential, the CCT is less profitable when the foreign-minus-home slope differential is larger.[4]

The model generates a positive slope coefficient, although smaller than its empirical counterpart. The intuition why the coefficient is positive is as follows. Suppose that the demand for foreign bonds by preferred-habitat investors is low. This pushes up foreign bond yields, raising the foreign-minus-home slope differential and causing the foreign currency to appreciate (Proposition 4.4). Hence, the future expected return on foreign currency declines. As found in the data, this predictability of slope is primarily only over short and medium maturities. For long maturities, the effects go away and UIP holds.

Overall, the model generates sizeable deviations from UIP, although the fit is not as good as for the bond predictability regressions. Nevertheless, the model replicates the key patterns shown in the absence of demand risk in Sections 3 and 4: UIP violations; LSV coefficient that declines with maturity; CM coefficient that rises to one as maturity increases.

5.4 Monetary Policy

We next explore the implications of our estimated model for the domestic and international transmission of monetary policy. We start with conventional monetary policy, and consider a cut to the short rate by the central bank. We assume that the cut is unanticipated and occurs at time zero. We set the size of the cut to 25 basis points (bps).

[4] With only one currency pair, the Lloyd and Marin (2020) regression results are never strongly significant, except at very long horizons where one may be concerned about the strong serial correlation due to overlapping observations. Our standard-errors are Newey-West corrected but with few genuine non-overlapping observations, they may still be artificially low.

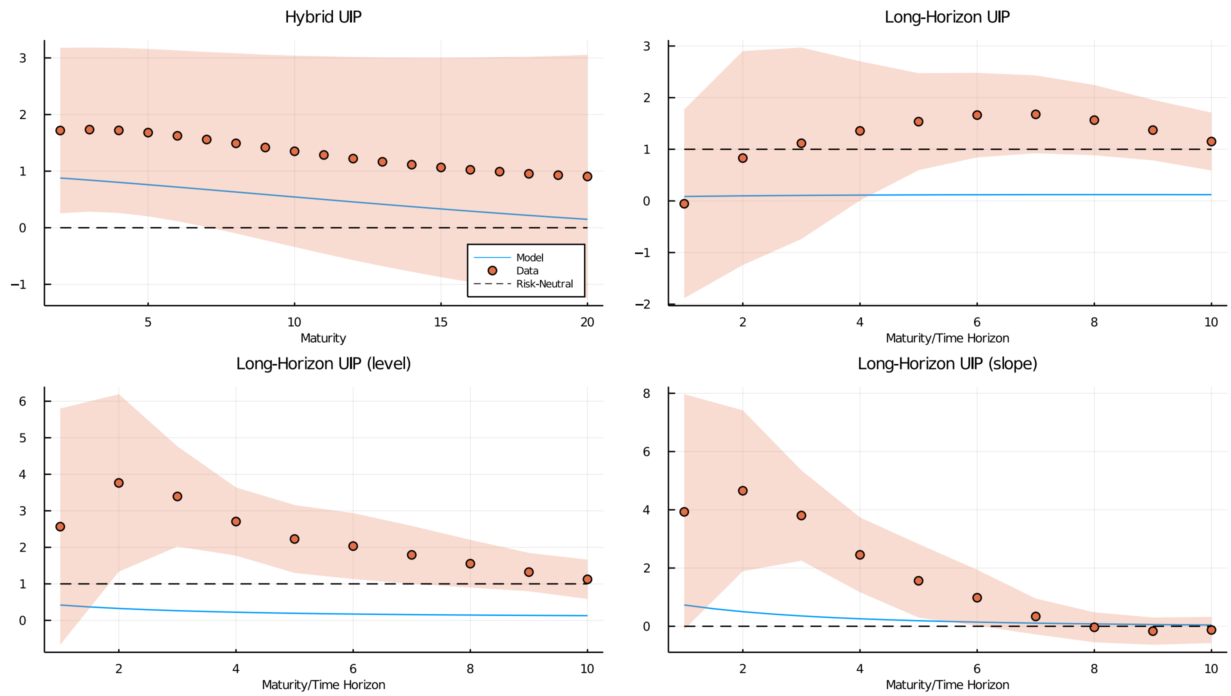

Figure 3: Generalized UIP Regression Coefficients

Figure 4 shows how a short-rate cut in the US (top row) or the Eurozone (bottom row) affects the term structures in the two countries at the time of impact (left column) and the exchange rate over time (right column). In the left column, the US term structure response is shown in blue, while the Eurozone response is shown in red.

The short-rate cut affects the term structure in the country where it originates, but has essentially no effect on the other country's term structure. Its effect on the exchange rate is modest and significantly smaller than under UIP. The dollar depreciates by 18bps (0.18%) following a US rate cut, and appreciates by 18bps following an ECB rate cut. Under UIP, by contrast, the dollar would depreciate by $25/\Gamma_{iH}$=217bps (2.17%) following a US rate cut, and appreciate by $25/\Gamma_{iF}$=333bps following an ECB rate cut. Our estimated model generates a small effect of the short-rate cut on the exchange rate because it attributes about half of exchange-rate volatility to the currency demand factor. Currency demand risk dissuades arbitrageurs from taking large positions in currency in response to short-rate changes.

We next turn to non-conventional monetary policy, and consider large-scale purchases of bonds by each of the central banks. We assume that the purchases are unanticipated, occur at time zero, and are unwound over time. We describe the net amount purchased by the central bank (purchases at time zero minus subsequent unwinding) by the same exponential specification as the demand

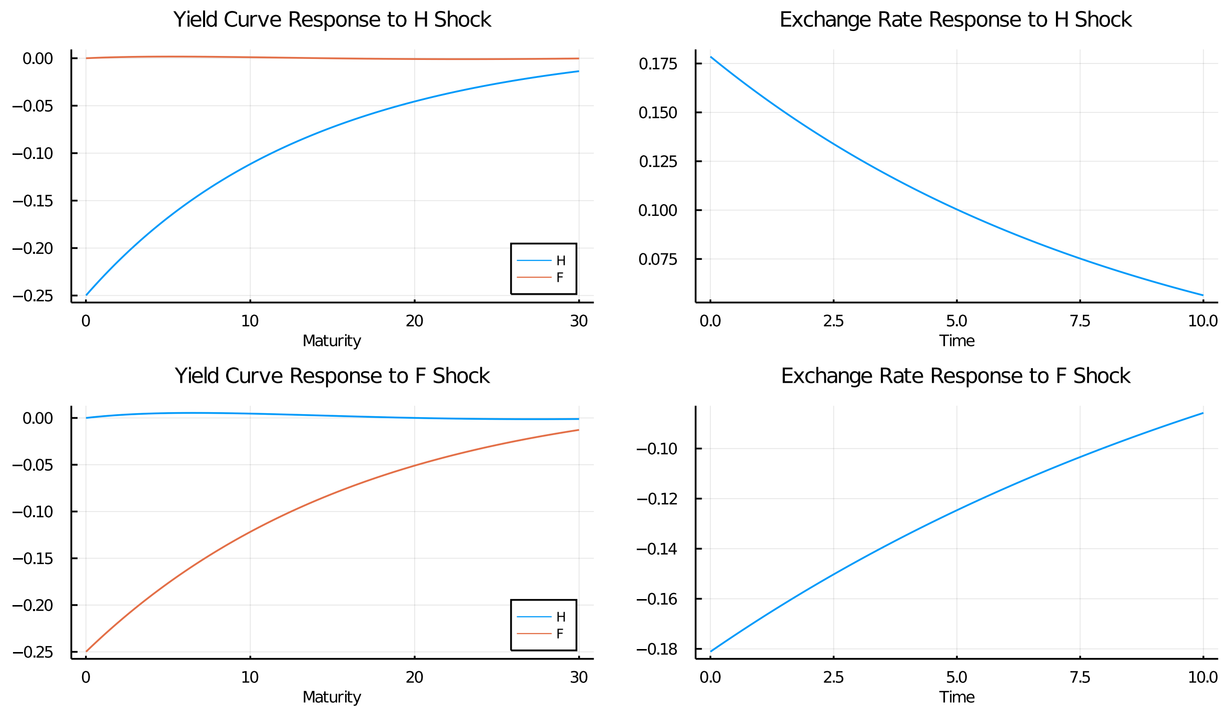

Figure 4: Conventional Monetary Policy – Short Rate Shock

intercept:

$$\theta_{jt}^{QE}(\tau) \equiv \theta_{j0}^{QE}\tau \exp\left(-\theta_{j1}^{QE}\tau\right) \exp\left(-\kappa_j^{QE} t\right).$$

The parameter θ_{j0}^{QE} characterizes the size of the purchases. The parameter θ_{j1}^{QE} characterizes the breakdown of purchases across maturities. We assume that both parameters are the same in the two countries to render the results more comparable. For the same reason, we assume that the parameter κ_j^{QE}, which describes the rate at which purchases are unwound, is the same across countries.

Omitting the subscript j, we set $(\theta_1^{QE}, \kappa^{QE})$ to (0.2,0.2). Thus, purchases in the cross-section are maximized at the five-year maturity ($1/\theta_1^{QE} = 5$), and their half life in the time-series is 3.47 years ($\log(2)/\kappa^{QE} = 3.47$). We pick values for θ_0^{QE} based on the size of time-zero purchases aggregated across maturities, which is

$$\int_0^\infty \theta_0^{QE}\tau \exp\left(-\theta_1^{QE}\tau\right) d\tau = \frac{\theta_0^{QE}}{\left(\theta_1^{QE}\right)^2}.$$

We assume that purchases represent 10% of US GDP. Using US GDP as the numeraire, we thus set

$$\theta^{QE}_{US,0} = 10\% \times 0.2^2 = 0.004.$$

To determine the effects of QE, we need to calibrate one remaining parameter, the arbitrageurs' risk-aversion coefficient a. The effects of a on moments of yields and exchange rates cannot be identified separately from the effects of the size of demand shocks (which is why our estimation determines a only up to its products with demand slope and intercept). To determine, however, the effects of a demand shock of a given size, we need a value for a. Since a corresponds to a coefficient of absolute risk aversion, it is equal to γ/W, where γ is the arbitrageurs' coefficient of relative risk aversion and W is their wealth. We set $\gamma = 2$, in line with common estimates. An estimate for W can be derived by identifying arbitrageurs with hedge funds. The assets of hedge funds in the fixed-income, macro and balanced categories in 2020 were about 5% of US GDP in that year.[5] We take 5% as a lower bound for W since arbitrageurs can include additional agents such as global banks and multinational corporations, and use 20% as an upper bound. The implied bounds for a are $2/5\% = 40$ and $2/20\% = 10$.

Figure 5 shows how QE purchases in the US (top row) or the Eurozone (bottom row) affect the term structures in both countries at the time of impact (left column) and the exchange rate over time (right column) when $a = 40$. In the left panels, the US term structure response is shown in blue, while the Eurozone response is shown in red. When $a = 10$, the effects are one-quarter of those in Figure 5.

QE purchases have pronounced effects on the term structure in the country where they originate. They reduce the ten-year yield by 50-60bps, and have even larger effects for longer maturities. These magnitudes are comparable to estimates in the literature: according to a summary of these estimates in Wiliams (2014), QE purchases of 10% of GDP reduce the ten-year yield by 35-65 basis points. Figure 5 indicates that QE has sizeable effects on the term structure of the other country as well: the effect on the ten-year yield is 30-40% of that in the country where QE originates, with the percentage rising for longer maturities. Our analysis reveals that conventional and non-conventional policy differ sharply in their international spillovers: non-existent for the former, and sizeable for the latter.

The effects of QE on the exchange rate are somewhat larger than those of conventional policy.

[5] https://www.barclayhedge.com/solutions/assets-under-management/hedge-fund-assets-under-management/

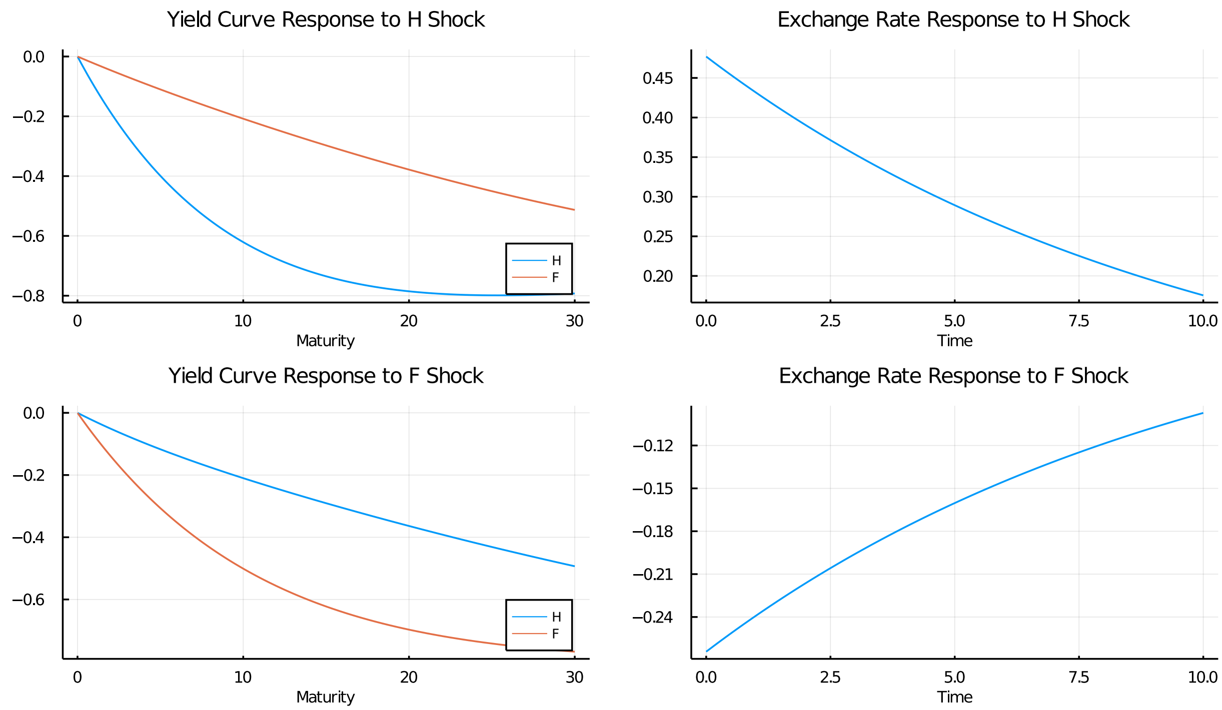

Figure 5: QE Shock Spillovers

QE in the US causes the Euro to appreciate by about 0.5%, while QE in the Eurozone causes the Euro to depreciate by about 0.25%.

6 Conclusion

We propose an integrated preferred-habitat model of bond and currency markets across two countries. Prices are determined by arbitrageurs trading with investors with preferences for specific assets. Risk premia vary over time in response to shocks to short rates and to bond and currency demand. This variation generates empirically documented violations of Expectations Hypothesis and Uncovered Interest Parity. Large-scale asset purchases in one country cause that country's currency to depreciate, bond yields in that country to drop, and yields in the other country to drop by a smaller amount. A short-rate cut in one country has the same qualitative effects, although our estimated model reveals that the spillovers to the other country's term structure are significantly smaller.

Appendix

A Proofs

Proof of Proposition 3.1: Equation (3.10) follows by identifying the linear terms in (i_{Ht}, i_{Ft}) in (3.9). Equation (3.11) follows by identifying the constant terms.

To show that the system of (3.10) and (3.11) has a unique solution for $(\{A_{ije}\}_{j=H,F}, C_e)$, we start with the system of two equations in $\{A_{ije}\}_{j=H,F}$ obtained by writing (3.10) for $j = H$ and $j = F$. A solution to the latter system must be positive, as can be seen by writing (3.10) as

$$\left[\kappa_{ij} + a_e \alpha_e \left(\sigma_{iH}^2 A_{iHe}^2 + \sigma_{iF}^2 A_{iFe}^2\right)\right] A_{ije} = 1. \tag{A.1}$$

Since $A_{ije} > 0$, the right-hand side of (3.10) is negative. Therefore, the left-hand side is negative as well, which implies $A_{ije} < \frac{1}{\kappa_{ij}}$. Dividing (3.10) written for $j = H$ by (3.10) written for $j = F$, we find

$$\frac{1 - \kappa_{iH} A_{iHe}}{1 - \kappa_{iF} A_{iFe}} = \frac{A_{iHe}}{A_{iFe}} \Leftrightarrow A_{iHe} = \frac{A_{iFe}}{1 + (\kappa_{iH} - \kappa_{iF}) A_{iFe}}. \tag{A.2}$$

Equation (A.2) determines A_{iHe} as an increasing function of $A_{iFe} \in \left[0, \frac{1}{\kappa_{iF}}\right]$, equal to zero for $A_{iFe} = 0$, and equal to $\frac{1}{\kappa_{iH}}$ for $A_{iFe} = \frac{1}{\kappa_{iF}}$. Substituting A_{iHe} as a function of A_{iFe} in (A.1) written for $j = F$, we find an equation in the single unknown A_{iFe}. The left-hand side of that equation is increasing in A_{iFe}, is equal to zero for $A_{iFe} = 0$, and is equal to a value larger than one for $A_{iFe} = \frac{1}{\kappa_{iF}}$. Hence, that equation has a unique solution A_{iFe}. Given that solution, (A.2) determines A_{iHe} uniquely, and (3.11) determines C_e uniquely. ∎

Proof of Corollary 3.1: When $a_e = 0$, (3.10) implies $A_{ije} = \frac{1}{\kappa_{ij}}$. Substituting into (3.11), we find (3.12). Substituting into (3.8), we find $\mu_{et} = i_{Ht} - i_{Ft}$.

When a_e goes to zero, (3.10) implies that A_{ije} converges to $\frac{1}{\kappa_{ij}}$. When, in addition, (3.12) does not hold, (3.11) implies that C_e converges to plus or minus infinity at the rate $\frac{1}{a_e}$, and (3.8) implies that μ_{et} does not converge to $i_{Ht} - i_{Ft}$. ∎

Proof of Proposition 3.2: Substituting μ_{Ht} and μ_{Ft} from (3.14) and (3.16), respectively, into (3.19), we find

$$\begin{aligned} &A'_{ij}(\tau)i_{jt} + C'_j(\tau) - A_{ij}(\tau)\kappa_{ij}(\bar{i}_j - i_{jt}) + \frac{1}{2}A_{ij}(\tau)\left(A_{ij}(\tau) - 2A_{iFe}1_{\{j=F\}}\right)\sigma_{ij}^2 - i_{jt} \\ &= a_j A_{ij}(\tau)\left(\int_0^T \left[\zeta_j(\tau) - \alpha_j(\tau)\left(A_{ij}(\tau)i_{jt} + C_j(\tau)\right)\right]A_{ij}(\tau)d\tau\right)\sigma_{ij}^2. \end{aligned} \tag{A.3}$$

Equation (3.20) follows by identifying the linear terms in i_{jt} in (A.3). Equation (3.21) follows by identifying the constant terms. The initial conditions $A_{ij}(0) = C_j(0) = 0$ follow because the price of a bond with zero maturity is its face value, which is one.

Solving (3.20) with the initial condition $A_{ij}(0) = 0$, we find

$$A_{ij}(\tau) = \frac{1 - e^{-\kappa_{ij}^*}}{\kappa_{ij}^*}, \tag{A.4}$$

with

$$\kappa_{ij}^* \equiv \kappa_{ij} + a_j\sigma_{ij}^2 \int_0^T \alpha_j(\tau)A_{ij}(\tau)^2 d\tau. \tag{A.5}$$

Substituting $A_{ij}(\tau)$ from (A.4) into (A.5), we find the equation

$$\kappa_{ij}^* - \kappa_{ij} + a_j\sigma_{ij}^2 \int_0^T \alpha_j(\tau)\left(\frac{1 - e^{-\kappa_{ij}^*}}{\kappa_{ij}^*}\right)^2 d\tau = 0 \tag{A.6}$$

in the single unknown κ_{ij}^*. The left-hand side of (A.6) is increasing in κ_{ij}^*, is negative for $\kappa_{ij}^* = \kappa_{ij}$, and goes to infinity when κ_{ij}^* goes to infinity. Hence, (A.6) has a unique solution $\kappa_{ij}^* > \kappa_{ij}$. Given κ_{ij}^*, (A.4) determines $A_{ij}(\tau)$ uniquely.

Solving (3.21) with the initial condition $C(\tau) = 0$, we find

$$C_j(\tau) = \kappa_{ij}^*\bar{i}_j^* \int_0^\tau A_{ij}(\tau)d\tau - \frac{1}{2}\sigma_{ij}^2 \int_0^\tau A_{ij}(\tau)^2 d\tau, \tag{A.7}$$

with

$$\kappa_{ij}^*\bar{i}_j^* \equiv \kappa_{ij}\bar{i}_j + a_j\sigma_{ij}^2 \int_0^T \left[\zeta_j(\tau) - \alpha_j(\tau)C_j(\tau)\right]A_{ij}(\tau)d\tau + \sigma_{ij}^2 A_{iFe}1_{\{j=F\}}. \tag{A.8}$$

Substituting $C_j(\tau)$ from (A.7) into (A.8), we find

$$\bar{i}_j^* = \frac{\kappa_{ij}\bar{i}_j + a_j\sigma_{ij}^2 \int_0^T \zeta_j(\tau)A_{ij}(\tau)d\tau + \sigma_{ij}^2 A_{iFe}1_{\{j=F\}} + \frac{1}{2}a_j\sigma_{ij}^4 \int_0^T \alpha_j(\tau)\left(\int_0^\tau A_{ij}(\tau')^2 d\tau'\right)A_{ij}(\tau)d\tau}{\kappa_{ij}^*\left[1 + a_j\sigma_{ij}^2\int_0^T \alpha_j(\tau)\left(\int_0^\tau A_{ij}(\tau')d\tau'\right)A_{ij}(\tau)d\tau\right]} \tag{A.9}$$

Given $\bar{i}_j^*$, (A.7) determines $C_j(\tau)$ uniquely. ∎

Proof of Corollary 3.2: When $a_j = 0$, (3.20) with the initial condition $A_{ij}(0) = 0$ implies $A_{ij}(\tau) = \frac{1-e^{-\kappa_{ij}\tau}}{\kappa_{ij}}$. Substituting into (3.19), we find $\mu_{jt}^{(\tau)} = i_{jt}$. The same results hold when $a_j \to 0$. ∎

Proof of Proposition 3.3: Equations (A.4) and $\kappa_{ij}^* > \kappa_{ij}$ imply $A_{ij}(\tau) < \frac{1-e^{-\kappa_{ij}\tau}}{\kappa_{ij}}$. Differentiating (3.19) with respect to i_{jt} implies

$$\frac{\partial\left(\mu_{jt}^{(\tau)} - i_{jt}\right)}{\partial i_{jt}} = -a_j\sigma_{ij}^2 A_{ij}(\tau)\int_0^T \alpha_j(\tau)A_{ij}(\tau)^2 d\tau < 0,$$

where the second step follows because (A.4) implies $A_{ij}(\tau) > 0$. ∎

Proof of Proposition 3.4: The property $A_{ije} < \frac{1}{\kappa_{ij}}$ is shown in the proof of Proposition 3.1. Differentiating (3.8) with respect to i_{Ht} and i_{Ft}, we find

$$\frac{\partial(\mu_{et} + i_{Ft} - i_{Ht})}{\partial i_{Ht}} = -a_e\alpha_e A_{iHe}\left(\sigma_{iH}^2 A_{iHe}^2 + \sigma_{iF}^2 A_{iFe}^2\right) < 0,$$

$$\frac{\partial(\mu_{et} + i_{Ft} - i_{Ht})}{\partial i_{Ft}} = a_e\alpha_e A_{iFe}\left(\sigma_{iH}^2 A_{iHe}^2 + \sigma_{iF}^2 A_{iFe}^2\right) > 0.$$

where the second step in each case follows because $A_{ije} > 0$. ∎

Proof of Proposition 3.5: Consider an one-off increase in β_{jt} at time zero, and denote by $\kappa_{\beta j}$ the rate at which β_{jt} reverts to its mean of zero. Bond prices in country j at time t are

$$P_{jt}^{(\tau)} = e^{-\left[A_{ij}(\tau)i_{jt} + A_{\beta j}(\tau)\beta_{jt} + C_j(\tau)\right]}, \tag{A.10}$$

where $(A_{ij}(\tau), A_{\beta j}(\tau), C_j(\tau))$ are functions of τ. The counterpart of (A.3) is

$$\begin{aligned}
&A'_{ij}(\tau)i_{jt} + A'_{\beta j}(\tau)\beta_{jt} + C'_j(\tau) - A_{ij}(\tau)\kappa_{ij}(\bar{i}_j - i_{jt}) + A_{\beta j}(\tau)\kappa_{\beta j}\beta_{jt} \\
&+ \frac{1}{2}A_{ij}(\tau)\left(A_{ij}(\tau) - 2A_{iFe}1_{\{j=F\}}\right)\sigma_{ij}^2 - i_{jt} \\
&= a_j A_{ij}(\tau)\left(\int_0^T \left[\zeta_j(\tau) + \theta_j(\tau)\beta_{jt} - \alpha_j(\tau)\left(A_{ij}(\tau)i_{jt} + A_{\beta j}(\tau)\beta_{jt} + C_j(\tau)\right)\right] A_{ij}(\tau)d\tau\right)\sigma_{ij}^2.
\end{aligned} \tag{A.11}$$

Identifying terms in r_t and constant terms, we find (3.20) and (3.21), respectively. Identifying terms in β_{jt}, we find

$$A'_{\beta j}(\tau) + \kappa_{\beta j}A_{\beta j}(\tau) = a_j\sigma_{ij}^2 A_{ij}(\tau)\int_0^T \left[\theta_j(\tau) - \alpha_j(\tau)A_{\beta j}(\tau)\right] A_{ij}(\tau)d\tau. \tag{A.12}$$

Solving (A.12) with the initial condition $A_{\beta j}(\tau) = 0$, we find

$$A_{\beta j}(\tau) = \lambda_{\beta j}\int_0^\tau A_{ij}(\tau')e^{-\kappa_{\beta j}(\tau-\tau')}d\tau', \tag{A.13}$$

with

$$\lambda_{\beta j} \equiv a_j\sigma_{ij}^2\int_0^T \left[\theta_j(\tau) - \alpha_j(\tau)A_{\beta j}(\tau)\right] A_{ij}(\tau)d\tau. \tag{A.14}$$

Substituting $A_{\beta j}(\tau)$ from (A.13) into (A.14), we find

$$\lambda_{\beta j} = \frac{a_j\sigma_{ij}^2\int_0^T \theta_j(\tau)A_{ij}(\tau)d\tau}{1 + a_j\sigma_{ij}^2\int_0^T \alpha_j(\tau)\left(\int_0^\tau A_{ij}(\tau')e^{-\kappa_{\beta j}(\tau-\tau')}d\tau'\right)A_{ij}(\tau)d\tau}. \tag{A.15}$$

Since $(\theta_j(\tau), A_{ij}(\tau))$ are positive, so is $\lambda_{\beta j}$ and $A_{\beta j}(\tau)$. Hence, (A.15) implies that an increase in β_{jt} raises bond yields in country j. Since the foreign currency and bonds in country j' are traded by different agents than those trading bonds in country j, their prices do not depend on β_{jt}.

Consider next an one-off increase in γ_t at time zero, and denote by κ_γ the rate at which γ_t reverts to its mean of zero. The exchange rate at time t is

$$e_t = e^{-\left[A_{iHe}i_{Ht} - A_{iFe}i_{Ft} + A_{\gamma e}\gamma_t + C_e + \frac{\psi_e}{\alpha_e}t\right]}, \tag{A.16}$$

where $(\{A_{ije}\}_{j=H,F}, A_{\gamma e}, C_e)$ are scalars. The counterpart of (3.9) is

$$- A_{iHe}\kappa_{iH}(\bar{i}_H - i_{Ht}) + A_{iFe}\kappa_{iF}(\bar{i}_F - i_{Ft}) + A_{\gamma e}\kappa_\gamma\gamma_t - \frac{\psi_e}{\alpha_e} + \frac{1}{2}A_{iHe}^2\sigma_{iH}^2 + \frac{1}{2}A_{iFe}^2\sigma_{iF}^2 + i_{Ft} - i_{Ht}$$

$$= a_e\left[\zeta_e + \theta_e\gamma_t + \psi_e t - \alpha_e\left(A_{iHe}i_{Ht} - A_{iFe}i_{Ft} + A_{\gamma e}\gamma_t + C_e + \frac{\psi_e}{\alpha_e}t\right)\right]\left(A_{iHe}^2\sigma_{iH}^2 + A_{iFe}^2\sigma_{iF}^2\right). \tag{A.17}$$

Identifying terms in (i_{Ht}, i_{Ft}) and constant terms, we find (3.10) and (3.11), respectively. Identifying terms in γ_t, we find

$$\kappa_\gamma A_{\gamma e} = a_e(\theta_e - \alpha_e A_{\gamma e})\left(A_{iHe}^2\sigma_{iH}^2 + A_{iFe}^2\sigma_{iF}^2\right)$$

$$\Rightarrow A_{\gamma e} = \frac{a_e\theta_e\left(A_{iHe}^2\sigma_{iH}^2 + A_{iFe}^2\sigma_{iF}^2\right)}{\kappa_\gamma + a_e\alpha_e\left(A_{iHe}^2\sigma_{iH}^2 + A_{iFe}^2\sigma_{iF}^2\right)}. \tag{A.18}$$

Since θ_e is positive, so is $A_{\gamma e}$. Hence, (A.18) implies that an increase in γ_t causes the foreign currency to depreciate. Since bonds in each country are traded by a separate set of agents than those trading foreign currency, their prices do not depend on γ_t. ■

Proof of Proposition 4.1: Applying Ito's Lemma to (4.1) for $j = H$, we find the following counterpart of (3.13):

$$\frac{dP_{Ht}^{(\tau)}}{P_{Ht}^{(\tau)}} = \mu_{Ht}^{(\tau)}dt - A_{iHH}(\tau)\sigma_{iH}dB_{iHt} - A_{iHF}(\tau)\sigma_{iF}dB_{iFt}, \tag{A.19}$$

where

$$\mu_{Ht}^{(\tau)} \equiv A'_{iHH}(\tau)i_{Ht} + A'_{iHF}(\tau)i_{Ft} + C'_H(\tau) - A_{iHH}(\tau)\kappa_{iH}(\bar{i}_H - i_{Ht}) - A_{iHF}(\tau)\kappa_{iF}(\bar{i}_F - i_{Ft})$$
$$+ \frac{1}{2}A_{iHH}(\tau)^2\sigma_{iH}^2 + \frac{1}{2}A_{iHF}(\tau)^2\sigma_{iF}^2. \tag{A.20}$$

Likewise, (4.1) for $j = F$ and (3.2) yield the following counterpart of (3.15):

$$\frac{d(P_{Ft}^{(\tau)}e_t)}{P_{Ft}^{(\tau)}e_t} - \frac{de_t}{e_t} = \mu_{Ft}^{(\tau)}dt - A_{iFH}(\tau)\sigma_{iH}dB_{iHt} - A_{iFF}(\tau)\sigma_{iF}dB_{iFt}, \tag{A.21}$$

where

$$\mu_{Ft}^{(\tau)} \equiv A'_{iFH}(\tau)i_{Ht} + A'_{iFF}(\tau)i_{Ft} + C'_F(\tau) - A_{iFH}(\tau)\kappa_{iH}(\bar{i}_H - i_{Ht}) - A_{iFF}(\tau)\kappa_{iF}(\bar{i}_F - i_{Ft})$$
$$+ \frac{1}{2}A_{iFH}(\tau)\left(A_{iFH}(\tau) + 2A_{iHe}\right)\sigma_{iH}^2 + \frac{1}{2}A_{iFF}(\tau)\left(A_{iFF}(\tau) - 2A_{iFe}\right)\sigma_{iF}^2. \tag{A.22}$$

Substituting the returns (3.4), (A.19) and (A.21) into the arbitrageurs' budget constraint (2.3), we can write their optimization problem (2.4) as

$$\max_{W_{Ft},\{X_{jt}^{(\tau)}\}_{\tau\in(0,T),j=H,F}} \left[W_{Ft}\left(\mu_{et}+i_{Ft}-i_{Ht}\right)+\sum_{j=H,F}\int_0^T X_{jt}^{(\tau)}\left(\mu_{jt}^{(\tau)}-i_{jt}\right)d\tau \right.$$

$$\left. -\frac{a}{2}\sum_{j=H,F}\left(W_{Ft}A_{ije}(-1)^{1_{\{j=F\}}}+\sum_{j'=H,F}\int_0^T X_{j't}^{(\tau)}A_{ij'j}(\tau)d\tau\right)^2\sigma_{ij}^2\right]. \tag{A.23}$$

The first-order condition with respect to W_{Ft} is (4.2), and the first-order condition with respect to $X_{jt}^{(\tau)}$ is (4.3).

Using (3.7) and (3.18), we can write λ_{ijt} as

$$\begin{aligned}
\lambda_{ijt} &= a\sigma_{ij}^2\left(-\sum_{j'=H,F}\int_0^T Z_{j't}^{(\tau)}A_{ij'j}(\tau)d\tau - Z_{et}A_{ije}(-1)^{1_{\{j=F\}}}\right)\\
&= a\sigma_{ij}^2\left(\sum_{j'=H,F}\int_0^T\left[\alpha_{j'}(\tau)\log\left(P_{j't}^{(\tau)}\right)+\zeta_{j'}(\tau)+\theta_{j'}(\tau)\beta_{j't}\right]A_{ij'j}(\tau)d\tau\right.\\
&\qquad\left.+\left[\alpha_e\log(e_t)+\zeta_e+\theta_e\gamma_t+\psi_e t\right]A_{ije}(-1)^{1_{\{j=F\}}}\right)\\
&= a\sigma_{ij}^2\left(\sum_{j'=H,F}\int_0^T\left[\zeta_{j'}(\tau)+\theta_{j'}(\tau)\beta_{j't}-\alpha_{j'}(\tau)\left(A_{ij'H}(\tau)i_{Ht}+A_{ij'F}(\tau)i_{Ft}+C_{j'}(\tau)\right)\right]A_{ij'j}(\tau)d\tau\right.\\
&\qquad\left.+\left[\zeta_e+\theta_e\gamma_t+\psi_e t-\alpha_e\left(A_{iHe}i_{Ht}-A_{iFe}i_{Ft}+C_e+\frac{\psi_e}{\alpha_e}t\right)\right]A_{ije}(-1)^{1_{\{j=F\}}}\right)\\
&= a\sigma_{ij}^2\left(\bar{\lambda}_{ijj}i_{jt}+\bar{\lambda}_{rj'j}i_{j't}+\bar{\lambda}_{ijC}\right),
\end{aligned} \tag{A.24}$$

where the second step follows from (2.5) and (2.7), the third step follows from (3.2) and (4.1), and the fourth step follows from $\beta_{Ht}=\beta_{Ft}=\gamma_t=0$ and the definitions of $(\bar{\lambda}_{ijj},\bar{\lambda}_{ijj'},\bar{\lambda}_{ijC})$ in the statement of the proposition. We next substitute $(\mu_{et},\{\mu_{jt}^{(\tau)},\lambda_{ijt}\}_{j=H,F})$ from (3.5), (A.20), (A.22) and (A.24) into the arbitrageurs' first-order condition. Substituting into (4.2) and identifying terms in (i_{Ht},i_{Ft}) and constant terms, we find (4.5) and (4.6), respectively. Substituting into (4.3) and identifying terms in i_{jt}, terms in $i_{j't}$ and constant terms, we find (4.7), (4.8) and (4.9), respectively.

∎

Proof of Corollary 4.1: When $a=0$, (4.5) implies $A_{ije}=\frac{1}{\kappa_{ij}}$, (4.7) with the initial condition $A_{ijj}(0)=0$ implies $A_{ijj}(\tau)=\frac{1-e^{-\kappa_{ij}\tau}}{\kappa_{ij}}$, and (4.8) with the initial condition $A_{ijj'}(0)=0$ implies $A_{ijj'}(\tau)=0$. Substituting into (4.6), we find (3.12). Substituting into (4.2), we find $\mu_{et}=i_{Ht}-i_{Ft}$, and substituting into (4.3), we find $\mu_{jt}^{(\tau)}=i_{jt}$.

When a goes to zero, (4.5) implies that A_{ije} converges to $\frac{1}{\kappa_{ij}}$, (4.7) with the initial condition $A_{ijj}(0)=0$ implies that $A_{ijj}(\tau)$ converges to $\frac{1-e^{-\kappa_{ij}\tau}}{\kappa_{ij}}$, and (4.8) with the initial condition $A_{ijj'}(0)=0$ implies that $A_{ijj'}(\tau)$ converges to zero. When, in addition, (3.12) does not hold, (4.6) and (4.12) imply that C_e converges to plus or minus infinity at the rate $\frac{1}{a_e}$, and (A.24) implies that λ_{ijt} converges to a non-zero limit for $j=H,F$. Hence, (4.2) implies that μ_{et} does not converge to $i_{Ht}-i_{Ft}$, and (4.3) implies that $\mu_{jt}^{(\tau)}$ does not converge to i_{jt}. ∎

Proof of Proposition 4.2: We start by proving a series of lemmas.

Lemma A.1. *The matrix*

$$M \equiv \begin{pmatrix} \kappa_{iH}-a\sigma_{iH}^2\bar{\lambda}_{rHH} & -a\sigma_{iF}^2\bar{\lambda}_{rHF} \\ -a\sigma_{iH}^2\bar{\lambda}_{rFH} & \kappa_{iF}-a\sigma_{iF}^2\bar{\lambda}_{rFF} \end{pmatrix} \tag{A.25}$$

has two positive eigenvalues.

Proof: The characteristic polynomial of M is

$$\Pi(\lambda) \equiv \left(\kappa_{iH}-a\sigma_{iH}^2\bar{\lambda}_{rHH}-\lambda\right)\left(\kappa_{iF}-a\sigma_{iF}^2\bar{\lambda}_{rFF}-\lambda\right)-a^2\sigma_{iH}^2\sigma_{iF}^2\bar{\lambda}_{rHF}\bar{\lambda}_{rFH}. \tag{A.26}$$

For $\lambda=0$, $\Pi(\lambda)$ takes the value

$$\begin{aligned}\Pi(0) &= \left(\kappa_{iH}-a\sigma_{iH}^2\bar{\lambda}_{rHH}\right)\left(\kappa_{iF}-a\sigma_{iF}^2\bar{\lambda}_{rFF}\right)-a\sigma_{iH}^2\sigma_{iF}^2\bar{\lambda}_{rHF}\bar{\lambda}_{rFH} \\ &> a^2\sigma_{iH}^2\sigma_{iH}^2\left(\bar{\lambda}_{rHH}\bar{\lambda}_{rFF}-\bar{\lambda}_{rHF}\bar{\lambda}_{rFH}\right) \\ &= a^2\sigma_{iH}^2\sigma_{iH}^2\Bigg[\left(\int_0^T\alpha_H(\tau)A_{iHH}(\tau)^2d\tau+\int_0^T\alpha_F(\tau)A_{iFH}(\tau)^2d\tau+\alpha_eA_{iHe}^2\right) \\ &\quad\times\left(\int_0^T\alpha_H(\tau)A_{iHF}(\tau)^2d\tau+\int_0^T\alpha_F(\tau)A_{iFF}(\tau)^2d\tau+\alpha_eA_{iFe}^2\right) \\ &\quad-\left(\int_0^T\alpha_H(\tau)A_{iHH}(\tau)A_{iHF}(\tau)d\tau+\int_0^T\alpha_F(\tau)A_{iFH}(\tau)A_{iFF}(\tau)d\tau-\alpha_eA_{iHe}A_{iFe}\right)^2\Bigg].\end{aligned} \tag{A.27}$$

The second step in (A.27) follows because $(\kappa_{iH}, \kappa_{iF})$ are positive and because (4.10) implies that $(\bar{\lambda}_{rHH}, \bar{\lambda}_{rFF})$ are non-positive. The third step in (A.27) follows from (4.10) and (4.11). The Cauchy-Schwarz inequality associated to the scalar product

$$X \cdot Y \equiv \int_0^T \alpha_H(\tau) X_H(\tau) Y_H(\tau) d\tau + \int_0^T \alpha_F(\tau) X_F(\tau) Y_F(\tau) d\tau + \alpha_e xy$$

where $X \equiv (X_H(\tau), X_F(\tau), x)$, $Y \equiv (Y_H(\tau), Y_F(\tau), y)$, $(X_H(\tau), X_F(\tau), Y_H(\tau), Y_F(\tau))$ are functions of τ, and (x, y) are scalars, implies that (A.27) is non-negative. Hence, $\Pi(0) > 0$.

For $\lambda = \kappa_{iH} - a\sigma_{iH}^2 \bar{\lambda}_{rHH}$ and $\lambda = \kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{rFF}$, $\Pi(\lambda)$ takes the value $-a^2 \sigma_{iH}^2 \sigma_{iF}^2 \bar{\lambda}_{rHF} \bar{\lambda}_{rFH}$, which is non-positive because (4.11) implies $\bar{\lambda}_{rHF} = \bar{\lambda}_{rFH}$. Since $(\kappa_{iH}, \kappa_{iF})$ are positive and $(\bar{\lambda}_{rHH}, \bar{\lambda}_{rFF})$ are non-positive, $\kappa_{iH} - a\sigma_{iH}^2 \bar{\lambda}_{rHH}$ and $\lambda = \kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{rFF}$ are positive. Since $\Pi(\lambda)$ is a quadratic function of λ, is positive for $\lambda = 0$, is non-positive for two positive values of λ, and converges to infinity when λ goes to infinity, it has two positive roots. ∎

The matrix M plays an important role in the determination of $(A_{iHH}(\tau), A_{iHF}(\tau), A_{iFH}(\tau), A_{iFF}(\tau))$ and (A_{iHe}, A_{iFe}). Equation (4.5) gives rise to the linear system

$$M \begin{pmatrix} A_{iHe} \\ A_{iFe} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{A.28}$$

Since M has two positive eigenvalues, it is invertible, and hence (A.28) can be solved for (A_{iHe}, A_{iFe}). Equations (4.7) and (4.8) give rise to the linear system

$$\begin{pmatrix} A_{iHH}(\tau) \\ A_{iHF}(\tau) \end{pmatrix}' + M \begin{pmatrix} A_{iHH}(\tau) \\ A_{iHF}(\tau) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{A.29}$$

for $(j, j') = (H, F)$, and to

$$\begin{pmatrix} A_{iFH}(\tau) \\ A_{iFF}(\tau) \end{pmatrix}' + M \begin{pmatrix} A_{iFH}(\tau) \\ A_{iFF}(\tau) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{A.30}$$

for $(j, j') = (F, H)$. Since M has two positive eigenvalues, the solutions $(A_{iHH}(\tau), A_{iHF}(\tau))$ to (A.29) and $(A_{iFH}(\tau), A_{iFF}(\tau))$ to (A.30) converge to finite limits when τ goes to infinity.

Lemma A.2. *The normalized factor prices $\bar{\lambda}_{rHF} = \bar{\lambda}_{rFH}$ are non-negative.*

Proof: Suppose, proceeding by contradiction, that $\bar{\lambda}_{rHF} = \bar{\lambda}_{rFH}$ are negative. The solution to (A.28) is

$$A_{iHe} = \frac{\kappa_{iH} - a\sigma_{iH}^2(\bar{\lambda}_{rHH} + \bar{\lambda}_{rFH})}{\left(\kappa_{iH} - a\sigma_{iH}^2\bar{\lambda}_{rHH}\right)\left(\kappa_{iF} - a\sigma_{iF}^2\bar{\lambda}_{rFF}\right) - a^2\sigma_{iH}^2\sigma_{iF}^2\bar{\lambda}_{rHF}\bar{\lambda}_{rFH}}, \tag{A.31}$$

$$A_{iFe} = \frac{\kappa_{iF} - a\sigma_{iF}^2(\bar{\lambda}_{rFF} + \bar{\lambda}_{rHF})}{\left(\kappa_{iH} - a\sigma_{iH}^2\bar{\lambda}_{rHH}\right)\left(\kappa_{iF} - a\sigma_{iF}^2\bar{\lambda}_{rFF}\right) - a^2\sigma_{iH}^2\sigma_{iF}^2\bar{\lambda}_{rHF}\bar{\lambda}_{rFH}}. \tag{A.32}$$

The denominator in (A.31) and (A.32) is $\Pi(0) > 0$. The numerators in (A.31) and (A.32) are positive because $(\kappa_{iH}, \kappa_{iF})$ are positive and $(a\bar{\lambda}_{rHH}, a\bar{\lambda}_{rFF}, a\bar{\lambda}_{rHF}, a\bar{\lambda}_{rFH})$ are non-positive. Hence, A_{iHe} and A_{iFe} are positive.

When $a = 0$, (4.8) with the initial conditions $A_{iHF}(0) = A_{iFH}(0) = 0$ implies $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$ for all $\tau > 0$. Since, in addition, $A_{iHe} > 0$ and $A_{iFe} > 0$, (4.11) implies $\bar{\lambda}_{rHF} = \bar{\lambda}_{rFH} \geq 0$, a contradiction.

When $a > 0$, (4.7) and (4.8) with the initial conditions $A_{iHH}(0) = A_{iFF}(0) = A_{iHF}(0) = A_{iFH}(0) = 0$ imply $A'_{iHH}(0) = A'_{iFF}(0) = 1$ and $A'_{iHF}(0) = A'_{iFH}(0) = 0$. Moreover, differentiating (4.8), we find $A''_{iHF}(0) = a\sigma_{iH}^2\bar{\lambda}_{rFH}A'_{iHH}(0) < 0$ and $A''_{iFH}(0) = a\sigma_{iF}^2\bar{\lambda}_{rHF}A'_{iFF}(0) < 0$. Hence, $A_{iHH}(\tau) > 0$, $A_{iFF}(\tau) > 0$, $A_{iHF}(\tau) < 0$ and $A_{iFH}(\tau) < 0$ for τ close to zero. We define τ_0 by

$$\tau_0 \equiv \sup_{\tau}\{A_{iHH}(\tau') > 0,\ A_{iFF}(\tau') > 0,\ A_{iHF}(\tau') < 0 \text{ and } A_{iFH}(\tau') < 0 \text{ for all } \tau' \in (0, \tau)\}.$$

If τ_0 is finite, then (i) $A_{iHH}(\tau_0) = 0$, $A'_{iHH}(\tau_0) \leq 0$, $A_{iFF}(\tau_0) \geq 0$, $A_{iHF}(\tau_0) \leq 0$ and $A_{iFH}(\tau_0) \leq 0$, or (ii) $A_{iHH}(\tau_0) > 0$, $A_{iFF}(\tau_0) = 0$, $A'_{iFF}(\tau_0) \leq 0$, $A_{iHF}(\tau_0) \leq 0$ and $A_{iFH}(\tau_0) \leq 0$, or (iii) $A_{iHH}(\tau_0) > 0$, $A_{iFF}(\tau_0) > 0$, $A_{iHF}(\tau_0) = 0$, $A'_{iHF}(\tau_0) \geq 0$ and $A_{iFH}(\tau_0) \leq 0$, or (iv) $A_{iHH}(\tau_0) > 0$, $A_{iFF}(\tau_0) > 0$, $A_{iHF}(\tau_0) < 0$, $A_{iFH}(\tau_0) = 0$ and $A'_{iFH}(\tau_0) \geq 0$. Case (i) yields a contradiction because (4.7) for $j = H$, $A_{iHH}(\tau_0) = 0$, $A_{iHF}(\tau_0) \leq 0$ and $\bar{\lambda}_{rHF} < 0$ imply $A'_{iHH}(\tau_0) \geq 1$. Case (ii) yields a contradiction by using the same argument as in Case (i) and switching H and F. Case (iii) yields a contradiction because (4.8) for $(j, j') = (H, F)$, $A_{iHH}(\tau_0) > 0$, $A_{iHF}(\tau_0) = 0$ and $\lambda_{rFH} < 0$ imply $A'_{iHF}(\tau_0) < 0$. Case (iv) yields a contradiction by using the same argument as in Case (iii) and switching H and F. Therefore, τ_0 is infinite, which means $A_{iHH}(\tau) > 0$, $A_{iFF}(\tau) > 0$, $A_{iHF}(\tau) < 0$ and $A_{iFH}(\tau) < 0$ for all $\tau > 0$. Since, in addition, $A_{iHe} > 0$ and $A_{iFe} > 0$, (4.11) implies $\bar{\lambda}_{rHF} = \bar{\lambda}_{rFH} \geq 0$, a contradiction. Hence, $\bar{\lambda}_{rHF} = \bar{\lambda}_{rFH}$ are non-negative. ∎

Lemma A.3. *The functions $A_{iHH}(\tau)$ and $A_{iFF}(\tau)$ are positive for all $\tau > 0$.*

- *When $a > 0$ and $\alpha_e > 0$, the functions $A_{iHF}(\tau)$ and $A_{iFH}(\tau)$ are positive for all $\tau > 0$.*

- *When $a = 0$ or $\alpha_e = 0$, the functions $A_{iHF}(\tau)$ and $A_{iFH}(\tau)$ are zero.*

Proof: Consider first the case $a > 0$ and $\alpha_e > 0$. If $\bar{\lambda}_{rHF} = \bar{\lambda}_{rFH} = 0$, then (4.8) with the initial conditions $A_{iHF}(0) = A_{iFH}(0) = 0$ implies $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$ for all $\tau > 0$. Since, in addition, (A.31) and (A.32) imply $A_{iHe} > 0$ and $A_{iFe} > 0$, (4.11) implies $\bar{\lambda}_{rHF} = \bar{\lambda}_{rFH} > 0$, a contradiction. Hence, Lemma A.2 implies $\bar{\lambda}_{rHF} = \bar{\lambda}_{rFH} > 0$.

Equations (4.7) and (4.8) with the initial conditions $A_{iHH}(0) = A_{iFF}(0) = A_{iHF}(0) = A_{iFH}(0) = 0$ imply $A'_{iHH}(0) = A'_{iFF}(0) = 1$ and $A'_{iHF}(0) = A'_{iFH}(0) = 0$. Moreover, differentiating (4.8), we find $A''_{iHF}(0) = a\sigma_{iH}^2\bar{\lambda}_{rFH}A'_{iHH}(0) > 0$ and $A''_{iFH}(0) = a\sigma_{iF}^2\bar{\lambda}_{rHF}A'_{iFF}(0) > 0$. Hence, $A_{iHH}(\tau) > 0$, $A_{iFF}(\tau) > 0$, $A_{iHF}(\tau) > 0$ and $A_{iFH}(\tau) > 0$ for τ close to zero. We define τ_0 by

$$\tau_0 \equiv \sup_{\tau}\{A_{iHH}(\tau') > 0,\, A_{iFF}(\tau') > 0,\, A_{iHF}(\tau') > 0 \text{ and } A_{iFH}(\tau') > 0 \text{ for all } \tau' \in (0,\tau)\}.$$

If τ_0 is finite, then (i) $A_{iHH}(\tau_0) = 0$, $A'_{iHH}(\tau_0) \leq 0$, $A_{iFF}(\tau_0) \geq 0$, $A_{iHF}(\tau_0) \geq 0$ and $A_{iFH}(\tau_0) \geq 0$, or (ii) $A_{iHH}(\tau_0) > 0$, $A_{iFF}(\tau_0) = 0$, $A'_{iFF}(\tau_0) \leq 0$, $A_{iHF}(\tau_0) \geq 0$ and $A_{iFH}(\tau_0) \geq 0$, or (iii) $A_{iHH}(\tau_0) > 0$, $A_{iFF}(\tau_0) > 0$, $A_{iHF}(\tau_0) = 0$, $A'_{iHF}(\tau_0) \leq 0$ and $A_{iFH}(\tau_0) \geq 0$, or (iv) $A_{iHH}(\tau_0) > 0$, $A_{iFF}(\tau_0) > 0$, $A_{iHF}(\tau_0) > 0$, $A_{iFH}(\tau_0) = 0$ and $A'_{iFH}(\tau_0) \leq 0$. Case (i) yields a contradiction because (4.7) for $j = H$, $A_{iHH}(\tau_0) = 0$, $A_{iHF}(\tau_0) \geq 0$ and $\bar{\lambda}_{rHF} > 0$ imply $A'_{iHH}(\tau_0) \geq 1$. Case (ii) yields a contradiction by using the same argument as in Case (i) and switching H and F. Case (iii) yields a contradiction because (4.8) for $(j, j') = (H, F)$, $A_{iHH}(\tau_0) > 0$, $A_{iHF}(\tau_0) = 0$ and $\lambda_{rFH} > 0$ imply $A'_{iHF}(\tau_0) > 0$. Case (iv) yields a contradiction by using the same argument as in Case (iii) and switching H and F. Therefore, τ_0 is infinite, which means $A_{iHH}(\tau) > 0$, $A_{iFF}(\tau) > 0$, $A_{iHF}(\tau) > 0$ and $A_{iFH}(\tau) > 0$ for all $\tau > 0$.

Consider next the case $a = 0$. The properties of $(A_{iHH}(\tau), A_{iFF}(\tau), A_{iHF}(\tau), A_{iFH}(\tau))$ follow from Corollary 4.1.

Consider finally the case $a > 0$ and $\alpha_e = 0$. Suppose, proceeding by contradiction, that $\bar{\lambda}_{rHF} = \bar{\lambda}_{rFH}$ are positive. The argument in the case $a > 0$ and $\alpha_e > 0$ implies $A_{iHH}(\tau) > 0$, $A_{iFF}(\tau) > 0$, $A_{iHF}(\tau) > 0$ and $A_{iFH}(\tau) > 0$ for all $\tau > 0$. Since $\alpha_e = 0$, (4.11) implies $\bar{\lambda}_{rHF} = \bar{\lambda}_{rFH} \leq 0$, a contradiction. Hence, Lemma A.2 implies $\bar{\lambda}_{rHF} = \bar{\lambda}_{rFH} = 0$.

Since $\bar{\lambda}_{rHF} = \bar{\lambda}_{rFH} = 0$, (4.8) with the initial conditions $A_{iHF}(0) = A_{iFH}(0) = 0$ implies $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$. Since $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$, (4.7) with the initial conditions $A_{iHH}(0) = A_{iFF}(0) = 0$ implies that $A_{iHH}(\tau)$ and $A_{iFF}(\tau)$ are positive for all $\tau > 0$. ∎

Lemma A.4. *The functions $A_{iHH}(\tau)$ and $A_{iFF}(\tau)$ are increasing. When $a > 0$ and $\alpha_e > 0$, the functions $A_{iHF}(\tau)$ and $A_{iFH}(\tau)$ are also increasing.*

Proof: Consider first the case $a > 0$ and $\alpha_e > 0$. Equations $A'_{iHH}(0) = A'_{iFF}(0) = 1$, $A'_{iHF}(0) = A'_{iFH}(0) = 0$, $A''_{iHF}(0) = a\sigma^2_{iH}\bar{\lambda}_{rFH}A'_{iHH}(0) > 0$ and $A''_{iFH}(0) = a\sigma^2_{iF}\bar{\lambda}_{rHF}A'_{iFF}(0) > 0$ imply $A'_{iHH}(\tau) > 0$, $A'_{iFF}(\tau) > 0$, $A'_{iHF}(\tau) > 0$ and $A'_{iFH}(\tau) > 0$ for τ close to zero. We define τ_0 by

$$\tau_0 \equiv \sup_{\tau}\{A'_{iHH}(\tau') > 0,\ A'_{iFF}(\tau') > 0,\ A'_{iHF}(\tau') > 0 \text{ and } A'_{iFH}(\tau') > 0 \text{ for all } \tau' \in (0,\tau)\}.$$

If τ_0 is finite, then (i) $A'_{iHH}(\tau_0) = 0$, $A''_{iHH}(\tau_0) \leq 0$, $A'_{iFF}(\tau_0) \geq 0$, $A'_{iHF}(\tau_0) \geq 0$ and $A'_{iFH}(\tau_0) \geq 0$, or (ii) $A'_{iHH}(\tau_0) > 0$, $A'_{iFF}(\tau_0) = 0$, $A''_{iFF}(\tau_0) \leq 0$, $A'_{iHF}(\tau_0) \geq 0$ and $A_{iFH}(\tau_0)' \geq 0$, or (iii) $A'_{iHH}(\tau_0) > 0$, $A'_{iFF}(\tau_0) > 0$, $A'_{iHF}(\tau_0) = 0$, $A''_{iHF}(\tau_0) \leq 0$ and $A'_{iFH}(\tau_0) \geq 0$, or (iv) $A'_{iHH}(\tau_0) > 0$, $A'_{iFF}(\tau_0) > 0$, $A'_{iHF}(\tau_0) > 0$, $A'_{iFH}(\tau_0) = 0$ and $A''_{iFH}(\tau_0) \leq 0$. To analyze Cases (i)-(iv), we use

$$A''_{ijj}(\tau) + \kappa_{ij}A'_{ijj}(\tau) = a\sigma^2_{ij}\bar{\lambda}_{ijj}A'_{ijj}(\tau) + a\sigma^2_{ij'}\bar{\lambda}_{ijj'}A'_{ijj'}(\tau), \tag{A.33}$$

$$A''_{ijj'}(\tau) + \kappa_{rj'}A'_{ijj'}(\tau) = a\sigma^2_{ij}\bar{\lambda}_{rj'j}A'_{ijj}(\tau) + a\sigma^2_{ij'}\bar{\lambda}_{rj'j'}A'_{ijj'}(\tau), \tag{A.34}$$

which follow from differentiating (4.7) and (4.8), respectively.

Case (i) yields a contradiction. Indeed, if $A''_{iHH}(\tau_0) = 0$, then (A.33) for $j = H$, $A'_{iHH}(\tau_0) = 0$ and $\bar{\lambda}_{rHF} > 0$ imply $A'_{iHF}(\tau_0) = 0$. The unique solution to the linear system of ODEs (A.33) for $j = H$ and (A.34) for $(j,j') = (H,F)$ with the initial condition $(A'_{iHH}(\tau_0), A'_{iHF}(\tau_0)) = (0,0)$ is the function that equals (0,0) for all τ. This yields a contradiction because $(A'_{iHH}(0), A'_{iHF}(0)) = (1,0)$. Hence, $A''_{iHH}(\tau_0) < 0$, which combined with (A.33) for $j = H$, $A'_{iHH}(\tau_0) = 0$ and $\bar{\lambda}_{rHF} > 0$ implies $A'_{iHF}(\tau_0) < 0$, again a contradiction. Case (ii) yields a contradiction by using the same argument as in Case (i) and switching H and F. Case (iii) yields a contradiction because (A.34) for $(j,j') = (H,F)$, $A_{iHH}(\tau_0) > 0$, $A_{iHF}(\tau_0) = 0$ and $\lambda_{rFH} > 0$ imply $A''_{iHF}(\tau_0) > 0$. Case (iv) yields a contradiction by using the same argument as in Case (iii) and switching H and F. Therefore, τ_0 is infinite, which means that $(A_{iHH}(\tau), A_{iFF}(\tau), A_{iHF}(\tau), A_{iFH}(\tau))$ are increasing.

In the case $a = 0$ or $\alpha_e = 0$, Lemma A.3 implies $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$. Since $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$, (4.7) with the initial conditions $A_{iHH}(0) = A_{iFF}(0) = 0$ implies that $A_{iHH}(\tau)$ and $A_{iFF}(\tau)$ are increasing. ∎

Lemma A.5. *The scalars A_{iHe} and A_{iFe} are positive.*

Proof: Consider first the case $a > 0$ and $\alpha_e > 0$. Since $\bar{\lambda}_{rHF} = \bar{\lambda}_{rFH} > 0$ and $A_{iHH}(\tau) > 0$, $A_{iFF}(\tau) > 0$, $A_{iHF}(\tau) > 0$ and $A_{iFH}(\tau) > 0$ for all $\tau > 0$ (Lemma A.3), (4.11) implies $A_{iHe}A_{iFe} > 0$. Hence, (A_{iHe}, A_{iFe}) are either both positive or both negative. Suppose, proceeding by contradiction, that they are both negative. Equations (A.31) and (A.32) imply

$$\kappa_{iH} - a\sigma_{iH}^2\bar{\lambda}_{rHH} < a\sigma_{iH}^2\bar{\lambda}_{rFH}, \tag{A.35}$$

$$\kappa_{iF} - a\sigma_{iF}^2\bar{\lambda}_{rFF} < a\sigma_{iF}^2\bar{\lambda}_{rHF}. \tag{A.36}$$

Since the left-hand side in each of (A.35) and (A.36) is positive, (A.35) and (A.36) imply

$$\Pi(0) = \left(\kappa_{iH} - a\sigma_{iH}^2\bar{\lambda}_{rHH}\right)\left(\kappa_{iF} - a\sigma_{iF}^2\bar{\lambda}_{rFF}\right) - a\sigma_{iH}^2\sigma_{iF}^2\bar{\lambda}_{rHF}\bar{\lambda}_{rFH} < 0,$$

a contradiction. Hence, A_{iHe} and A_{iFe} are positive.

Consider next the case $a = 0$. Corollary 4.1 implies that A_{iHe} and A_{iFe} are positive. Consider finally the case $\alpha_e = 0$ and $a > 0$. Since $\bar{\lambda}_{rHF} = \bar{\lambda}_{rFH} = 0$ and $(\bar{\lambda}_{rHH}, \bar{\lambda}_{rFF})$ are non-positive, (A.31) and (A.32) imply that A_{iHe} and A_{iFe} are positive. ∎

Lemma A.6. *The functions $A_{iHH}(\tau) - A_{iFH}(\tau)$ and $A_{iFF}(\tau) - A_{iHF}(\tau)$ are positive for all $\tau > 0$.*

Proof: In the case $a = 0$ or $\alpha_e = 0$, the lemma follows from Lemma A.3. To prove the lemma in the case $a > 0$ and $\alpha_e > 0$, we proceed in two steps. In Step 1, we show that $A_{iHH}(\tau) - A_{iFH}(\tau)$ and $A_{iFF}(\tau) - A_{iHF}(\tau)$ are positive in the limit when τ goes to infinity. In Step 2, we show that $A_{iHH}(\tau) - A_{iFH}(\tau)$ and $A_{iFF}(\tau) - A_{iHF}(\tau)$ are either increasing in τ, or increasing and then decreasing. The lemma follows by combining these properties with $A_{iHH}(0) - A_{iFH}(0) = A_{iFF}(0) - A_{iHF}(0) = 0$.

Step 1: Limit at infinity. Since the matrix M has two positive eigenvalues, the functions $(A_{iHH}(\tau),\ A_{iFF}(\tau), A_{iHF}(\tau), A_{iFH}(\tau))$ converge to finite limits when τ goes to infinity. These limits solve the system of equations

$$\kappa_{ij} A_{ijj}(\infty) - 1 = a\sigma_{ij}^2 \bar{\lambda}_{ijj} A_{ijj}(\infty) + a\sigma_{ij'}^2 \bar{\lambda}_{ijj'} A_{ijj'}(\infty), \tag{A.37}$$

$$\kappa_{rj'} A_{ijj'}(\infty) = a\sigma_{ij}^2 \bar{\lambda}_{rj'j} A_{ijj}(\infty) + a\sigma_{ij'}^2 \bar{\lambda}_{rj'j'} A_{ijj'}(\infty), \tag{A.38}$$

which are derived from (4.7) and (4.8) by setting the derivatives to zero. Subtracting (A.38) for $(j, j') = (F, H)$ from (A.37) for $j = H$, we find

$$\begin{aligned}&\kappa_{iH}(A_{iHH}(\infty) - A_{iFH}(\infty)) - 1 \\ &= a\sigma_{iH}^2 \bar{\lambda}_{rHH}(A_{iHH}(\infty) - A_{iFH}(\infty)) + a\sigma_{iF}^2 \bar{\lambda}_{rHF}(A_{iHF}(\infty) - A_{iFF}(\infty)).\end{aligned} \tag{A.39}$$

Subtracting (A.38) for $(j, j') = (H, F)$ from (A.37) for $j = F$, we similarly find

$$\begin{aligned}&\kappa_{iF}(A_{iFF}(\infty) - A_{iHF}(\infty)) - 1 \\ &= a\sigma_{iH}^2 \bar{\lambda}_{rFH}(A_{iHF}(\infty) - A_{iHH}(\infty)) + a\sigma_{iF}^2 \bar{\lambda}_{rFF}(A_{iFF}(\infty) - A_{iHF}(\infty)).\end{aligned} \tag{A.40}$$

The solution to the system of (A.39) and (A.40) is

$$A_{iHH}(\infty) - A_{iFH}(\infty) = \frac{\kappa_{iH} - a\sigma_{iH}^2(\bar{\lambda}_{rHH} + \bar{\lambda}_{rFH})}{\left(\kappa_{iH} - a\sigma_{iH}^2 \bar{\lambda}_{rHH}\right)\left(\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{rFF}\right) - a^2 \sigma_{iH}^2 \sigma_{iF}^2 \bar{\lambda}_{rHF} \bar{\lambda}_{rFH}} = A_{iHe}, \tag{A.41}$$

$$A_{iFF}(\infty) - A_{iHF}(\infty) = \frac{\kappa_{iF} - a\sigma_{iF}^2(\bar{\lambda}_{rFF} + \bar{\lambda}_{rHF})}{\left(\kappa_{iH} - a\sigma_{iH}^2 \bar{\lambda}_{rHH}\right)\left(\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{rFF}\right) - a^2 \sigma_{iH}^2 \sigma_{iF}^2 \bar{\lambda}_{rHF} \bar{\lambda}_{rFH}} = A_{iFe}, \tag{A.42}$$

where the second equality in (A.41) and (A.42) follows from (A.31) and (A.32), respectively. Since (A_{iHe}, A_{iFe}) are positive (Lemma A.5), so are $(A_{iFF}(\infty) - A_{iHF}(\infty), A_{iFF}(\infty) - A_{iHF}(\infty))$.

Step 2: Monotonicity. Equations (4.7) and (4.8) with the initial conditions $A_{iHH}(0) = A_{iFF}(0) = A_{iHF}(0) = A_{iFH}(0) = 0$ imply $A'_{iHH}(0) = A'_{iFF}(0) = 1 > 0$ and $A'_{iHF}(0) = A'_{iFH}(0) = 0$. Hence, $A'_{iHH}(\tau) - A'_{iFH}(\tau) > 0$ and $A'_{iFF}(\tau) - A'_{iHF}(\tau) > 0$ for τ close to zero. We define τ_0 by

$$\tau_0 \equiv \sup_{\tau} \{A'_{iHH}(\tau') - A'_{iFH}(\tau') > 0 \text{ and } A'_{iFF}(\tau') - A'_{iHF}(\tau') > 0 \text{ for all } \tau' \in (0, \tau)\}.$$

If τ_0 is infinity, then $A_{iHH}(\tau) - A_{iFH}(\tau)$ and $A_{iFF}(\tau) - A_{iHF}(\tau)$ are increasing in τ. Suppose instead that τ_0 is finite. Then, either (i) $A'_{iHH}(\tau_0) - A'_{iFH}(\tau_0) = 0$, $A''_{iHH}(\tau_0) - A''_{iFH}(\tau_0) \leq 0$ and $A'_{iFF}(\tau_0) - A'_{iHF}(\tau_0) \geq 0$, or (ii) $A'_{iHH}(\tau_0) - A'_{iFH}(\tau_0) > 0$, $A'_{iFF}(\tau_0) - A'_{iHF}(\tau_0) = 0$ and $A''_{iFF}(\tau_0) - A''_{iHF}(\tau_0) \leq 0$. To analyze Cases (i) and (ii), we use

$$\begin{aligned}&A'_{iHH}(\tau) - A'_{iFH}(\tau) + \kappa_{iH}(A_{iHH}(\tau) - A_{iFH}(\tau)) - 1 \\ &= a\sigma_{iH}^2 \bar{\lambda}_{rHH}(A_{iHH}(\tau) - A_{iFH}(\tau)) + a\sigma_{iF}^2 \bar{\lambda}_{rHF}(A_{iHF}(\tau) - A_{iFF}(\tau)),\end{aligned} \tag{A.43}$$

which follows by subtracting (4.8) for $(j, j') = (F, H)$ from (A.37) for $j = H$, and

$$A'_{iFF}(\tau) - A'_{iHF}(\tau) + \kappa_{iF}(A_{iFF}(\tau) - A_{iHF}(\tau)) - 1$$
$$= a\sigma_{iH}^2\bar{\lambda}_{rFH}(A_{iHF}(\tau) - A_{iHH}(\tau)) + a\sigma_{iF}^2\bar{\lambda}_{rFF}(A_{iFF}(\tau) - A_{iHF}(\tau)), \tag{A.44}$$

which follows by subtracting (A.38) for $(j, j') = (H, F)$ from (A.37) for $j = F$. Differentiating (A.43) and (A.44), we find

$$A''_{iHH}(\tau) - A''_{iFH}(\tau) + \kappa_{iH}(A'_{iHH}(\tau) - A'_{iFH}(\tau))$$
$$= a\sigma_{iH}^2\bar{\lambda}_{rHH}(A'_{iHH}(\tau) - A'_{iFH}(\tau)) + a\sigma_{iF}^2\bar{\lambda}_{rHF}(A'_{iHF}(\tau) - A'_{iFF}(\tau)) \tag{A.45}$$

and

$$A''_{iFF}(\tau) - A''_{iHF}(\tau) + \kappa_{iF}(A'_{iFF}(\tau) - A'_{iHF}(\tau))$$
$$= a\sigma_{iH}^2\bar{\lambda}_{rFH}(A'_{iHF}(\tau) - A'_{iHH}(\tau)) + a\sigma_{iF}^2\bar{\lambda}_{rFF}(A'_{iFF}(\tau) - A'_{iHF}(\tau)), \tag{A.46}$$

respectively. Equations (A.45) and (A.46) are a linear system of ODEs in the functions $(A'_{iHH}(\tau) - A'_{iFH}(\tau), A'_{iFF}(\tau) - A'_{iHF}(\tau))$.

Consider first Case (i). If $A''_{iHH}(\tau_0) - A''_{iFH}(\tau_0) = 0$, then (A.45), $A'_{iHH}(\tau_0) - A'_{iFH}(\tau_0) = 0$ and $\bar{\lambda}_{rHF} > 0$ imply $A'_{iFF}(\tau_0) - A'_{iHF}(\tau_0) = 0$. The unique solution to the linear system of ODEs (A.45) and (A.46) with the initial condition $(A'_{iHH}(\tau_0) - A'_{iFH}(\tau_0), A'_{iFF}(\tau_0) - A'_{iHF}(\tau_0)) = (0, 0)$ is the function that equals (0,0) for all τ. This yields a contradiction because $(A'_{iHH}(0) - A'_{iFH}(0), A'_{iFF}(0) - A'_{iHF}(0)) = (1, 1)$. Hence, $A''_{iHH}(\tau_0) - A''_{iFH}(\tau_0) < 0$, which combined with (A.45), $A'_{iHH}(\tau_0) - A'_{iFH}(\tau_0) = 0$ and $\bar{\lambda}_{rHF} > 0$ implies $A'_{iFF}(\tau_0) - A'_{iHF}(\tau_0) > 0$. Since $A'_{iHH}(\tau_0) - A'_{iFH}(\tau_0) = 0$ and $A''_{iHH}(\tau_0) - A''_{iFH}(\tau_0) < 0$, $A'_{iHH}(\tau) - A'_{iFH}(\tau) < 0$ for τ larger than and close to τ_0. We define τ'_0 by

$$\tau'_0 \equiv \sup_{\tau}\{A'_{iHH}(\tau') - A'_{iFH}(\tau') < 0 \text{ and } A'_{iFF}(\tau') - A'_{iHF}(\tau') > 0 \text{ for all } \tau' \in (\tau_0, \tau)\}.$$

If τ'_0 is finite, then either (ia) $A'_{iHH}(\tau_0) - A'_{iFH}(\tau_0) = 0$, $A''_{iHH}(\tau_0) - A''_{iFH}(\tau_0) \geq 0$ and $A'_{iFF}(\tau_0) - A'_{iHF}(\tau_0) \geq 0$, or (ib) $A'_{iHH}(\tau_0) - A'_{iFH}(\tau_0) < 0$, $A'_{iFF}(\tau_0) - A'_{iHF}(\tau_0) = 0$ and $A''_{iFF}(\tau_0) - A''_{iHF}(\tau_0) \leq 0$. In Case (ia), the same argument as for τ_0 implies $A''_{iHH}(\tau'_0) - A''_{iFH}(\tau'_0) > 0$, which combined with (A.45), $A'_{iHH}(\tau_0) - A'_{iFH}(\tau_0) = 0$ and $\bar{\lambda}_{rHF} > 0$ implies $A'_{iFF}(\tau'_0) - A'_{iHF}(\tau'_0) < 0$, a contradiction. In Case (ib), the same argument as for τ_0 implies $A''_{iFF}(\tau'_0) - A''_{iHF}(\tau'_0) < 0$, which combined with (A.46), $A'_{iFF}(\tau_0) - A'_{iHF}(\tau_0) = 0$ and $\bar{\lambda}_{rFH} > 0$ implies $A'_{iHH}(\tau'_0) - A'_{iFH}(\tau'_0) > 0$, a contradiction.

Therefore, τ_0' is infinite, which means that $A_{iFF}(\tau) - A_{iHF}(\tau)$ is increasing, and $A_{iHH}(\tau) - A_{iFH}(\tau)$ is increasing in $(0, \tau_0)$ and decreasing in (τ_0, ∞).

Consider next Case (ii). A symmetric argument by switching H and F implies that $A_{iHH}(\tau) - A_{iFH}(\tau)$ is increasing, and $A_{iFF}(\tau) - A_{iHF}(\tau)$ is increasing in $(0, \tau_0)$ and decreasing in (τ_0, ∞). ∎

Using Lemmas A.1-A.6, we next prove the proposition. Since (A_{iHe}, A_{iFe}) are positive (Lemma A.5), (3.2) implies $\frac{\partial e_t}{\partial i_{Ht}} < 0$ and $\frac{\partial e_t}{\partial i_{Ft}} > 0$. When $a > 0$ and $\alpha_e > 0$, (4.10) implies that $(\bar{\lambda}_{rHH}, \bar{\lambda}_{rFF})$ are negative, and the proof of Lemma A.3 implies that $(\bar{\lambda}_{rHF}, \bar{\lambda}_{rFH})$ are positive. Hence,

$$a\sigma_{iH}^2 \bar{\lambda}_{rHH} A_{iHe} - a\sigma_{iF}^2 \bar{\lambda}_{rHF} A_{iFe} < 0, \tag{A.47}$$

$$a\sigma_{iF}^2 \bar{\lambda}_{rFF} A_{iFe} - a\sigma_{iH}^2 \bar{\lambda}_{rFH} A_{iHe} < 0. \tag{A.48}$$

Combining (A.47) and (A.48) with (4.5), we find $A_{iHe} < \frac{1}{\kappa_{iH}} \equiv A_{iHe}^{UIP}$ and $A_{iFe} < \frac{1}{\kappa_{iF}} \equiv A_{iFe}^{UIP}$. Combining (A.47) and (A.48) with (4.2) and (A.24), we find $\frac{\partial(\mu_{et} + i_{Ft} - i_{Ht})}{\partial i_{Ht}} < 0$ and $\frac{\partial(\mu_{et} + i_{Ft} - i_{Ht})}{\partial i_{Ft}} > 0$. This establishes the first bullet point of the proposition.

Since $(A_{iHH}(\tau), A_{iFF}(\tau))$ are positive for all $\tau > 0$ (Lemma A.3), (2.1) and (4.1) imply that $(\frac{\partial y_{Ht}^{(\tau)}}{\partial i_{Ht}}, \frac{\partial y_{Ft}^{(\tau)}}{\partial i_{Ft}})$ are positive. When $a > 0$ and $\alpha_e > 0$, Lemma A.3 implies that $(A_{iHF}(\tau), A_{iFH}(\tau))$ are positive for all $\tau > 0$, and Lemma A.4 implies that $(A_{iHF}(\tau), A_{iFH}(\tau))$ are increasing. Equation (4.8) for $(j, j') = (H, F)$ implies

$$a\sigma_{iH}^2 \bar{\lambda}_{rFH} A_{iHH}(\tau) + a\sigma_{iF}^2 \bar{\lambda}_{rFF} A_{iHF}(\tau) > 0. \tag{A.49}$$

Multiplying both sides of (A.49) by $\frac{\bar{\lambda}_{rHH}}{\bar{\lambda}_{rFH}} < 0$, we find

$$a\sigma_{iH}^2 \bar{\lambda}_{rHH} A_{iHH}(\tau) + a\sigma_{iF}^2 \frac{\bar{\lambda}_{rHH}\bar{\lambda}_{rFF}}{\bar{\lambda}_{rFH}} A_{iHF}(\tau) < 0$$

$$\Rightarrow a\sigma_{iH}^2 \bar{\lambda}_{rHH} A_{iHH}(\tau) + a\sigma_{iF}^2 \bar{\lambda}_{rHF} A_{iHF}(\tau) < 0, \tag{A.50}$$

where the second step follows from $A_{iHF}(\tau) > 0$ and from the inequality $\bar{\lambda}_{rHH}\bar{\lambda}_{rFF} - \bar{\lambda}_{rHF}\bar{\lambda}_{rFH} < 0$ established in the proof of Lemma A.1. We likewise find

$$a\sigma_{iF}^2 \bar{\lambda}_{rHF} A_{iFF}(\tau) + a\sigma_{iH}^2 \bar{\lambda}_{rHH} A_{iFH}(\tau) > 0, \tag{A.51}$$

$$a\sigma_{iF}^2 \bar{\lambda}_{rFF} A_{iFF}(\tau) + a\sigma_{iH}^2 \bar{\lambda}_{rFH} A_{iFH}(\tau) < 0, \tag{A.52}$$

by switching H and F. Equations (A.50) and (A.52) hold also when $a > 0$, $\alpha_e = 0$ and $(\alpha_H(\tau), \alpha_F(\tau))$ are positive in a positive measure set of $(0, T)$. Indeed, the proof of Lemma A.3 implies $\bar{\lambda}_{rHF} = \bar{\lambda}_{rFH} = 0$, and since $(A_{iHH}(\tau), A_{iFF}(\tau))$ are positive, (4.10) implies that $(\bar{\lambda}_{rHH}, \bar{\lambda}_{rFF})$ are negative. Combining (A.50) and (A.52) with (4.7), we find $A_{iHH}(\tau) < \frac{1-e^{-\kappa_{iH}\tau}}{\kappa_{iH}} \equiv A^{EH}_{iHH}(\tau)$ and $A_{iFF}(\tau) < \frac{1-e^{-\kappa_{iF}\tau}}{\kappa_{iF}} \equiv A^{EH}_{iFF}(\tau)$. Combining (A.50) and (A.52) with (4.3) and (A.24), we find $\frac{\partial\left(\mu^{(\tau)}_{Ht} - i_{Ht}\right)}{\partial i_{Ht}} < 0$ and $\frac{\partial\left(\mu^{(\tau)}_{Ft} - i_{Ft}\right)}{\partial i_{Ft}} < 0$. This establishes the second bullet point of the proposition.

When $a > 0$ and $\alpha_e > 0$, $(A_{iHF}(\tau), A_{iFH}(\tau))$ are positive for all $\tau > 0$, and hence (2.1) and (4.1) imply that $(\frac{\partial y^{(\tau)}_{Ht}}{\partial i_{Ft}}, \frac{\partial y^{(\tau)}_{Ft}}{\partial i_{Ht}})$ are positive. Moreover, combining (A.49) and (A.51) with (4.3) and (A.24), we find $\frac{\partial\left(\mu^{(\tau)}_{Ht} - i_{Ht}\right)}{\partial i_{Ft}} > 0$ and $\frac{\partial\left(\mu^{(\tau)}_{Ft} - i_{Ft}\right)}{\partial i_{Ht}} > 0$. This establishes the third bullet point of the proposition. The fourth bullet point follows from Lemma A.6, (2.1) and (4.1). ∎

Proof of Proposition 4.3: Using (3.4), (4.2), (4.3), (4.13), (A.19) and (A.21), we can write the expected return of the hybrid CCT as

$$\mu^{(\tau)}_{hCCTt} \equiv \lambda_{iHt}(A_{iHe} + A_{iFH}(\tau) - A_{iHH}(\tau)) - \lambda_{iFt}(A_{iFe} + A_{iHF}(\tau) - A_{iFF}(\tau)). \tag{A.53}$$

Using (A.24), we find

$$\frac{\partial \mu^{(\tau)}_{hCCTt}}{\partial i_{Ht}} = a\sigma^2_{iH}\bar{\lambda}_{rHH}(A_{iHe} + A_{iFH}(\tau) - A_{iHH}(\tau)) - a\sigma^2_{iF}\bar{\lambda}_{rHF}(A_{iFe} + A_{iHF}(\tau) - A_{iFF}(\tau)), \tag{A.54}$$

$$\frac{\partial \mu^{(\tau)}_{hCCTt}}{\partial i_{Ft}} = a\sigma^2_{iH}\bar{\lambda}_{rFH}(A_{iHe} + A_{iFH}(\tau) - A_{iHH}(\tau)) - a\sigma^2_{iF}\bar{\lambda}_{rFF}(A_{iFe} + A_{iHF}(\tau) - A_{iFF}(\tau)). \tag{A.55}$$

When $a > 0$, and $\alpha_e > 0$ or $\alpha_j(\tau) > 0$, $(\bar{\lambda}_{rHH}, \bar{\lambda}_{rFF})$ are negative. Since, in addition, $(\bar{\lambda}_{rHF}, \bar{\lambda}_{rFH})$ are non-negative, (A_{iHe}, A_{iFe}) are positive and $A_{iHH}(0) - A_{iFH}(0) = A_{iFF}(0) - A_{iHF}(0) = 0$, (A.54) and (A.55) imply that there exists a threshold $\tau^* > 0$ such that $\frac{\partial \mu^{(\tau)}_{hCCTt}}{\partial i_{Ht}} < 0$ and $\frac{\partial \mu^{(\tau)}_{hCCTt}}{\partial i_{Ft}} > 0$ for all $\tau \in (0, \tau^*)$. Since at least one of $(A_{iHH}(\tau) - A_{iFH}(\tau), A_{iFF}(\tau) - A_{iHF}(\tau))$ is increasing (proof of Lemma A.4), they are both increasing when countries are symmetric. Since, in addition, $(A_{iHH}(\infty) - A_{iFH}(\infty), A_{iFF}(\infty) - A_{iHF}(\infty)) = (A_{iHe}, A_{iFe})$ (proof of Lemma A.6), (A.54) and (A.55) imply that when countries are symmetric, $\frac{\partial \mu^{(\tau)}_{hCCTt}}{\partial i_{Ht}} < 0$ and $\frac{\partial \mu^{(\tau)}_{hCCTt}}{\partial i_{Ft}} > 0$ for all $\tau > 0$, which means $\tau^* = \infty$.

Since (4.2) implies that the expected return of the basic CCT is

$$\mu_{CCTt} \equiv \mu_{et} + i_{Ft} - i_{Ht} = \lambda_{iHt}A_{iHe} - \lambda_{iFt}A_{iFe},$$

(A.24), (A.54) and (A.55) imply

$$\frac{\partial\left(\mu^{(\tau)}_{hCCTt} - \mu_{CCTt}\right)}{\partial i_{Ht}} = \bar{\lambda}_{rHH}(A_{iFH}(\tau) - A_{iHH}(\tau)) - \bar{\lambda}_{rHF}(A_{iHF}(\tau) - A_{iFF}(\tau)) > 0, \tag{A.56}$$

$$\frac{\partial\left(\mu^{(\tau)}_{hCCTt} - \mu_{CCTt}\right)}{\partial i_{Ft}} = \bar{\lambda}_{rFH}(A_{iFH}(\tau) - A_{iHH}(\tau)) - \bar{\lambda}_{rFF}(A_{iHF}(\tau) - A_{iFF}(\tau)) < 0, \tag{A.57}$$

where the inequalities follow because $(\bar{\lambda}_{rHH}, \bar{\lambda}_{rFF})$ are negative, $(\bar{\lambda}_{rHF}, \bar{\lambda}_{rFH})$ are non-negative, and $(A_{iHH}(\tau) - A_{iFH}(\tau), A_{iFF}(\tau) - A_{iHF}(\tau))$ are positive for all $\tau > 0$ (Lemma A.6). Hence, the sensitivity of the hybrid CCT's expected return to (i_{Ht}, i_{Ft}) is smaller (less negative in the case of i_{Ht} and less positive in the case of i_{Ft}) than for the basic CCT. Since $(A_{iHH}(\infty) - A_{iFH}(\infty), A_{iFF}(\infty) - A_{iHF}(\infty)) = (A_{iHe}, A_{iFe})$, (A.53) implies that $\mu^{(\tau)}_{hCCTt}$ goes to zero when τ goes to infinity, and (A.54) and (A.55) imply the same for $\frac{\partial \mu^{(\tau)}_{hCCTt}}{\partial i_{Ht}}$ and $\frac{\partial \mu^{(\tau)}_{hCCTt}}{\partial i_{Ft}}$.

Using (3.2), (4.1) and (4.14), we can write the return of the long-horizon CCT as

$$\begin{aligned} &A_{iHe}i_{Ht} - A_{iFe}i_{Ft} + C_e - (A_{iHe}i_{H,t+\tau} - A_{iFe}i_{F,t+\tau} + C_e) \\ &+ A_{iFF}(\tau)i_{Ft} + A_{iFH}(\tau)i_{Ht} + C_F(\tau) - (A_{iHH}(\tau)i_{Ht} + A_{iHF}(\tau)i_{Ft} + C_H(\tau)). \end{aligned}$$

Hence, (3.1) implies that the expected return of the long-horizon CCT is

$$\begin{aligned} \mu^{(\tau)}_{\ell CCTt} \equiv\ & A_{iHe}(1 - e^{-\kappa_{iH}\tau})(i_{Ht} - \bar{i}_H) - A_{iFe}(1 - e^{-\kappa_{iF}\tau})(i_{Ft} - \bar{i}_F) \\ & + A_{iFF}(\tau)i_{Ft} + A_{iFH}(\tau)i_{Ht} + C_F(\tau) - (A_{iHH}(\tau)i_{Ht} + A_{iHF}(\tau)i_{Ft} + C_H(\tau)), \end{aligned}$$

and its sensitivity to (i_{Ht}, i_{Ft}) is

$$\frac{\partial \mu^{(\tau)}_{\ell CCTt}}{\partial i_{Ht}} = A_{iHe}(1 - e^{-\kappa_{iH}\tau}) + A_{iFH}(\tau) - A_{iHH}(\tau), \tag{A.58}$$

$$\frac{\partial \mu^{(\tau)}_{\ell CCTt}}{\partial i_{Ft}} = -A_{iFe}(1 - e^{-\kappa_{iF}\tau}) + A_{iFF}(\tau) - A_{iHF}(\tau). \tag{A.59}$$

When $a > 0$, and $\alpha_e > 0$ or $\alpha_j(\tau) > 0$, $A_{iHe} < \frac{1}{\kappa_{iH}}$ and $A_{iFe} < \frac{1}{\kappa_{iF}}$. (These properties are shown in Proposition 4.2 for $a > 0$ and $\alpha_e > 0$. They also hold for $a > 0$ and $\alpha_j(\tau) > 0$ since $(\bar{\lambda}_{rHH}, \bar{\lambda}_{rFF})$

are negative and $(\bar{\lambda}_{rHF}, \bar{\lambda}_{rFH})$ are non-negative.) Since, in addition, $A'_{iHH}(0) = A'_{iFF}(0) = 1$ and $A'_{iHF}(0) = A'_{iFH}(0) = 0$, the derivative of (A.58) with respect to τ at $\tau = 0$ is negative, and the derivative of (A.59) with respect to τ at $\tau = 0$ is positive. Hence, there exists a threshold $\tau^* > 0$ such that $\frac{\partial \mu^{(\tau)}_{\ell CCTt}}{\partial i_{Ht}} < 0$ and $\frac{\partial \mu^{(\tau)}_{\ell CCTt}}{\partial i_{Ft}} > 0$ for all $\tau \in (0, \tau^*)$. When countries are symmetric, we set $\kappa_r \equiv \kappa_{iH} = \kappa_{iF}$, $\sigma_r \equiv \sigma_{iH} = \sigma_{iF}$, $A_{ie} \equiv A_{iHe} = A_{iFe}$, $\Delta A(\tau) \equiv A_{iHH}(\tau) - A_{iFH}(\tau) = A_{iFF}(\tau) - A_{iHF}(\tau)$, $\Delta\bar{\lambda} \equiv \bar{\lambda}_{rHH} - \bar{\lambda}_{rFH} = \bar{\lambda}_{rFF} - \bar{\lambda}_{rHF} < 0$. Taking the difference between (4.7) and (4.8) yields

$$\Delta A'(\tau) + \kappa_r \Delta A(\tau) - 1 = a\sigma_r^2 \Delta\bar{\lambda} \Delta A(\tau),$$

which integrates to

$$\Delta A(\tau) = A_{ie}\left(1 - e^{-(\kappa_r - a\sigma_r^2 \Delta\bar{\lambda})\tau}\right)$$

since $\Delta A(0) = 0$ and $\Delta A(\infty) = A_{ie}$. Substituting into (A.58) and (A.59), we find

$$\frac{\partial \mu^{(\tau)}_{\ell CCTt}}{\partial i_{Ht}} = -\frac{\partial \mu^{(\tau)}_{\ell CCTt}}{\partial i_{Ft}} = A_{ie}(e^{-(\kappa_r - a\sigma_r^2 \Delta\bar{\lambda})\tau} - e^{-\kappa_r \tau}) < 0. \tag{A.60}$$

Hence, $\tau^* = \infty$.

The expected return of the sequence of basic CCTs is

$$\mu^{(\tau)}_{CCTt} \equiv \mathbb{E}_t \int_t^{t+\tau} (\lambda_{rHt'} A_{iHe} - \lambda_{rFt'} A_{iFe})\, dt'.$$

Using (3.1) and (A.24), we find

$$\begin{aligned} \frac{\partial \mu^{(\tau)}_{CCTt}}{\partial i_{Ht}} &= \frac{1 - e^{-\kappa_{iH}\tau}}{\kappa_{iH}} \left(a\sigma_{iH}^2 \bar{\lambda}_{rHH} A_{iHe} - a\sigma_{iF}^2 \bar{\lambda}_{rHF} A_{iFe}\right) \\ &= \frac{1 - e^{-\kappa_{iH}\tau}}{\kappa_{iH}} (\kappa_{iH} A_{iHe} - 1), \end{aligned} \tag{A.61}$$

where the second step follows from (4.5). We likewise find

$$\frac{\partial \mu^{(\tau)}_{CCTt}}{\partial i_{Ft}} = -\frac{1 - e^{-\kappa_{iF}\tau}}{\kappa_{iF}} (\kappa_{iF} A_{iFe} - 1). \tag{A.62}$$

Combining (A.58) and (A.61), we find

$$\frac{\partial \left(\mu^{(\tau)}_{\ell CCTt} - \mu^{(\tau)}_{CCTt}\right)}{\partial i_{Ht}} = \frac{1 - e^{-\kappa_{iH}\tau}}{\kappa_{iH}} + A_{iFH}(\tau) - A_{iHH}(\tau) > 0,$$

where the inequality sign follows from (A.43) by noting that the left-hand side of (A.43) is negative. Combining (A.59) and (A.62), we likewise find

$$\frac{\partial\left(\mu_{\ell CCTt}^{(\tau)}-\mu_{CCTt}^{(\tau)}\right)}{\partial i_{Ft}}=-\frac{1-e^{-\kappa_{iF}\tau}}{\kappa_{iF}}+A_{iFF}(\tau)-A_{iHF}(\tau)<0.$$

Hence, the sensitivity of the long-horizon CCT's expected return to (i_{Ht}, i_{Ft}) is smaller (less negative in the case of i_{Ht} and less positive in the case of i_{Ft}) than for the corresponding sequence of basic CCTs. Since $(A_{iHH}(\infty)-A_{iFH}(\infty), A_{iFF}(\infty)-A_{iHF}(\infty))=(A_{iHe}, A_{iFe})$, (A.58) and (A.59) imply that $\frac{\partial\mu_{\ell CCTt}^{(\tau)}}{\partial i_{Ht}}$ and $\frac{\partial\mu_{\ell CCTt}^{(\tau)}}{\partial i_{Ft}}$ go to zero when τ goes to infinity. ∎

We next prove a lemma that we use in subsequent proofs.

Lemma A.7. *When $a>0$ and $\alpha_e>0$, the functions $\left(\frac{A_{iHF}(\tau)}{A_{iHH}(\tau)}, \frac{A_{iFH}(\tau)}{A_{iFF}(\tau)}\right)$ are increasing.*

Proof: The functions $(A_{iHH}(\tau), A_{iHF}(\tau))$ solve the system (A.29) of linear ODEs with constant coefficients. The solution is an affine function of $(e^{-\nu_1\tau}, e^{-\nu_2\tau})$, where (ν_1, ν_2) are the eigenvalues of the matrix M. Because of the initial conditions $A_{iHH}(0)=A_{iHF}(0)=0$, we can write the solution as a linear function of $\left(\frac{1-e^{-\nu_1\tau}}{\nu_1}, \frac{1-e^{-\nu_2\tau}}{\nu_2}\right)$. Because $(A'_{iHH}(0), A'_{iHF}(0))=(1,0)$, the coefficients of the linear terms sum to one for $A_{iHH}(\tau)$ and to zero for $A_{iHF}(\tau)$. Hence, we can write the solution as

$$A_{iHH}(\tau)=\frac{1-e^{-\nu_1\tau}}{\nu_1}+\phi_{HH}\left(\frac{1-e^{-\nu_2\tau}}{\nu_2}-\frac{1-e^{-\nu_1\tau}}{\nu_1}\right), \tag{A.63}$$

$$A_{iHF}(\tau)=\phi_{HF}\left(\frac{1-e^{-\nu_2\tau}}{\nu_2}-\frac{1-e^{-\nu_1\tau}}{\nu_1}\right), \tag{A.64}$$

for scalars (ϕ_{HH}, ϕ_{HF}). The eigenvalues (ν_1, ν_2) are positive (Lemma A.1), and without loss of generality we can set $\nu_1>\nu_2$. Since $A_{iFH}(\tau)$ is positive when $a>0$ and $\alpha_e>0$ (Lemma A.3), $\phi_{HF}>0$. Since

$$\frac{A_{iHH}(\tau)}{A_{iHF}(\tau)}=\frac{\frac{1-e^{-\nu_1\tau}}{\nu_1}}{\phi_{HF}\left(\frac{1-e^{-\nu_2\tau}}{\nu_2}-\frac{1-e^{-\nu_1\tau}}{\nu_1}\right)}+\frac{\phi_{HH}}{\phi_{HF}}=\frac{1}{\phi_{HF}\left(\frac{\nu_1}{\nu_2}\frac{1-e^{-\nu_2\tau}}{1-e^{-\nu_1\tau}}-1\right)}+\frac{\phi_{HH}}{\phi_{HF}},$$

and the function $(\nu_1, \nu_2, \tau)\longrightarrow\frac{1-e^{-\nu_2\tau}}{1-e^{-\nu_1\tau}}$ increases in τ because its derivative has the same sign as $\frac{e^{\nu_1\tau}-1}{\nu_1}-\frac{e^{\nu_2\tau}-1}{\nu_2}$, the function $\frac{A_{iHH}(\tau)}{A_{HF}(\tau)}$ is decreasing. Hence, the inverse function $\frac{A_{iHF}(\tau)}{A_{HH}(\tau)}$ is increasing. A similar argument using (A.30) establishes that $\frac{A_{iFH}(\tau)}{A_{FF}(\tau)}$ is increasing. ∎

Proof of Proposition 4.4: We prove the proposition in the case $j = H$. The proof for the case $j = F$ is symmetric. Consider a one-off increase in β_{Ht} at time zero, and denote by $\kappa_{\beta H}$ the rate at which β_{Ht} reverts to its mean of zero. Bond prices in country $j = H, F$ at time t are

$$P_{jt}^{(\tau)} = e^{-\left[A_{ijj}(\tau)i_{jt}+A_{ijj'}(\tau)i_{j't}+A_{\beta jH}(\tau)\beta_{Ht}+C_j(\tau)\right]}, \tag{A.65}$$

and the exchange rate is

$$e_t = e^{-\left[A_{iHe}i_{Ht}-A_{iFe}i_{Ft}+A_{\beta He}\beta_{Ht}+C_e\right]}, \tag{A.66}$$

where $(\{A_{ijj'}(\tau)\}_{j,j'=H,F}, \{A_{\beta jH}(\tau), C_j(\tau)\}_{j=H,F})$ are functions of τ, and $(\{A_{ije}\}_{j=H,F}, A_{\beta He}, C_e)$ are scalars.

The arbitrageurs' first-order condition (4.2) and (4.3) remains the same, with $(\mu_{et}, \mu_{Ht}^{(\tau)}, \mu_{Ft}^{(\tau)}, \lambda_{ijt})$ taking the values

$$\mu_{et} = - A_{iHe}\kappa_{iH}(\bar{i}_H - i_{Ht}) + A_{iFe}\kappa_{iF}(\bar{i}_F - i_{Ft}) + A_{\beta He}\kappa_{\beta H}\beta_{Ht} + \frac{1}{2}A_{iHe}^2\sigma_{iH}^2 + \frac{1}{2}A_{iFe}^2\sigma_{iF}^2, \tag{A.67}$$

$$\begin{aligned}\mu_{Ht}^{(\tau)} =& A'_{iHH}(\tau)i_{Ht} + A'_{iHF}(\tau)i_{Ft} + A'_{\beta HH}(\tau)\beta_{Ht} + C'_H(\tau) - A_{iHH}(\tau)\kappa_{iH}(\bar{i}_H - i_{Ht})\\ & - A_{iHF}(\tau)\kappa_{iF}(\bar{i}_F - i_{Ft}) + A_{\beta HH}(\tau)\kappa_{\beta H}\beta_{Ht} + \frac{1}{2}A_{iHH}(\tau)^2\sigma_{iH}^2 + \frac{1}{2}A_{iHF}(\tau)^2\sigma_{iF}^2,\end{aligned} \tag{A.68}$$

$$\begin{aligned}\mu_{Ft}^{(\tau)} =& A'_{iFH}(\tau)i_{Ht} + A'_{iFF}(\tau)i_{Ft} + A'_{\beta FH}(\tau)\beta_{Ht} + C'_F(\tau) - A_{iFH}(\tau)\kappa_{iH}(\bar{i}_H - i_{Ht})\\ & - A_{iFF}(\tau)\kappa_{iF}(\bar{i}_F - i_{Ft}) + A_{\beta FH}(\tau)\kappa_{\beta H}\beta_{Ht} + \frac{1}{2}A_{iFH}(\tau)\left(A_{iFH}(\tau) + 2A_{iHe}\right)\sigma_{iH}^2\\ & + \frac{1}{2}A_{iFF}(\tau)\left(A_{iFF}(\tau) - 2A_{iFe}\right)\sigma_{iF}^2,\end{aligned} \tag{A.69}$$

$$\lambda_{ijt} = a\sigma_{ij}^2\left(\bar{\lambda}_{ijj}i_{jt} + \bar{\lambda}_{rj'j}i_{j't} + \bar{\lambda}_{\beta Hj}\beta_{Ht} + \bar{\lambda}_{ijC}\right), \tag{A.70}$$

instead of those in (3.5), (A.20), (A.22) and (A.24), and $\lambda_{\beta Hj}$ taking the value

$$\bar{\lambda}_{\beta Hj} \equiv \int_0^T [\theta_H(\tau) - \alpha_H(\tau)A_{\beta HH}(\tau)]A_{iHj}(\tau)d\tau - \int_0^T \alpha_F(\tau)A_{\beta FH}(\tau)A_{iFj}(\tau)d\tau - \alpha_e A_{\beta He}A_{ije}(-1)^{1_{j=F}}. \tag{A.71}$$

We next substitute $(\mu_{et}, \mu_{Ht}^{(\tau)}, \mu_{Ft}^{(\tau)}, \lambda_{ijt})$ from (A.67)-(A.70) into the arbitrageurs' first-order condition. Substituting into (4.2) and identifying terms in β_{Ht}, we find

$$\kappa_{\beta H}A_{\beta He} = a\sigma_{iH}^2\bar{\lambda}_{\beta HH}A_{iHe} - a\sigma_{iF}^2\bar{\lambda}_{\beta HF}A_{iFe}. \tag{A.72}$$

Substituting into (4.3) and identifying terms in β_{Ht}, we find

$$A'_{\beta jH}(\tau) + \kappa_{\beta H} A_{\beta jH}(\tau) = a\sigma_{iH}^2 \bar{\lambda}_{\beta HH} A_{ijH}(\tau) + a\sigma_{iF}^2 \bar{\lambda}_{\beta HF} A_{ijF}(\tau), \tag{A.73}$$

which integrates to

$$A_{\beta jH}(\tau) = a\sigma_{iH}^2 \bar{\lambda}_{\beta HH} \int_0^\tau A_{ijH}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau' + a\sigma_{iF}^2 \bar{\lambda}_{\beta HF} \int_0^\tau A_{ijF}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau', \tag{A.74}$$

since $A_{\beta jH}(0) = 0$. Substituting $A_{\beta He}$ from (A.72) and $\{A_{\beta jH}(\tau)\}_{j=H,F}$ from (A.74) into (A.71), we find

$$(1 + a\sigma_{iH}^2 z_{HH})\bar{\lambda}_{\beta HH} + a\sigma_{iF}^2 z_{HF} \bar{\lambda}_{\beta HF} = \int_0^T \theta_H(\tau) A_{iHH}(\tau) d\tau, \tag{A.75}$$

$$a\sigma_{iH}^2 z_{FH} \bar{\lambda}_{\beta HH} + (1 + a\sigma_{iF}^2 z_{FF})\bar{\lambda}_{\beta HF} = \int_0^T \theta_H(\tau) A_{iHF}(\tau) d\tau, \tag{A.76}$$

where

$$\begin{aligned}
z_{HH} &= \int_0^T \alpha_H(\tau) A_{iHH}(\tau) \left[\int_0^\tau A_{iHH}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau'\right] d\tau \\
&\quad + \int_0^T \alpha_F(\tau) A_{iFH}(\tau) \left[\int_0^\tau A_{iFH}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau'\right] d\tau + \frac{\alpha_e}{\kappa_{\beta H}} A_{iHe}^2, \\
z_{HF} &= \int_0^T \alpha_H(\tau) A_{iHH}(\tau) \left[\int_0^\tau A_{iHF}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau'\right] d\tau \\
&\quad + \int_0^T \alpha_F(\tau) A_{iFH}(\tau) \left[\int_0^\tau A_{iFF}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau'\right] d\tau - \frac{\alpha_e}{\kappa_{\beta H}} A_{iHe} A_{iFe}, \\
z_{FH} &= \int_0^T \alpha_H(\tau) A_{iHF}(\tau) \left[\int_0^\tau A_{iHH}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau'\right] d\tau \\
&\quad + \int_0^T \alpha_F(\tau) A_{iFF}(\tau) \left[\int_0^\tau A_{iFH}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau'\right] d\tau - \frac{\alpha_e}{\kappa_{\beta H}} A_{iHe} A_{iFe}, \\
z_{FF} &= \int_0^T \alpha_H(\tau) A_{iHF}(\tau) \left[\int_0^\tau A_{iHF}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau'\right] d\tau \\
&\quad + \int_0^T \alpha_F(\tau) A_{iFF}(\tau) \left[\int_0^\tau A_{iFF}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau'\right] d\tau + \frac{\alpha_e}{\kappa_{\beta H}} A_{iFe}^2.
\end{aligned}$$

Equations (A.75) and (A.76) form a linear system of two equations in the two unknowns $(\bar{\lambda}_{\beta HH}, \bar{\lambda}_{\beta HF})$.

Its solution is

$$\bar{\lambda}_{\beta HH} = \frac{1}{\Delta_z}\left[(1+a\sigma_{iF}^2 z_{FF})\int_0^T \theta_H(\tau)A_{iHH}(\tau)d\tau - a\sigma_{iF}^2 z_{HF}\int_0^T \theta_H(\tau)A_{iHF}(\tau)d\tau\right] \quad \text{(A.77)}$$

$$\bar{\lambda}_{\beta HF} = \frac{1}{\Delta_z}\left[(1+a\sigma_{iH}^2 z_{HH})\int_0^T \theta_H(\tau)A_{iHF}(\tau)d\tau - a\sigma_{iH}^2 z_{FH}\int_0^T \theta_H(\tau)A_{iHH}(\tau)d\tau\right], \quad \text{(A.78)}$$

where

$$\Delta_z \equiv (1+a\sigma_{iH}^2 z_{HH})(1+a\sigma_{iF}^2 z_{FF}) - a^2\sigma_{iH}^2\sigma_{iF}^2 z_{HF}z_{FH}.$$

The statements in the proposition concern the signs of $(A_{\beta HH}(\tau), A_{\beta FH}(\tau), A_{\beta He})$. To determine these signs, we proceed in four steps. In Step 1, we show that Δ_z is positive. In Step 2, we show that (z_{HF}, z_{FH}) are non-negative, and are zero when $\alpha_e = 0$. In Step 3, we show that $A_{\beta HH}(\tau)$ is positive, and that $A_{\beta FH}(\tau)$ is positive when $\alpha_e > 0$ and zero when $\alpha_e = 0$. In Step 4, we show that $A_{\beta He}$ is positive. The first statement in the proposition follows from the first result in Step 3. The second statement follows from the second result in Step 3. The third statement follows from the result in Step 4.

Step 1: Δ_z is positive. Since (z_{HH}, z_{FF}) are non-negative, Δ_z is positive under the sufficient condition

$$z_{HH}z_{FF} \geq z_{HF}z_{FH}. \quad \text{(A.79)}$$

The function

$$\begin{aligned}
F(\mu) &\equiv z_{HH} + \mu(z_{HF}+z_{FH}) + \mu^2 z_{FF} \\
&= \int_0^T \alpha_H(\tau)\left[A_{iHH}(\tau)+\mu A_{iHF}(\tau)\right]\left[\int_0^T \left[A_{iHH}(\tau)+\mu A_{iHF}(\tau)\right]e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'\right]d\tau \\
&\quad + \int_0^T \alpha_F(\tau)\left[A_{iFH}(\tau)+\mu A_{iFF}(\tau)\right]\left[\int_0^T \left[A_{iFH}(\tau)+\mu A_{iFF}(\tau)\right]e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'\right]d\tau \\
&\quad + \frac{\alpha_e}{\kappa_{\beta H}}\left(A_{iHe}-\mu A_{iFe}\right)^2
\end{aligned}$$

is non-negative for all μ if

$$F_0 \equiv \int_0^T \alpha(\tau)A(\tau)\left[\int_0^\tau A(\tau')e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'\right]d\tau$$

is non-negative for a non-negative and non-increasing $\alpha(\tau)$. Since

$$F_0 = \int_0^T \phi(\tau)\Phi(\tau)\left[\int_0^\tau \Phi(\tau')d\tau'\right]d\tau,$$

where

$$\phi(\tau) \equiv \alpha(\tau)e^{-2\kappa_{\beta H}\tau},$$

$$\Phi(\tau) \equiv A(\tau)e^{\kappa_{\beta H}\tau},$$

integration by parts implies

$$F_0 = \frac{1}{2}\phi(T)\left[\int_0^T \Phi(\tau)d\tau\right]^2 - \frac{1}{2}\int_0^T \phi'(\tau)\left[\int_0^\tau \Phi(\tau')d\tau'\right]^2 d\tau. \tag{A.80}$$

The first term in the right-hand side of (A.80) is non-negative because $\alpha(\tau)$ is non-negative, and the first term is non-positive because $\alpha(\tau)$ is non-increasing. Therefore, F_0 is non-negative. Since $F(\mu)$ is quadratic in μ, its non-negativity for all μ implies

$$4z_{HH}z_{FF} \geq (z_{HF}+z_{FH})^2$$

$$\Rightarrow z_{HH}z_{FF} \geq \frac{1}{4}(z_{HF}+z_{FH})^2 = z_{HF}z_{FH} + \frac{1}{4}(z_{HF}-z_{FH})^2 \geq z_{HF}z_{FH}.$$

Therefore, (A.79) holds.

Step 2: (z_{HF}, z_{FH}) are non-negative, and are zero when $\alpha_e = 0$. Since Lemma A.3 implies that $A_{iHH}(\tau)$ is positive and $A_{iFH}(\tau)$ is non-negative, and Lemma A.4 implies that $A_{iHF}(\tau)$ is non-decreasing and $A_{iFF}(\tau)$ is increasing,

$$\begin{aligned}
z_{HF} \leq & \int_0^T \alpha_H(\tau)A_{iHH}(\tau)\left[\int_0^\tau A_{iHF}(\tau)e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'\right]d\tau \\
& + \int_0^T \alpha_F(\tau)A_{iFH}(\tau)\left[\int_0^\tau A_{iFF}(\tau)e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'\right]d\tau - \frac{\alpha_e}{\kappa_{\beta H}}A_{iHe}A_{iFe} \\
\leq & \int_0^T \alpha_H(\tau)A_{iHH}(\tau)\frac{A_{iHF}(\tau)}{\kappa_{\beta H}}d\tau + \int_0^T \alpha_F(\tau)A_{iFH}(\tau)\frac{A_{iFF}(\tau)}{\kappa_{\beta H}} - \frac{\alpha_e}{\kappa_{\beta H}}A_{iHe}A_{iFe} \\
= & -\frac{\bar{\lambda}_{rHF}}{\kappa_{\beta H}} \leq 0,
\end{aligned}$$

where the second step follows because $(A_{iHH}(\tau), A_{iFF}(\tau))$ are positive and $(A_{iFH}(\tau), A_{iFH}(\tau))$ are non-negative, the third step follows from (4.11), and the fourth step follows from Lemma A.2. The inequality $z_{FH} \leq 0$ follows similarly.

When $\alpha_e = 0$, Lemma A.3 implies $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$. Therefore, $z_{HF} = z_{FH} = 0$.

Step 3: $A_{\beta HH}(\tau)$ is positive, and $A_{\beta FH}(\tau)$ is positive when $\alpha_e > 0$ and zero when $\alpha_e = 0$. Since $(\Delta_z, \theta_H(\tau), A_{iHH}(\tau))$ are positive, $(A_{iHF}(\tau), z_{FF})$ are non-negative, and z_{HF} is non-positive, (A.77) implies that $\bar{\lambda}_{\beta HH}$ is positive. When $\alpha_e > 0$, $A_{iHF}(\tau)$ is positive. Since, in addition, z_{HH} is non-negative and z_{FH} is non-positive, (A.78) implies that $\bar{\lambda}_{\beta HF}$ is positive. When $\alpha_e = 0$, (A.78) and $A_{iHF}(\tau) = z_{FH} = 0$ imply $\bar{\lambda}_{\beta HF} = 0$.

Since $(\bar{\lambda}_{\beta HH}, A_{iHH}(\tau))$ are positive and $(\bar{\lambda}_{\beta HF}, A_{iHF}(\tau))$ are non-negative, (A.73) implies that $A_{\beta HH}(\tau)$ is positive. When $\alpha_e > 0$, $A_{iFH}(\tau)$ is positive. Since, in addition, $(\bar{\lambda}_{\beta HF}, A_{iFF}(\tau))$ are positive, (A.73) implies that $A_{\beta FH}(\tau)$ is positive. When $\alpha_e = 0$, (A.73) and $A_{iFH}(\tau) = \bar{\lambda}_{\beta HF} = 0$ imply $A_{\beta FH}(\tau) = 0$.

Step 4: $A_{\beta He}(\tau)$ is positive. Substituting $(\bar{\lambda}_{\beta HH}, \bar{\lambda}_{\beta HF})$ from (A.77) and (A.78) into (A.72), and using the definitions of $(z_{HH}, z_{HF}, z_{FH}, z_{FF})$, we find that $A_{\beta He}$ is positive if

$$Z_H \int_0^T \theta_H(\tau) A_{iHH}(\tau) d\tau - Z_F \int_0^T \theta_H(\tau) A_{iHF}(\tau) d\tau > 0, \tag{A.81}$$

where

$$\begin{aligned}
Z_H &\equiv \sigma_{iH}^2 (1 + a\sigma_{iF}^2 z_{FF}) A_{iHe} + a\sigma_{iH}^2 \sigma_{iF}^2 z_{FH} A_{iFe} \\
&= \sigma_{iH}^2 A_{iHe} \\
&\quad + a\sigma_{iH}^2 \sigma_{iF}^2 \int_0^T \alpha_H(\tau) A_{iHF}(\tau) \left[\int_0^\tau [A_{iHe} A_{iHF}(\tau') + A_{iFe} A_{iHH}(\tau')] e^{-\kappa_{\beta H}(\tau - \tau')} d\tau' \right] d\tau \\
&\quad + a\sigma_{iH}^2 \sigma_{iF}^2 \int_0^T \alpha_F(\tau) A_{iFF}(\tau) \left[\int_0^\tau [A_{iHe} A_{iFF}(\tau') + A_{iFe} A_{iFH}(\tau')] e^{-\kappa_{\beta H}(\tau - \tau')} d\tau' \right] d\tau, \\
Z_F &\equiv a\sigma_{iH}^2 \sigma_{iF}^2 z_{HF} A_{iHe} + \sigma_{iF}^2 (1 + a\sigma_{iH}^2 z_{HH}) A_{iFe} \\
&= \sigma_{iF}^2 A_{iFe} \\
&\quad + a\sigma_{iH}^2 \sigma_{iF}^2 \int_0^T \alpha_H(\tau) A_{iHH}(\tau) \left[\int_0^\tau [A_{iHe} A_{iHF}(\tau') + A_{iFe} A_{iHH}(\tau')] e^{-\kappa_{\beta H}(\tau - \tau')} d\tau' \right] d\tau \\
&\quad + a\sigma_{iH}^2 \sigma_{iF}^2 \int_0^T \alpha_F(\tau) A_{iFH}(\tau) \left[\int_0^\tau [A_{iHe} A_{iFF}(\tau') + A_{iFe} A_{iFH}(\tau')] e^{-\kappa_{\beta H}(\tau - \tau')} d\tau' \right] d\tau.
\end{aligned}$$

Since $(\theta(\tau), A_{iHH}(\tau))$ are positive, $A_{iHF}(\tau)$ is non-negative, and $\frac{A_{iHF}(\tau)}{A_{iHH}(\tau)}$ is non-decreasing (increasing when $a > 0$ and $\alpha_e > 0$ from Lemma A.7, and zero when $a = 0$ or $\alpha_e = 0$), the ratio

$\frac{\int_0^T \theta_H(\tau) A_{iHF}(\tau) d\tau}{\int_0^T \theta_H(\tau) A_{iHH}(\tau) d\tau}$ is bounded above by $\frac{A_{iHF}(\infty)}{A_{iHH}(\infty)}$. Since, in addition (Z_H, Z_F) are positive, (A.81) holds for all positive functions $\theta_H(\tau)$ under the sufficient condition

$$Z_H A_{iHH}(\infty) - Z_F A_{iHF}(\infty) > 0. \tag{A.82}$$

Using the definitions of (Z_H, Z_F), we can write (A.82) as

$$\begin{aligned}
&\sigma_{iH}^2 A_{iHe} A_{iHH}(\infty) - \sigma_{iF}^2 A_{iFe} A_{iHF}(\infty) \\
&+ a\sigma_{iH}^2 \sigma_{iF}^2 \int_0^T \alpha_H(\tau) \left[A_{iHF}(\tau) A_{iHH}(\infty) - A_{iHH}(\tau) A_{iHF}(\infty)\right] \\
&\times \left[\int_0^\tau [A_{iHe} A_{iHF}(\tau') + A_{iFe} A_{iHH}(\tau')] e^{-\kappa_{\beta H}(\tau-\tau')} d\tau'\right] d\tau \\
&+ a\sigma_{iH}^2 \sigma_{iF}^2 \int_0^T \alpha_F(\tau) \left[A_{iFF}(\tau) A_{iHH}(\infty) - A_{iFH}(\tau) A_{iHF}(\infty)\right] \\
&\times \left[\int_0^\tau [A_{iHe} A_{iFF}(\tau') + A_{iFe} A_{iFH}(\tau')] e^{-\kappa_{\beta H}(\tau-\tau')} d\tau'\right] d\tau > 0.
\end{aligned} \tag{A.83}$$

Equation (4.8) for $(j, j') = (H, F)$ implies

$$A_{iHF}(\tau) = \frac{a\sigma_{iH}^2 \bar{\lambda}_{rFH} A_{iHH}(\tau)}{\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{rFF}} - \frac{A'_{iHF}(\tau)}{\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{rFF}}, \tag{A.84}$$

which for $\tau = \infty$ becomes

$$A_{iHF}(\infty) = \frac{a\sigma_{iH}^2 \bar{\lambda}_{rFH} A_{iHH}(\infty)}{\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{rFF}}. \tag{A.85}$$

Equation (4.7) for $j = F$ implies

$$A_{iFF}(\tau) = \frac{a\sigma_{iH}^2 \bar{\lambda}_{rFH} A_{iFH}(\tau)}{\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{rFF}} + \frac{1 - A'_{iFF}(\tau)}{\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{rFF}}. \tag{A.86}$$

Using (A.84)-(A.86) to simplify the terms in the first, second and fourth lines of (A.83), and dividing throughout by $\frac{a\sigma_{iH}^2\sigma_{iF}^2 A_{iHH}(\infty)}{\kappa_{iF}-a\sigma_{iF}^2\bar{\lambda}_{rFF}} > 0$, we find that (A.83) is equivalent to

$$\begin{aligned}
&\left(\frac{\kappa_{iF}}{a\sigma_{iF}^2} - \bar{\lambda}_{rFF}\right) A_{iHe} - \bar{\lambda}_{rFH} A_{iFe} \\
&- \int_0^T \alpha_H(\tau) A'_{iHF}(\tau) \left[\int_0^\tau [A_{iHe}A_{iHF}(\tau') + A_{iFe}A_{iHH}(\tau')] e^{-\kappa_{\beta H}(\tau-\tau')} d\tau'\right] d\tau \\
&+ \int_0^T \alpha_F(\tau)(1 - A'_{iFF}(\tau)) \left[\int_0^\tau [A_{iHe}A_{iFF}(\tau') + A_{iFe}A_{iFH}(\tau')] e^{-\kappa_{\beta H}(\tau-\tau')} d\tau'\right] d\tau > 0.
\end{aligned} \tag{A.87}$$

Equations (4.10) and (4.11) imply

$$\begin{aligned}
&- \bar{\lambda}_{rFF} A_{iHe} - \bar{\lambda}_{rFH} A_{iFe} \\
&= \int_0^T \alpha_H(\tau) A_{iHF}(\tau) [A_{iHe}A_{iHF}(\tau) + A_{iFe}A_{iHH}(\tau)] d\tau \\
&\quad + \int_0^T \alpha_F(\tau) A_{iFF}(\tau) [A_{iHe}A_{iFF}(\tau) + A_{iFe}A_{iFH}(\tau)] d\tau.
\end{aligned} \tag{A.88}$$

We next substitute (A.88) into (A.87). Noting that $1 - A'_{iFF}(\tau) > 0$, which follows from (4.7) for $j = F$ and (A.52), and that $(A_{iHH}(\tau), A_{iFF}(\tau), A_{iHe}, A_{iFe})$ are positive and $(A_{iFH}(\tau), A_{iFH}(\tau))$ are non-negative, we find that (A.87) holds under the sufficient condition

$$\begin{aligned}
&\int_0^T \alpha_H(\tau) \Bigg\{ A_{iHF}(\tau) [A_{iHe}A_{iHF}(\tau) + A_{iFe}A_{iHH}(\tau)] d\tau \\
&- A'_{iHF}(\tau) \left[\int_0^\tau [A_{iHe}A_{iHF}(\tau') + A_{iFe}A_{iHH}(\tau')] e^{-\kappa_{\beta H}(\tau-\tau')} d\tau'\right] \Bigg\} d\tau \geq 0,
\end{aligned}$$

which, in turn, holds under the sufficient condition

$$\int_0^T \alpha_H(\tau) \Bigg\{ A_{iHF}(\tau) [A_{iHe}A_{iHF}(\tau) + A_{iFe}A_{iHH}(\tau)] d\tau \tag{A.89}$$

$$- A'_{iHF}(\tau) \left[\int_0^\tau [A_{iHe}A_{iHF}(\tau') + A_{iFe}A_{iHH}(\tau')] d\tau'\right] \Bigg\} d\tau \geq 0. \tag{A.90}$$

Equation (A.90) holds under the sufficient condition that the function

$$G(\tau) \equiv \frac{A_{iHF}(\tau)}{\int_0^\tau [A_{iHe}A_{iHF}(\tau') + A_{iFe}A_{iHH}(\tau')] d\tau'}$$

is non-increasing because the term in curly brackets in (A.90) is the negative of the numerator of $G'(\tau)$. The function $G'(\tau)$ is non-increasing under the sufficient condition that the function

$$G_1(\tau) \equiv \frac{A'_{iHF}(\tau)}{A_{iHe}A_{iHF}(\tau) + A_{iFe}A_{iHH}(\tau)}$$

is non-increasing. Equation (4.8) for $(j, j') = (H, F)$ implies

$$\begin{aligned} G_1(\tau) &= \frac{a\sigma_{iH}^2\bar{\lambda}_{rFH}A_{iHH}(\tau) + (a\sigma_{iF}^2\bar{\lambda}_{rFF} - \kappa_{iF})A_{iHF}(\tau)}{A_{iHe}A_{iHF}(\tau) + A_{iFe}A_{iHH}(\tau)} \\ &= \frac{a\sigma_{iH}^2\bar{\lambda}_{rFH} + (a\sigma_{iF}^2\bar{\lambda}_{rFF} - \kappa_{iF})\frac{A_{iHF}(\tau)}{A_{iHH}(\tau)}}{A_{iHe}\frac{A_{iHF}(\tau)}{A_{iHH}(\tau)} + A_{iFe}}. \end{aligned}$$

Since $\bar{\lambda}_{rFH} \geq 0$, $\bar{\lambda}_{rFF} \leq 0$ and $\frac{A_{iHF}(\tau)}{A_{iHH}(\tau)}$ is non-decreasing, $G_1(\tau)$ is non-increasing. ∎

Proof of Proposition 4.5: Consider a one-off increase in γ_t at time zero, and denote by κ_γ the rate at which γ_t reverts to its mean of zero. Bond prices in country $j = H, F$ at time t are

$$P_{jt}^{(\tau)} = e^{-\left[A_{ijj}(\tau)i_{jt} + A_{ijj'}(\tau)i_{j't} + A_{\gamma j}(\tau)\gamma_t + C_j(\tau)\right]}, \tag{A.91}$$

and the exchange rate is

$$e_t = e^{-[A_{iHe}i_{Ht} - A_{iFe}i_{Ft} + A_{\gamma e}\gamma_t + C_e]}, \tag{A.92}$$

where $(\{A_{ijj'}(\tau)\}_{j,j'=H,F}, \{A_{\gamma j}(\tau), C_j(\tau)\}_{j=H,F})$ are functions of τ, and $(\{A_{ije}\}_{j=H,F}, A_{\gamma e}, C_e)$ are scalars.

The counterparts of (A.72) and (A.74) are

$$\kappa_\gamma A_{\gamma e} = a\sigma_{iH}^2\bar{\lambda}_{\gamma H}A_{iHe} - a\sigma_{iF}^2\bar{\lambda}_{\gamma F}A_{iFe} \tag{A.93}$$

and

$$A_{\gamma j}(\tau) = a\sigma_{iH}^2\bar{\lambda}_{\gamma H}\int_0^\tau A_{ijH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau' + a\sigma_{iF}^2\bar{\lambda}_{\gamma F}\int_0^\tau A_{ijF}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau', \tag{A.94}$$

respectively, where

$$\bar{\lambda}_{\gamma j} \equiv -\int_0^T \alpha_H(\tau)A_{\gamma H}(\tau)A_{iHj}(\tau)d\tau - \int_0^T \alpha_F(\tau)A_{\gamma F}(\tau)A_{iFj}(\tau)d\tau + (\theta_e - \alpha_e A_{\gamma e})A_{ije}(-1)^{1_{j=F}}.$$

$$(A.95)$$

The counterparts of (A.77) and (A.78) are

$$\bar{\lambda}_{\gamma H} = \frac{\theta_e}{\Delta_z}\left[(1 + a\sigma_{iF}^2 z_{FF})A_{iHe} + a\sigma_{iF}^2 z_{HF} A_{iFe}\right] \tag{A.96}$$

$$\bar{\lambda}_{\gamma F} = -\frac{\theta_e}{\Delta_z}\left[(1 + a\sigma_{iH}^2 z_{HH})A_{iFe} + a\sigma_{iH}^2 z_{FH} A_{iHe}\right], \tag{A.97}$$

respectively, where

$$\hat{z}_{HH} = \int_0^T \alpha_H(\tau)A_{iHH}(\tau)\left[\int_0^\tau A_{iHH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau$$
$$+ \int_0^T \alpha_F(\tau)A_{iFH}(\tau)\left[\int_0^\tau A_{iFH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau + \frac{\alpha_e}{\kappa_{\beta H}}A_{iHe}^2,$$

$$\hat{z}_{HF} = \int_0^T \alpha_H(\tau)A_{iHH}(\tau)\left[\int_0^\tau A_{iHF}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau$$
$$+ \int_0^T \alpha_F(\tau)A_{iFH}(\tau)\left[\int_0^\tau A_{iFF}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau - \frac{\alpha_e}{\kappa_{\beta H}}A_{iHe}A_{iFe},$$

$$\hat{z}_{FH} = \int_0^T \alpha_H(\tau)A_{iHF}(\tau)\left[\int_0^\tau A_{iHH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau$$
$$+ \int_0^T \alpha_F(\tau)A_{iFF}(\tau)\left[\int_0^\tau A_{iFH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau - \frac{\alpha_e}{\kappa_{\beta H}}A_{iHe}A_{iFe},$$

$$\hat{z}_{FF} = \int_0^T \alpha_H(\tau)A_{iHF}(\tau)\left[\int_0^\tau A_{iHF}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau$$
$$+ \int_0^T \alpha_F(\tau)A_{iFF}(\tau)\left[\int_0^\tau A_{iFF}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau + \frac{\alpha_e}{\kappa_{\beta H}}A_{iFe}^2.$$

To complete the proof, we proceed in two steps. In Step 1, we show that $A_{\gamma e}$ is positive. This proves the first statement in the proposition. In Step 2, we show that $A_{\gamma H}(\tau)$ is positive and $A_{\gamma F}(\tau)$ is negative. This proves the second and third statements in the proposition.

Step 1: $A_{\gamma e}(\tau)$ is positive. Substituting $(\bar{\lambda}_{\gamma H}, \bar{\lambda}_{\gamma F})$ from (A.96) and (A.97) into (A.93), , and using the definitions of $(\hat{z}_{HH}, \hat{z}_{HF}, \hat{z}_{FH}, \hat{z}_{FF})$, we find that $A_{\gamma e}$ is positive if

$$\hat{Z}_H A_{iHe} + \hat{Z}_F A_{iFe} > 0, \tag{A.98}$$

where

$$\begin{aligned}
\hat{Z}_H &\equiv \sigma_{iH}^2 \left[(1 + a\sigma_{iF}^2 z_{FF})A_{iHe} + a\sigma_{iF}^2 z_{HF} A_{iFe}\right] \\
&= \sigma_{iH}^2 A_{iHe} \\
&\quad + a\sigma_{iH}^2\sigma_{iF}^2 \int_0^T \alpha_H(\tau)[A_{iHe}A_{iHF}(\tau) + A_{iFe}A_{iHH}(\tau)] \left[\int_0^\tau A_{iHF}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right] d\tau \\
&\quad + a\sigma_{iH}^2\sigma_{iF}^2 \int_0^T \alpha_F(\tau)[A_{iHe}A_{iFF}(\tau) + A_{iFe}A_{iFH}(\tau)] \left[\int_0^\tau A_{iFF}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right] d\tau, \\
\hat{Z}_F &\equiv \sigma_{iF}^2 \left[(1 + a\sigma_{iH}^2 z_{HH})A_{iFe} + a\sigma_{iH}^2 z_{FH} A_{iHe}\right] \\
&= \sigma_{iF}^2 A_{iFe} \\
&\quad + a\sigma_{iH}^2\sigma_{iF}^2 \int_0^T \alpha_H(\tau)[A_{iHe}A_{iHF}(\tau) + A_{iFe}A_{iHH}(\tau)] \left[\int_0^\tau A_{iHH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right] d\tau \\
&\quad + a\sigma_{iH}^2\sigma_{iF}^2 \int_0^T \alpha_F(\tau)[A_{iHe}A_{iFF}(\tau) + A_{iFe}A_{iFH}(\tau)] \left[\int_0^\tau A_{iFH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right] d\tau.
\end{aligned}$$

Since $(A_{iHe}, A_{iFe}, Z_H, Z_F)$ are positive, (A.98) holds.

Step 2: $A_{\gamma H}(\tau)$ is positive and that $A_{\gamma F}(\tau)$ is negative. We prove that $A_{\gamma H}(\tau)$ is positive. The proof that $A_{\gamma F}(\tau)$ is negative is symmetric. Substituting $(\bar{\lambda}_{\gamma H}, \bar{\lambda}_{\gamma F})$ from (A.96) and (A.97) into (A.94) for $j = H$, and using the definitions of $(\hat{z}_{HH}, \hat{z}_{HF}, \hat{z}_{FH}, \hat{z}_{FF})$, we find that $A_{\gamma H}(\tau)$ is positive if

$$\hat{Z}_H \int_0^\tau A_{iHH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau' - \hat{Z}_F \int_0^\tau A_{iHF}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau' > 0. \tag{A.99}$$

Since $(A_{iHH}(\tau), \hat{Z}_H, \hat{Z}_F)$ are positive, $A_{iHF}(\tau)$ is non-negative and $\frac{A_{iHF}(\tau)}{A_{iHH}(\tau)}$ is non-decreasing, (A.81) holds under the sufficient condition

$$\hat{Z}_H A_{iHH}(\infty) - \hat{Z}_F A_{iHF}(\infty) > 0. \tag{A.100}$$

Using the definitions of $(\hat{Z}_H, \hat{Z}_F)$, we can write (A.100) as

$$\begin{aligned}
&\sigma_{iH}^2 A_{iHe} A_{iHH}(\infty) - \sigma_{iF}^2 A_{iFe} A_{iHF}(\infty) \\
&+ a\sigma_{iH}^2\sigma_{iF}^2 \int_0^T \alpha_H(\tau)[A_{iHe}A_{iHF}(\tau) + A_{iFe}A_{iHH}(\tau)] \\
&\times \left[\int_0^\tau \left[A_{iHF}(\tau')A_{iHH}(\infty) - A_{iHH}(\tau')A_{iHF}(\infty)\right] e^{-\kappa_\gamma(\tau-\tau')} d\tau'\right] d\tau \\
&+ a\sigma_{iH}^2\sigma_{iF}^2 \int_0^T \alpha_F(\tau)[A_{iHe}A_{iFF}(\tau) + A_{iFe}A_{iFH}(\tau)] \\
&\times \left[\int_0^\tau \left[A_{iFF}(\tau')A_{iHH}(\infty) - A_{iFH}(\tau')A_{iHF}(\infty)\right] e^{-\kappa_\gamma(\tau-\tau')} d\tau'\right] d\tau > 0.
\end{aligned} \tag{A.101}$$

Using (A.84)-(A.86) to simplify the terms in the first, second and fourth lines of (A.101), and dividing throughout by $\frac{a\sigma_{iH}^2\sigma_{iF}^2 A_{iHH}(\infty)}{\kappa_{iF} - a\sigma_{iF}^2\bar{\lambda}_{rFF}} > 0$, we find that (A.101) is equivalent to

$$\begin{aligned}
&\left(\frac{\kappa_{iF}}{a\sigma_{iF}^2} - \bar{\lambda}_{rFF}\right) A_{iHe} - \bar{\lambda}_{rFH} A_{iFe} \\
&- \int_0^T \alpha_H(\tau)[A_{iHe}A_{iHF}(\tau) + A_{iFe}A_{iHH}(\tau)] \left[\int_0^\tau A'_{iHF}(\tau') e^{-\kappa_\gamma(\tau-\tau')} d\tau'\right] d\tau \\
&+ \int_0^T \alpha_F(\tau)[A_{iHe}A_{iFF}(\tau) + A_{iFe}A_{iFH}(\tau)] \left[\int_0^\tau (1 - A'_{iFF}(\tau')) e^{-\kappa_\gamma(\tau-\tau')} d\tau'\right] d\tau > 0.
\end{aligned} \tag{A.102}$$

We next substitute (A.88) into (A.102). Noting that $1 - A'_{iFF}(\tau) > 0$ and that $(A_{iHH}(\tau), A_{iFF}(\tau), A_{iHe}, A_{iFe})$ are positive and $(A_{iFH}(\tau), A_{iFH}(\tau))$ are non-negative, we find that (A.87) holds under the sufficient condition

$$\int_0^T \alpha_H(\tau)[A_{iHe}A_{iHF}(\tau) + A_{iFe}A_{iHH}(\tau)] \left[A_{iHF}(\tau) - \int_0^\tau A'_{iHF}(\tau') e^{-\kappa_\gamma(\tau-\tau')} d\tau'\right] d\tau \geq 0,$$

which, in turn, holds because

$$A_{iHF}(\tau) - \int_0^\tau A'_{iHF}(\tau) e^{-\kappa_\gamma(\tau-\tau')} d\tau' \geq A_{iHF}(\tau) - \int_0^\tau A'_{iHF}(\tau') d\tau' = A_{iHF}(0) = 0.$$

■

Proof of Proposition 5.1: Applying Ito's Lemma to (5.1), we find the following counterpart of

(3.4):

$$\frac{de_t}{e_t} = \mu_{et}dt - A_e^\top \Sigma dB_t, \tag{A.103}$$

where

$$\mu_{et} \equiv -A_e^\top \Gamma(\bar{q} - q_t) - \frac{\psi_e}{\alpha_e} + \frac{1}{2} A_e^\top \Sigma\Sigma^\top A_e. \tag{A.104}$$

Applying Ito's Lemma to (5.2) for $j = H$, we find the following counterpart of (A.19):

$$\frac{dP_{Ht}^{(\tau)}}{P_{Ht}^{(\tau)}} = \mu_{Ht}^{(\tau)}dt - A_H(\tau)^\top \Sigma dB_t, \tag{A.105}$$

where

$$\mu_{Ht}^{(\tau)} \equiv A'_H(\tau)^\top q_t + C'_H(\tau) - A_H(\tau)^\top \Gamma(\bar{q} - q_t) + \frac{1}{2} A_H(\tau)^\top \Sigma\Sigma^\top A_H(\tau). \tag{A.106}$$

Likewise, (5.2) for $j = F$ and (5.1) yield the following counterpart of (A.21):

$$\frac{d(P_{Ft}^{(\tau)} e_t)}{P_{Ft}^{(\tau)} e_t} - \frac{de_t}{e_t} = \mu_{Ft}^{(\tau)}dt - A_F(\tau)^\top \Sigma dB_t, \tag{A.107}$$

where

$$\mu_{Ft}^{(\tau)} \equiv A'_F(\tau)^\top q_t + C'_F(\tau) - A_F(\tau)^\top \Gamma(\bar{q} - q_t) + \frac{1}{2} A_j(\tau)^\top \Sigma\Sigma^\top \left(A_j(\tau) + 2A_e\right). \tag{A.108}$$

Substituting the returns (A.103), (A.105) and (A.107) into the arbitrageurs' budget constraint (2.3), we can write their optimization problem (2.4) as

$$\max_{W_{Ft}, \{X_{jt}^{(\tau)}\}_{\tau \in (0,T), j=H,F}} \Bigg[W_{Ft}\left(\mu_{et} + i_{Ft} - i_{Ht}\right) + \sum_{j=H,F} \int_0^T X_{jt}^{(\tau)} \left(\mu_{jt}^{(\tau)} - i_{jt}\right) d\tau$$
$$- \frac{a}{2} \left(W_{Ft} A_e + \sum_{j=H,F} \int_0^T X_{jt}^{(\tau)} A_j(\tau) d\tau \right)^\top \Sigma\Sigma^\top \left(W_{Ft} A_e + \sum_{j=H,F} \int_0^T X_{jt}^{(\tau)} A_j(\tau) d\tau \right) \Bigg]. \tag{A.109}$$

The first-order condition with respect to W_{Ft} is (5.3), and the first-order condition with respect to $X_{jt}^{(\tau)}$ is (5.4).

Using (3.7) and (3.18), we can write λ_t as

$$\lambda_t = a\Sigma\Sigma^\top \left(- \sum_{j=H,F} \int_0^T Z_{jt}^{(\tau)} A_j(\tau) d\tau - Z_{et} A_e \right)$$

$$= a\Sigma\Sigma^\top \left(\sum_{j=H,F} \int_0^T \left[\alpha_j(\tau) \log\left(P_{jt}^{(\tau)}\right) + \zeta_j(\tau) + \theta_j(\tau)\beta_{jt} + \left(\zeta_e(\tau) + \theta_e(\tau)\gamma_t\right)(-1)^{1_{\{j=H\}}} \right] A_j(\tau) d\tau \right.$$

$$\left. + \left[\alpha_e \log(e_t) + \zeta_e + \theta_e \gamma_t + \psi_e t + \int_0^T \left(\zeta_e(\tau) + \theta_e(\tau)\gamma_t\right) d\tau \right] A_e \right)$$

$$= a\Sigma\Sigma^\top \left(\sum_{j=H,F} \int_0^T \left[\zeta_j(\tau) + \theta_j(\tau)\beta_{jt} + \left(\zeta_e(\tau) + \theta_e(\tau)\gamma_t\right)(-1)^{1_{\{j=H\}}} \right. \right.$$

$$\left. - \alpha_j(\tau) \left(A_j(\tau)^\top q_t + C_j(\tau) \right) \right] A_j(\tau) d\tau$$

$$\left. + \left[\zeta_e + \theta_e \gamma_t + \psi_e t + \int_0^T \left(\zeta_e(\tau) + \theta_e(\tau)\gamma_t\right) d\tau - \alpha_e \left(A_e^\top q_t + C_e + \frac{\psi_e}{\alpha_e} t \right) \right] A_e \right)$$

$$= a\Sigma\Sigma^\top \left(\sum_{j=H,F} \int_0^T A_j(\tau) \left(\theta_j(\tau)\mathcal{E}_{\beta j} + \theta_e(\tau)\mathcal{E}_\gamma (-1)^{1_{\{j=H\}}} - \alpha_j(\tau) A_j(\tau) \right)^\top d\tau \right.$$

$$\left. + A_e \left(\theta_e \mathcal{E}_\gamma + \int_0^T \theta_e(\tau)\mathcal{E}_\gamma d\tau - \alpha_e A_e \right)^\top \right) q_t$$

$$+ a\Sigma\Sigma^\top \left(\sum_{j=H,F} \int_0^T \left(\zeta_j(\tau) + \zeta_e(\tau)(-1)^{1_{\{j=H\}}} - \alpha_j(\tau) C_j(\tau) \right) A_j(\tau) \right.$$

$$\left. + \left(\zeta_e + \int_0^T \zeta_e(\tau) d\tau - \alpha_e C_e \right) A_e \right)$$

$$= -(M - \Gamma^\top)^\top q_t + \lambda_C, \tag{A.110}$$

where the second step follows from (2.5) and (2.7), the third step follows from (5.1) and (5.2), and the fifth step follows from the definitions of (M, λ_C) in the statement of the proposition. We next substitute $(\mu_{et}, \{\mu_{jt}^{(\tau)}\}_{j=H,F}, \lambda_t)$ from (A.104), (A.106), (A.108) and (A.110) into the arbitrageurs' first-order condition. Substituting into (5.3) and identifying terms in q_t and constant terms, we find (5.6) and (5.7), respectively. Substituting into (5.4) and identifying terms in q_t and constant terms, we find (5.8) and (5.9), respectively. ∎

B Numerical Solution

B.1 Model Dynamics

Stack the J state variables in a vector $\mathbf{y}_t$, which include the H and F short rates i_{Ht}, i_{Ft}, and all the demand factors. Dynamics:

$$\mathrm{d}\mathbf{y}_t = -\boldsymbol{\Gamma}\left(\mathbf{y}_t - \overline{\mathbf{y}}\right)\mathrm{d}t + \boldsymbol{\sigma}\,\mathrm{d}\mathbf{B}_t \tag{A1}$$

where $\boldsymbol{\Gamma}$ is a $J \times J$ matrix determines the mean reversion of the state, and $\boldsymbol{\sigma}$ determines the stochastic properties. Define $\boldsymbol{\Sigma} = \boldsymbol{\sigma}\boldsymbol{\sigma}^\top$.

Write the habitat demand factors as

$$\beta_{jt}^{(\tau)} = \zeta_{jt}(\tau) + \mathbf{y}_t^\top \boldsymbol{\Theta}_j(\tau)$$

$$\gamma_{et} = \zeta_{et} + \mathbf{y}_t^\top \boldsymbol{\Theta}_e$$

Note that the vector functions $\boldsymbol{\Theta}_j(\tau)$ will typically be zero in most elements.

B.2 Characterizing the Solution

Conjecture that all (log) prices are affine in the state variables:

$$-\log P_{jt}^{(\tau)} = \mathbf{y}_t^\top \mathbf{A}_j(\tau) + C_j(\tau)$$

$$-\log e_t = \mathbf{y}_t^\top \mathbf{A}_e + C_e$$

Define the following matrix

$$\begin{aligned}\mathbf{M} = \boldsymbol{\Gamma}^\top - a\Bigg\{ &\int_0^T \left[-\alpha_H(\tau)\mathbf{A}_H(\tau) + \boldsymbol{\Theta}_H(\tau)\right]\mathbf{A}_H(\tau)^\top\,\mathrm{d}\tau \\ &+ \int_0^T \left[-\alpha_F(\tau)\mathbf{A}_F(\tau) + \boldsymbol{\Theta}_F(\tau)\right]\mathbf{A}_F(\tau)^\top\,\mathrm{d}\tau \\ &+ \left[-\alpha_e\mathbf{A}_e + \boldsymbol{\Theta}_e\right]\mathbf{A}_e^\top \Bigg\}\boldsymbol{\Sigma}\end{aligned} \tag{A2}$$

Then the solution to the affine functions $\mathbf{A}_j(\tau), \mathbf{A}_e$:

$$\mathbf{A}_j'(\tau) + \mathbf{M}\mathbf{A}_j(\tau) - \mathbf{e}_j = \mathbf{0} \tag{A3}$$

$$\mathbf{M}\mathbf{A}_e - (\mathbf{e}_H - \mathbf{e}_F) = \mathbf{0} \tag{A4}$$

with initial conditions $\mathbf{A}_j(0) = \mathbf{0}$.

Hence equations (A2), (A3), and (A4) implicitly characterize the solution to the model (although in general, the solution is not available in closed form).

B.3 Laplace Transforms

In order to solve the model numerically, we need to take a stance on the functional forms of the elasticity and demand functions $\alpha, \boldsymbol{\Theta}$. A numerically tractable approach is to assume that $T \to \infty$ and use Laplace transforms. Assume that

$$\alpha(\tau; \alpha_0, \alpha_1) \equiv \alpha_0 \exp(-\alpha_1 \tau)$$

$$\theta(\tau; \theta_0, \theta_1) \equiv \theta_0 \theta_1^2 \tau \exp(-\theta_1 \tau)$$

and note this implies $\int_0^\infty \theta(\tau; \theta_0, \theta_1)\, \mathrm{d}\tau = \theta_0$.

Eq. (A3) is a differential equation characterizing the coefficient functions $\mathbf{A}_j(\tau)$. Define the Laplace transform $\mathcal{A}_j(s) \equiv \mathcal{L}\left\{\mathbf{A}_j(\tau)\right\}(s)$. Then Eq. (A3) implies:

$$s\mathcal{A}_j(s) + \mathbf{M}\mathcal{A}_j(s) - \frac{1}{s}\mathbf{e}_j = \mathbf{0}$$

$$\implies \mathcal{A}_j(s) = [s\mathbf{I} + \mathbf{M}]^{-1}\left[\frac{1}{s}\mathbf{e}_j\right]$$

Additionally, define $\mathbf{X}_j(\tau) \equiv \mathbf{A}_j(\tau)\mathbf{A}_j(\tau)^\top$. Note that from Eq. (A3) we can write

$$\mathbf{A}_j'(\tau)\mathbf{A}_j(\tau)^\top + \mathbf{A}_j(\tau)\mathbf{A}_j'(\tau)^\top + \mathbf{M}\mathbf{X}_j(\tau) + \mathbf{X}_j(\tau)\mathbf{M}^\top - \mathbf{e}_j\mathbf{A}_j(\tau)^\top - \mathbf{A}_j(\tau)\mathbf{e}_j^\top = \mathbf{0}$$

$$\iff \mathbf{X}_j'(\tau) + \mathbf{M}\mathbf{X}_j(\tau) + \mathbf{X}_j(\tau)\mathbf{M}^\top - \mathbf{e}_j\mathbf{A}_j(\tau)^\top - \mathbf{A}_j(\tau)\mathbf{e}_j^\top = \mathbf{0}$$

Define the Laplace transform $\mathcal{X}_j(s) \equiv \mathcal{L}\left\{\mathbf{X}_j(\tau)\right\}(s)$. Then we have

$$\left[\frac{1}{2}s\mathbf{I} + \mathbf{M}\right]\mathcal{X}_j(s) + \mathcal{X}_j(s)\left[\frac{1}{2}s\mathbf{I} + \mathbf{M}\right]^\top = \mathbf{e}_j\mathcal{A}_j(s)^\top + \mathcal{A}_j(s)\mathbf{e}_j^\top$$

This is a Lyapunov equation. A sufficient conditions for a unique solution $\mathcal{X}_j(s)$ is if all the eigenvalues of $\left[\frac{1}{2}s\mathbf{I}+\mathbf{M}\right]$ have positive real parts. The solution can be written

$$\begin{aligned} vec\mathcal{X}_j(s) &= \left[\mathbf{I}\otimes\left[\frac{1}{2}s\mathbf{I}+\mathbf{M}\right]+\left[\frac{1}{2}s\mathbf{I}+\mathbf{M}\right]\otimes\mathbf{I}\right]^{-1} vec\left[\mathbf{e}_j\mathcal{A}_j(s)^\top+\mathcal{A}_j(s)\mathbf{e}_j^\top\right] \\ &\equiv \left[\left[\frac{1}{2}s\mathbf{I}+\mathbf{M}\right]\oplus\left[\frac{1}{2}s\mathbf{I}+\mathbf{M}\right]\right]^{-1} vec\left[\mathbf{e}_j\mathcal{A}_j(s)^\top+\mathcal{A}_j(s)\mathbf{e}_j^\top\right] \end{aligned}$$

However, for numerically computing the solution, more efficient algorithms exist.

With this notation, we have that

$$\int_0^T \alpha_j(\tau)\mathbf{A}_j(\tau)\mathbf{A}_j(\tau)^\top \,\mathrm{d}\tau = \alpha_{j0}\mathcal{X}_j(\alpha_{j1}) \equiv \tilde{\mathcal{X}}_j$$

$$\int_0^T \theta_{jk}(\tau)\mathbf{A}_j(\tau)^\top \,\mathrm{d}\tau = -\theta_{j0k}\theta_{j1k}^2\mathcal{A}_j'(\theta_{j1k})^\top$$

$$\implies \int_0^T \mathbf{\Theta}_j(\tau)\mathbf{A}_j(\tau)^\top \,\mathrm{d}\tau = \begin{bmatrix} \vdots \\ -\theta_{j0k}\theta_{j1k}^2\mathcal{A}_j'(\theta_{j1k})^\top \\ \vdots \end{bmatrix} \equiv \tilde{\mathcal{Y}}_j$$

and note that the n^{th} derivative is given recursively by

$$\mathcal{A}_j^{(n)}(s) = [s\mathbf{I}+\mathbf{M}]^{-1}\left[\frac{(-1)^n n!}{s^{n+1}}\mathbf{e}_j - n\mathcal{A}_j^{(n-1)}(s)\right]$$

Finally, define the exchange rate terms

$$\tilde{\mathcal{Z}} = \left[-\alpha_e\mathbf{A}_e - \mathbf{\Theta}_e\right]\mathbf{A}_e^\top$$

where recall $\mathbf{A}_e = \mathbf{M}^{-1}(\mathbf{e}_H - \mathbf{e}_F)$.

The terms $\tilde{\mathcal{X}}_j, \tilde{\mathcal{Y}}_j, \tilde{\mathcal{Z}}$ are all determined by $\mathbf{M}$. Hence we can write the equation characterizing $\mathbf{M}$, Eq. (A2), as the solution of a root-finding problem:

$$F(\mathbf{M}) = \mathbf{0}$$

$$F(\mathbf{M}) = \mathbf{\Gamma}^\top - a\left\{\tilde{\mathcal{Y}}_H - \tilde{\mathcal{X}}_H + \tilde{\mathcal{Y}}_F - \tilde{\mathcal{X}}_F + \tilde{\mathcal{Z}}\right\}\mathbf{\Sigma} - \mathbf{M}$$

The advantage of this approach is the solution does not require computing the eigen-decomposition and computing exponentials of the eigenvalues, which can lead to numerical instability.

B.4 Continuation Solution Algorithm

Given model dynamics parameters $\mathbf{\Gamma}, \boldsymbol{\sigma}$, the habitat elasticity parameters $\alpha_{j,0}, \alpha_{j,1}$, the habitat demand parameters $\theta_{j,0}, \theta_{j,1}$, and risk aversion a, the following continuation algorithm solves for the endogenous parameters $\mathbf{M}$:

1. Keeping all other parameters fixed, set risk aversion $a^{(0)} = 0$. The solution to this simplified model is $\mathbf{M}^{(0)} = \mathbf{\Gamma}^\top$.
2. Use the solution $\mathbf{M}^{(i)}$ to the model in the i step with risk aversion $a^{(i)}$ as the initial point for a local root-finding algorithm for $a^{(i+1)} = a^{(i)} + s^{(i+1)}$ for some small stepsize $s^{(i+1)}$.
3. If $a^{(i+1)} = a$, stop. Otherwise, return to step 2.

The algorithm selects the unique equilibrium (if it exists) that persists when tracing the solution as risk aversion falls to zero.

C Predictive Regression Coefficients

[[[**TO BE ADDED**]]]

D Method of Simulated Moments

Let $\boldsymbol{\rho}$ be the set of parameters to estimate. Set $\hat{\boldsymbol{\rho}}$ in order to minimize the following loss function:

$$L(\boldsymbol{\rho}) = \sum_{n=1}^{N} w_n (\hat{m}_n - m_n(\boldsymbol{\rho}))^2$$

where $\hat{m}_n$ and $m_n(\boldsymbol{\rho}$ are the empirical and model-implied covariances involving yields and exchange rates described below.

Given the dynamics in equation (A1), the long-run (unconditional) variance and autocovariance of the state is given by:

$$Var[\mathbf{y}_t] = vec^{-1}\left[(\mathbf{\Gamma} \oplus \mathbf{\Gamma})^{-1} vec(\mathbf{\Sigma})\right] \equiv \mathbf{\Sigma}^\infty \tag{B1}$$

$$Cov[\mathbf{y}_{t+s}, \mathbf{y}_t] = \exp(-\mathbf{\Gamma} s)\mathbf{\Sigma}^\infty \tag{B2}$$

Hence, moments involving yields and the exchange rate are straight-forward to compute. For instance, the covariance of H and F τ yields is given by

$$Cov(y_{Ht}^{(\tau)}, y_{Ft}^{(\tau)}) = [\mathbf{A}_H(\tau)/\tau)]^\top \mathbf{\Sigma}^\infty [\mathbf{A}_F(\tau)/\tau)]$$

Note that computing these moments involves first solving the model for any choice of $\boldsymbol{\rho}$ (using the continuation algorithm defined above).

D.1 Baseline Calibration

D.1.1 Model Specifics

The baseline calibration model is a 5-factor model: H and F short rates i_{Ht}, i_{Ft}; H and F bond demand factors β_{Ht}, β_{Ft}, and a currency demand factor γ_{et}. The state vector is therefore

$$\mathbf{y}_t = \begin{bmatrix} i_{Ht} \\ i_{Ft} \\ \beta_{Ht} \\ \beta_{Ft} \\ \gamma_{et} \end{bmatrix}$$

The corresponding demand vector functions are:

$$\mathbf{\Theta}_H(\tau) = \begin{bmatrix} 0 \\ 0 \\ \theta_H(\tau) \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{\Theta}_F(\tau) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \theta_F(\tau) \\ 0 \end{bmatrix}, \ \mathbf{\Theta}_e = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \theta_e \end{bmatrix}$$

We allow for the following correlation structure:

$$\mathbf{\Gamma} = \begin{bmatrix} \Gamma_{i_H} & 0 & 0 & 0 & 0 \\ 0 & \Gamma_{i_F} & 0 & 0 & 0 \\ 0 & 0 & \Gamma_{\beta_H} & 0 & 0 \\ 0 & 0 & 0 & \Gamma_{\beta_F} & 0 \\ 0 & 0 & 0 & 0 & \Gamma_{\gamma_e} \end{bmatrix}, \quad \mathbf{\Sigma} = \begin{bmatrix} \Sigma_{i_H} & \Sigma_{i_H, i_F} & 0 & 0 & 0 \\ \Sigma_{i_H, i_F} & \Sigma i_F & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence, we have the following 15 parameters to estimate:

1. 5 parameters in the $\mathbf{\Gamma}$ matrix.

2. 3 parameters in the $\boldsymbol{\sigma}$ matrix.

3. 3 elasticity size parameters: $\alpha_{0H}, \alpha_{0F}, \alpha_e$.

4. 3 demand size parameters: $\theta_{0H}, \theta_{0F}, \theta_e$.

5. 1 shape parameter $\delta = \alpha_{1j} = \theta_{1j}$.

D.1.2 Target Moments

We use the US as the Home country and Germany (Eurozone) as the Foreign country. The zero-coupon yield curve data is from Wright (2011) (frequency: monthly, from 1986).

Table B1: Targeted variances and covariances

Row	Variable	Maturity
1	$\sigma\left(y_{jt}^{(1)}\right)$	
2	$\sigma\left(\Delta y_{jt}^{(1)}\right)$	
3	$\rho\left(\Delta y_{Ht}^{(1)}, \Delta y_{Ft}^{(1)}\right)$	
4	$\sigma\left(\Delta \log e_t\right)$	
5	$\rho\left(\Delta \log e_t, \Delta^2 \log e_t\right)$	
6	$\rho\left(\Delta y_{Ht}^{(1)} - \Delta y_{Ft}^{(1)}, \Delta \log e_t\right)$	
7	$Vol_H(\tau \leq 3)$	
8	$\sigma\left(y_{jt}^{(\tau)}\right)$	✓
9	$\sigma\left(\Delta y_{jt}^{(\tau)}\right)$	✓
10	$\rho\left(\Delta y_{jt}^{(\tau)}, \Delta y_{jt}^{(1)}\right)$	✓

The Δ prefix denotes the 12-month forward difference. So $\Delta x_t = x_{t+12} - x_t$ for any variable x_t. Δ^2 denotes a "long" 24-month difference. The first seven rows refer to 9 scalar moments, while the bottom 3 rows refer to collections of moments (as a function of maturities, up to a maximum maturity of 20 years).

Note that, with the exception of the rows 1 and 8, all of the moments are either time differences or country differentials. The variances of the levels yields (H and F) are the only exception. We remove a common linear time trend from these series before computing this variance in the data.

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