Intermediated Asymmetric Information, Compensation, and Career Prospects *

Ron Kaniel

University of Rochester and CEPR

Dmitry Orlov

University of Wisconsin

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Abstract

Adverse selection harms workers, but benefits firms able to identify talent. An informed intermediary expropriates its agents' ability by threatening to fire and expose them to undervaluation of their skill. Agents' track record gradually reduces the intermediary's information advantage. We show that in response, the intermediary starts churning well-performing agents she knows to be less skilled. Despite leading to an accelerated reduction in information advantage, such selectivity boosts profits as retained agents accept below-reservation wages to build a reputation faster. Agents prefer starting their careers working for an intermediary, as benefits from building reputation faster more than offsets expropriation costs. We derive implications of this mechanism for pay-for-performance sensitivity, bonuses, and turnover. Our analysis applies to professions where talent is essential, and performance is publicly observable, such as asset management, legal partnerships, and accounting firms.

Keywords: dynamic signaling, dynamic adverse selection, compensation, career concerns, real options.

1 Introduction

Productive ability is the cornerstone of a successful enterprise. Investors allocate assets to a mutual fund if they believe the asset manager has skill. Corporations retain legal firms for

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the perceived wits of their lawyers. Businesses choose reputable accounting firms to perform audits and bookkeeping. These professions share a common theme – a significant subset of workers are employed by intermediaries who, in turn, sell workers' services to clients. The asset manager oversees investors' capital but is employed by a fund family, such as Fidelity. Legal associates serve clients but report to partners in their firm. It is surprising that many workers possessing such general industry skills do not contract directly with clients to sell their services, especially in professions where agents' performance is observable. Economists have, however, long understood that asymmetric information prevents skilled workers from charging the actual value of their services and enables informed intermediaries to step into the market and contract with clients on the agent's behalf. In this paper, we analyze the dynamic profitability of such intermediation and show how compensation and turnover are jointly shaped by asymmetric information and performance over the life cycle of an agent's career. We do so in a setting where both the agent's outside option while working for the intermediary and the value of the intermediary firm are endogenously determined in equilibrium.

Our analysis reveals an intriguing dynamic interplay between performance, turnover pattern, and evolving compensation. The equilibrium consists of quiet periods with limited departures of agents from the intermediary and limited wage dispersion, followed by periods with increased turnover and differentiated compensation. The intermediary is a monopsonist for the agent's labor when the information asymmetry is high. She screens the agents at the hiring stage but then retains them as long as they generate good performance, regardless of their ability to generate good performance in the future. Importantly, the quiet period is not induced by the intermediary's desire to learn about the agent, but instead the desire to maintain high information asymmetry. Over time, the agent's public track record reduces this information advantage, and the intermediary starts churning lower-skilled agents. While this reduces the intermediary's information advantage, her profitability increases in these periods as higher-skilled agents pay to build their reputation with clients and separate from lower-skilled agents. In equilibrium, the intermediary serves as a reputation building conduit for the hired agents, making it optimal for them to start their careers with the intermediary. While most models of labor market signaling assume exogenous costs of attending school, our model points to rich compensation dynamics arising endogenously when the agent is already employed, but still trying to signal his skill to the market.

In our model, the intermediary owns a long-term business and employs a sequence of agents to operate it. Each agent can work for the intermediary, but can, at any point, quit and open an independent firm to contract directly with clients.[1] The agent is privately informed

[1] We assume the agent faces no dead-weight costs of opening the firm, but our findings can easily incor-

about his skill relative to clients. The intermediary acquires this information at the time the agent is hired, while the clients remain uninformed.[2] We assume this private information is imperfect and that all parties further learn about the agent's skill from his publicly-observed performance. In every period, the intermediary either pays the agent enough to retain him or the agent separates from the intermediary, and the intermediary hires a new one at a cost. The decision to replace an agent depends on the intermediary's profitability relative to the benefit of hiring a new agent. The intermediary's revenue is determined by the clients' belief about the quality of the agent's services, i.e., by the belief about the ability of the retained agents.[3] The intermediary's cost is the compensation necessary to retain the agent and is unobserved by clients, as consistent with practice. The agent's compensation depends on the private information of the intermediary-agent pair as well as the clients' perception of the agents who are let go by the intermediary. The difference in skill between agents who are retained and those who quit allows the intermediary to pay the agent only a fraction of the revenues collected from clients. Private information makes the intermediary a transient monopsonist for the agent's labor, making it optimal for him to start his career working for the intermediary.

The intermediary favors employing higher-skilled agents for two reasons. First, even though the immediate revenue is determined by the clients' belief and is not sensitive to the residual private information, a better agent has higher performance prospects, resulting in improved revenues in the future. Second, the intermediary can bargain more effectively with a higher skilled agent, as being fired and pooled with lower-skilled agents presents a more severe punishment for a higher-skilled agent, who is more sensitive to clients' beliefs about him as he expects to stay in the industry longer. Consequently, the intermediary's threat of early termination is more effective when negotiating the compensation with a better agent, leading to greater profitability of employing him.

When information asymmetry is high, the intermediary retains all agents, as long as they perform sufficiently well. The intuition is that the revenues are pinned down by clients' belief about the retained agents, while compensation costs are determined by the worst remaining agent. Even an intermediary employing the worst agent can collect substantial revenues from pooling with higher-skilled agents before letting go of the agent and paying

porate such friction. Moreover, the agent's positive reservation value from leaving the industry results in an opportunity cost of him opening his firm even in our current setting.

[2] Consistent with survey evidence in Behrenz (2001), who shows that most of the private information of the firm accrues at the interview stage.

[3] It is common that the client is the residual claimant of the quality of the provided service. In the context of mutual funds, investors pay a percentage fee of assets under management to the fund family but are the residual claimants of the manager's performance. The fund family rarely invests its capital into its funds. In the context of legal services, it is common for a client to retain a law firm for a fee. The law firm then represents the client, but does not bear residual claim to the outcome of the trial or negotiation.

the resampling cost. We term this as the quiet period of the employment relationship. It is characterized by low turnover and a lack of wage dispersion. The agent's performance track-record gradually reduces information asymmetry, making retention of the lower-skilled agent expensive, eroding the intermediary's profitability of retaining all agents, and eventually ending the quiet period.

When the information asymmetry is low, the intermediary cannot profitably retain all agents. Dropping lower-skilled agents improves the pool of retained workers, leading to an increase in revenues from clients, but also a further reduction in information asymmetry. The intermediary can affect the rate of this decline by strategically setting her retention policy. One might naturally conjecture that she would refrain as much as possible from letting go of lower-skilled agents in order to maintain a steady profit wedge. We show that, on the contrary, she accelerates reduction in information asymmetry by churning low-skilled agents at a faster rate, and highlight that the higher churning rate increases the intermediary's profits. The key to understanding this is to keep in mind that being retained by the intermediary conveys a positive signal about the agent's ability to clients and improves his future career prospects. A higher skilled agent is willing to be under-compensated to the extent that he accepts even less than the compensation of the separating low-skilled agent in order to capture the deferred benefit of such reputation building. Such willingness allows the intermediary to differentially underpay higher-skilled agents during churning periods, boosting her profits despite diminishing information asymmetry. Such dynamics lead her net profits to be non-monotone in elapsed time and performance – they decline while in a quiet region, but then increase as the intermediary starts churning lower-skilled agents. Churning periods are associated with more wage dispersion, as higher-skilled agents differentially pay for reputation – a result that highlights the importance of private compensation contracts, contrary to prior literature. In equilibrium, better agents are retained for longer but are more underpaid while working for the intermediary. These results hold regardless of whether the intermediary can commit to long-term contracts with the agent.

We show that the pay-for-reputation mechanism is robust to a number of extensions of the model. First, we allow the agent to move laterally across different, but symmetrically-informed intermediaries. Such a possibility gives the agent bargaining power and reduces the intermediary's profits, but does not alter the equilibrium structure. Second, we allow the agent to signal his ability to clients by selling his services at a persistent discount when he opens his own firm. We show that, as long as the agent's performance is reasonably informative, the agent prefers to build a reputation by working for the intermediary, and does not rely on independent signaling. The intuition is that, while a higher-skilled agent benefits more from a better reputation, he also suffers a greater dead-weight cost of signaling his ability independently. Third, we contrast the role of reputation building and general training. We show that the intermediary has an interest in training the agent only when the information asymmetry is high, and she can capture part of the incremental surplus. As the information asymmetry declines due to publicly-observable performance signals, the intermediary substitutes training with profitable turnover, in which higher skilled agents pay for building a reputation.

To illustrate the economic mechanism, we first express it in a parsimonious model where performance signals stem from a perfectly informative negative Poisson process.[4] This sharpens the economic intuition but, because elapsed time is the only state variable, makes it difficult to separate the effects of performance and residual uncertainty about the agent in determining compensation and turnover. To remedy this, we develop a novel learning model that combines a general distribution of private information and conditionally normal Brownian performance signals.[5] We are then able to characterize the equilibrium of this more general model as a solution to a multi-dimensional real-option problem, demonstrate that the equilibrium structure is very similar to the case of the Poisson model, and also derive a number of additional distinguishing results highlighting the effect of past performance on the agent's compensation and turnover.

In the Brownian model, the intermediary's profitability endogenously increases if the agent performs well, as clients' more dispersed beliefs about the retained agent lead them to put more weight on performance signals relative to their conditional belief about the worst remaining agent. The intermediary's revenue thus increases more with good performance than does the agent's reservation wage, leading to her profit wedge to be increasing with the agent's performance, absent any churning. This observation implies an intuitive equilibrium structure - the intermediary lets go of lower-skilled agents when their performance drops below a certain threshold. This churning threshold and the corresponding retention decisions depend on the residual uncertainty about the agent and belief about the worst remaining agent. We characterize equilibrium dynamics given three dynamic states: elapsed time, cumulative performance, and the worst agent retained by the intermediary.

In equilibrium, the agent's compensation is increasing in performance. When it drops below the churning threshold, however, and in contrast to the Poisson version of the model, compensation of all retained agents suffers a discontinuous downward drop as the intermediary churns lower-skilled agents at a strictly positive rate, and all retained agents pay for reputation. The intuition is that an agent is retained at a discount during downsizing but benefits

[4]The tractability of this approach has been emphasized in Hörner and Skrzypacz (2018).

[5]The model is a natural analog of Brownian performance signals in the setting of Fuchs and Skrzypacz (2010), and continuous types in the setting of Daley and Green (2012).

from a better reputation going forward if he survives with the current intermediary. Interestingly, as residual uncertainty about the agent declines over time, the intermediary increases the churning threshold and corresponding turnover rate, as the option value of retaining a lower-skilled agent in the hope he performs well declines, leading to all agents being let go in finite time.

Our analysis applies to professions in which the agent's talent is essential, his performance is observable with reasonable frequency, and the agent can contract directly with clients. In this environment, an intermediary able to identify ability can enter the market and sell the agent's services to clients. The prospect of generating a track record attracts the worker to the intermediary, while the intermediary's private information allows her to bargain with the worker profitably. Some of the occupations we have in mind are a mutual fund manager employed by a fund family to run one of its funds, a non-partner lawyer in a law firm, a non-partner physician or architect, an accountant working for one of the big accounting firms.

1.1 Related Literature

Our model contributes to a couple of strands of the literature: dynamic signaling and adverse selection, compensation and turnover in the presence of dynamic performance signals, and delegation through intermediation.

In our model, the intermediary is able to extract rents from the agent by being informed about his ability, similar to early works on asymmetric information in the labor markets, such as Greenwald (1986) and Gibbons and Katz (1991). Greenwald (1986) shows, in a three-period model, that in equilibrium, lower-skilled workers separate from the firm first; in the period they separate, their wages are higher than the wage of retained workers who more than make up the gap in subsequent period wages. A key contribution of our framework is that, in addition to retention decisions, we incorporate dynamic and publicly observable performance signals generated by the agent. The interplay between the dynamic evolution of asymmetric information, impacted by both performance signals and retention decisions, and reputation considerations is a key driver of our results. This leads to novel turnover dynamics including an interplay between quiet periods where all retained agents earn the same wage, and the intermediary does not strategically let agents go, and churning periods that emerge when asymmetric information is low in which retained agents' compensation is tightly linked to their skill level. This is in contrast to prior work where all retained agents receive the same wage.

Higher skilled agents may signal their ability to prospective employers by becoming educated, as shown by the seminal work of Spence (1973). We show that the incentive of higher-skilled agents to signal their ability to the market shape their compensation and turnover dynamics even if they are already employed. Waldman (1984) and Bernhardt (1995) study the role of promotions in determining worker compensation. These papers assume that worker compensation is publicly observed, leading to higher-skilled agents being promoted first. We show that, when compensation is unobserved by the outside, the employer can charge the agent for building a reputation. Strobl and Van Wesep (2013) show that a worker may accept lower compensation if his employer commits to disclosing his performance if the latter is privately observed. Our findings show that a strategic employer employer can assist with reputation building by way of conveying her private information about the agents' skill even if performance is publicly observed. Tervio (2009) shows that, in the absence of private information about the agents' skill, firms may prefer proven workers to young ones even if the latter have more upside potential. We show that it is precisely the private information about the ability of young workers that may lead an employer to choose them over established ones. There is significant literature studying equilibrium selection in dynamic signaling games. Cho and Sobel (1990) show in a static setting, that, as long as the players' preferences satisfy certain monotonicity conditions, the divinity criterion is equivalent to the independence from never weak-best responses of Kohlberg and Mertens (1986) and leads to a unique signaling equilibrium. Noldeke and Van Damme (1990) show in a dynamic game with two types that divinity[6] leads to the Riley outcome of the Spence (1973) signaling model. Introducing such refinements directly in a continuous-time model presents a significant challenge. We identify the unique equilibrium in which the clients positively update about the ability of the agent while he is employed, similar to the motivation of the divinity criterion in Cho and Sobel (1990).

Farber and Gibbons (1996) consider a model of the public learning about the agent's ability, but abstract away from asymmetric information. The resulting wage of the agent is equal to his expected marginal product. We show that adverse selection alters wage dynamics in two fundamental ways. First, during quiet periods, clients learn about skill only from performance signals, and all agents are paid the marginal product of the worst retained type. Second, during periods of churning, clients infer the agent's skill based on both performance and retention; consequently, higher-skilled agents pay for reputation.

Quiet periods also arise in Kremer and Skrzypacz (2007), Daley and Green (2012), and Zryumov (2018), where the possibility of a pooling offer in the future discourages early trade. The economics behind the quiet period is, however, different in our paper – delays occur because an intermediary employing a lower-skilled agent can collect revenues corresponding

[6]They use the Independence of Never Weak Best Responses, which is equivalent in the setting.

to the average remaining agent, resulting in a transient pooling period, before the agent leaves and accepts a separating offer from clients. These quiet periods resemble probation stages when the employer keeps turnover at a minimum, and there is little wage dispersion. However, the key driver is not the objective of the intermediary to learn about the agent's ability from his performance, but, instead, that pooling is optimal as long as information asymmetry is sufficiently high. Once information asymmetry is low, the intermediary both starts churning agents and wages of retained agents become inversely related to their private skill level.

Reputation contributes to the revenues an agent generates and can be regarded as a form of general human capital. Prior work has shown that when labor markets are imperfect, firm-sponsored general skills training can emerge in equilibrium; see Acemoglu and Pischke (1999) for a survey. Specifically, Acemoglu and Pischke (1998) show this occurs when there is asymmetric information between current and potential employers about workers' skill since the current employer can capture part of the incremental surplus. We confirm the intermediary has an interest in training the agent when information asymmetry is high. However, our analysis reveals that as information asymmetry declines, due to all parties observing performance signals, the intermediary substitutes training with profitable turnover, in which higher skilled agents pay for building a reputation. Higher turnover is detrimental to training incentives, as shown in Acemoglu and Pischke (1998), but is beneficial for accumulating reputation as we show in this paper.

Identifying talent is essential in the asset management industry. Berk and Green (2004) appeal to it to explain the relationship between fund size and performance. Moreover, Berk, Van Binsbergen, and Liu (2017) identify private information a fund family has about the skill of the managers it employs. While there has been extensive theoretical work on delegation contracts between investors and asset management entities, a centerpiece that has been mostly ignored is that investors sign contracts with the intermediary, for example, a mutual fund family, and not directly with the asset manager. In this paper, we focus on contracting implications for this important and under-researched second layer. An exception in the literature is the work of Gervais, Lynch, and Musto (2005), who show that a fund family can add value by committing to fire a fixed percentage of the managers it perceives to be the worst. We show that a fund family does not need to commit to long term contracts and termination policies as lower-skilled agents become too expensive to retain anyway. Our findings are also consistent with the observed behavior of mutual fund managers who, at times, are allowed to open a separate hedge fund to run on the side.[7]

[7]See Nohel, Wang, and Zheng (2010) and Deuskar, Pollet, Wang, and Zheng (2011) for details.

Our work also relates to the literature on dynamic certification. In that literature the certifier, for example, a rating agency, issues a formal stamp of approval, where typically two types of certifiers are assumed: a type committed to being honest and an opportunistic type; see for example Mathis, McAndrews, and Rochet (2009).[8] In contrast, in our setting retention serves as an indirect stamp of approval, and the intermediary acts optimally.

Similar to Fuchs and Skrzypacz (2010), Kremer and Skrzypacz (2007), and Bonatti and Hörner (2017), in our base model we take advantage of the tractability benefits of modeling performance as a perfectly informative Poisson arrival process.[9] However, a limitation of this structure is that it co-mingles the effects of residual uncertainty about an agent's skill and his performance in his retention and compensation. We are able to disentangle these two forces in a Brownian model of performance while showing that the equilibrium structure remains unchanged.

Our method to solving the Brownian model provides methodological contributions on structuring and numerically solving dynamic adverse selection models with learning about a continuum of types.[10] First, we obtain the intermediary's churning strategy by solving a set of parallel, but independent, non-stationary optimal stopping problems. Second, we note the tractability benefit of combining a truncated normal distribution of private information about the agent type with dynamic Brownian signals of the type. The benefit stems from the class of truncated normal distributions being a conjugate of itself with respect to the Gaussian likelihood function. Finally, we perform a Girsanov change of measure and integrate out the latent information conditional on a public history. We are then able to obtain the intermediary's equilibrium expected value as a solution to a single dynamic program. This reduces the required number of calculations dramatically as we have integrated out private information - so much so that it enables solving problems that otherwise would just take too long even with significant computation resources. This is a novel dynamic framework that can be applied to study the implications of dynamics asymmetric information in other economic settings.

The rest of the paper is organized as follows. Section 2 introduces the baseline model. We derive the equilibrium and characterize its properties in Section 3. Section 4 considers imperfect competition among intermediaries, the possibility of the agent independently signaling ability, and the differences between reputation building and training. We consider the model of Brownian performance in Section 5. Section 6 concludes. Formal proofs and additional analyses are in Online Appendix A and Online Appendix B provides theoretical steps necessary

[8] For a survey see Dranov and Zhe Jin (2010).

[9] For a discussion of the tractability benefits see Hörner and Skrzypacz (2018) for a survey of the experimentation literature.

[10] To our knowledge, Daley and Green (2012) and Daley and Green (2014) are the only models exploring Brownian performance signals, yet they focus on the binary nature of asymmetric information.

for the numerical evaluation of the Brownian model.

2 Setup

Our setting stems from four building blocks. First, the employee provides clients a service with publicly observable outcomes, which we denote by X_t; for example, the success or failure of a trial lawyer in court. Second, the public attributes a significant part of the performance to that individual's skill, which we denote by θ. For example, the performance of a mutual fund is attributed to a large extent to the skill of the fund manager and not the family the fund belongs to. Third, while the employing firm knows more about the skill of the employee than the public, this information asymmetry eventually declines as a result of all parties learning from performance. Finally, it is not prohibitively costly for the agent to cut out the intermediary and provide his services directly to clients by opening his own firm. In what follows, we refer to the worker as the agent and the firm as the intermediary. We cast the game in continuous time $t \in \mathbb{R}_+$ due to its significant tractability advantages in analyzing both games of asymmetric information and real option problems.

Service Technology. An agent has unknown skill $\theta \in \{0, 1\}$, that is not directly observable by any of the players, and affects the cumulative performance of his services sold to clients, given by a publicly observable process

$$X_t = \mu t - N_t^\theta. \tag{1}$$

Process $N^\theta = (N_t^\theta)_{t\geq 0}$ is Poisson with a constant arrival intensity $\lambda(1-\theta)$. [11] A skilled, $\theta = 1$, agent never performs a bad service, generating a flow payoff of μ to clients. An unskilled, $\theta = 0$, agent exposes clients to a possibility of a bad service, such as bad investment returns or legal penalties, generating an expected flow payoff of $\mu - \lambda$ to clients. The expected value of the agent's service at time t is $\mu + \lambda(\mathrm{E}_t[\theta] - 1)$ which is increasing in the agent's expected skill. If the agent performed poorly prior to time t, i.e., $X_t < \mu t$ (which is equivalent to $N_t^\theta > 0$), then all players correctly identify him as unskilled.

Clients. The clients are willing to pay $A(\mathrm{E}_t[\theta])$ for services rendered by the agent in period t if his perceived skill is $\mathrm{E}_t[\theta]$, where $A(\cdot)$ is increasing in θ. For expositional simplicity, we assume this willingness-to-pay is the same whether the agent works for the intermediary, or contracts directly with clients.[12] While we model the clients' demand for services in reduced

[11] We show in Section 5 that the results hold in a Brownian model of performance.

[12] We allow the agent to sell services at a discount in Section 4.2. Our results and methods are directly extendable to settings in which the demand for agent's services is different if he contracts with clients directly.

form to fit a broad set of applications, we assume the clients rationally update their beliefs about the agent's skill-based both on performance and retention outcomes.

In equilibrium, the agent leaves the intermediary at a, possibly infinite, time τ observable by all players. We denote by q_t the clients' equilibrium belief about the skill of the agent retained by the intermediary up to time t

$$q_t \overset{def}{=} \mathrm{E}\left[\theta \,|\, (X_s)_{s \le t}, t < \tau\right] = \mathrm{E}\left[\theta \,|\, X_t, t < \tau\right]. \tag{2}$$

Belief q_t determines the revenue $A(q_t)$ obtained by the intermediary in period t.

The time at which the agent separates from the intermediary is, potentially, informative about his skill. We denote by k_t the clients' equilibrium belief about the agent who leaves the intermediary at time t. If he leaves the intermediary with good performance $X_t = \mu t$, then

$$k_t \overset{def}{=} \mathrm{E}\left[\theta \,|\, X_t = \mu t, \tau = t\right]. \tag{3}$$

If the agent separates from the intermediary after bad performance, then all players identify him as unskilled, resulting in $k_t = 0$. Belief k_t determines the starting revenue $A(k_t)$ obtained by the agent were he to open his own firm and, thus, influences his career prospects were he to separate from the intermediary. We term k_t as the agent's (outside) reputation.

Agent. The intermediary-agent pair is endowed with an initial private signal about θ. We identify this signal with their private posterior $\tilde{p}_0 \sim F(\cdot)$ at $t = 0$,[13] and require $F(\cdot)$ to be continuously distributed with full support on $[\underline{p}, \overline{p}]$. If the agent performs well up to time t, the intermediary-agent pair update their private posterior belief about his skill to

$$\tilde{p}_t = \pi(\tilde{p}_0, t) \overset{def}{=} \mathrm{P}\big(\theta = 1 \,\big|\, X_t = \mu t,\, \tilde{p}_0\big) = \frac{\tilde{p}_0}{\tilde{p}_0 + (1 - \tilde{p}_0) \cdot e^{-\lambda t}}. \tag{4}$$

If the agent performed a bad service before time t, i.e., $X_t < \mu t$, then $\tilde{p}_t = 0$. We refer to $\tilde{p}_t$ as the agent's private type at time t, or, simply, as the agent's type when it is unambiguous.

Suppose the agent leaves the intermediary after good performance up to time t and opens his firm.[14] Once he does so, subsequent learning about θ is driven solely by public performance signals X_t,[15] meaning that after a history of good performance between t and s, the clients' posterior belief becomes $\pi(k_t, s - t)$. The agent is risk-neutral and discounts cash flows at a rate ρ. His expected value from separating from the intermediary at time t given (outside)

[13] If $\tilde{s}$ is the signal privately observed by the intermediary-agent pair, then $\tilde{p}_0 = \mathrm{E}\left[\theta \,|\, \tilde{s}\right]$.

[14] We assume that opening the firm is costless, but our results are unaffected if such a cost is present. We show in Section 4.1 that the results are unchanged if the agent can switch laterally between intermediaries.

[15] We show in Section 4.2 that the results are unchanged if the agent can independently signal his ability.

reputation k_t and private type $\tilde{p}_t$ is the expected discounted sum of revenues

$$U(\tilde{p}_t, k_t) \overset{def}{=} \max_{\hat{\eta}} \mathrm{E}_{\tilde{p}_t} \left[\int_t^{\hat{\eta}} e^{-\rho(s-t)} A\Big(\pi(k_t, s-t)\Big)\, ds + e^{-\rho(\hat{\eta}-t)} \cdot L \right]. \tag{5}$$

We denote by η the stopping time maximizing (5), which denotes the time when the agent chooses to leave the industry altogether. When $\eta > t$, the agent opens his own firm upon quitting the intermediary and when $\eta = t$ he leaves the industry immediately upon being let go. His reservation value $L \geq A(0)/\rho$ captures the agent's prospects outside of the industry.[16] Reputation k_t determines the agent's expected value from opening his firm at time t, providing an outside option when negotiating compensation with the intermediary. Process $(k_t)_{t \geq 0}$ governs the endogenous dynamics of this outside option.

Intermediary. The intermediary's revenue at time t is given by $A(q_t)$ which is determined by the clients' belief about the agent in her employment. The profit of the intermediary, however, is the revenue net of the cost of retaining the agent. We denote by $\tilde{w}_t$ the wage paid by the intermediary to the agent and note that it may depend on the private type $\tilde{p}_t$. Consistent with all of our applications, wage $\tilde{w}_t$ is unobservable by clients, allowing the intermediary to condition it on her private information. The agent of type $\tilde{p} = (\tilde{p}_t)_{t \geq 0}$ accepts a "sequence" of wages $\tilde{w} = (\tilde{w}_t)_{t \in [0,\tau]}$ if staying with the intermediary until time τ is weakly better than leaving immediately given prevailing reputation k_t

$$\mathrm{E}_{\tilde{p}_t} \left[\int_t^{\tau} e^{-\rho(s-t)} \tilde{w}_s\, ds + e^{-\rho(\tau-t)} \cdot U\left(\tilde{p}_\tau,\, k_\tau\right) \right] \;\geq\; U(\tilde{p}_t, k_t) \tag{6}$$

for all $t \in [0, \tau]$ and the expectations is taken with respect to future performance, conditional on the agent's type $\tilde{p}_t$.

The intermediary sets wages strategically, understanding the adverse-selection frictions faced by the agent, manifested by the difference between his true type $\tilde{p}_t$ and his (outside) reputation k_t, were he to leave. We assume the intermediary cannot commit to long-term contracts to illustrate how the agent's reputation k_t alone can act as a commitment device for deferred compensation.[17] The intermediary is risk-neutral, and discounts the future at rate $r \leq \rho$, resulting in an expected profit

$$\mathrm{E}_{\tilde{p}_0} \left[\int_0^{\tau} e^{-rt} (A(q_t) - \tilde{w}_t)\, dt + e^{-r\tau} \cdot V \right],$$

where V is the endogenous continuation value of the intermediary when she lets go of the agent and, potentially, hires a new agent. In most applications, the intermediary runs a long-term

[16] Mathematically, such normalization is without loss of generality. Economically, it states that it is more efficient for unskilled agents to leave the industry and is similar to the favorable selection argument in Jovanovic (1982).

[17] Our findings are robust to giving commitment power to the intermediary as we show in Lemma A.2.

business and can replace the outgoing agent with a new one, at a cost. The intermediary's continuation value V is pinned down as a solution to the fixed point equation

$$V = \max \left[e^{-r\Delta} \cdot \mathrm{E} \left[\int_0^\tau e^{-rt} \big(A(q_t) - \tilde{w}_t \big) \, dt + e^{-r\tau} \cdot V \right] - I, \ 0 \right] \tag{7}$$

where I is the fixed cost and Δ is the delay to replace the agent.[18]

Application to Money-Management. Consider a mutual fund family offering a fund to its investors. It charges investors a percentage fee f on its assets under management, while privately compensating the manager it employs. Investors know the name of the manager who oversees their wealth but do not know his investment skill and learn about it from observing the returns he generates. Moreover, the fund family is likely to know more about its manager than the clients, as shown by Berk, Van Binsbergen, and Liu (2017). The realized per-dollar return is identified with $dX_t - g(S_t)\, dt$, where S_t represents assets under management and function $g(\cdot)$ captures decreasing returns to scale due to increased market impact, as modeled by the seminal work of Berk and Green (2004). Investors provide capital competitively, and invest in the fund until its expected return equates their opportunity cost r_I

$$\mathrm{E}_t \left[dX_t - g(S_t)\, dt \right] = \big(\mu - \lambda \, (1 - q_t) - g(S_t) - f \big) \, dt \overset{(i)}{=} r_I \, dt.$$

The resulting revenue $A(\cdot)$ of the intermediary (fund family in this case) is given by

$$A(q_t) = f \cdot S_t = f \cdot g^{-1} \left(\lambda q_t + \mu - \lambda - f - r_I \right).$$

The manager's outside option is to open his own fund by attracting investment capital. He can attract more capital and, thus, collect more fees, if he has a better reputation in the money-management industry.[19]

Equilibrium definition. Our solution concept is a Perfect Bayesian Equilibrium adapted to our continuous-time setting with frequent actions and asymmetric information.

Definition 1. *A (monotone) Perfect Bayesian Equilibrium is a public termination time τ, a collection of private wage processes $\tilde{w} = (\tilde{w}_t)_{t \geq 0}$ for each agent type $(\tilde{p})_{t \geq 0}$, and the clients' belief processes $(q_t)_{t \geq 0}$ and $(k_t)_{t \geq 0}$ such that*

(i) Stopping time τ and wage process $\tilde{w} = (\tilde{w}_t)_{t \geq 0}$ solve the intermediary's retention prob-

[18] If I or Δ is very large, the intermediary may choose to not to hire a new agent, leading to $V = 0$. By setting $\Delta = +\infty$ and $I < 0$, such specification captures the case of an exogenous outside option.

[19] If the manager opens a hedge fund, his revenues may be different from those obtained in a mutual fund, but our setting is robust to such extension.

lem given the clients' beliefs

$$\{\tau,\, \tilde{w}\} \in \underset{\{\hat{\tau},\hat{w}\}}{\arg\max}\ \mathrm{E}_{\tilde{p}_0}\left[\int_0^{\hat{\tau}} e^{-rt}\left(A(q_t) - \hat{w}_t\right)\, dt + e^{-r\hat{\tau}} \cdot V\right],$$

subject to the retention constraint (6) *being satisfied with equality for every* $t \in [0, \tau)$ *and the expected firm value* V *satisfying* (7)*.*

(ii) Belief k_t *is monotone in initial types:* $\pi(k_t, -t)$ *is weakly increasing in* t *along the path of good performance.*

- *On-path beliefs:* $k_t = \mathrm{E}\left[\tilde{p}_t | X_t, \tau = t\right]$ *and* $q_t = \mathrm{E}\left[\tilde{p}_t \,|\, X_t, \tau > t\right]$ *if* $t \in$ *support*(τ)*.*
- *Off-path beliefs:* $k_t \in$ *support*$(\tilde{p}_t | \tau > t)$ *if* $t \notin$ *support*(τ) *and* $\mathrm{P}(\tau > t) \neq 0$*. If* $\mathrm{P}(\tau > t) = 0$*, then* $k_t = q_t = \pi(\bar{p}, t)$*.*

The intermediary chooses when to let the agent go while satisfying the agent's retention constraint (6). In a subgame perfect equilibrium, the intermediary pays the agent just enough for him to stay in the next period, implying that (6) must be satisfied with equality along the path of play. While it is difficult to introduce subgame-perfection in a game with frequent observable actions directly in continuous time,[20] requiring that the retention constraint (6) is binding captures the subgame-perfect wage-setting by the intermediary. Equilibrium wages $\tilde{w}$ are a function of performance, private information, as well as public beliefs, and the intermediary solves the optimal retention problem for the manager of every skill level $\tilde{p}$ separately.

We require that the clients' beliefs are consistent with the intermediary's strategy so that, along the equilibrium path, belief processes $(q_t)_{t \geq 0}$ and $(k_t)_{t \geq 0}$ satisfy (2) and 3 respectively. In addition, we require that the agent does not get penalized for working for the intermediary, which corresponds to the outside reputation k_t increasing weakly faster along the path of good performance than just stemming from public news.[21] We also require that once all agents leave the intermediary, the clients' belief is that the remaining agent is the highest possible type, following the intuition underlying the D_1 criterion that the deviating type must be the one obtaining the greatest gains from a deviation. This is identical to the equilibrium refinement used in the dynamic signaling game of Noldeke and Van Damme (1990), but applied directly to a signaling game featuring a continuum of types, dynamic performance signals, and cast directly in continuous time.

[20]See Simon and Stinchcombe (1989) for pathological cases that may arise in continuous time games with frequent actions. In our model, the binding constraint (6) can be derived from considering a limit of equilibria in which the intermediary sets a fixed wage for a small, but discrete time interval.

[21]A weaker assumption, which is also sufficient for our purposes, is that the support of types remaining employed by the intermediary is convex. While this does not rule out higher skilled agents from leaving the intermediary first, the intermediary would not be able to retain other agents, implying an atom of exits.

3 Equilibrium Analysis

To characterize the equilibrium, we proceed in three steps. First, we characterize the agent's endogenous dynamic outside option $U(\tilde{p}_t, k_t)$ if he leaves the intermediary at time t given clients' belief k_t. This determines the agent's reservation wage when working for the intermediary. Second, we characterize the intermediary's decision of retaining the agent as a function of his skill level $\tilde{p}_t$ and the intermediary's continuation value V. Finally, we complete the characterization by pinning down V.

3.1 Agent's Dynamic Outside Option

The agent's outside option comprises of either starting his own firm or leaving the industry. The decision to start his own firm at time t depends on starting revenues $A(k_t)$ and their expected growth rate, which is determined by the agent's private type $\tilde{p}_t$. As higher skill implies a higher likelihood of continued good performance, it also leads to greater expected revenues in the future.

Define by $u_\theta(k)$ to be the expected value to an agent of opening a firm conditional on $\theta \in \{0, 1\}$ given reputation k and operating it as long as he performs well. If $\theta = 1$, then the agent never performs poorly and

$$u_1(k) \stackrel{def}{=} \int_0^\infty e^{-\rho\, t} \cdot A\left(\pi(k,t)\right)\, dt \tag{8}$$

is the discounted sum of revenues in perpetuity. If, however, $\theta = 0$ then at an exponentially distributed random time he performs poorly, is revealed to be unskilled, and consequently leaves the industry. His conditional expected value

$$u_0(k) \stackrel{def}{=} \int_0^\infty e^{-(\rho+\lambda)t} \cdot A\left(\pi(k,t)\right)\, dt + \frac{\lambda}{\rho+\lambda} \cdot L, \tag{9}$$

is the expected discounted sum of revenues until he performs poorly and the payoff L from leaving the industry.

Since good performance increases posterior beliefs of both the clients and the agent, if it is optimal for the agent to open his own firm, then it is best to operate it along the path of good performance. The agent, then, leaves the industry only once he performs poorly.

Lemma 1. *The agent's expected value of leaving the intermediary given his own belief $\tilde{p}$ and the clients' belief k is given by*

$$U(\tilde{p}, k) = \max\Big[\tilde{p} \cdot u_1(k) + (1-\tilde{p}) \cdot u_0(k),\ L\Big]. \tag{10}$$

It is weakly increasing in his skill $\tilde{p}$ and client's belief k about his ability.

Equation (10) provides a tractable solution for the agent's dynamic reservation value and allows us to establish an important property of the agent's outside option.

Proposition 1. *Manager's value function $U(\tilde{p}, k)$ satisfies single crossing: a higher skilled agent is more sensitive to changes in clients' beliefs than a lower skilled agent*

$$\frac{\partial}{\partial k} U\left(\tilde{p}', k\right) > \frac{\partial}{\partial k} U\left(\tilde{p}, k\right) \qquad \textit{for any} \qquad \tilde{p}' > \tilde{p}. \tag{11}$$

The result of Proposition 1 can be obtained by differentiating (10) with respect to $\tilde{p}$ and, then, differentiating (8) and (9) with respect to k under the sign of the integral

$$\frac{\partial^2}{\partial \tilde{p} \partial k} U(\tilde{p}, k) = u_1'(k) - u_0'(k) = \int_0^\infty \underbrace{\left(e^{-\rho t} - e^{-(\rho+\lambda)t}\right)}_{>0} \cdot \underbrace{\left(A'\left(\pi(k,t)\right) \cdot \partial_1 \pi(k,t)\right)}_{\geq 0} dt \geq 0.$$

A higher-skilled agent expects to remain in the industry in the future with a higher probability, making him, effectively, more patient. Thus, he is more sensitive to changes in his current reputation, since it entails a longer-term impact on his revenues. An important consequence of Proposition 1 is that a higher-skilled agent would be willing to sacrifice more short term revenues in favor of building a reputation.

We denote by $w_R(\tilde{p}, k)$ to be the reservation wage of the agent as the flow utility he receives if he pursues his outside option. Since he is risk-neutral and discounts cash flows at rate ρ, we can think about him receiving ρL in perpetuity when he leaves the industry. The agent's reservation wage can be expressed as

$$w_R(\tilde{p}, k) = \begin{cases} \rho L & \text{if} \qquad U(\tilde{p}, k) = L, \\ A(k) & \text{if} \qquad U(\tilde{p}, k) > L. \end{cases} \tag{12}$$

The agent's reservation wage depends on his skill $\tilde{p}$ (i.e., his belief about his ability θ) only via the public decision to start his firm. The agent values future revenue growth, and as such may decide to open his own firm, instead of leaving the industry, with initial revenues below ρL in order to obtain higher revenues in the future. Moreover, as shown in the proof of Proposition 1, a higher-skilled agent values this growth option more than a low-skilled agent. As a result, reservation wage $w_R(\tilde{p}, k)$ is weakly decreasing in $\tilde{p}$.

3.2 Equilibrium Turnover and Compensation

The intermediary's profits stems from the fact that clients are at an information disadvantage and she retains an agent of a given skill only if it is more profitable than to replace him. The difference between revenues $A(q_t)$ and reservation wage $w_R(\tilde{p}_t, k_t)$ is weakly increasing in the agent's skill $\tilde{p}_t$, implying that lower skilled agents are the first to be let go. Such ordering, intuitively, implies that agents of types $\tilde{p}_t \geq k_t$ are still employed by the intermediary at time t and we refer to $\tilde{p}_t = k_t$ as the cutoff agent.[22] The average type of the agent retained by the intermediary is then given by $q_t = \mathrm{E}[\tilde{p}_t \,|\, \tilde{p}_t \geq k_t]$. In order for the agent's posterior type $\tilde{p}_t$ at time t to weakly exceed k_t, it must be that his initial type $\tilde{p}_0$ is such that $\pi(\tilde{p}_0, t) > k_t$ or, equivalently, $\tilde{p}_0 \geq \pi(k_t, -t)$.[23] The average type of the agent retained by the intermediary given good performance is then obtained by Law of Iterated Expectation

$$q_t = Q(k_t, t) \stackrel{def}{=} \mathrm{E}\left[\theta \,\middle|\, \tilde{p}_0 \geq \pi(k_t, -t), X_t = \mu t\right] = \pi\Big(\mathrm{E}\left[\tilde{p}_0 \,\middle|\, \tilde{p}_0 \geq \pi(k_t, -t)\right],\, t\Big).$$

If the difference $A(q_t) - w_R(k_t, k_t)$ is sufficiently large, then the intermediary optimally retains all agents and beliefs change only as a result of observing performance signals X. If, however, $A(q_t) - w_R(k_t, k_t)$, is low, then the intermediary finds lower-skilled agents unprofitable, relative to replacing them, and lets them go. Churning the cutoff agent at time t, despite his history of good performance that is indistinguishable from other well-performing agents, increases the expected skill of the worst agent employed next period, k_t, as well as the clients' belief about the average agent still employed, q_t, but may further reduce the information advantage $q_t - k_t$. At first glance, such erosion of the information advantage, which is the source of the intermediary's profit, seems to be detrimental to her. However, churning lower-skilled agents generates a positive signal about agents who are retained by the intermediary. Consequently, in periods when the intermediary churns lower-skilled agents, higher-skilled agents are willing to pay for reputation building. Consequently, they are willing to work for the intermediary at below their reservation wage, thus increasing the intermediary's profits. Figure 1a illustrates that, along the path of good performance, the belief about the cutoff type k_t exceeds the posterior belief $\pi(k_0, t)$ about the worst type initially hired, due to selective retention by the intermediary.

Proposition 2. *The equilibrium is characterized by a churning set $\mathbb{T} \subseteq \mathbb{R}_+$ comprised of a finite union of intervals.*

(i) The intermediary lets the agent go either after bad performance, or if he is the cutoff type $\tilde{p}_t = k_t$ during the churning period $t \in \mathbb{T}$.

(ii) If the agent is not identified as unskilled, i.e., $k_t > 0$, his reputation grows faster during churning periods

[22] This is an informal argument. In what follows, we first construct an equilibrium satisfying this property and then establish its uniqueness.

[23] The domain of $\pi(x, t)$ can be extended on $\mathbb{R} \times [0, 1]$ using the algebraic definition in (4). Then $\pi(\pi(p, -t), t) \equiv p$ for any p.

$$dk_t = \underbrace{\lambda k_t(1-k_t)\,dt + k_t\big(dX_t - \mu\,dt\big)}_{\textit{learning from performance}} + \underbrace{\begin{cases} 0 & \textit{if} \quad t \notin \mathbb{T}, \\ \gamma(k_t, t)\,dt & \textit{if} \quad t \in \mathbb{T}, \end{cases}}_{\textit{learning from churning}} \tag{13}$$

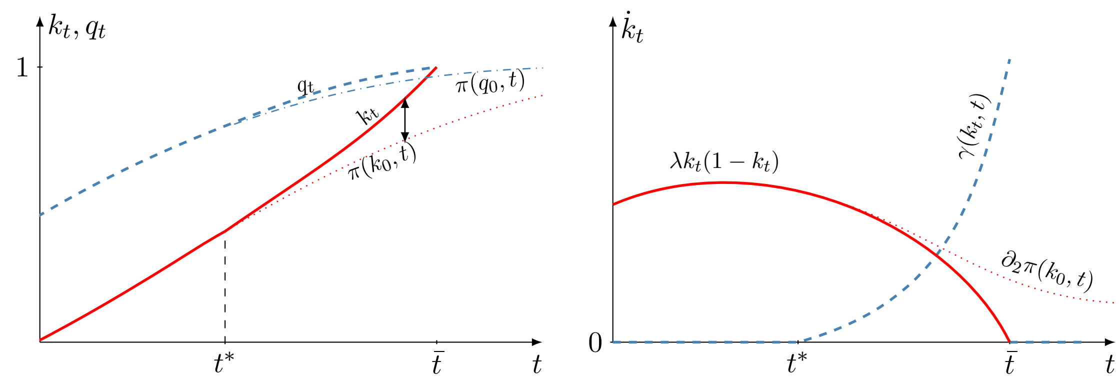
Incremental growth rate $\gamma(k_t, t)$ is determined by the ratio of the profitability of the cutoff agent and his marginal value of reputation[24]

$$\gamma(k_t, t) \stackrel{def}{=} \frac{\Big[rV - \Big(A\big(Q(k_t,t)\big) - w_R(k_t,k_t)\Big)\Big]^+}{\partial_2 U(k_t, k_t)} \geq 0. \tag{14}$$

(iii) Agents pay for reputation building by accepting lower compensation

$$w(\tilde{p}_t, k_t) = w_R(\tilde{p}_t, k_t) - \underbrace{\begin{cases} 0 & \textit{if} \quad t \notin \mathbb{T}, \\ \gamma(k_t, t) \cdot \partial_2 U(\tilde{p}_t, k_t) & \textit{if} \quad t \in \mathbb{T}. \end{cases}}_{\textit{pay for reputation}} \tag{15}$$

(iv) If $A'(p)p(1-p)$ is decreasing in $p \geq \underline{p}$, then $\mathbb{T} = [t^*, \infty)$ where t^* is the first time $\gamma(k_t, t) > 0$.



(a) Belief about cutoff type k_t (solid) exceeds belief based solely on performance $\pi(k_0, t)$ (dotted), due to churning of lower skilled agents.

(b) Belief about the cutoff agent is the sum of learning from performance $\lambda k_t(1-k_t)$ (solid) and learning from churning γ_t (dashed).

Figure 1: Equilibrium learning dynamics if $\mathbb{T} = [t^*, \bar{t}]$. For $t \in [0, t^*]$ there is no churning and clients only learn from performance. For $t \in [t^*, \bar{t}]$ the intermediary gradually churns lower skilled agents resulting in faster learning.

To illustrate the economic mechanism, suppose, for simplicity, that L is sufficiently low so that for all agents $\tilde{p}_0 > \underline{p}$ the option of starting their own firm dominates leaving the industry.[25] Define by γ_t the incremental reputation the agent gets if he stays with the intermediary

[24] For notational convenience $\partial_2 U(\tilde{p}, k) \equiv \frac{\partial}{\partial k} U(\tilde{p}, k)$. In particular, $\partial_2 U(\tilde{p}, k) \equiv \frac{\partial}{\partial k} U(k, k)\big|_{\tilde{p}=k}$.

[25] In this case the reservation wage $w_R(\tilde{p}_t, k_t)$ is simply equal to $A(k_t)$.

between t and $t + dt$ instead of opening his own firm, along the path of good performance

$$\gamma_t \stackrel{def}{=} \dot{k}_t - \lambda k_t(1 - k_t).$$

Without churning, reputation change is driven purely by the observable performance and $\gamma_t = 0$. When the intermediary churns lower-skilled agents, the incremental reputation change γ_t is strictly positive. The agent values this reputation growth as it increases his expected revenues of starting a firm in the future. His private value of gaining $\gamma_t \, dt$ units of reputation is, by definition, equal to $\gamma_t \cdot \partial_2 U(\tilde{p}, k) \, dt$. It is optimal for him to stay with the intermediary in period t if the combined value of his compensation $\tilde{w}_t = w(\tilde{p}_t, k_t)$ and reputation building exceeds his reservation wage

$$w(\tilde{p}_t, k_t) + \gamma_t \cdot \partial_2 U(\tilde{p}_t, k_t) \geq w_R(\tilde{p}_t, k_t). \tag{16}$$

Inequality (16) is binding in equilibrium because the intermediary chooses the lowest possible wage to retain the agent in each period. We see that the agent is willing to forgo short-term compensation only if $\gamma_t > 0$ since working for the intermediary facilitates building reputation and acts as a deferred compensation device. The intermediary can pay higher skilled agents less as long as she can commit to selective retention $\gamma_t > 0$ since higher skilled agents value reputation more as shown in Proposition 1.

Churning rate γ_t affects the compensation of all agents but is determined by the incentives of the intermediary to retain the cutoff agent $\tilde{p}_t = k_t$ in period t. The intermediary's revenue $A(q_t)$ is public and is pinned down by clients' belief about the agent's ability $q_t = Q(k_t, t)$. The intermediary's flow profit of employing the cutoff agent $\tilde{p}_t = k_t$ is equal to

$$A(Q(k_t, t)) - w(k_t, k_t) = A(Q(k_t, t)) - w_R(k_t, k_t) + \gamma_t \cdot \partial_2 U(k_t, k_t).$$

Turnover comes either as a result of the agent generating bad performance, and being fired, or as a result of his retention wage $w(k_t, k_t)$ being so high that the intermediary would rather replace him. In the latter case, the optimal churning time τ makes the intermediary exactly indifferent between retaining the cutoff agent and replacing him in that instance

$$\begin{aligned} \overbrace{A\big(Q(k_\tau, \tau)\big) - w_R(k_\tau, k_\tau)}^{net\,profit} &= \overbrace{rV}^{\substack{opportunity \\ cost}} \\ \Rightarrow \qquad \gamma(k_\tau, \tau) &= \frac{rV - \Big(A\big(Q(k_\tau, \tau)\big) - w_R(k_\tau, k_\tau)\Big)}{\partial_2 U(k_\tau, k_\tau)}. \end{aligned} \tag{17}$$

If it were the case that $A(Q(k_\tau, \tau)) - w_R(k_\tau, k_\tau) > rV$, then the intermediary would profit by retaining the cutoff agent for a bit longer. By similar logic, if $A(Q(k_\tau, \tau)) - w_R(k_\tau, k_\tau) < rV$, then the intermediary would have profited by letting go of the agent $\tilde{p}_t = k_t$ strictly before time τ. Equation (17) characterizes the reputation growth $\gamma(k_t, t)$ at every time t in which there is churning after good performance. By definition, $\gamma(k_t, t) = 0$ in all other periods. While (17) is a necessary first-order condition for churning time τ to be optimal, and we still need to identify the set $\mathbb{T}$ constituting the support of churning times when the intermediary lets go of the cutoff agent given good performance.

To identify the stopping set $\mathbb{T}$ we consider the intermediary's "autarky" problem of retaining the cutoff agent $k_t = \tilde{p}_t$ absent reputation building dynamics, i.e., if $k_t = \tilde{p}_t$ evolves solely based on performance (4), and the intermediary must pay the agent reservation wage $w_R(\tilde{p}_t, \tilde{p}_t)$ to retain him

$$\sup_{\hat{\tau}} \mathrm{E}_{\tilde{p}_t}\left[\int_t^{\hat{\tau}} e^{-r(s-t)}\Big[A\big(Q(\tilde{p}_s, s)\big) - w_R\big(\tilde{p}_s, \tilde{p}_s\big)\Big]\, ds + e^{-r(\hat{\tau}-t)} \cdot V\right]. \tag{18}$$

Surprisingly, the equilibrium decision to churn a $\tilde{p}_t' < k_t$ agent at time t is determined by the incentive to fire the agent in (18) and is unaffected by the pay-for-reputation dynamics. The heuristic argument is that reputation building rate $\gamma(k_\tau, \tau)$ makes the intermediary employing cutoff agent $\tilde{p}_\tau = k_\tau$ exactly indifferent between keeping and firing her agent, as can be seen in (17), and is strictly insufficient for any $\tilde{p}_t' < k_t$ agent to be retained going forward.

3.2.1 Churning Set $\mathbb{T}$

First consider the case where the profit wedge of the intermediary declines as performance signals are observed. Specifically, suppose that for each belief q about the retained agent and type k of the cutoff agent, the difference $A(\pi(q, t)) - A(\pi(k, t))$ is declining in t. A sufficient condition for it is $A'(p)p(1-p)$ be decreasing for $p \geq \underline{p}$.[26] Define t^* to be the first time when the initial cutoff agent $\tilde{p}_0 = k_0$ becomes unprofitable for the intermediary

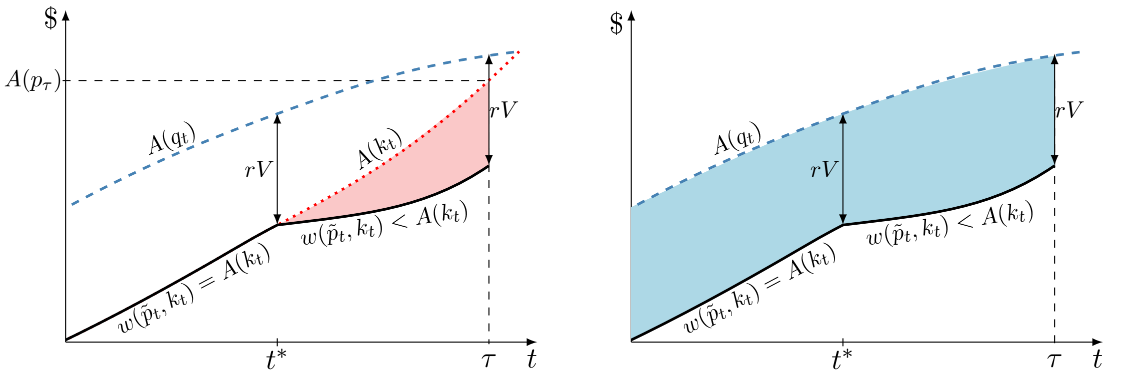
$$t^* \stackrel{def}{=} \inf\big\{t \geq 0 :\ A(\pi(q_0, t)) - A(\pi(k_0, t)) < rV\big\}. \tag{19}$$

The profit wedge is decreasing with performance signals and, once an agent becomes unprofitable, he remains unprofitable in the future.[27] Figure 2 illustrates the equilibrium wage and profitability dynamics.

For $t < t^*$ the profit wedge $A(q_t) - A(k_t)$ is sufficiently high so that the intermediary retains all well performing agents and only lets go of agents who have generated bad performance. We refer to this as the "quiet" period. The intermediary's profit wedge decreases as clients learn from performance and departures of agents who perform poorly and reveal themselves as unskilled. At $t = t^*$ the profit from retaining the lowest skilled agent is exactly equal to

[26] $A'(p)p(1-p)$ is decreasing for $p > \underline{p}$ for all continuously differentiable functions $A(\cdot)$ such that $A'(\cdot) \in [0, +\infty)$ as long as $\underline{p}$ is close enough to 1. For example, if $A(\cdot)$ is linear, then $\underline{p} \geq \frac{1}{2}$ is sufficient.

[27] We show in Lemma 2 that $w_R(k_t, k_t)$ in Equation (14) can be replaced by $A(k_t)$.



(a) Equilibrium wage $w(\tilde{p}_t, k_t)$ (solid line) and reservation wage $w_R(\tilde{p}_t, k_t) = A(k_t)$ (dotted line). The difference $w_R(\tilde{p}_t, k_t) - w(\tilde{p}_t, k_t)$ (filled area) is the agent paying for reputation.

(b) The intermediary's contemporaneous profit is the difference between her revenues $A(q_t)$ (dashed line) and the agent's compensation $w(\tilde{p}_t, t)$ (solid line). Profit wedge equals rV at t^* and $\bar{t}$.

Figure 2: Wage and revenue dynamics if $A(p) = p$ and $F(\cdot) \sim U\left[\frac{1}{3}, 1\right]$.

the intermediary's opportunity cost, i.e., $A(q_{t^*}) - A(k_{t^*}) = rV$. For $t > t^*$ the intermediary gradually churns lower-skilled agents by letting them go at an increasing rate $\gamma(k_t, t)$ (See Figure 1b), even though they have generated good performance up to time t, until all types are let go at a finite time $\bar{t}$. The resulting churning set is given by $\mathbb{T} = [t^*, \bar{t}]$. Churning is based on the intermediary's private information and is indicative of the agent's skill, further reducing the profit wedge $A(q_t) - A(k_t)$. Yet, the intermediary can profit from this as better agents are willing to accept lower pay to build reputation, as can be seen in Figure 2a. For $t \in [t^*, \bar{t}]$ the intermediary lets go of the lowest skilled agents at an increasing rate $\gamma(k_t, t)$.

Corollary 1. *For a general churning set $\mathbb{T}$ the equilibrium dynamics exhibit four robust properties*

(i) higher-skilled agents, as measured by their private information, have longer careers with the intermediary;

(ii) intermediary's flow profit decreases before the start of the churning period and increases after;

(iii) higher-skilled agents pay for building a reputation in periods when the intermediary churns lower-skilled agents, i.e., $t \in \mathbb{T}$;

(iv) if the agent is churned, then he quits the intermediary after good performance and obtains a positive jump in compensation at the time of opening his own firm. The compensation increase is higher if the agent leaves when information asymmetry $q_\tau - k_\tau$ is lower.

Results $(i)-(iii)$ of Corollary 1 follow from our previous arguments. Reputation building dynamics are critical for these results – absent it, the agent would be bound to his reservation wage before and after he leaves the intermediary. Figure 2 illustrates these properties. They are robust to the shape of the revenue function $A(\cdot)$ and the distribution of private information $F(\cdot)$. Result (iv) stems from the fact that if the asymmetry is low, the intermediary is churning agents at a higher rate and, hence, they pay more for reputation just before quitting, leading to a bigger increase in compensation right after as illustrated in Figure 2a. An additional empirical implication of our analysis is that wage dispersion within the intermediary is higher in periods of churning, as in quite periods, agents' wage does not depend on their skill level, whereas in churning periods wage is tightly linked to the agent's skill.

Non-monotone profit wedge. If $A'(p)p(1-p)$ is not decreasing in p, the profit wedge $A(q_t) - A(k_t)$ may (locally) increase in response to good performance signals[28]. Identifying the churning set $\mathbb{T}$ requires a more subtle understanding of the optimal turnover decision of the intermediary: when dynamically determining the optimal retention decision (i.e., when solving (18)) churning time τ still satisfies (17), however, the prospect of a greater profit wedge in the future implies that it may be sub-optimal to churn the agent the first time the local indifference condition (17) is met. In other words, the intermediary may strategically lose money on the agent in the short-run in the hope that his good performance leads to high profits in the future. The intermediary switches between (positive length) periods in which all agents are retained and periods in which lower-skilled agents are gradually churned, pinning down the life-cycle of an agent's career. Eventually, even the highest skilled agents leave the intermediary to contract directly with clients. Thus, the churning set $\mathbb{T}$ is a natural generalization of the simple case of $[t^*, \bar{t}]$. Our key insights and the economic mechanism driving them do not depend on the specific shape of the distribution of asymmetric information $F(\cdot)$ or the shape of the revenue function $A(\cdot)$. The solution approach and equilibrium structure also naturally extend to a Brownian model of performance, as shown in Section 5.

Initial k_0. It may not be profitable for the intermediary to hire all agents initially. $\underline{p}$ may be very low so that the worst agents would not value staying in the industry. Their reservation wage ρL may, as a result, be high relative to the revenues $A(Q(\underline{p}, t))$ were they to be hired. The intermediary is, thus, selective at $t = 0$, and agents who are not hired at $t = 0$ leave the industry.

Lemma 2. *The intermediary hires all agents who prefer to stay in the industry under full information, i.e., all $\tilde{p}_0$ such that $U(\tilde{p}_0, \tilde{p}_0) > L$. However, she hires weakly fewer agents than*

[28] This may occur due to the nonlinear nature of binary learning $\pi(p, t)$, or due to potential convexity of the revenue function $A(\cdot)$

would have opened their firm in the absence of the intermediary. Higher skilled agents may obtain strictly positive rents from going to work for the intermediary.

Lemma 2 is important in understanding the role of the intermediary in modulating entry into the industry. The agents willing to open their firm under full information are willing to pay the intermediary for reputation, and, as a result, she is interested in hiring them even if it is only to extract this value from them. In the absence of the intermediary, lower-skilled agents can pool with higher skilled agents until their performance reveals their skill. They have a greater incentive to enter into the industry than the intermediary has in hiring them, as she faces an additional opportunity cost rV of hiring a new agent. This leads to the intermediary's incentive to hire a lower-skilled agent lower than the incentive of that agent to pool with higher skilled agents until bad performance is realized. The intermediary, thus, serves an additional purpose of initial selection. The agents who are not hired by the intermediary are locked out of the market and leave the industry. Higher skilled agents find the initial certification valuable and may find it *strictly* optimal to start their careers with the intermediary.

Uniqueness given V. For a given expected continuation value $V > 0$ of the intermediary, the equilibrium dynamics are uniquely pinned down, and we show that there exists a unique equilibrium limit to a sequence of games in which V converges to 0.

Lemma 3. *Proposition 2 specifies the unique pure-strategy equilibrium if $V > 0$. Moreover, it specifies the limiting equilibrium corresponding to $V = 0$.*

We first establish that there cannot be an atom of agents leaving the intermediary at a given time t along the equilibrium path. If this were the case, then the monotonicity of process k requires that it is lower-skilled agents that are being let go. If a positive mass of lower-skilled agents leave the intermediary, then the skill of the worst remaining agent increases discretely and belief consistency (on- and off-path) implies that k_t experiences a positive jump at t.[29] If k_t experiences a positive jump at time t, i.e., $k_{t+} - k_t > 0$, then it is sub-optimal to let go of the agent at this time, as the intermediary would like to charge the agent for building reputation in that period, leading to a contradiction that there can be an atom of quits at any time. We, thus, focus on continuous belief process $(k_t)_{t\geq 0}$ with the requirement that for an off-equilibrium path $t > \tau$ the clients' beliefs satisfy $k_t = q_t = \pi(\bar{p}, t)$. Together with the

[29] In the event that $\mathrm{P}(\tau < t) = 1$, the *independence of never weak best-responses*, as in Noldeke and Van Damme (1990), puts off-path beliefs on types who gain most from the deviation, which are the best ones, i.e., $\pi(\bar{p}, t)$. If $V = 0$, then the independence of never weak best-responses has no effect, and we focus on the unique limiting equilibrium. Monotonicity in initial types is sufficient but not necessary - our results hold if we were to restrict attention to the *best or worst* agents leaving the intermediary at any point in time.

monotonicity and consistency requirements, this leads to a unique process $(k_t)_{t\geq 0}$ derived in above.

Equilibrium value V. In most applications our model is intended to capture, the intermediary can replace the agent, justifying the endogenous determination of V via (7). Explicit characterization of the equilibrium value V, however, is challenging due to the dependency of the revenue and wage processes $A(q_t)$ and $w(\tilde{p}_t, k_t)$ on V, especially for a general revenue function $A(\cdot)$ and distribution of private information $F(\cdot)$.[30]

Lemma 4. *Assume the intermediary has an outside option of* 0 *and denote by V_n to be her expected value if she can sequentially hire at most n agents. Then*

(i) there exists a unique limit $V = \lim_{n\to\infty} V_n$ and, if $\Delta > 0$, then V is a solution to (7)*;*

(ii) if the replacement costs I or Δ are relatively large, then V is the unique solution to (7)*;*

(iii) When both I and Δ are small, there may be other equilibria, corresponding to the multiple solutions to (7)*.*

An equilibrium value V can be obtained as the unique limit to the intermediary's equilibrium payoff in games in which she can sequentially employ a finite number of agents. By continuity, V must satisfy (7). If costs I or Δ are very large, then the intermediary would rather not look for a new agent at all, implying that $V = 0$. In this special case the intermediary never churns lower-skilled agents since $A(q_t) - A(k_t) > 0 = rV$. If the intermediary's value V is positive but small, the quiet period $[0, t^*]$,[31] over which the intermediary's revenues are independent of V, is large. It implies that the intermediary's continuation payoff on the right-hand side of (7) is less sensitive to V, implying a single root of the fixed point equation (7). If costs I or Δ are sufficiently large, then V is small enough to guarantee this unique solution. We observe numerically that (7) admits a unique solution for a broad range of replacement costs I and Δ. Finally, if replacement costs are low, there may be multiple equilibria, ranked by the intermediary's expected payoff V. These equilibria correspond to multiple solutions to (7), which we illustrate in Figure A.1 in Online Appendix A. The intuition is that a higher equilibrium value V can be self-enforcing as it acts as a commitment device to churn agents at a high rate, leading to more pay-for-reputation dynamics and, in turn, justifying the higher expected value.

[30] It is useful to note that (7) does not specify a contraction operator for the expected value V.

[31] The definition of t^* in (19) extends to a general churning set $\mathbb{T}$ as $t^* \stackrel{def}{=} \inf\{t : t \in \mathbb{T}\}$.

3.3 Intermediary's Profitability

Adverse selection is bad for workers but is beneficial for employers who can identify talent. The intermediary's cost of compensating an agent is driven by the worst remaining agent, while her revenues are pinned down by the average agent employed.

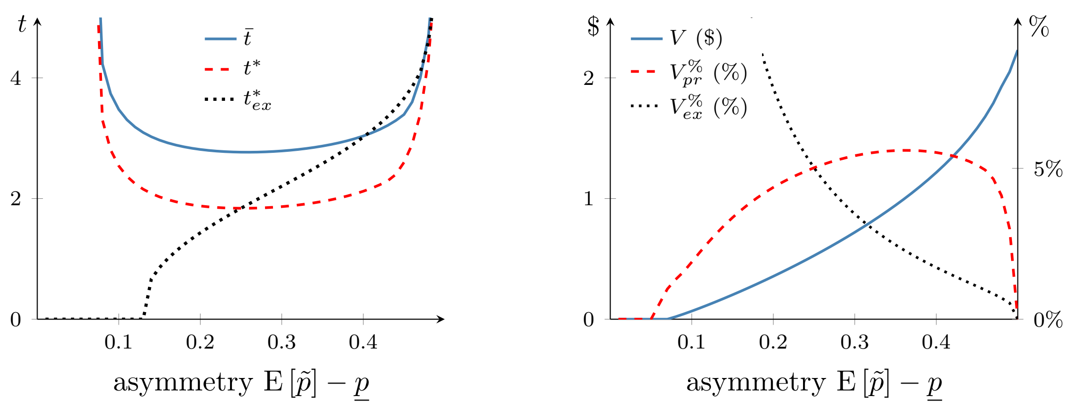
Lemma 5. *Suppose replacement costs I, or Δ are reasonably large. The intermediary's expected value V is weakly*

(i) decreasing in the cutoff agent's expected skill $\underline{p}$ and outside option L;

(ii) increasing in the average quality of the agent's average skill $\mathrm{E}[\tilde{p}]$;

(iii) decreasing in the informativeness λ of performance signals for a sufficiently low r or a sufficiently high λ.

The dependence of V on the support of beliefs and performance informativeness are illustrated by solid lines in Figures 3b and 4b respectively. Value V not only captures the overall profitability of the intermediary, but also the continuation value she obtains if she were to replace the current agent. Such a feedback effect has profound implications on the dynamics of the agent's career in response to changes in the information environment.



(a) Churning start and end times.

(b) Equilibrium value V and percentage of revenues obtained from pay-for-reputation.

Figure 3: Comparative statics with respect to information asymmetry $\mathrm{E}[\tilde{p}] - \underline{p}$. For the parameters considered, the intermediary hires all initial types, i.e., $\underline{p} = k_0$, and the churning set is given by $\mathbb{T} = [t^*, \bar{t}]$.

Figure 3a plots the churning interval $\mathbb{T} = [t^*, \bar{t}]$ as a function of the initial information asymmetry, measured by $\mathrm{E}[\tilde{p}] - \underline{p}$. As depicted in the figure, the intermediary starts churning agents earlier (lower t^*) and for longer (larger interval $\bar{t} - t^*$) for intermediate levels of asymmetry. If asymmetry is low then, as can be seen in Figure 3b, the value V of hiring a new agent is low, implying the agent is retained for longer (larger t^*). If asymmetry is large, then V is greater, but belief about the worst agent $\underline{p}$ is also quite low. Consequently, it takes a long performance track-record to improve the reputation of the worst agent enough to induce churning by the intermediary. The intermediary, thus, retains agents for a long time, as long as they generate good performance.[32]

For every $t \in \mathbb{T}$, higher-skilled agents pay the intermediary in order to build their reputation faster. Total profits of the intermediary obtained through this channel are also maximized for intermediate levels of asymmetry. Define $V_{pr}^{\%}$ to be the fraction of the intermediary's expected profits arising from the agents paying to build reputation

$$V_{pr}^{\%} \stackrel{def}{=} \frac{\mathrm{E}\left[\int_0^\tau e^{-rt}(w_R(\tilde{p}_t, k_t) - w(\tilde{p}_t, k_t))\, dt\right]}{\mathrm{E}\left[\int_0^\tau e^{-rt}(A(q_t) - w(\tilde{p}_t, k_t))\, dt\right]}.$$

We see in Figure 3b that $V_{pr}^{\%}$ (dashed line) exhibits an inverse U-shape. For extreme levels of asymmetry there is very little churning and, thus, $V_{pr}^{\%}$ is close to 0. For intermediate levels of asymmetry the churning set $\mathbb{T}$ starts sooner and lasts for longer, implying that $V_{pr}^{\%}$ is higher.

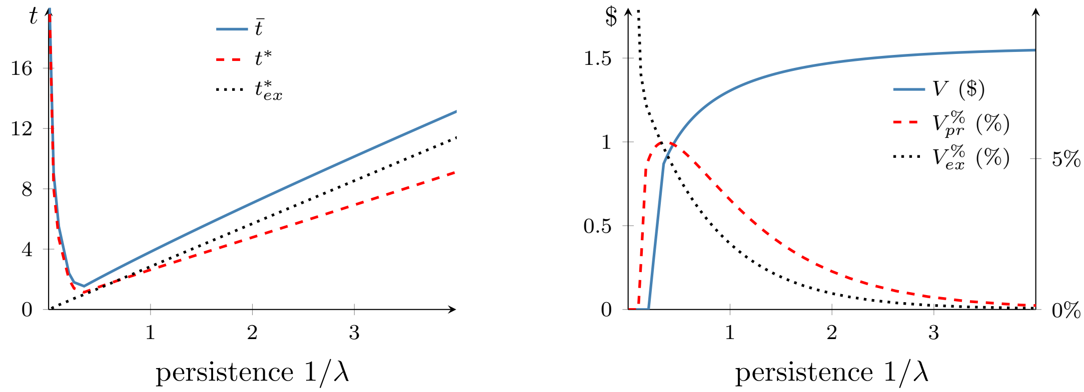
Both panels of Figure 3 highlight the importance of the intermediary's value V being determined in equilibrium, embedding the inter-temporal spillover effects across the sequence of employed agents. We plot the start of the churning time t_{ex}^* and the pay-for-reputation percentage $V_{ex}^{\%}$ if V were taken as exogenous in Figures 3a and 3b respectively. The differences are stark: t_{ex}^* is increasing in the initial asymmetry, while $V_{ex}^{\%}$ is decreasing, leading to misleading predictions when the information asymmetry is low.

Corollary 2. *The fraction of the intermediary's profits $V_{pr}^{\%}$ obtained from the agents paying for reputation is greatest for intermediate levels of the initial information asymmetry* $\mathrm{E}[\tilde{p}] - \underline{p}$.

Similar forces are in play when we consider implications of informativeness λ of performance on the value V of the intermediary. We plot the comparative statics in Figure 4 as a function of the information persistence $1/\lambda$ in order for the x-axis to increase in the measure of information asymmetry and making it economically comparable to Figure 3.

For low information persistence, the profitability of the new agent is so low that the intermediary would rather not churn a currently employed agent. For high information persistence, beliefs about the current agent are slow to change, implying little benefit of replacing a given agent. The churning set $\mathbb{T}$ is U-shaped in persistence, and the percent of profits due stemming from pay-for-reputation $V_{pr}^{\%}$ is maximized for interior levels of persistence. The contrast to how results would look if the intermediary's continuation value V were exogenous is telling

[32] This argument holds exactly if $L = A(0)/\rho$, but the same forces are in play if L strictly exceeds it.



(a) Churning start and end times.

(b) Equilibrium value V and percentage of revenues obtained from pay-for-reputation.

Figure 4: Comparative statics with respect to information persistence $1/\lambda$.

and highlights the importance of accounting for V being determined in equilibrium; with an exogenous V, both the churning start time and the fraction of present value of profits are monotone.

Corollary 3. *The fraction of the intermediary's profits $V^{\%}_{pr}$ obtained from the agents paying for reputation is greatest for intermediate levels of information persistence $1/\lambda$.*

4 Competition, Signaling, and Training

We show that the pay-for-reputation mechanism applies even if the agent has bargaining power, stemming from either imperfect competition among informed intermediaries, or from the possibility of independently signaling his ability to clients. Moreover, we highlight the interactions and differences between training and reputation building, showing the former is concentrated in quiet periods when the information asymmetry is large, while the latter occurs during churning periods when information asymmetry is small.

4.1 Competition for Agents

The intermediary can underpay the agent since his outside option is to either open a firm, but be perceived as a lower type, or leave the industry altogether. Suppose, if the agent leaves the current intermediary, he can finds one new intermediary to work for with probability ζ_1, and finds two or more intermediaries to work for with probability ζ_2.[33] For tractability, we assume the intermediary at the new firm has the same information $\tilde{p}$ about the agent's

[33] If the agent approached only one intermediary at a time, the intermediary would retain all of the bargaining power, economic behavior known as the Diamond paradox and shown in Diamond (1971).

ability. The possibility of finding multiple intermediaries is beneficial to the agent as they compete for his services by offering a signing bonus.[34] With probability $1-\zeta_1-\zeta_2$, the agent does not find a new intermediary and either starts his own firm or leaves the industry. The risk of not finding a new intermediary and having to start his firm prematurely is still costly for the agent if he chooses to leave the incumbent intermediary.[35]

Lemma 6. *For a given intermediary's expected value V, the equilibrium churning set $\mathbb{T}$ and belief processes $(k_t, q_t)_{t\geq 0}$ are unchanged. The agent's compensation is determined by his endogenous bargaining power $\zeta = \zeta_2/(\zeta_1+\zeta_2)$. Moreover, if the principal and the agent are equally patient, $r=\rho$, then*

$$w_\zeta(\tilde{p}_t, t) = \underbrace{(1-\zeta)w_R(\tilde{p}_t, k_t) + \zeta\big(A(Q(k_t,t)) - rV\big)}_{\text{reservation wage with bargaining power } \zeta} - \underbrace{(1-\zeta)\gamma(k_t,t)\mathbb{1}\left\{t\in\mathbb{T}\right\}\partial_2 U(\tilde{p}_t, k_t)}_{\text{pay for reputation}}.$$

The intermediary's equilibrium value V is weakly decreasing in ζ.

For a given expected value of V, the churning set $\mathbb{T}$ is derived in Proposition 2. The agent's bargaining power, however, increases the wages he obtains from the intermediary and lowers her equilibrium value V. Thus, in equilibrium, the intermediary churns agents less, even though they require greater compensation.

4.2 Signaling via Discount

We show that the agent values building reputation by working for the intermediary even if he is able to signal his ability to clients at a cost. While such a possibility limits the intermediary's ability to underpay the agent, the pay-for-reputation dynamics still hold. Moreover, higher-skilled agents are now willing to pay the intermediary for the possibility to generate a track record of performance before contracting with clients directly. In equilibrium, agents pay more for generating performance signals, but pay less for building reputation, relative to if they had no opportunity to signal independently.

Suppose, when opening his firm, the agent can offer services at a percentage discount β, resulting in revenue $\beta\cdot A(\cdot)$. For tractability, we assume that β is chosen once and does not change, although the argument applies to any setting in which repricing of services is sufficiently costly.[36]

[34] In the absence of commitment to long-term contracts multiple intermediaries compete a-la Bertrand by offering an up-front payment to the agent, but, subsequently, keeping the agent at his outside option.

[35] The main model corresponds to the case of $\zeta_2=\zeta_1=0$.

[36] In the context of asset management, management fees are specified as a percentage of assets under management, justifying a percentage discount in pricing. These contracts are signed with a large number of investors and are costly to renegotiate.

Lemma 7. *Suppose the cutoff agent's type is k_t. A higher skilled agent $\tilde{p}_t > k_t$ can signal his ability to clients by offering a discount $\beta(\tilde{p}_t, k_t)$ on his services given by*

$$\beta(\tilde{p}_t, k_t) = exp\left(- \int_{k_t}^{\tilde{p}_t} \frac{\partial_2 U(x,x)}{U(x,x) - \frac{\lambda L(1-x)}{\rho+\lambda}} \, dx \right). \tag{20}$$

Discount $\beta(\tilde{p}_t, k_t)$ is a function of both the agent's private type $\tilde{p}_t$, as well as the past history, summarized by the cutoff type k_t in that period. The agent's expected value if he can signal his ability is then given by

$$U_\beta(\tilde{p}, k) \stackrel{def}{=} \tilde{p} \cdot \beta(\tilde{p}, k) \cdot u_1(\tilde{p}) + (1 - \tilde{p}) \cdot \left[\beta(\tilde{p}, k) \cdot u_0(\tilde{p}) + (1 - \beta(\tilde{p}, k)) \cdot \frac{\lambda L}{\rho + \lambda} \right].$$

The possibility to signal the agent's skill improves his dynamic outside option, but it comes at a cost which increases in his skill, as can be seen from (20). Higher skilled agents suffer this discount to revenues for longer, implying greater overall cost of signaling for higher skilled agents. This increased cost implies that higher skilled agents preserves the single-crossing property of the modified outside option $U_\beta(\tilde{p}, k)$. Proposition 3 establishes that, in equilibrium, the agent does not signal his ability independently to clients, yet this possibility affects his compensation and tenure dynamics.

Proposition 3. *Higher-skilled agents value reputation more even if they are able to signal their ability to clients*

$$\partial_2 U_\beta(\tilde{p}', k) > \partial_2 U_\beta(\tilde{p}, k) \quad \text{for any} \quad \tilde{p}' > \tilde{p}.$$

If λ/ρ is sufficiently large,[37] *then the agent starts by working for the intermediary, opens his firm when clients correctly assesses his skill, i.e., $\tilde{p}_\tau = k_\tau$, and does not offer a discount when opening his firm on equilibrium path. The intermediary's payoff is lower than if the agent were unable to signal his ability, and, consequently, churning starts later.*

A higher skilled agent suffers a greater dead-weight cost of signaling his skill to clients. Working for the intermediary provides two advantages. First, the agent generates a track-record of performance. Second, due to the intermediary's churning, the agent can build a reputation relatively quickly. When the agent is sufficiently patient, or performance is sufficiently informative, the agent does not signal his skill with fees but instead builds reputation only by working for the intermediary. The option of independent signaling, however, has an overall increase in the agent's payoff and reduces the intermediary's expected profits.

For tractability, we have assumed that the agent chooses $\beta(\tilde{p}, k)$ once at the time of opening his firm. If the agent were able to change it subsequently, then a high-skilled agent can

[37] The precise sufficient condition is $A'(p) \geq \frac{A(1)-A(p)}{1-p} \cdot \frac{(\rho/\lambda+1)\rho/\lambda}{\rho/\lambda+p}$ for every $p \geq \underline{p}$.

credibly signal his skill by starting with a high discount, i.e., a low β, and then reduce the discount once his performance is sufficiently good. Such dynamic contracts are preferable by the high-skilled agents as it allows them to condition profits on their good performance and separate efficiently from lower-skilled agents.[38] Formally, if β_t can take values in $[\underline{\beta}, 1]$, then the most efficient signaling contract is to offer $\beta_t = \underline{\beta}$ until the first time when $\tilde{p}_t = k_t$, and set $\beta_t = 1$ subsequently. Whether reputation-building is more efficient independently or via the intermediary depends on the ranking of

$$\underbrace{\frac{(1-\underline{\beta}) \cdot A(k_t)}{\partial_2 U(k_t, k_t)}}_{\substack{\text{independent} \\ \text{reputation building}}} \qquad versus \qquad \underbrace{\frac{\left[rV - A(q_t) + A(k_t)\right]^+}{\partial_2 U(k_t, k_t)}}_{\substack{\text{intermediated} \\ \text{reputation building}}}. \tag{21}$$

If $\underline{\beta}$ is not too low or the intermediary's expected value V is sufficiently high, the right hand side of (21) may be higher than the left, implying that intermediated reputation building may be more efficient even if the agent can change the discount rate frequently. Importantly, allowing a low $\underline{\beta}$ implies the agent must have substantial capital of his own to sustain himself for an extended period of time, which may not be the case in many economic settings.

The intermediary does not have an incentive to signal the agent's ability to clients – any such signaling would benefit the agent at her expense. This preference for opacity may help rationalize the low variation in fees among mutual funds and hedge funds in the finance industry.

4.3 Paying for Reputation versus Paying for Training

Agents are willing to accept below reservation wages to establish a reputation with clients. Reputation does not affect worker productivity but can be thought of as a "quality" of a worker, enabling him to generate greater revenues. We show, however, that building a reputation is distinct from training and occurs at different times of an agent's career.

Suppose skill is subject to depreciation at rate δ, such that a skilled, $\theta = 1$, agent becomes an unskilled, $\theta = 0$, agent with intensity δ. Assume the intermediary can spend a private convex flow cost $c(a)$ to reduce this depreciation intensity by a. Furthermore, assume training also benefits an unskilled, $\theta = 0$, agent by making him skilled with intensity a. To avoid corner solutions, we assume $c'(\delta) = +\infty$, so that $a \leq \delta$. Under such specification the belief process follows

$$d\tilde{p}_t = \underbrace{\lambda\tilde{p}_t(1-\tilde{p}_t)\,dt + \tilde{p}_t(dX_t - \mu\,dt)}_{learning\ from\ performance} - \underbrace{(\delta - \tilde{a}_t)\tilde{p}_t\,dt}_{skill\ depreciation} + \underbrace{\tilde{a}_t(1-\tilde{p}_t)\,dt.}_{skill\ accumulation} \tag{22}$$

[38] See Laffont and Martimort (2009), Chapter 3.

where $\tilde{a}_t$ is the training provided to type $\tilde{p}_t$ agent, given the cutoff type equal to k_t. The depreciation of skill makes agents value training regardless of their ability $\tilde{p}_t$ and provides a tractable setting to contrast pay-for-reputation with pay-for-training.[39] Denote by $\tilde{a}_t = a(\tilde{p}_t, k_t)$ to be the training offered by the intermediary to an agent with skill $\tilde{p}_t$, given clients' belief k_t about the cutoff agent. The equilibrium wage of the agent is then given by

$$w_T(\tilde{p}_t, k_t) = w_R(\tilde{p}_t, k_t) - \underbrace{a(\tilde{p}_t, k_t) \cdot \partial_1 U(\tilde{p}_t, k_t)}_{pay\ for\ training} - \underbrace{\big(\gamma(t, k_t) + a(k_t, k_t)\big) \cdot \partial_2 U(\tilde{p}_t, k_t)}_{pay\ for\ reputation}.$$

Agents value being skilled as it increases their chances of staying in the industry. They are willing to be underpaid for an increase in their ability, consistent with the logic of Becker (1962) and Acemoglu and Pischke (1999). That increase, however, is not directly observed by clients, who have to conjecture the amount of training received by the workers. This affects the agent's reputation in equilibrium, as clients infer that the ability of the worst remaining agent increases not only due to the turnover, but also due to training. As a result, training not only improves the agent's ability, but also facilitates reputation building.

The intermediary's incentive to invest in general training, as pointed out by Acemoglu and Pischke (1999), is determined by the frictions in the agent's labor market. Denote by $V(\tilde{p}_t, k_t)$ the continuation value of the intermediary from employing an agent $\tilde{p}_t$ given market beliefs k_t. The agent observes the training provided to him and is willing to be under-compensated in exchange. The amount of training is, then, pinned down by equating the marginal cost of training to the marginal value of skill to both the agent and the intermediary

$$c'\big(a(\tilde{p}_t, k_t)\big) = \underbrace{\partial_1 U(\tilde{p}_t, k_t)}_{(i)} + \underbrace{\partial_1 V(\tilde{p}_t, k_t)}_{(ii)}. \tag{23}$$

Term (i) in (23) captures the agent's incentive to increase his ability and, thus, pay for his training. Because of the difference between $\tilde{p}_t$ and k_t, the agent does not capture all of the surplus generated by his training. Term (ii) in (23) captures the intermediary's marginal value of employing a higher skilled agent. The intermediary values higher-skilled agents more since they are more likely to generate good performance, and are willing to pay more for reputation building. As the information advantage of the intermediary declines, however, her incentive to train the worker also declines.

Lemma 8. *The intermediary's incentive to train the agent declines as he approaches termination, i.e.,* $\partial_1 V(k_t, k_t) = 0$ *for* $t \in \mathbb{T}$.

[39] Acemoglu and Pischke (1998) provide a model in which higher-skilled agents value training more than lower-skilled agents, which creates an additional reason for wage compression, which can be easily embedded in (22). In our model, however, wage compression is present even if skill and training are substitutes, as long as the intermediary's cost of training is sufficiently large.

The intermediary has no interest in training the agent when he is close to being let go, while this is precisely the time when the agent builds reputation, as can be seen from the increase in reputation growth in Figure 1b. Churning occurs when information asymmetry is low, training, in contrast, occurs exactly when information asymmetry is high and reputation growth $\gamma(k_t, t)$ is low. In addition, Acemoglu and Pischke (1998) show that there may multiple equilibria characterized by different turnover rates, with greater turnover leading to less training. Multiple equilibria also arise in our model, as shown in Lemma 4, with turnover being beneficial for reputation building: greater turnover improves intermediary's profitability, increases her selectivity at the hiring stage, and speeds up reputation building by higher-skilled agents.

4.4 Further robustness

Long-term commitment. We conduct our analysis assuming the intermediary cannot commit to long-term contracts. This way, the agent's incentives to forgo short-term compensation are solely driven by his reputation building motives and not deferred compensation promised by the intermediary. We show in Lemma A.1 of the Online Appendix that, under the optimal long-term contract chosen by the intermediary, the agent's continuation value is pinned down by the binding retention constraint (6), implying equivalent reputation and compensation dynamics.

Limited liability. We do not impose limited liability in the baseline formulation of the model. This is justified by the fact that the agent pays for reputation out of his strictly positive reservation wage $w_R(\tilde{p}_t, k_t)$, implying that equilibrium wages $w(\tilde{p}_t, k_t)$ are positive if V is not very big. If V is big, then the rate of churning may be quite high, and limited liability may bind when the agent pays for reputation building. In this case, the agent gets a strictly positive benefit during the reputation building stage. Foreseeing this, the intermediary can underpay the agent even during the quiet period, extracting the expected gain obtained by the agent during the churning period.

Positive Poisson performance signals. We model performance signals as a perfectly informative negative Poisson process. The tractability of using a perfectly informative Poisson learning technology is well described in the survey of the experimentation literature of Hörner and Skrzypacz (2018). In the professions we have in mind, it is easy to reveal a lack of skill, but it likely requires a long track record to convince clients of true ability. As a result, highly skilled agents cannot rely on outstanding performance to convince clients of their ability, increasing their reliance on reputation. If the performance signals stemmed from a positive

Poisson process, then our economic argument would still hold as long as the agent's skill when working for the intermediary is imperfectly correlated with his skill when he opens his own firm. The intuition is similar to Holmström (1999), in which residual public uncertainty is essential to making agents care about their reputation via performance.

5 Brownian Model of Performance

While the Poisson structure for performance signals is highly tractable, the resulting model co-mingles the effects of residual uncertainty about an agent's skill and his performance in his retention and compensation. Our objective in this section is to disentangle the two. We do so by introducing Brownian performance signals. We show that good performance endogenously increases information asymmetry, leading the intermediary to retain well-performing agents for longer. Importantly, the equilibrium structure remains similar, exhibiting a combination of quiet periods with periods in which the intermediary churns agents to entice higher-skilled agents to pay for reputation.

Introducing Brownian performance signals is a nontrivial task both in setting up and analyzing the model. If we were to, simply, replace the Poisson process N^{θ} in (1) with a Brownian motion while keeping $\tilde{\theta}$ binary, we would inherit the restrictive property of binary learning linking posterior belief about the agent's ability with the residual uncertainty about it. In particular, after a history of good performance, the clients' belief becomes less and less sensitive to performance, making a well-performing agent, potentially, care less about his future performance. We view this as an artifact of the binary learning technology, and contrary to the economic intuition that agents with a better reputation care more to protect it, as shown in Section 3.1. To remedy this problem, we consider a model in which θ takes a continuum of values, and the players learn about it by observing Brownian performance signals. This leads to a rich learning environment, allowing us to distinguish between the effects of the agent's past performance and residual uncertainty about his ability on compensation and turnover outcomes.

5.1 Brownian Model Information Structure

The agent's skill $\tilde{\theta}$ follows a normal distribution $\mathcal{N}(\tilde{p}_0, \sigma_{\theta}^2)$, conditional on a random variable $\tilde{p}_0 \sim F(\cdot)$. Both the agent and the intermediary privately observe $\tilde{p}_0$, while the clients do not. Similar to the Poisson model, we assume $F(\cdot)$ is continuous and has full support over $[\underline{p}, \overline{p}]$, with $\underline{p} > -\infty$ so that the severity of the initial adverse selection problem is bounded.[40]

[40]This would arise endogenously in our model anyway as agents of very low skill would exit the industry. The necessary methodological step is to obtain a tractable model of Brownian learning about a truncated

For expositional convenience, we assume $F(\cdot)$ has a log-concave density function $f(\cdot)$.[41] All parties observe performance process X_t given by

$$X_t = \theta t + \sigma B_t, \tag{24}$$

where process $B = (B_t)_{t \geq 0}$ is a standard Brownian motion independent of θ. It is convenient to define the informativeness of the performance signal as

$$\phi \overset{def}{=} \sigma_\theta^2/\sigma^2.$$

The private posterior of the intermediary-agent pair after observing a performance path $(X_s)_{s \leq t}$ is given by a combination of performance signals and their private information

$$\tilde{p}_t = \Pi(\tilde{p}_0, t, X_t) \overset{def}{=} \mathrm{E}\left[\theta \,\middle|\, X_t, \tilde{p}_0\right] = \frac{\phi}{\phi t + 1} \cdot X_t + \frac{1}{\phi t + 1} \cdot \tilde{p}_0. \tag{25}$$

Function $\Pi(\tilde{p}_0, t, X_t)$ is the analogue of $\pi(\tilde{p}_0, t)$, but distinguishes the effects of elapsed time t and realized performance X_t on the posterior belief of the intermediary-agent pair. Denote by k_t the clients' belief about the agent who separates from the intermediary and starts his firm at time t:

$$k_t \overset{def}{=} \mathrm{E}\left[\theta \,\middle|\, X_t, \tau = t\right] = \mathrm{E}\left[\tilde{p}_t \,\middle|\, X_t, \tau = t\right].$$

Similarly, denote by q_t the clients' belief about the agent retained by the intermediary. The intermediary retains higher skilled agents for longer, implying that q_t is the average belief about the agents who are more skilled than the cutoff agent[42]

$$q_t = \mathrm{E}\left[\theta \,\middle|\, (X_s)_{s \leq t}, t < \tau\right] = \mathrm{E}\left[\theta \,\middle|\, X_t, t < \tau\right] = \mathrm{E}\left[\tilde{p}_t \,\middle|\, X_t, \tilde{p}_t > k_t\right].$$

The equilibrium definition is the same as the one used in Section 2 with the additional requirement that the clients correctly identify the agent's type when he opens his own firm. In the Poisson setting, the unique equilibrium is separating, and similar arguments can be applied to this Brownian model. The incremental difficulty of ruling out other equilibria in the Brownian model stems from characterizing the agent's subgame if an atom of agents of different types were to leave the principal. This leads to a dynamic signaling subgame, very similar to the one we already analyzed in Section A.4 of the Online Appendix and one that we could, potentially, tackle.[43]

distribution of private information.

[41] Most commonly used distributions, such as uniform, normal, and exponential satisfy this property. See Burdett (1996) for the properties of log-concave distributions.

[42] We show in Lemma A.25 that cumulative performance X_t is a sufficient statistic for $(X_s)_{s \leq t}$ for a general distribution of private information $F(\cdot)$. This drastically simplifies our filtering problem and permits us to derive the equilibrium structure for a general distribution $F(\cdot)$.

[43] In it we show that even after the agent opens his own firm, a more skilled agent can credibly convey his private information to clients by keeping his firm open after poor performance.

5.2 Agent's Dynamic Outside Option given Brownian Signals

In a separating equilibrium the agent quits the intermediary when clients correctly identify his skill, i.e., $\tilde{p}_\tau = k_\tau$. Subsequently, both the agent and the clients update their beliefs for $s > t$ based on subsequent performance, resulting in a posterior belief process about θ given by

$$k_{t,s} = \mathrm{E}\left[\theta \,|\, X_s - X_t, k_t\right] = (\phi t + 1) \cdot \left(\frac{\phi}{\phi s + 1} \cdot (X_s - X_t) + \frac{1}{\phi s + 1} \cdot k_t \right). \tag{26}$$

The clients' belief increases in response to the agent's good performance and he obtains greater profits. If the agent performs poorly, the belief about him declines, and he may find staying in the industry unprofitable, relative to his outside option L.

As we've seen in Sections 3 and 4, the agent's compensation while working for the intermediary is determined by his outside option of opening his own firm. Suppose the agent was to refuse the compensation offered by the intermediary and open his firm when his type $\tilde{p}_t$ exceeds the cutoff type k_t. Consequently, he is relatively more optimistic about future performance since, from his perspective, future performance X_t has better prospects, increasing his value of staying in the industry. In this case, he finds it optimal to stay in the industry longer than an agent with a lower private belief. For the sake of concision, since $\tilde{p}_t > k_t$ occurs off-equilibrium path, we assume that the clients continue to update their beliefs according to (26) and do not make additional positive inferences about the agent when they observe him staying in the industry after bad performance. We also formulate and solve the subgame equilibrium if the clients update positively if the agent remains in the industry following bad performance in Section A.4 of the Online Appendix A. This requires extra care and notation in defining the clients' inferences, but the implications for the agent's compensation and turnover are qualitatively unchanged.[44] To economize on space, we focus on the simpler subgame equilibrium specification in the main text.

The expected value to the agent who leaves the intermediary is given by

$$U(\tilde{p}_t, k_t, t) \stackrel{def}{=} \sup_{\hat{\eta}} \mathrm{E}_{\tilde{p}_t} \left[\int_t^{\hat{\eta}} e^{-\rho(s-t)} A\left(k_{t,s}\right) \, ds + e^{-\rho(\hat{\eta}-t)} \cdot L \right]. \tag{27}$$

It depends on time t explicitly as the latter governs the reduction in the posterior variance $Var(\theta \,|\, \tilde{p}_t)$ about the agent's skill.

Lemma 9. *There exists a subgame-perfect equilibrium in which clients' belief follows* (26) *and the agent leaves the industry the first time his belief $\tilde{p}_s$ falls below a stopping boundary*

[44] This analysis requires additional expositional space in order to formulate the equilibrium definition for the agent's dynamic subgame in which clients' positively interpret the agent's deviations. We focus on the more concise version in the main text in order to have more space to explore the Brownian model.

$b(k_{t,s}, s)$, *i.e.*,

$$\eta = \inf\{s:\ \tilde{p}_s \le b(k_{t,s}, s)\}.$$

Moreover, if $A(\cdot)$ *is weakly convex, then higher skilled agents value reputation more than lower skilled agents:* $\partial_2 U(\tilde{p}', k, t) > \partial_2 U(\tilde{p}, k, t)$, *for any* $\tilde{p}' > \tilde{p}$.

Lemma 9 establishes that higher-skilled agents value reputation more under the sufficient condition that $A(\cdot)$ is weakly convex. First, higher-skilled agents stay longer in the industry, as clearly seen from the optimality condition $\tilde{p}_t \ge b(k_{t,s}, s)$. This is the same economic mechanism as the one behind Proposition 1 of the Poisson model. Second, a higher-skilled agent is more likely to generate good performance and have a high reputation. Weak convexity of $A(\cdot)$ ensures that revenues remain sensitive to the clients' perceptions even when the agent's reputation takes high values. While weak convexity is sufficient to ensure the agent's single-crossing holds, it is not necessary, as the first channel implies our results even if $A(\cdot)$ is linear. The agent's reservation wage $w_R(\tilde{p}_t, k_t, t)$ is given by

$$w_R(\tilde{p}_t, k_t, t) \overset{def}{=} \begin{cases} \rho L & \text{if} \quad U(\tilde{p}_t, k_t, t) = L, \\ A(k_t) & \text{if} \quad U(\tilde{p}_t, k_t, t) > L. \end{cases}$$

This is the direct analogue of the agent's reservation value in (12). We are now ready to characterize the agent's compensation and employment while working for the intermediary.

5.3 Equilibrium given Brownian Signals

The agent's performance X_t affects the intermediary's profitability and, therefore, her incentive to retain him. The clients' posterior belief about the retained agent working for the intermediary is given by

$$q_t = Q(k_t, t, X_t) \overset{def}{=} \underbrace{\frac{\phi}{\phi t + 1} \cdot X_t}_{(i)} + \underbrace{\frac{1}{\phi t + 1} \cdot \mathrm{E}\left[\tilde{p}_t \,\middle|\, X_t,\, \tilde{p}_t > k_t\right]}_{(ii)}. \tag{28}$$

Belief about the average type q_t depends on past performance in two ways. First, conditional on the realization of $\tilde{p}_0$, the incremental performance dX_t changes the posterior belief by $\frac{\phi}{\phi t+1} \cdot dX_t$, implying that term (i) in (28) is the same as the first term in the right-hand-side of (25). A second, and more subtle, inference is that better performance X_t is more likely to come from a better private type $\tilde{p}_t$ of the retained agent, implying that term (ii) in (28) is also increasing in X_t. This second channel implies that the posterior belief about the average agent $\{\tilde{p}_t \ge k_t\}$ is more sensitive to performance signals than the posterior belief about a given agent $\tilde{p}_t$. Informally stated, clients put more weight on performance signals than the intermediary-agent pair, because they have a more dispersed belief about θ, making the performance process X relatively more informative.[45] The economic implication of this differential learning is that the intermediary's revenues are more sensitive to performance than the agent's reservation wage. Good performance, leading up to time t, increases the intermediary's profit wedge, before accounting for reputation dynamics.

Retention real option. Similar to the Poisson model, consider a retention problem of the intermediary who employs the cutoff agent $\tilde{p}_t = k_t$ and chooses when to let him go. This cutoff agent has no room to build reputation, implying that the intermediary needs to pay him the reservation wage $w_R(\tilde{p}_t, \tilde{p}_t, t)$. Since this agent is being pooled with better agents while he works for the intermediary, she may keep him employed if the profit wedge $A(q_t) - w_R(\tilde{p}_t, \tilde{p}_t, t)$ is high enough for $q_t = Q(\tilde{p}_t, t, X_t)$. As explained earlier, this is true if past performance X_t has been relatively good. The intermediary's expected value of employing the cutoff agent is

$$\sup_{\hat{\tau}} \mathrm{E}_{\tilde{p}} \left[\int_0^{\hat{\tau}} e^{-rt} \Big[A\big(Q\left(\tilde{p}_t, t, X_t\right)\big) - w_R\big(\tilde{p}_t, \tilde{p}_t, t\big) \Big] \, dt + e^{-r\hat{\tau}} \cdot V \right]. \tag{29}$$

Lemma 10. *There exists a stopping surface $B(\tilde{p}, t)$ such that the intermediary optimally lets go of the cutoff agent when performance X_t falls below $B(\tilde{p}_t, t)$, i.e.,*

$$\tau = \inf\big\{ t \geq 0 : \; X_t < B(\tilde{p}_t, t) \big\}. \tag{30}$$

Moreover, the stopping surface $B(\tilde{p}, t)$ satisfies two properties:

(i) $\partial_1 B(\tilde{p}, t) < \phi t + 1$ for every $\tilde{p}$ and t, implying that the difference $X_t - B\big(\Pi(\tilde{p}_0, t, X_t), t\big)$ is increasing in performance X_t.

(ii) If $V > 0$, then $B(\tilde{p}, t) \to \infty$ as $t \to \infty$ for every $\tilde{p}$, leading to all agents being let go in finite time.

Lemma 10 characterizes the intermediary's decision to let go of the cutoff agent absent any reputation considerations. The profit wedge of the intermediary in (29) depends on performance X_t and private type $\tilde{p}_t$, which is equal to $\Pi(\tilde{p}_0, t, X_t)$ and also depends on X_t. The intermediary's decision to let go of the agent in (29) thus depends on dynamic states t and X_t, as well as the initial type $\tilde{p}_0$ of the cutoff agent at $t = 0$. Because the intermediary's profit wedge is increasing in performance X_t, then, for each initial $\tilde{p}_0$, there exists a boundary $\hat{B}(\tilde{p}_0, t)$ such that the optimal time to fire the agent is when performance X_t falls below it. By solving (29) for each initial type $\tilde{p}_0 \in [\underline{p}, \overline{p}]$ we construct the entire stopping surface $B(\tilde{p}_0, t)$ and pin down the intermediary's decision to let go of the cutoff agent. While such construction is mathematically convenient, it requires us to appeal to the space of initial

[45] This is easy to see if, for instance, $\sigma_\theta = 0$. Then the intermediary and the agent know his skill perfectly and do not need to update on performance signals, while the latter still influence the clients' beliefs.

types $\tilde{p}_0$, even if the decision is made at time t, complicating the narrative. If $F(\cdot)$ has a log-concave density, as we have assumed, then the intermediary's turnover decision can be expressed via a stopping surface $B(\tilde{p}_t, t)$ defined as

$$B(\tilde{p}_t, t) \stackrel{def}{=} \sup\Big\{x :\ x \leq \hat{B}\big[(\phi t + 1) \cdot \tilde{p}_t - \phi \cdot x,\ t\big]\Big\}.$$

The intermediary lets go of the agent if $X_t \leq B(\tilde{p}_t, t)$, where $B(\tilde{p}_t, t)$ differs from $\hat{B}(\tilde{p}_0, t)$ in that it depends on the agent's posterior type $\tilde{p}_t$ at time t, rather than his initial type $\tilde{p}_0$. In the Brownian setting, process X_t introduces an independent performance dimension, splitting the three-dimensional state space (k, t, X) into the quiet region $\{X > B(k, t)\}$ and the churning region $\{X \leq B(k, t)\}$. The key step, which we establish in the formal proof of Proposition 4, is that the intermediary employing the cutoff agent does not benefit from the future reputation dynamics leading to her expected value being exactly equal to (29). This allows us to characterize the equilibrium by solving for $B(k, t)$ separately and, then, construct the cutoff type dynamics $(k_t)_{t\geq 0}$ forward using the on-path churning threshold $B(k_t, t)$.

Churning rate. The equilibrium churning rate is pinned down by the intermediary employing the cutoff agent k_t. If X_t is large, then there is no churning. If, however, X_t declines and reaches $B(k_t, t)$, the intermediary lets go of the cutoff agent $\tilde{p}_t = k_t$. Once in the churning region, the intermediary gradually lets go of the cutoff agents if they find reputation building valuable, i.e., if $k_t \geq b(k_t, t)$. The equilibrium churning rate $\gamma(k_t, t)$ is such that the intermediary is marginally indifferent between retaining and firing the agent once in the churning region. The equilibrium churning rate $\gamma(k_t, t)$ is pinned down by the intermediary's local indifference condition to retain the cutoff type given this history

$$A(Q(k_t, t, X_t)) - w_R(k_t, k_t, t) + \gamma(k_t, t) \cdot \partial_2 U(k_t, k_t, t) = rV. \tag{31}$$

This is the Brownian equivalent to the optimal stopping condition (17) of the Poisson model, and only applies once the intermediary finds herself in the churning region $X_t \leq B(k_t, t)$ and the cutoff agent finds reputation building valuable, i.e., $k_t \geq b(k_t, t)$. If $k_t < b(k_t, t)$, then the cutoff agent does not value reputation at all as he plans to leave the industry upon being let go. In this case, the intermediary cannot elicit pay-for-reputation dynamics and needs to fire a discrete set of agents $\delta(k_t, t, X_t)$ to either exit the churning region, or increase the cutoff type by enough so that they begin to value reputation themselves.

Proposition 4. *The equilibrium is characterized by the churning boundary $B(k_t, t)$, which depends on the cutoff type k_t and elapsed time t.*

(i) The intermediary lets go of agent k_t when performance X_t drops below $B(k_t, t)$.

(ii) The agent's reputation grows faster during churning periods

$$dk_t = \underbrace{\frac{\phi}{\phi t + 1}\big(dX_t - k_{t-}dt\big)}_{\textit{learning from performance}} + \underbrace{\begin{cases} 0 & \textit{if } X_t \geq B(k_{t-}, t), \\ \gamma(k_t, t, X_t)dt & \textit{if } X_t < B(k_{t-}, t),\ k_{t-} \geq b(k_{t-}, t), \\ \delta(k_{t-}, t, X_t) & \textit{if } X_t < B(k_{t-}, t),\ k_{t-} < b(k_{t-}, t), \end{cases}}_{\textit{learning from churning}} \quad (32)$$

where $\gamma(k_t, t, X_t)$ is determined by the profitability of the cutoff agent

$$\gamma(k_t, t, X_t) \overset{def}{=} \frac{\Big[rV - \Big(A\big(Q(k_t, t, X_t)\big) - w_R(k_t, k_t, t)\Big)\Big]^+}{\partial_2 U(k_t, k_t, t)} \geq 0. \quad (33)$$

and $\delta(k_t, t, X_t)$ is the minimal jump necessary to either exit the churning region or increase the cutoff type to a point where he stays in the industry were he to be let go

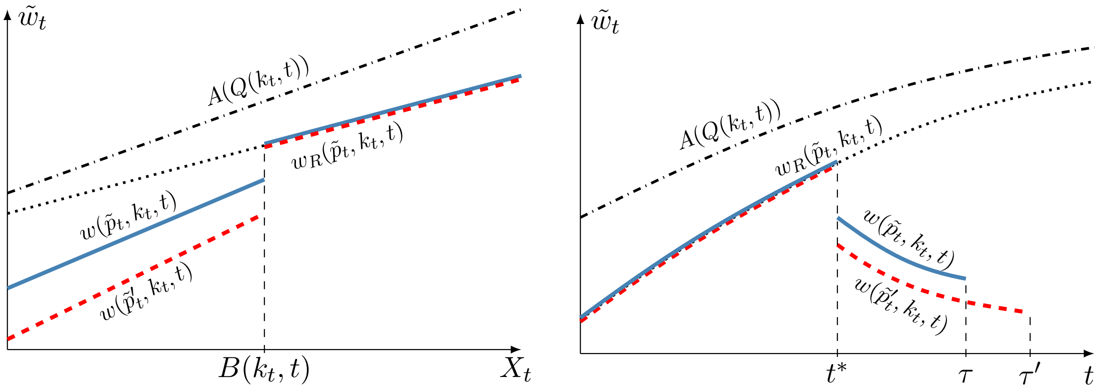
$$\delta(k_t, t, X_t) \overset{def}{=} \inf\Big\{\delta > 0 : X_t \geq B\big(k_t + \delta, t\big) \;\; or \;\; k + \delta = b\big(k_t + \delta, t\big)\Big\}. \quad (34)$$

(iii) Agents pay for reputation building by accepting lower compensation. The cumulative compensation process of the agent of skill $\tilde{p}_t$ is given by

$$d\tilde{W}_t = w_R\big(\tilde{p}_t, k_t, t\big)\, dt - \underbrace{\begin{cases} 0 & \textit{if } \;\; X_t \geq B(k_t, t), \\ \gamma(k_t, t, X_t)\partial_2 U(\tilde{p}_t, k_t, t)dt & \textit{if } X_t < B(k_t, t),\ k_{t-} > b(k_{t-}, t), \\ U(\tilde{p}_t, k_t, t) - U(\tilde{p}_t, k_{t-}, t) & \textit{if } X_t < B(k_t, t),\ k_{t-} \leq b(k_{t-}, t). \end{cases}}_{\textit{pay for reputation}}$$

Proposition 4 is the counterpart of Proposition 2; the similarities in the equilibrium structure are evident. Implications of the equilibrium in Proposition 4 on agent's compensation are illustrated in Figure 5. In order to track the same agent across the state-space, we fix initial types $\tilde{p}_0' > \tilde{p}_0$ and consider posteriors influenced by time and performance, given by (25). Figure 5a considers the agent's compensation as performance X_t changes "quickly". For simplicity, we assume that $k_t \geq b(k_t, t)$ and the reputation building in (32) occurs gradually, allowing us to keep the cutoff type constant, implying that k_t is also given by (25) for a fixed initial $\tilde{p}_0$. In this case, the agent collects his reservation wage when performance exceeds the churning threshold $B(k_t, t) = \hat{B}(\tilde{p}_0, t)$. When performance declines, however, the principal churns lower-skilled agents at a strictly positive rate, as manifested by $\gamma(k_t, t, X_t)$ being bounded away from 0.[46] The agent, thus, pays a positive amount for reputation the moment he enters the churning region, leading to a discontinuity in his flow compensation. The fact

[46] This stems from the optimality in (29) of letting go of the cutoff agent when his profitability is strictly below the opportunity cost rV.



(a) Compensation as a function of performance X_t. We assume the cutoff type is the same, i.e., $k_t = \Pi(l,t,X_t)$, implying that the churning threshold $B(k_t,t) \equiv \hat{B}(l,t)$ for every X_t.

(b) Compensation as a function of elapsed time t, along a constant performance path $X_t = \mu t$. Time t^* is the first time in the churning region and the cutoff type k_t is increasing.

Figure 5: Effects of past performance X_t and tenure t on compensation. To track the same agent across the state space (k_t,t,X_t), we keep the initial types constant: $\tilde{p}'_t = \Pi(\tilde{p}'_0,t,X_t)$ and $\tilde{p}_t = \Pi(\tilde{p}_0,t,X_t)$ for fixed initial $\tilde{p}'_0 > \tilde{p}_0$. $A(\cdot)$ is assumed to be linear.

that the intermediary cuts wages of *all* retained agents discontinuously as she enters the churning region in which some employees are let go is distinct from the model based on the Poisson information process. The stark difference stems from the fact that in the current setting, there is an embedded option value to the intermediary of retaining the agent even if the profit flow is currently below the intermediary's flow opportunity cost rV. The same does not hold for the Poisson model, where the future profit path from retaining an agent of a given skill conditional on that agent not being revealed as low-skilled is known in advance. Once an agent is revealed as low ability, separation from the intermediary is immediate.

Figure 5b depicts the agent's compensation along a (simple) performance path $X_t = \mu t$, where we assume that $\mu > \underline{p}$. Initially, the information asymmetry is high, and the principal can retain all agents paying them their reservation wage. Since $\mu > \underline{p}$, the beliefs about the cutoff agent increase over time, resulting in greater compensation and eroding the intermediary's profitability. Eventually, the intermediary starts churning lower-skilled agents whom she finds unprofitable to employ, resulting in a decrease in compensation, as illustrated in Figure 5b. In order to avoid entering the churning region, the agent must perform increasingly well, which is unsustainable in the long-run. The intermediary, eventually, churns all agents.

As we see in Figure 5, the agent's compensation increases in performance, but, eventually, decreases in time. The Poisson model bundles these two channels - Figure 2a is unable to distinguish between effects of performance and time, which are decoupled in Figures 5a and 5b. Corollary 4 summarizes the equilibrium properties under Brownian performance signals. If however, $k_t > b(k_t, t)$, then the churned agent stays in the industry, and the rate of reputation building exhibits smooth behavior.[47]

Corollary 4. *The equilibrium dynamics exhibit five properties*

(i) higher-skilled agents, as measured by their private information, have longer employment spells with the intermediary;

(ii) intermediary's flow profit decreases as X_t approaches $B(k_t, t)$ from above, but then increases as X_t declines further as higher-skilled agents pay to build a reputation.

(iii) higher-skilled agents pay for building a reputation in periods when the intermediary churns lower-skilled agents, i.e., $X_t < B(k_t, t)$;

(iv) if the agent leaves the intermediary and chooses to open his own firm, he obtains a positive jump in compensation.

(v) agents with a better performance history have longer employment spells with the intermediary; their compensation is increasing in performance outside of the churning region but suffers a discontinuous downward drop once the intermediary starts churning lower-skilled agents.

Results $(i)-(iv)$ are direct analogs of Corollary 1. Point (ii) highlights that the intermediary's profits increase in performance when outside of the churning region. Once in the churning region, however, the intermediary's flow profit may increase if the agent performs poorly, as a higher high-skilled agent pays to build a reputation. Point (v) shows that past performance can be a substitute for skill in determining the agent's retention by the intermediary. As illustrated in Figure 5, the agent is paid discretely less when in the churning region. This can be thought of as the intermediary cutting bonuses during periods of downsizing. Moreover, almost all agents pay for reputation just before leaving the intermediary and, thus, obtain a positive jump in compensation after opening their own firm.

We also note that similar to the Poisson setting, the model suggests that wage dispersion within the intermediary should be higher in churning periods. Since in the current model churning periods are associated with times where returns have recently been low, we further predict that wage dispersion should be higher in such times.

The expected value of the intermediary is pinned down by the fixed-point condition

$$V = \max\left[e^{-r\Delta} \cdot \mathrm{E}\left[\int_0^{\tau} e^{-rt}\Big(A\big(Q(k_t, t, X_t)\big) - w(\tilde{p}_t, k_t, t)\Big)\, dt + e^{-r\tau} \cdot V\right] - I,\ 0\right]. \quad (35)$$

[47]Similar behavior also occurs in the Poisson model when, at $t = 0$, the intermediary lets go of some agents, manifested by the, potentially positive, difference $k_0 - \underline{p}$.

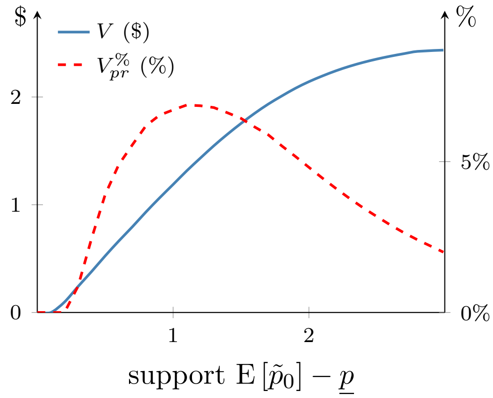
The equilibrium is unique for relatively large I or Δ, but, similar to the Poisson model, there may exist multiple equilibria for intermediate resampling costs.

5.4 Comparative Statics given Brownian Signals

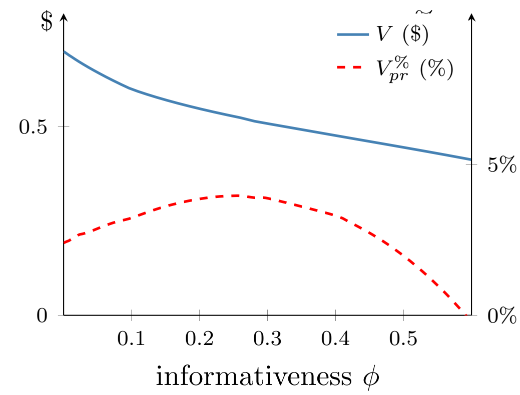
We derive the equilibrium under mild assumptions on the revenue function $A(\cdot)$ and private information $F(\cdot)$. Such generality ensures the robustness of the economic mechanism but makes it challenging to derive comparative statics. For our numerical calculations, we assume the private information $\tilde{p}_0$ follows a normal distribution, truncated at $\underline{p} > -\infty$ from below and at $\overline{p}$ from above. The tractability benefit stems from the class of truncated normal distribution being a conjugate of itself with respect to the Gaussian likelihood function, arising from the conditionally normal performance process $(X_t)_{t\geq 0}$. Such specification allows for a semi-analytic expression for $Q(k_t, t, X_t)$. To simplify calculations further, we assume revenues are exponential, i.e., $A(x) = e^{\alpha x}$, for $\alpha > 0$, and the agent always stays in the industry, i.e., $L = 0$. This allows to derive the agent's dynamic outside option $U(\tilde{p}_t, k_t, t)$ in semi-closed form and focus on the intermediary's churning region.

For a given intermediary value V, the stopping surface $B(k, t)$ specifies the forward dynamics of the cutoff process $(k_t)_{t\geq 0}$ via Proposition 4. This permits simulation of the equilibrium paths (k_t, t, X_t) forward, but does not lend itself to an easy evaluation of (35). The difficulty stems from simultaneously evaluating the expected value over three independent sources of uncertainty: private information $\tilde{p}_0$, residual uncertainty of θ given $\tilde{p}_0$, and the Brownian motion $(B_t)_{t\geq 0}$. The simulations necessary to evaluate the intermediary's expected value in (35) with a satisfactory precision are in the trillions. It turns out that we can significantly simplify the problem by utilizing the fact that the equilibrium dynamics of process $(k_t, t, X_t)_{t\geq 0}$ are pinned down by the stopping surface $B(k_t, t)$ via (32). We perform a Girsanov change of measure with respect to the drift of the performance process X to isolate the intermediary's private information $\tilde{p}_0$ and latent skill θ. We then compute the expected values of the Girsanov densities with respect to θ conditional on the three-dimensional state (k_t, t, X_t).[48] This leaves only the Brownian motion $(B_t)_{t\geq 0}$ as the source of dynamic uncertainty. We can evaluate (35) via a dynamic program in three state variables (k_t, t, X_t), in which X_t follows a standard Brownian motion, and k_t is described in Proposition 4. Evaluating the right-hand side of (35) for a given V can be performed on a personal computer in a few hours. We identify the fixed points of (35) and perform comparative statics, depicted in Figure 6 by utilizing parallel calculations on a university computer cluster. The technical details of the method are provided in Appendix B.

[48]The strategic considerations pertaining to θ affecting the drift of the performance process $(X_t)_{t\geq 0}$ are present via the churning surface $B(k_t, t)$.



(a) Equilibrium value V and percentage of revenues $V_{pr}^{\%}$ obtained from pay-for-reputation as a function of the support of asymmetric information.



(b) Equilibrium value V and percentage of revenues $V_{pr}^{\%}$ obtained from pay-for-reputation as a function of the informativeness ϕ of the performance signal.

Figure 6: Value and pay-for-reputation comparative statics under Brownian signals.

Figure 6 depicts the intermediary's expected value V and the fraction $V_{pr}^{\%}$ of the intermediary's profits stemming from the agent paying for reputation. The pay-for-reputation percent $V_{pr}^{\%}$ as a function of the initial magnitude and persistence of asymmetric information. Similar to the Poisson model, the intermediary's expected value increases in the support of the private information and decreases in the informativeness ϕ of the performance signal, as can be seen in Figures 6a and 6b respectively. Moreover, similar to the Poisson model, $V_{pr}^{\%}$ is maximized for interior levels of information asymmetry and persistence.

6 Conclusion

Markets for services are plagued by uncertainty about agents' underlying skill. Our analysis highlights the reputation-building role of informed intermediaries, and how clients dynamical observation of the agent's performance shapes turnover dynamics and the interplay between turnover and compensation dynamics. Our focus is on professions with three important characteristics: agents' talent is essential, their performance is observable by clients with reasonable frequency, and they are able to contract directly with clients to provide their services. A few of examples of such professions are fund managers employed by a fund family, non-partner lawyers in a law firm, and accountants working in one of the big accounting firms.

Our investigation provides profound implications for turnover and compensation dynamics, providing a novel rationale for the observed practice of low wage dispersion accompanied by low turnover frequency at the initial phase of employment by an intermediary for such workers. Importantly, the channel we highlight is distinct from the intermediary's desire to evaluate the skills of the employee, but instead is driven by the desire to maintain the informational advantage she has relative to clients. Since clients dynamically observe agent's performance, informational advantage dissipates over time and, when it shrinks sufficiently, the intermediary starts churning low skilled agents, consequently expediting depreciation of her informational advantage: a key insight is that strategic churning enables the intermediary to retain high-skilled agents below market costs, and increases profits relative to the period preceding churning. Churning periods are associated with increased wage dispersion of agents retained by the intermediary. We also endogenize the market value of the intermediary, a feature absent from common dynamic adverse selection models and signaling models. Interestingly, with endogenous intermediary value, the length of quiet and churning intervals becomes non-monotone in the level of initial asymmetric information and informativeness of public performance signals. An important next step is to augment moral hazard considerations, in conjunction with allowing for performance-sensitive compensation contracts, as well as to understand how our mechanism interacts with other rationals for the existence of an intermediary firm such as economies of scale and benefits of working in teams.

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A Online Appendix (Not for Printed Publication)

To facilitate access to the the proofs we provide a table of contents for Online Appendix A.

Contents

A.1 Equilibrium Fixed Point V A.1

A.2 Implications of firm-specific skill, team production, and rival services A.3

A.3 Robustness to Dynamic Contracts A.3

A.4 Alternative Specification of Agent's Outside Option given Brownian Signals. A.4

A.5 Main Text Proofs A.8

Proof of Lemma 1 (agent's outside option) A.8

Proof of Proposition 1 (agent's single-crossing) A.10

Proof of Proposition 2 (equilibrium verification) A.11

Proof of Lemma 2 (determination of $k_0 \geq \underline{p}$) A.16

Proof of Lemma 3 (equilibrium uniqueness) A.17

Proof of Lemma 4 (equilibrium value V) A.20

Proof of Lemma 5 (comparative statics) A.21

Proof of Lemma 6 (imperfect competition among intermediaries) A.24

Proof of Lemma 7 (signaling outside option $U_\beta(p,k)$) A.26

Proof of Lemma 8 (training incentives) A.27

Proof of Proposition 3 (equilibrium given signaling) A.27

Proof of Lemma 9 (agent's outside option under Brownian signals) A.33

Proof of Lemma 10 (boundary $B(p,t)$) A.34

Proof of Proposition 4 (equilibrium verification given Brownian signals) A.39

A.1 Equilibrium Fixed Point V

Define the incremental value of a new agent at the start of his employment to the intermediary as

$$\begin{aligned} G(V) &\stackrel{def}{=} \mathrm{E}\left[\int_0^\tau e^{-r\tau}(A(q_t) - w(\tilde{p}_t, k_t))\,dt + e^{-r\tau}V\right] - V \qquad \text{(A.1)} \\ &= \mathrm{E}\left[\int_0^\tau e^{-r\tau}(A(q_t) - w(\tilde{p}_t, k_t) - rV)\,dt\right] \end{aligned}$$

Fixed point equation (7) can then be rewritten for $V > 0$ as

$$V = e^{-r\Delta}\cdot \mathrm{E}\left[\int_0^\tau e^{-rt}(A(q_t) - \tilde{w}_t)\,dt + e^{-r\tau}\cdot V\right] - I$$

$$Ie^{r\Delta} + V\left(e^{r\Delta} - 1\right) = \mathrm{E}\left[\int_0^\tau e^{-rt}(A(q_t) - \tilde{w}_t)\,dt + e^{-r\tau}\cdot V\right] - V$$

$$Ie^{r\Delta} + V\left(e^{r\Delta} - 1\right) = G(V). \tag{A.2}$$

The intuition behind (A.2) is that the expected gains from employing a given agent must equal to the fixed costs of replacing him.

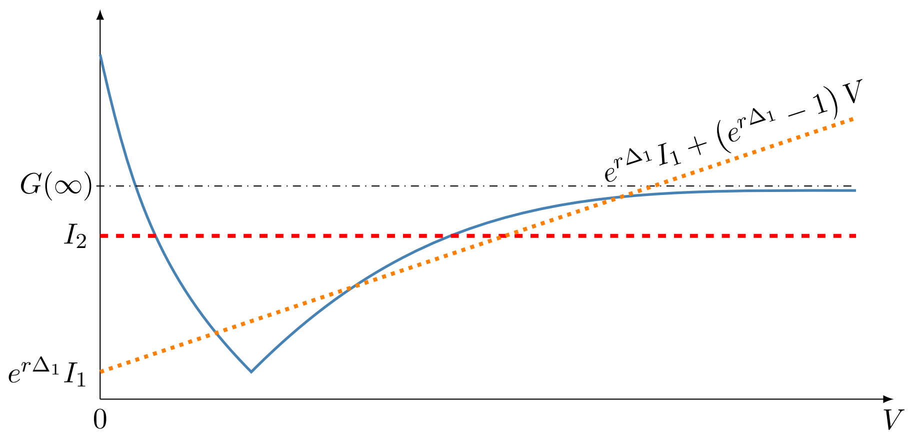

Figure 7: Graphical representation for the fixed point equation (A.2) for V. Function $G(V)$ is depicted in solid. Replacement cost parameters are $I_2 > I_1$, and $\Delta_2 = 0 < \Delta_1$. Ex-ante private information $\tilde{p}_0$ is distributed uniformly on $\left[\frac{1}{3}, 1\right]$ and revenues $A(p) = p$. The asymptotic value is $G(\infty) = \mathrm{E}\left[U(\tilde{p}_0, \tilde{p}_0) - U(\tilde{p}_0, \underline{p})\right]$.

Figure 7 plots $G(V)$ as a solid line. It is decreasing for low values of V as the profitability of each agent is offset by a greater opportunity cost in periods when there is no churning. A higher value of V, however, leads the intermediary to churn lower skilled agents at a higher rate, as can be seen from (13). This allows the intermediary to charge higher skilled agents more for building reputation, which increases the gain to the intermediary. The slope of the overall employment gain of the intermediary balances the two economic forces and is, in part, determined by the relative value of employing the agent in times when there is and isn't churning.

Lemma 4 first shows that there exists a unique equilibrium limit as the number of agents the intermediary can consequently employ increases. If the search costs are sufficiently large, the equilibrium corresponds to the unique root of (A.2). On the other hand, if the replacement cost is small, then a large fraction of the intermediary's profits comes from the agents paying for reputation building. It is a surprising result that in this case there may arise additional equilibrium values of V corresponding to either low or high profitability of the intermediary. As a result, the intermediary's reputation to be selective with its agents can be self-enforcing and result in multiple equilibria with different expected values V. This is a novel and surprising feature in both the dynamic employment and adverse selection literature. It arises from endogenizing the gains from trade by imposing the employment capacity constraint, and is natural in our context since the intermediary is able to hire a new agent. The intuition is similar to a market for goods under adverse selection: higher gains from trade increase market liquidity, which lead to goods being produced at a higher rate, feeding back into higher gains from trade. A good example is a of a contractor building houses and having inventory constraints, for instance due to credit availability. A more liquid market increases the rate at which houses are built and sold and increases his equilibrium continuation value.

A.2 Implications of firm-specific skill, team production, and rival services

Firm specific skill. The agent's skill may be specific to the firm. This may not be a first-order force in settings such as accounting or money management as they rely on general human capital, but may be more relevant in other contexts. If the intermediary finds an agent who has good synergies with the firm, then she may not wish to churn him. As a result, clients identify agents who are churned by the intermediary by their expected skill but also the fact that they do not have synergies with the current employer.

Team production. In the context of money management, funds within the family can be overseen by teams of two to five money managers. Naturally, this introduces a problem for the markets to infer which of the team members is truly skilled. The analysis in this paper can be applied to the team as a whole where the entire team can leave the family and start a fund on their own. It also, potentially, introduces the possibility of a staggered exit in which the first party that leaves the team is less skilled. As such, the longer the team member stays with the fund, the higher his perceived level of skill is. While this paper focuses on the case when a single manager governs the fund, the same mechanism applies to team production.

Rival services. Suppose an outgoing agent may compete with the intermediary after starting his firm.[49] The economic idea of letting go of a well-established agent because he becomes too expensive is still at play, but it comes at the cost of competing with him in the future. If the intermediary does not let go of him after a long time, however, her profit wedge collapses to 0 due to learning from either good or bad performance. This implies that, as long as the agent is unable to capture the entire market, the intermediary is, eventually, better of replacing him with a new agent. Our solution approach extends to this setting. We would first solve backward for the intermediary's non-stationary expected value from employing the next agent and factor it into the optimal stopping problem determining set $\mathbb{T}$.

A.3 Robustness to Dynamic Contracts

Lemma A.1. *Suppose the intermediary can commit to a long-term contract. Then, if $\rho > r$ then under the optimal contract, the agent's continuation value is given by $U(\tilde{p}_t, k_t)$. If $\rho = r$, then the contract in which the agent's continuation value is equal to $U(\tilde{p}_t, k_t)$ up until termination is one of the optimal contracts.*

Proof. Suppose the intermediary can commit to a cumulative compensation process $(\tilde{C}_t)_{t\geq 0}$. The agent's continuation value from staying with the intermediary is given by

$$\hat{W}_t = \mathrm{E}_{\tilde{p}_t}\left[\int_t^{\tau\wedge\eta} e^{-\rho(s-t)}\, d\hat{C}_s + e^{-\rho(\tau\wedge\eta - t)} \cdot U\big(\tilde{p}_\tau, k_\tau\big)\right].$$

[49]Such dynamic has been studied in Glode, Green, and Lowery (2012) in the context of the financial service industry.

On the other hand, define wage $\tilde{w}_t$ satisfying

$$U(\tilde{p}_t, k_t) = \mathrm{E}_{\tilde{p}_t}\left[\int_t^{\tau\wedge\eta} e^{-\rho(s-t)}\, dC_s + e^{-\rho(\tau\wedge\eta-t)} \cdot U\big(\tilde{p}_\tau, k_\tau\big)\right].$$

Due to the agent's risk-neutrality, it is without loss that he is let go after generating poor performance and gets paid nothing in that event. The agent is retained by the intermediary as long as $\tilde{W}_t \geq U(\tilde{p}_t, k_t)$ and at time τ when the agent is let go, $\tilde{W}_\tau = U(\tilde{p}_\tau, k_\tau)$.

Suppose there exists a t such that $\hat{W}_t > U(\tilde{p}_t, k_t)$. Consider an alternative contract in which the intermediary follows contract $(\hat{C}_s)_{s\leq t}$, makes the agent a fixed payment $\hat{W}_t - U(\tilde{p}_t, k_t)$ and, subsequently, compensate the agent according to process $(C_s)_{s\in[t,\tau]}$. The net continuation benefit to the intermediary of the new contract relative to the original contract is

$$\begin{aligned}
&\mathrm{E}\left[\int_t^\tau e^{-r(s-t)}\, d\hat{C}_s\right] - \hat{W}_t + U(\tilde{p}_t, k_t) - \mathrm{E}\left[\int_t^\tau e^{-r(s-t)}\, dC_s\right]\\
=&\mathrm{E}\left[\int_t^\tau e^{-r(s-t)}\, d\hat{C}_s\right] - \mathrm{E}\left[\int_t^\tau e^{-\rho(s-t)}\, d\hat{C}_s\right] + \mathrm{E}\left[\int_t^\tau e^{-\rho(s-t)}\, dC_s\right] - \mathrm{E}\left[\int_t^\tau e^{-r(s-t)}\, dC_s\right]\\
=&\mathrm{E}\left[\int_t^\tau \Big(e^{-\rho(s-t)} - e^{-r(s-t)}\Big) \cdot \Big(dC_s - d\hat{C}_s\Big)\right]\\
=&\mathrm{E}\left[\int_t^\tau \Big(1 - e^{(\rho-r)(s-t)}\Big) \cdot e^{-\rho(s-t)}\Big(dC_s - d\hat{C}_s\Big)\right]\\
=&\mathrm{E}\left[\int_t^\tau \Big(1 - e^{(\rho-r)(s-t)}\Big) \cdot \Big(-dU\big(\tilde{p}_s, k_s\big) + d\hat{W}_s\Big)\right]\\
=&\mathrm{E}\left[\Big(1 - e^{(\rho-r)(s-t)}\Big)\Big(\hat{W}_s - U\big(\tilde{p}_s, k_s\big)\Big)\Big|_{s=t}^{s=\tau} - \int_t^\tau \Big(\hat{W}_s - U\big(\tilde{p}_s, k_s\big)\Big)\, d\Big(1 - e^{(\rho-r)(s-t)}\Big)\right]\\
=&\mathrm{E}\left[\int_t^\tau \Big(\hat{W}_s - U\big(\tilde{p}_s, k_s\big)\Big)\, de^{(\rho-r)(s-t)}\right]\\
=&\mathrm{E}\left[(\rho - r) \cdot \int_t^\tau e^{(\rho-r)(s-t)}\Big(\hat{W}_s - U\big(\tilde{p}_s, k_s\big)\Big)\, ds\right] \overset{(i)}{\geq} 0,
\end{aligned}$$

where inequality (i) is strict unless $\hat{W}_s = U(\tilde{p}_s, k_s)$ P-a.s. This implies that under the optimal contract it must be the case that $\tilde{W}_t = U(\tilde{p}_t, k_t)$. □

Lemma A.2. *The equilibrium dynamics are unaffected if the intermediary can commit to a long-term contract with the agent.*

Proof. Under the optimal dynamic contract, the intermediary keeps the agent exactly at his outside option for any equilibrium process of beliefs. As a result, commitment does not expand the set of feasible contracts offered to the agent implying equivalent dynamics to the setting where the intermediary cannot commit. □

A.4 Alternative Specification of Agent's Outside Option given Brownian Signals.

With Brownian signals, defining and deriving the equilibrium of the agent's subgame is a nontrivial task. Because the agent's off-equilibrium outside option determines his bargaining power with the intermediary via the retention constraint (6), we characterize the agent's expected value of opening a firm if he is let go off the equilibrium path. The main result of this section is that similar to before, in equilibrium, higher-skilled agents value reputation more than lower-skilled agents since they stay in the industry longer, under the sufficient condition that revenue $A(\cdot)$ is weakly convex.

Clients assign belief k_t to the agent who opens his firm at time t. They subsequently update their beliefs based on his performance

$$\mathrm{E}\left[\theta \,|\, X_s, k_t\right] = \frac{\phi}{\phi s + 1} \cdot (X_s - X_t) + \frac{\phi t + 1}{\phi s + 1} \cdot k_t. \tag{A.3}$$

If the agent performs well, clients' belief increases and the agent obtains greater profits. If, however, he performs poorly, clients' belief declines, and he may find staying in the industry unprofitable, relative to his outside option L. The agent decides when to exit the industry given the clients' belief process and solves

$$U(k_t, k_t, t) \overset{def}{=} \max_{\hat{\eta}} \mathrm{E}\left[\int_0^{\hat{\eta}} e^{-\rho(s-t)} A\left(\frac{\phi(X_s - X_t) + (\phi t + 1)k_t}{\phi s + 1}\right) ds + e^{-\rho\hat{\eta}} L \,\middle|\, X_t, k_t\right] \tag{A.4}$$

This decision depends on past performance X_t as well as elapsed time t, which governs the residual uncertainty about his ability

$$\Sigma_t \overset{def}{=} \mathrm{Var}(\theta \,|\, X_t) = \frac{\sigma_\theta^2 \sigma^2}{\sigma_\theta^2 t + \sigma^2},$$

If Σ_t is very low, then the estimate of the agent's ability is so precise that the optimal exit rule can be obtained by, essentially, comparing the current flow profit and the reservation utility ρL. If Σ_t is high, however, then the agent values additional learning about his ability and is willing to stay in the industry even if current profits are low.

Lemma A.3. *Suppose when the agent opens his own firm he shares the same belief as clients, i.e., $\tilde{p}_t = k_t$. Then he optimally leaves the industry when his belief $\tilde{p}_s$ drops below a deterministic boundary $b(s)$. Boundary $b(s) \to \rho L$ as $s \to \infty$.*

Proof. Follows the steps to constructing boundary $b(k,t)$ in Lemma 9. □

The agent's compensation while working for the intermediary is determined by his outside option captured by opening the firm in that instance. Suppose he opens his firm at time t when his belief is given by $\tilde{p}_t$ exceeds that of clients, i.e., k_t. Consequently, he is relatively more optimistic about future performance since, from his perspective, X_t has a higher drift, increasing his value of staying in the industry. In this case, he finds it optimal to stay in the industry longer than an agent with a lower private belief. In particular, the agent with $\tilde{p}_t > k_t$ does not leave the industry if X_s reaches $b(s)$. This poses a conundrum for clients as it violates the incentive compatibility about the agent they expect to be dealing with.[50] The clients need to interpret this deviation. On the one hand, they may assume that the agent delayed his exit from the industry by mistake, in which case beliefs should continue to follow (A.3). On the other hand, they may think that the agent started his firm with a higher belief than expected, and his delayed exit

[50] In the Poisson model of Section 2 and 3 this did not arise as the agent either generated good performance and stayed in the industry, or received a negative shock, which revealed him to be unskilled. This resulted in clients not needing to update based on off-equilibrium behavior by the agent.

from the industry is informative about his ability. In both cases, higher-skilled agents $\tilde{p}_t > k_t$ stay in the industry longer, leading to very similar wage dynamics in the employment relationship. We find it more natural that the agent opens his firm early due to a disagreement with the intermediary over compensation, and the clients assign the lowest belief about the agent consistent with him staying in the industry given that history. Such assumption on off-equilibrium beliefs makes our analysis more difficult, but comes very close to nesting the case in which investors do not update on the exit times.[51]

Definition 2. *A Perfect Bayesian Equilibrium of the agent's subgame starting at time t is a belief process* $(k_{t,s})_{s\geq 0}$ *and a collection of exit times* $(\eta_t(\hat{p}))_{\hat{p}\geq k_t}$ *adapted process* $(X_s - X_t)_{s\geq t}$ *such that*

(i) $\eta_t(\hat{p})$ *maximizes the agent's time of leaving the industry given clients' belief*

$$\eta_t(\hat{p}) = \arg\max_{\hat{\eta}} \mathrm{E}\left[\int_t^{\hat{\eta}} e^{-\rho(s-t)} A\left(k_{t,s}\right)\, ds + e^{-\rho(\hat{\eta}-t)} \cdot L \,\Big|\, \tilde{p}_t = \hat{p}\right].$$

(ii) the clients' belief process $k_{t,s}$ *follows Bayes rule as long as* $s < \eta_t(k_t)$*. If they expect an agent's exit but do not observe it, then they revise their beliefs to the lowest skilled agent for whom such behavior is incentive compatible*

$$k_{t,s} = \frac{\phi}{\phi s+1} \cdot (X_s - X_t) + \frac{\phi t+1}{\phi s+1} \cdot \max\Big[k_t,\ \eta_t^{-1}(s)\Big].$$

The equilibrium expected value to the agent if he leaves the intermediary is given by

$$U(\tilde{p}_t, k_t, t) \stackrel{def}{=} \mathrm{E}\left[\int_t^{\eta_t(\tilde{p}_t)} e^{-\rho(s-t)} A\left(k_{t,s}\right)\, ds + e^{-\rho(\eta_t(\tilde{p}_t)-t)} \cdot L \,\Big|\, \tilde{p}_t\right].$$

For $\tilde{p}_t = k_t$ the agent leaves the industry along the equilibrium path and the above expression coincides with one shown earlier for $\tilde{p}_t = k_t$. This results in higher skilled agents leaving the industry later, and leads them to put more value on reputation.

Proposition 5. *There exists a Perfect Bayesian Equilibrium characterized by the on-path stopping boundary* $b(s)$*.*

(i) The agent optimally leaves the industry when his private belief drops below $b(s)$*:*

$$\eta_t(\hat{p}) = \inf\left\{s \geq t: \quad \left[\frac{\phi}{\phi s+1} \cdot (X_s - X_t) + \frac{\phi t+1}{\phi s+1} \cdot \hat{p}\right] < b(s)\right\}. \tag{A.5}$$

(ii) The clients' belief equals the posterior belief of the lowest skilled agent for whom it is incentive compatible to stay in the industry given the past performance history

$$k_{t,s} = \frac{\phi(X_s - X_t)}{\phi s+1} + \frac{\phi t+1}{\phi s+1} \cdot \max\left[k_t,\ \frac{\inf_{s'\in[t,s]}\{\phi(X_{s'} - X_t) - (\phi s'+1)b(s')\}}{\phi t+1}\right].$$

Moreover, if $A(\cdot)$ *is weakly convex, then higher skilled agents value reputation more than lower skilled agents:* $\partial_2 U(\tilde{p}', k, t) > \partial_2 U(\tilde{p}, k, t)$*, for any* $\tilde{p}' > \tilde{p}$*.*

[51] This does not change the equilibrium dynamics even though this gives the agent a tool to signal his ability. We have characterized both cases and they lead to very similar results. At the cost of notation, we present the more difficult case here.

Proof. It is without loss to set $t=0$ and show that $\partial_{12}U(\tilde{p},k,0)>0$. Then

$$k_{0,t}=\frac{\phi}{\phi t+1}\cdot X_t+\frac{1}{\phi t+1}\cdot \max[k_0,\eta_0^{-1}(t)].$$

Since $k_{0,t}$ Define $l_{0,t}=\max[k_0,\eta_0^{-1}(t)]$ to be the initial type. Moreover, we can express the agent's private information as

$$X_t=\tilde{p}_0 t+\xi t+\sigma B_t,$$

where $\theta=\tilde{p}_0+\xi$ for $\xi\sim\mathcal{N}(0,\sigma_\theta^2)$ independent from other model variables. The expected payoff of the agent can be written as

$$U(\tilde{p}_0,l_0,0)=\mathrm{E}\left[\int_0^\tau e^{-\rho t}A\left(\frac{\phi}{\phi t+1}\cdot X_t+\frac{1}{\phi t+1}\cdot l_{0,t}\right)dt+e^{-\rho\tau}\cdot L\right].$$

Suppose that $\tilde{p}_0>\hat{l}_0=l_0+\varepsilon$. Define

$$\begin{aligned}
\tau&=\inf\left\{\frac{\phi}{\phi t+1}\cdot X_t+\frac{1}{\phi t+1}\cdot l_0=b(t)\right\},\\
\hat{\tau}&=\inf\{t:\hat{l}_{0,t}=\hat{l}_0\}.
\end{aligned}$$

Due to the dynamics specified for process $l_{0,t}$ and $\hat{l}_{0,t}$ it follows that $d\hat{l}_{0,t}>0$ only if $l_{0,t}=\hat{l}_{0,t}$ Then

$$\begin{aligned}
U(\tilde{p}_0,\hat{l}_0,0)-U(\tilde{p}_0,l_0,0)&=\mathrm{E}\left[\int_0^{\hat{\tau}} e^{-\rho t}A\left(\frac{\phi}{\phi t+1}\cdot\left(\tilde{p}_0 t+\xi t+\sigma B_t\right)+\frac{1}{\phi t+1}\cdot\hat{l}_0\right)dt+e^{-\rho\hat{\tau}}\cdot L\right]\\
&-\mathrm{E}\left[\int_0^{\hat{\tau}} e^{-\rho t}A\left(\frac{\phi}{\phi t+1}\cdot\left(\tilde{p}_0 t+\xi t+\sigma B_t\right)+\frac{1}{\phi t+1}\cdot l_{0,t}\right)dt+e^{-\rho\hat{\tau}}\cdot L\right]\\
&=\mathrm{E}\left[\int_0^{\tau} e^{-\rho t}A\left(\frac{\phi}{\phi t+1}\cdot\left(\tilde{p}_0 t+\xi t+\sigma B_t\right)+\frac{1}{\phi t+1}\cdot\hat{l}_0\right)dt\right]\\
&-\mathrm{E}\left[\int_0^{\tau} e^{-\rho t}A\left(\frac{\phi}{\phi t+1}\cdot\left(\tilde{p}_0 t+\xi t+\sigma B_t\right)+\frac{1}{\phi t+1}\cdot l_0\right)dt\right]\\
&+\mathrm{E}\left[\int_\tau^{\hat{\tau}} e^{-\rho t}A\left(\frac{\phi}{\phi t+1}\cdot\left(\tilde{p}_0 t+\xi t+\sigma B_t\right)+\frac{1}{\phi t+1}\cdot\hat{l}_0\right)dt\right]\\
&-\mathrm{E}\left[\int_\tau^{\hat{\tau}} e^{-\rho t}A\left(\frac{\phi}{\phi t+1}\cdot\left(\tilde{p}_0 t+\xi t+\sigma B_t\right)+\frac{1}{\phi t+1}\cdot l_{0,t}\right)dt\right]\\
&\overset{(i)}{=}\mathrm{E}\left[\int_0^{\tau} e^{-\rho t}A'\left(\frac{\phi}{\phi t+1}\cdot\left(\tilde{p}_0 t+\xi t+\sigma B_t\right)+\frac{1}{\phi t+1}\cdot l_0\right)\frac{\varepsilon}{\phi t+1}dt\right]+O(\varepsilon^2)\\
&+\mathrm{E}\left[\int_\tau^{\hat{\tau}} e^{-\rho t}A'\left(\frac{\phi}{\phi t+1}\cdot\left(\tilde{p}_0 t+\xi t+\sigma B_t\right)+\frac{1}{\phi t+1}\cdot l_0\right)\frac{\varepsilon}{\phi t+1}dt\right]+O(\varepsilon^2)\\
&\overset{(ii)}{=}\varepsilon\mathrm{E}\left[\int_0^{\tau} e^{-\rho t}A'\left(\frac{\phi}{\phi t+1}\cdot\left(\tilde{p}_0 t+\xi t+\sigma B_t\right)+\frac{1}{\phi t+1}\cdot l_0\right)\frac{1}{\phi t+1}dt\right]+\overline{o}(\varepsilon)+O(\varepsilon^2),
\end{aligned}$$

where (i) holds by Taylor expansion and (ii) holds since $\lim_{\varepsilon\to 0}(\hat{\tau}-\tau)=0$. By this argument (not the same as the Envelope theorem) we can express the partial derivative of the agent's continuation payoff in, almost, closed form

$$\frac{\partial}{\partial l_0}U(\tilde{p}_0,l_0,0)=\mathrm{E}\left[\int_0^{\tau} e^{-\rho t}A'\left(\frac{\phi}{\phi t+1}\cdot\left(\tilde{p}_0 t+\xi t+\sigma B_t\right)+\frac{1}{\phi t+1}\cdot l_0\right)\frac{1}{\phi t+1}dt\right].$$

Taking the cross-partial we obtain

$$\begin{aligned}
\frac{\partial^2}{\partial \tilde{p}_0 \partial l_0} U(\tilde{p}_0, l_0, 0) &= \mathrm{E}\left[\int_0^\tau e^{-\rho t} A'\left(\frac{\phi}{\phi t+1}\cdot\left(\tilde{p}_0 t+\xi t+\sigma B_t\right)+\frac{1}{\phi t+1}\cdot l_0\right)\frac{1}{\phi t+1}\,dt\right] \\
&= \mathrm{E}\left[\int_0^\tau e^{-\rho t} A''\left(\frac{\phi}{\phi t+1}\cdot\left(\tilde{p}_0 t+\xi t+\sigma B_t\right)+\frac{1}{\phi t+1}\cdot l_0\right)\frac{\phi t}{\phi t+1}\cdot\frac{1}{\phi t+1}\,dt\right] \\
&\quad + \mathrm{E}\left[e^{-\rho\tau} A'\left(\frac{\phi}{\phi\tau+1}\cdot\left(\tilde{p}_0\tau+\xi\tau+\sigma B_\tau\right)+\frac{1}{\phi\tau+1}\cdot l_0\right)\frac{1}{\phi\tau+1}\frac{\partial\tau}{\partial\tilde{p}_0}\right] \geq 0,
\end{aligned}$$

since $A''(\cdot) \geq 0$ and τ is, by definition, increasing (strictly) in $\tilde{p}_0$. This concludes the proof. $\square$

Proposition 5 establishes that higher-skilled agents value reputation more under the sufficient condition that $A(\cdot)$ is weakly convex. First, higher-skilled agents stay longer in the industry, as manifested by $\eta_t(\hat{p})$ being an increasing function of $\hat{p}$ as can be seen from (A.5). This is the same economic mechanism derived in Proposition 1. Second, higher-skilled agents are more likely to generate good performance and to have a high reputation. Weak convexity of $A(\cdot)$ ensures that the revenues remain sensitive to clients' perceptions even when reputation is high. While weak convexity is sufficient to ensure the agent's single-crossing, it is not necessary, as the first channel is present and ensures our results arise if $A(\cdot)$ is linear. It is also worthwhile to note that the agent leaves the industry when his private belief $\tilde{p}_s$ coincides with the clients' belief $k_{t,s}$ even when the subgame is started off equilibrium path, $\tilde{p}_t > k_{t,t}$. The agent's reservation wage in this case requires an adjustment since, whenever $k_t < b(t)$, beliefs experience a jump. The agent's cumulative reservation wage is given by

$$d\tilde{W}_t = \begin{cases} \rho L dt & if \quad U(\tilde{p}_t, k_t, t) = L, \\ w_R(\tilde{p}_t, k_t, t)dt & if \quad U(\tilde{p}_t, k_t, t) > L \quad and \quad k_t > b(t), \\ U(\tilde{p}_t, b(t), t) - U(\tilde{p}_t, k_t, t) & if \quad U(\tilde{p}_t, b(t), t) > 0 \quad and \quad k_t < b(t). \end{cases} \tag{A.6}$$

which features an adjustment stemming from the fact that, were he to leave, he would be able to independently signal his ability to clients. If the agent's reservation value L is low, then $b(t)$ is also low, and this adjustment to the reservation wage does not alter the equilibrium behavior of the main model.

A.5 Main Text Proofs

Proof of Lemma 1 (agent's outside option)

If the agent leaves with reputation k, his continuation utility is given by

$$u_\theta(k) = \mathrm{E}_\theta\left[\int_0^\eta e^{-\rho s} A(\pi(k,s))\,ds + e^{-\rho\eta}\cdot L\right] \tag{A.7}$$

where $\eta \stackrel{def}{=} \inf\left\{t : N_t^\theta > 0\right\}$. The realized value to the agent of type θ at time t can be expressed as

$$Z_t \stackrel{def}{=} \int_0^{\eta\wedge t} e^{-\rho s} A(\pi(k,s))\,ds + e^{-\rho t}\cdot U(\theta,\pi(k,s))\cdot \mathbb{1}\{t<\eta\} + e^{-\rho\eta}\cdot L\cdot\mathbb{1}\{\eta\le t\}. \tag{A.8}$$

By Ito's lemma

$$
\begin{aligned}
dZ_t =& e^{-\rho t} A(\pi(k,t))\,dt - re^{-\rho t} u_\theta(\pi(k,t))\,dt \\
&+ e^{-\rho t} \lambda \pi(k,t)(1-\pi(k,t)) \cdot \partial_2 u_\theta(\pi(k,t))\,dt + e^{-\rho t}(L - u_\theta(\pi(k,t)) \cdot dN_t^\theta.
\end{aligned}
\tag{A.9}
$$

Process $(Z_t)_{t\geq 0}$ is a Levy martingale with respect to filtration of type θ agent, implying that the expected drift of process $(Z_t)_{t\geq 0}$ is 0. Multiplying both sides of (A.9) by $e^{\rho t}$ obtain

$$
\rho u_\theta(\pi(k,t)) = A(\pi(k,t)) + \lambda \pi(k,t)(1-\pi(k,t)) \cdot u_\theta'(\pi(k,t)) + (L - u_\theta(\pi(k,t))) \cdot \lambda(1-\theta).
$$

Define $x = \pi(k,t)$. Then

$$
\rho u_\theta(x) = A(x) + \lambda x(1-x) \cdot u_\theta(x) + (L - u_\theta(x)) \cdot \lambda(1-\theta). \tag{A.10}
$$

The above is a linear differential equation. The general solution at $t = 1$ is defined as the solution to the first order linear differential equation

$$
(\rho + \lambda(1-\theta)) \cdot u_\theta^G(x) = \lambda x(1-x) \cdot \partial_2 U_\theta^G(x).
$$

It is given by

$$
u_\theta^G(x) = \left(\frac{x}{1-x}\right)^{\frac{\rho}{\lambda}+1-\theta}.
$$

Function $u_\theta(k)$ is well defined as integral (A.7). Thus, the bounded solution to (A.10) exists. The fact that $u_\theta^G(1) = \infty$ implies that this solution is unique. The unique bounded solution satisfies

$$
u_\theta(1) = \frac{A(1)}{\rho + \lambda(1-\theta)} + \frac{\lambda(1-\theta)L}{\rho + \lambda(1-\theta)}.
$$

The agent's value function $U(\tilde{p}, k)$ is defined in (5), which we reiterate here

$$
U(\tilde{p}_t, k_t) \stackrel{def}{=} \max_{\hat{\eta}} \mathrm{E}_{\tilde{p}_t} \left[\int_t^{\hat{\eta}} e^{-\rho(s-t)} A\Big(\pi(k_t, s-t)\Big)\,ds + e^{-\rho(\hat{\eta}-t)} \cdot L \right].
$$

Lemma A.4. *$U(\tilde{p}, k)$ is increasing in p.*

Proof. The optimal time for the agent to leave the industry is

$$
\eta^p = \arg\max_{\hat{\eta}} \mathrm{E} \left[\tilde{p} \int_0^{\hat{\eta}} e^{-\rho t} \Big(A(\pi(k,t)) - \rho L \Big)\,dt + (1-\tilde{p}) \int_0^{\hat{\eta}} e^{-(\rho+\lambda)t} \Big(A(\pi(k,t) - \rho L \Big)\,dt + L \right].
$$

Subtract the value of stopping L. Taking expectations and applying the Envelope theorem with respect to $\tilde{p}$ obtain

$$
\int_0^\eta e^{-\rho t} \Big(A(\pi(k,t)) - \rho L \Big)\,dt - \int_0^\eta e^{-(\rho+\lambda)t} \Big(A(\pi(k,t)) - \rho L \Big)\,dt. \tag{A.11}
$$

Suppose (A.11) is negative. Then

$$
\max_{\hat{\eta}} \mathrm{E} \left[\tilde{p} \int_0^{\hat{\eta}} e^{-\rho t} \Big(A(\pi(k,t) - \rho L \Big)\,dt + (1-\tilde{p}) \int_0^{\hat{\eta}} e^{-(\lambda+\rho)t} \Big(A(\pi(k,t)) - \rho L \Big)\,dt \right]
$$

$$
\begin{aligned}
&= \max_{\hat\eta} \mathrm{E}\left[\int_0^{\hat\eta} \Big(\tilde p e^{-\rho t} + (1-\tilde p)e^{-(\rho+\lambda)t}\Big)\Big(A(\pi(k,t) - \rho L\Big)\, dt\right] \\
&< \max_{\hat\eta} \mathrm{E}\left[\int_0^{\hat\eta} e^{-(\lambda+\rho)t}\Big(A(\pi(k,t) - \rho L\Big)\, dt\right] = \mathrm{E}\left[\int_0^{\eta^0} e^{-(\lambda+\rho)t}\Big(A(\pi(k,t) - \rho L\Big)\, dt\right] \\
&= \mathrm{E}\left[\int_0^{\infty} e^{-\rho t}\cdot e^{-\lambda t}\cdot \mathrm{P}\big(\eta^0 > t \,|\, X_t = \mu t\big)\,\Big(A(\pi(k,t) - \rho L\Big)\, dt\right]
\end{aligned}
$$

For a given p, there exists a "mixed-strategy" stopping time $\hat\eta^p$ given by the conditional probability distribution

$$
\mathrm{P}(\hat\eta^p > t \,|\, X_t = \mu t) = e^{-\lambda t}\cdot \mathrm{P}\big(\eta^0 > t \,|\, X_t = \mu t\big)\,.
$$

This stopping time is feasible for a given p and generates the same exact same payoff under $p > 0$ as it does for $p = 0$. It implies that the ex-post distribution that can be achieved for $p = 0$ can be achieved for any $p > 0$. □

Lemma A.5. *Agent's outside option $U(\tilde p, k)$ is weakly increasing in k.*

Proof. Subtracting L from (5) and differentiating with respect to k, obtain

$$
\begin{aligned}
\frac{\partial}{\partial k}\Big[U(p,k) - L\Big] &= \frac{\partial}{\partial k}\mathrm{E}_{\tilde p_t}\left[\int_t^{\hat\eta} e^{-\rho(s-t)}\Big[A\Big(\pi(k_t, s-t)\Big) - \rho L\Big]\, ds\right] \\
&= \mathrm{E}\left[\int_0^{\eta^p} e^{-\rho t} A'(\pi(k,t))\partial_1\pi(k,t)\, dt\right] \overset{(i)}{\geq} 0,
\end{aligned}
$$

where inequality (i) is strictly positive whenever $\eta^p > 0$ with positive probability and $k \in (0,1)$. □

If the agent finds it optimal to stay in the industry for a given k, then he finds it optimal to stay in the industry for any $k' > k$. The dynamic counterpart to the argument is that he finds it optimal to remain in the industry until the arrival of the Poisson shock if he chose to remain in the industry when clients' belief was k. The expected payoff from staying in the industry until the arrival of the Poisson shock is, simply, given by $pu_1(k) + (1-p)u_0(k)$. As a result, (10) holds.

Lemma A.6. *The agent's reservation wage $w_R(\tilde p, k)$ is given by* (12) *and is weakly decreasing in $\tilde p$.*

Proof. Suppose $A(k) > \rho L$. Then the reservation wage of all agents is equal to $A(k)$, since they can collect the revenues until the arrival of the Poisson shock. If $A(k) < \rho L$, then, due to higher skilled agents being more likely to stay in the industry, they are willing to accept it, due to higher profits from surviving long enough. As a result, the willingness of the agent to accept $A(k) < \rho L$ is increasing in $\tilde p$. □

Proof of Proposition 1 (agent's single-crossing)

Self-contained in the main text.

Proof of Proposition 2 (equilibrium verification)

The argument is comprised of several small lemmas, but is conceptually very simple. First, we show that absent any pay-for-reputation dynamics, the intermediary prefers to retain higher skilled agents for longer. In the churning region the cutoff type pays the intermediary just enough to make her indifferent between keeping him and letting him go. This implies that all agents below the cutoff type are strictly unprofitable to the intermediary while all agents above the cutoff type are strictly profitable, due to the single-crossing condition of the agent.
For formal proofs, it is convenient to work with the initial agent types. I.e., for every type $\tilde{p}_t$ at time t we identify type $\tilde{p}_0$ such that $\tilde{p}_t = \pi(\tilde{p}_0, t)$. This is equivalent to $\tilde{p}_0 = \pi(\tilde{p}_t, -t)$. Similarly, denote by l_t the initial cutoff type

$$l_t \stackrel{def}{=} \pi(k_t, -t). \tag{A.12}$$

It is more convenient to work in the space of initial types l_t, rather than posterior types k_t, as to separate the learning stemming from churning versus performance. Define

$$Q(l) \stackrel{def}{=} \mathrm{E}\left[\tilde{p}_0 \,|\, \tilde{p}_0 > l\right] = \frac{1}{1-F(l)} \cdot \int_l^1 x \, dF(x).$$

to be the average initial type above l. Similarly, define

$$\hat{Q}(l,t) \stackrel{def}{=} \pi(Q(l), t).$$

Define by η to be the first time of arrival of process N^θ

$$\eta \stackrel{def}{=} \inf\{t : \, X_t < \mu t\}.$$

For each initial p_0 define $\mathbb{T}\left(\tilde{p}_0, l\right) \in \mathbb{R}$ to be the set of subgame-perfect times at which it is optimal to let go of the agent $\tilde{p}_0$ if the clients belief the cutoff agent is of type l if the intermediary has to pay the agent exactly his reservation wage

$$\mathbb{T}\left(\tilde{p}_0, l\right) \stackrel{def}{=} \bigcup_{t \geq 0} \left\{ \arg\max_{T \geq t} \mathrm{E}_{p_0} \left[\int_t^{T \wedge \eta} e^{-r(s-t)} \left[A\Big(\hat{Q}(l,s)\Big) - w_R\Big(\pi(\tilde{p}_0, s), \pi(l,s)\Big) - rV \right] ds + V \right] \right\} \tag{A.13}$$

where we used the identity

$$\begin{aligned} &\int_t^{T \wedge \eta} e^{-r(s-t)} \left[A\Big(\hat{Q}(l,s)\Big) - w_R\Big(\pi(\tilde{p}_0, s), \pi(l,s)\Big) \right] ds + e^{-r\hat{\tau} \wedge \eta} \cdot V \\ \equiv &\int_t^{T \wedge \eta} e^{-r(s-t)} \left[A\Big(\hat{Q}(l,s)\Big) - w_R\Big(\pi(\tilde{p}_0, s), \pi(l,s)\Big) - rV \right] ds + V. \end{aligned}$$

Lemma A.7. *Stopping set $\mathbb{T}\left(\tilde{p}_0, l\right)$ is weakly decreasing in $\tilde{p}_0$, i.e.,*

$$\mathbb{T}\left(\tilde{p}_0, l\right) \subseteq \mathbb{T}\left(\tilde{p}_0', l\right) \quad \forall \tilde{p}_0' \leq \tilde{p}_0.$$

Proof. For any stopping time $\hat{\tau}$

$$\begin{aligned}&\mathrm{E}_{\tilde{p}_0}\left[\int_t^{T\wedge\eta} e^{-r(s-t)}\left[A\Big(\hat{Q}(l,s)\Big)-w_R\Big(\pi(\tilde{p}_0,s),\pi(l,s)\Big)-rV\right]\,ds\right]+V \qquad (A.14)\\ =&\mathrm{E}\left[\int_t^{T\wedge\eta} e^{-r(s-t)}\left[\tilde{p}_0+(1-\tilde{p}_0)e^{-\lambda(s-t)}\right]\cdot\left[A\Big(\hat{Q}(l,s)\Big)-w_R\Big(\pi(\tilde{p}_0,s),\pi(l,s)\Big)-rV\right]\,ds\right]+V.\end{aligned}$$

Since $w(\tilde{p},k)$ is weakly decreasing in $\tilde{p}$, applying the Envelope theorem obtain

$$\begin{aligned}&\frac{\partial}{\partial\tilde{p}_0}\left[\mathrm{E}_{p_0}\left[\int_t^{T\wedge\eta} e^{-r(s-t)}\left[A\Big(\hat{Q}(l,s)\Big)-w_R\Big(\pi(\tilde{p}_0,s),\pi(l,s)\Big)-rV\right]\,ds\right]+V\right]\\ \leq&\mathrm{E}\left[\int_t^{T\wedge\eta} e^{-r(s-t)}\Big(1-e^{-\lambda(s-t)}\Big)\left[A\Big(\hat{Q}(l,s)\Big)-w_R\Big(\pi(\tilde{p}_0,s),\pi(l,s)\Big)-rV\right]\,ds\right]. \qquad (A.15)\end{aligned}$$

Suppose (A.15) is negative. Then, the expected payoff (A.14) for a given T satisfies

$$\begin{aligned}&\mathrm{E}_{\tilde{p}_0}\left[\int_t^{T\wedge\eta} e^{-r(s-t)}\left[A\Big(\hat{Q}(l,s)\Big)-w_R\Big(\pi(\tilde{p}_0,s),\pi(l,s)\Big)-rV\right]\,ds\right]+V\\ \overset{(i)}{<}&\mathrm{E}_{\tilde{p}_0=0}\left[\int_t^{T\wedge\eta} e^{-r(s-t)}\left[A\Big(\hat{Q}(l,s)\Big)-w_R\Big(\pi(\tilde{p}_0,s),\pi(l,s)\Big)-rV\right]\,ds\right]+V. \qquad (A.16)\end{aligned}$$

Inequality (i) cannot be strict, however, since the intermediary can replicate the payoff by employing a mixed-stopping rule, similar to the argument used in the proof of Lemma 1. Thus, the intermediary's expected optimal stopping payoff is weakly increasing in p_0, which implies the ranking of the stopping sets. □

Denote by $V(\tilde{p}_0,l,t)$ the intermediary's expected continuation value from time t onwards from employing the agent of initial skill $\tilde{p}_0$ while clients believe the cutoff type is l and absent any reputation-building considerations

$$V(\tilde{p}_0,l,t)\overset{def}{=}\sup_{\hat{\tau}}\mathrm{E}_{\tilde{p}_0}\left[\int_t^{\hat{\tau}\wedge\eta} e^{-r(s-t)}\left[A\Big(\hat{Q}(l,s)\Big)-w_R\Big(\pi(\tilde{p}_0,s),\pi(l,s)\Big)-rV\right]\,ds\right]+V. \qquad (A.17)$$

Using Ito's Lemma for $t\notin\mathbb{T}(\tilde{p}_0,l)$, function $V(\tilde{p}_0,l,t)$ satisfies

$$rV(\tilde{p}_0,l,t)=A\Big(\hat{Q}(l,s)\Big)-w_R\Big(\pi(\tilde{p}_0,s),\pi(l,s)\Big)+\frac{\partial}{\partial t}V(\tilde{p}_0,l,t)+\lambda\Big(1-\pi(\tilde{p}_0,t)\Big)\Big(V-V(\tilde{p}_0,l,t)\Big).$$

For $t\in\mathbb{T}(\tilde{p}_0,l)$ function $V(\tilde{p}_0,l,t)=V$. We can combine the two cases by writing

$$\begin{aligned}rV(\tilde{p}_0,l,t)=&\frac{\partial}{\partial t}V(\tilde{p}_0,l,t)+\lambda\Big(1-\pi(\tilde{p}_0,t)\Big)\Big(V-V(\tilde{p}_0,l,t)\Big)\\ &+\underbrace{\begin{cases}A\Big(\hat{Q}(l,s)\Big)-w_R\Big(\pi(\tilde{p}_0,s),\pi(l,s)\Big) & \textit{if}\quad t\notin\mathbb{T}(\tilde{p}_0,l),\\ rV & \textit{if}\quad t\in\mathbb{T}(\tilde{p}_0,l).\end{cases}}_{\textit{flow payoff}} \qquad (A.18)\end{aligned}$$

First, we show that if the agent is initially hired by the intermediary, then the churning occurs only in the region where he opens his own firm upon being churned.

Lemma A.8. *Define by $t^*(\tilde{p}_0,l)$ the first time when the intermediary is willing to let go of the agent $\tilde{p}_0$ given clients'* *belief* l

$$t^*(\tilde{p}_0, l) \overset{def}{=} \inf\{t \geq 0 : \quad t \in \mathbb{T}(\tilde{p}_0, l)\}.$$

If $t^*(\tilde{p}_0, l) > 0$, *then* $U\big(\pi(\tilde{p}_0, t), \pi(l, t)\big) > L$ *for every* $t \in \mathbb{T}(\tilde{p}_0, l)$, *i.e., the agent opens his firm when he leaves the intermediary.*

Proof. Suppose, from the contrary, that $U\big(\pi(\tilde{p}_0, t), \pi(l, t)\big) = L$ for some $\hat{t} \in \mathbb{T}(\tilde{p}_0, l)$. Due to Lemma A.5

$$w_R\Big(\pi(\tilde{p}_0, t), \pi(l, t)\Big) = \rho L \qquad \forall t \in [0, \hat{t}].$$

This implies that for every $t \leq \hat{t}$

$$A\Big(\hat{Q}(l, t)\Big) - w_R\Big(\pi(\tilde{p}_0, t), \pi(l, t)\Big) - rV = A\Big(\hat{Q}(l, t)\Big) - \rho L - rV \leq A\Big(\hat{Q}(l, \hat{t})\Big) - \rho L - rV \overset{(i)}{=} 0,$$

where (i) stems from the necessary optimality condition (17) for stopping at time $\hat{t}$. By definition of $t^*(\tilde{p}_0, l)$ it holds that $\hat{t} \geq t^*(\tilde{p}_0, l)$ implying that

$$\begin{aligned} V(\tilde{p}_0, l, 0) &= \mathrm{E}_{\tilde{p}_0}\left[\int_0^{t^*(\tilde{p}_0, l)} e^{-rt}\left[A\Big(\hat{Q}(l, \hat{t})\Big) - w_R\Big(\pi(\tilde{p}_0, t), \pi(l, t)\Big) - rV\right] dt + V\right] \\ &= \mathrm{E}_{\tilde{p}_0}\left[\int_0^{t^*(\tilde{p}_0, l)} e^{-rt}\left[A\Big(\hat{Q}(l, \hat{t})\Big) - \rho L - rV\right] dt + V\right] \overset{(ii)}{\leq} V, \end{aligned}$$

where inequality (ii) is strict whenever $t^*(\tilde{p}_0, l) > 0$. This implies that, if $t^*(\tilde{p}_0, l) > 0$, then the intermediary is strictly better of replacing the agent at $t = 0$, posing a contradiction with the optimality of the intermediary stopping at $t^*(\tilde{p}_0, l) > 0$. □

Lemma A.9. *Suppose process* $(l_t)_{t\geq 0}$ *is weakly increasing subject to* $dl_t > 0$ *only for* $t \in \mathbb{T}(\tilde{p}_0, l_t)$.[52] *Then, the intermediary does not benefit from the increase in process* $(l_t)_{t\geq 0}$ *and value function* $V(\tilde{p}_0, l_t, t)$ *can be written as*

$$V(\tilde{p}_0, l_t, t) = \mathrm{E}_{\tilde{p}_0}\left[\int_t^{\infty} e^{-r(s-t)}\left[A\Big(\hat{Q}(l_s, s)\Big) - w_R\Big(\pi(\tilde{p}_0, s), \pi(l_s, s)\Big) - rV\right] \mathbb{1}\{s \in \mathbb{T}(\tilde{p}_0, l_s)\}\, ds\right] + V. \tag{A.19}$$

Proof. Without loss, we prove (A.19) for $t = 0$. The same approach holds for any $t > 0$. Define the stochastic process $Z = (Z_t)_{t\geq 0}$ as

$$Z_t \overset{def}{=} \int_0^t e^{-rs}\left[A\Big(\hat{Q}(l_s, s)\Big) - w_R\Big(\pi(\tilde{p}_0, s), \pi(l, s)\Big) - rV\right] \mathbb{1}\{s \notin \mathbb{T}(\tilde{p}_0, l_s)\}\, ds + e^{-rt}\Big[V(\tilde{p}_0, l_t, t) - V\Big].$$

Suppose $t \notin \mathbb{T}(\tilde{p}_0, l_t)$. Then

$$\begin{aligned} dZ_t &= e^{-rt}\left[A\Big(\hat{Q}(l_t, t)\Big) - w_R\Big(\pi(\tilde{p}_0, t), \pi(l, t)\Big) - rV\right] dt - re^{-rt}\Big[V(\tilde{p}_0, l_t, t) - V\Big] dt + e^{-rt} dV(\tilde{p}_0, l_t, t) \\ &= e^{-rt}\left[A\Big(\hat{Q}(l_t, t)\Big) - w_R\Big(\pi(\tilde{p}_0, t), \pi(l, t)\Big) - rV\right] dt - re^{-rt}\Big[V(\tilde{p}_0, l_t, t) - V\Big] dt \\ &+ e^{-rt}\frac{\partial}{\partial t} V(\tilde{p}_0, l_t, t)\, dt + e^{-rt}\Big[V - V(\tilde{p}_0, l_t, t)\Big] dN_t^{\theta} \end{aligned}$$

[52]In the case of $l_t - l_{t-} > 0$ this requires that $t \in \cap_{l \in [l_{t-}, l_t]} \mathbb{T}(\tilde{p}_0, l)$.

$$
\begin{aligned}
&= e^{-rt}\left[A\Big(\hat{Q}(l_t,t)\Big) - w_R\Big(\pi(\tilde{p}_0,t),\pi(l,t)\Big)\right]dt - re^{-rt}V(\tilde{p}_0,l_t,t)\,dt \\
&+ e^{-rt}\frac{\partial}{\partial t}V(\tilde{p}_0,l_t,t)\,dt + e^{-rt}\Big[V - V(\tilde{p}_0,l_t,t)\Big]dN_t^\theta \\
&= e^{-rt}\cdot\Big[V - V(\tilde{p}_0,l_t,t)\Big]\cdot\Big[dN_t^\theta - \lambda\Big(1-\pi(\tilde{p}_0,t)\Big)\,dt\Big].
\end{aligned} \tag{A.20}
$$

For $t \in \mathbb{T}(\tilde{p}_0, l_t)$ we have $V(\tilde{p}_0,l_t,t) \equiv V$, implying that $Z_t \equiv 0$ regardless of the value of l_t. Thus, in addition to (A.20) that process $(Z_t)_{t\geq 0}$ is an L^1 martingale. This implies that

$$
\begin{aligned}
V(\tilde{p}_0,l_0,0) - V &= Z_0 = \mathrm{E}\left[Z_\infty\right] \\
&= \mathrm{E}_{\tilde{p}_0}\left[\int_0^\infty e^{-rt}\left[A\Big(\hat{Q}(l,t)\Big) - w_R\Big(\pi(\tilde{p}_0,t),\pi(l,t)\Big) - rV\right]\mathbb{1}\left\{t \in \mathbb{T}(\tilde{p}_0,l_t)\right\}\,dt\right]
\end{aligned}
$$

which implies (A.19) for $t=0$. The extension to $t>0$ is identical to the analysis above. □

Lemma A.10 (Retention Wage)**.** *Consider an increasing process $l = (l_t)_{t\geq 0}$. The cumulative compensation necessary to retain the agent of initial skill $\tilde{p}_0$, i.e., posterior skill $(\pi(\tilde{p}_0,t))_{t\geq 0}$, denoted by $(\tilde{C}_t^{\tilde{p}_0})_{t\geq 0}$ is given by*

$$
\begin{aligned}
d\tilde{C}_t^{\tilde{p}_0} = w_R\Big(\pi(\tilde{p}_0,t),\pi(l_t,t)\Big)\,dt &- \partial_2 U\Big(\pi(\tilde{p}_0,t),\pi(l_t,t)\Big)\partial_1\pi(l_t,t)\,\cdot\mathbb{1}\left\{l_t - l_{t-} = 0\right\}dl_t \\
&- \Big[U\Big(\pi(\tilde{p}_0,t),\pi(l_t,t)\Big) - U\Big(\pi(\tilde{p}_0,t),\pi(l_{t-},t)\Big)\Big].
\end{aligned} \tag{A.21}
$$

Moreover, if process $(l_t)_{t\geq 0}$ is weakly increasing, then $\tilde{C}_t^{\tilde{p}_0} \geq \tilde{C}_t^{\tilde{p}_0'}$ for $\tilde{p}_0 \leq \tilde{p}_0'$.

Proof. Without loss, we prove the result for $t=0$. The same approach holds for any $t>0$. Define process $(Z_t)_{t\geq 0}$ as

$$
Z_t \stackrel{def}{=} \int_0^t e^{-\rho s}\,d\tilde{C}_s + e^{-\rho t}\cdot U\Big(\pi(\tilde{p}_0,t),\pi(l_t,t)\Big).
$$

Applying Ito's lemma obtain

$$
\begin{aligned}
dZ_t &= e^{-\rho t}\,d\tilde{C}_t - \rho e^{-\rho t}U\Big(\pi(\tilde{p}_0,t),\pi(l_t,t)\Big)\,dt + e^{-\rho t}dU\Big(\pi(\tilde{p}_0,t),\pi(l_t,t)\Big) \\
&= e^{-\rho t}\,d\tilde{C}_t - \rho e^{-\rho t}U\Big(\pi(\tilde{p}_0,t),\pi(l_t,t)\Big)\,dt + e^{-\rho t}\partial_1 U\Big(\pi(\tilde{p}_0,t),\pi(l_t,t)\Big)\partial_2\pi(\tilde{p}_0,t)\,dt \\
&+ e^{-\rho t}\partial_2 U\Big(\pi(\tilde{p}_0,t),\pi(l_t,t)\Big)\Big[\partial_1\pi(l_t,t)\cdot\mathbb{1}\left\{l_t - l_{t-} = 0\right\}dl_t + \partial_2\pi(l_t,t)\,dt\Big] \\
&+ e^{-\rho t}\Big[U\Big(\pi(\tilde{p}_0,t),\pi(l_t,t)\Big) - U\Big(\pi(\tilde{p}_0,t),\pi(l_{t-},t)\Big)\Big] \\
&+ e^{-\rho t}\Big[L - U\Big(\pi(\tilde{p}_0,t),\pi(l_t,t)\Big)\Big]\,dN_t^\theta \\
&= e^{-\rho t}\Big[L - U\Big(\pi(\tilde{p}_0,t),\pi(l_t,t)\Big)\Big]\cdot\Big[dN_t^\theta - \lambda(1-\pi(\tilde{p}_0,t))\,dt\Big].
\end{aligned}
$$

Using the definition of $\tilde{C}$ and $U(\tilde{p},k)$, it is easy to see that $\mathrm{E}\left[dZ_t\right] = 0$. Due to the bounded flow payoffs, it implies that $(Z_t)_{t\geq 0}$ is a martingale. This implies that

$$
U(\tilde{p}_0,l_0) = Z_0 = \mathrm{E}\left[Z_{\hat{\tau}}\right] = \mathrm{E}_{\tilde{p}_0}\left[\int_0^\eta e^{-\rho t}\,d\tilde{C}_t + e^{-\rho\eta}L\right].
$$

The agent is, thus, indifferent at $t=0$ between obtaining his reservation wage and compensation $\tilde{C}_t$ up to any time $\hat{t}$. The monotonicity $\tilde{C}_t^{\tilde{p}_0} < \tilde{C}_t^{\tilde{p}'_0}$ for $\tilde{p}_0 < \tilde{p}'_0$ stems directly from the agent's outside option $U(\tilde{p}, k)$ satisfying single-crossing, i.e., $\frac{\partial^2}{\partial p \partial k} U(\tilde{p}, k) \geq 0$. □

Rewriting the dynamics for process $(k_t)_{t\geq 0}$ from (13), the conjectured equilibrium dynamics for process $(l_t)_{t\geq 0}$ are given by

$$\dot{l}_t = \begin{cases} 0 & \text{if} \quad t \notin \mathbb{T}(l_t, l_t), \\ \dfrac{rV + w_R\Big(\pi(\tilde{p}_0, t), \pi(l_t, t)\Big) - A\Big(\hat{Q}(l_t, t)\Big)}{\partial_1 \pi(l_t, t) \cdot \partial_2 U(\pi(l_t, t), \pi(l_t, t))} & \text{if} \quad t \in \mathbb{T}(l_t, l_t). \end{cases} \tag{A.22}$$

subject to the initial condition $l_0 = k_0$.

Lemma A.11 (Verification)**.** *Given dynamics $(l_t)_{t\geq 0}$ given by* (A.22)*, it is incentive compatible for the intermediary who employing agent $l_t = \pi(k_t, -t)$ to let go of the agent when $t \in \mathbb{T} = \{t : t \in \mathbb{T}(l_t, l_t)\}$.*

Proof. The minimum retention wage necessary to retain the agent if $dl_t > 0$ is

$$\hat{w}\Big(\pi(\tilde{p}_0, t), \pi(l_t, t)\Big) = w_R\Big(\pi(\tilde{p}_0, t), \pi(l_t, t)\Big) - \Big[rV + w_R\Big(\pi(\tilde{p}_0, t), \pi(l_t, t)\Big) - A\Big(\hat{Q}(l_t, t)\Big)\Big] \frac{\partial_2 U\Big(\pi(\tilde{p}_0, t), \pi(l_t, t)\Big)}{\partial_2 U\Big(\pi(l_t, t), \pi(l_t, t)\Big)}.$$

An important observation is

$$\frac{\partial_2 U(\pi(\tilde{p}_0, t), \pi(l_t, t))}{\partial_2 U(\pi(l_t, t), \pi(l_t, t))} > 1 \qquad \Leftrightarrow \qquad \tilde{p}_0 > l_t.$$

First, we show that it is sub-optimal for the intermediary to retain the agent of initial skill $\tilde{p}_0$ if $\tilde{p}_0 < l_t$. The intermediary's flow payoff is

$$\begin{aligned}
&A\Big(\hat{Q}(l_t, t)\Big) - w_R\Big(\pi(\tilde{p}_0, t), \pi(l_t, t)\Big) - rV \\
&-\Big[rV + w_R\Big(\pi(\tilde{p}_0, t), \pi(l_t, t)\Big) - A\Big(\hat{Q}(l_t, t)\Big)\Big] \cdot \frac{\partial_2 U\Big(\pi(\tilde{p}_0, t), \pi(l_t, t)\Big)}{\partial_2 U\Big(\pi(l_t, t), \pi(l_t, t)\Big)} \cdot \mathbb{1}\{t \in \mathbb{T}(l_t, l_t)\} \\
\leq &A\Big(\hat{Q}(l_t, t)\Big) - w_R\Big(\pi(\tilde{p}_0, t), \pi(l_t, t)\Big) - rV - \Big[rV + w_R\Big(\pi(\tilde{p}_0, t), \pi(l_t, t)\Big) - A\Big(\hat{Q}(l_t, t)\Big)\Big] \cdot \mathbb{1}\{t \in \mathbb{T}(l_t, l_t)\} \\
= &\Big[A\Big(\hat{Q}(l_t, t)\Big) - w_R\Big(\pi(\tilde{p}_0, t), \pi(l_t, t)\Big) - rV\Big] \cdot \mathbb{1}\{t \notin \mathbb{T}(l_t, l_t)\} \\
= &\Big[A\Big(\hat{Q}(l_t, t)\Big) - w_R\Big(\pi(\tilde{p}_0, t), \pi(l_t, t)\Big) - rV\Big] \cdot \mathbb{1}\{t \notin \mathbb{T}(\tilde{p}_0, l_t)\} \\
+ &\Big[A\Big(\hat{Q}(l_t, t)\Big) - w_R\Big(\pi(\tilde{p}_0, t), \pi(l_t, t)\Big) - rV\Big] \cdot \mathbb{1}\{t \in \mathbb{T}(\tilde{p}_0, l_t) \setminus \mathbb{T}(l_t, l_t)\} \\
\leq &\Big[A\Big(\hat{Q}(l_t, t)\Big) - w_R\Big(\pi(\tilde{p}_0, t), \pi(l_t, t)\Big) - rV\Big] \cdot \mathbb{1}\{t \notin \mathbb{T}(\tilde{p}_0, l_t)\},
\end{aligned}$$

since for every $t \in \mathbb{T}(\tilde{p}_0, l_t) \setminus \mathbb{T}(l_t, l_t)$ we have

$$A\Big(\hat{Q}(l_t, t)\Big) - w_R\Big(\pi(\tilde{p}_0, t), \pi(l_t, t)\Big) - rV \leq 0.$$

This implies that for $\tilde{p}_0 \leq l_t$ it follows that for every $t \in \mathbb{T}(\tilde{p}_0, l_t)$

$$\begin{aligned}
V &= \sup_{\tau} \left[\mathrm{E}_{\tilde{p}_0} \left[\int_t^{\tau} e^{-r(s-t)} \left[A\Big(\hat{Q}(l,s)\Big) - w_R\Big(\pi(\tilde{p}_0, s), \pi(l_s, s)\Big) - rV \right] \cdot \mathbb{1}\left\{ s \in \mathbb{T}(\tilde{p}_0, l_s) \right\} ds \right] + V \right] \\
&\geq \sup_{\tau} \left[\mathrm{E}_{\tilde{p}_0} \left[\int_t^{\tau} e^{-r(s-t)} \left[A\Big(\hat{Q}(l,s)\Big) - w\Big(\pi(\tilde{p}_0, s), \pi(l, s)\Big) - rV \right] \mathbb{1}\left\{ s \in \mathbb{T}(\tilde{p}_0, l_s) \right\} ds \right] + V \right].
\end{aligned}$$

This implies that for $p_0 < l_t$ it is weakly optimal for the intermediary to replace the agent the first time when $t \in \mathbb{T}(\tilde{p}_0, l_t)$.

Suppose, now, that $\tilde{p}_0 > l_t$. We reverse the previous argument by, similarly, ranking flow payoffs of the intermediary

$$\begin{aligned}
& A\Big(\hat{Q}(l_t,t)\Big) - w_R\Big(\pi(\tilde{p}_0, t), \pi(l_t, t)\Big) - rV \\
& - \left[rV + w_R\Big(\pi(\tilde{p}_0, t), \pi(l_t, t)\Big) - A\Big(\hat{Q}(l_t,t)\Big) \right] \cdot \frac{\partial_2 U\Big(\pi(\tilde{p}_0, t), \pi(l_t, t)\Big)}{\partial_2 U\Big(\pi(l_t, t), \pi(l_t, t)\Big)} \cdot \mathbb{1}\left\{ t \in \mathbb{T}(l_t, l_t) \right\} \\
\geq & A\Big(\hat{Q}(l_t,t)\Big) - w_R\Big(\pi(\tilde{p}_0, t), \pi(l_t, t)\Big) - rV - \left[rV + w_R\Big(\pi(\tilde{p}_0, t), \pi(l_t, t)\Big) - A\Big(\hat{Q}(l_t,t)\Big) \right] \cdot \mathbb{1}\left\{ t \in \mathbb{T}(l_t, l_t) \right\} \\
= & \left[A\Big(\hat{Q}(l_t,t)\Big) - w_R\Big(\pi(\tilde{p}_0, t), \pi(l_t, t)\Big) - rV \right] \cdot \mathbb{1}\left\{ t \notin \mathbb{T}(l_t, l_t) \right\} \\
= & \left[A\Big(\hat{Q}(l_t,t)\Big) - w_R\Big(\pi(\tilde{p}_0, t), \pi(l_t, t)\Big) - rV \right] \cdot \mathbb{1}\left\{ t \notin \mathbb{T}(\tilde{p}_0, l_t) \right\} \\
\geq & \left[A\Big(\hat{Q}(l_t,t)\Big) - w_R\Big(\pi(\tilde{p}_0, t), \pi(l_t, t)\Big) - rV \right] \cdot \mathbb{1}\left\{ t \notin \mathbb{T}(\tilde{p}_0, l_t) \right\}
\end{aligned}$$

since for every $t \in \mathbb{T}(\tilde{p}_0, l_t) \setminus \mathbb{T}(l_t, l_t)$ we have

$$A\Big(\hat{Q}(l_t,t)\Big) - w_R\Big(\pi(\tilde{p}_0, t), \pi(l_t, t)\Big) - rV \leq 0.$$

This implies that for $\tilde{p}_0 \geq l_t$ it follows that for every $t \in \mathbb{T}\left(\tilde{p}_0, l_t\right)$, which coincides with the first when $\tilde{p}_0 = l_t$.

$$\begin{aligned}
V &= \sup_{\tau} \left[\mathrm{E}_{\tilde{p}_0} \left[\int_t^{\tau} e^{-r(s-t)} \left[A\Big(\hat{Q}(l,s)\Big) - w_R\Big(\pi(\tilde{p}_0, s), \pi(l_s, s)\Big) - rV \right] \mathbb{1}\left\{ s \in \mathbb{T}\left(\tilde{p}_0, l_s\right) \right\} ds \right] + V \right] \\
&\leq \sup_{\tau} \left[\mathrm{E}_{\tilde{p}_0} \left[\int_t^{\tau} e^{-r(s-t)} \left[A\Big(\hat{Q}(l,s)\Big) - w\Big(\pi(\tilde{p}_0, s), \pi(l, s)\Big) - rV \right] \mathbb{1}\left\{ s \in \mathbb{T}\left(\tilde{p}_0, l_s\right) \right\} ds \right] + V \right].
\end{aligned}$$

This implies that for $\tilde{p}_0 > l_t$ it is weakly optimal for the intermediary to retain the agent. □

Proof of Lemma 2 (determination of $k_0 \geq \underline{p}$)

Lemma A.12. *Given V there exists a unique k_0 such that the intermediary retains all agents such that $\tilde{p}_0 > k_0$.*

Proof. Define $\tau(\tilde{p}_0)$ to be the solution to

$$\tau(\tilde{p}_0) \stackrel{def}{=} \arg\max_{\hat{\tau}} \mathrm{E}_{\tilde{p}_0} \left[\int_0^{\hat{\tau}} e^{-rt} \left[A\Big(\hat{Q}(\tilde{p}_0, t)\Big) - w_R\Big(\pi(\tilde{p}_0, t), \pi(\tilde{p}_0, t)\Big) - rV \right] dt \right].$$

Define $k_0 \overset{def}{=} \sup\{\tilde{p}_0 : \tau(\tilde{p}_0) = 0\}$. Consider $\hat{k}_0 < k_0$. Define

$$\varepsilon = \frac{1}{\lambda}\left[\log\left(\frac{k_0}{1-k_0}\right) - \log\left(\frac{\hat{k}_0}{1-\hat{k}_0}\right)\right].$$

Then

$$\begin{aligned}
&\mathrm{E}_{\hat{k}_0}\left[\int_0^{\hat{\tau}} e^{-rt}\left[A\Big(\hat{Q}(\hat{k}_0,t)\Big) - w_R\Big(\pi(\hat{k}_0,t),\pi(\hat{k}_0,t)\Big) - rV\right] dt\right] \\
=&\mathrm{E}_{\hat{k}_0}\left[\int_0^{\varepsilon\wedge\hat{\tau}} e^{-rt}\left[A\Big(\hat{Q}(\hat{k}_0,t)\Big) - w_R\Big(\pi(\hat{k}_0,t),\pi(\hat{k}_0,t)\Big) - rV\right] dt\right] \\
+&\mathrm{E}_{\hat{k}_0}\left[\int_{\varepsilon\wedge\hat{\tau}}^{\hat{\tau}} e^{-rt}\left[A\Big(\hat{Q}(\hat{k}_0,t)\Big) - w_R\Big(\pi(\hat{k}_0,t),\pi(\hat{k}_0,t)\Big) - rV\right] dt\right] < V \\
\le&\frac{\hat{k}_0}{k_0}\cdot \mathrm{E}_{k_0}\left[\int_{\varepsilon}^{\tau+\varepsilon} e^{-rt}\left[A\Big(\hat{Q}(\hat{k}_0,t)\Big) - w_R\Big(\pi(\hat{k}_0,t),\pi(\hat{k}_0,t)\Big) - rV\right] dt\right] \\
<&\mathrm{E}_{k_0}\left[\int_0^{\tau} e^{-rt}\left[A\Big(\hat{Q}(\hat{k}_0,t)\Big) - w_R\Big(\pi(\hat{k}_0,t),\pi(\hat{k}_0,t)\Big) - rV\right] dt\right] = V.
\end{aligned}$$

This implies that for any $\tilde{p}_0 < k_0$ the intermediary does not find it profitable to hire the agent. □

Proof of Lemma 3 (equilibrium uniqueness)

Lemma A.13. *Suppose that $V > 0$ and $\underline{p} > 0$. Then, in any equilibrium, there exists time $\bar{T} \in \mathbb{R}_+$ such that $\mathrm{P}\big(\tau \le \bar{T}\big) = 1$.*

Proof. Along the path of good performance there exists a finite threshold T such that for any initial type $\tilde{p}_0$ the profit of the intermediary is

$$\begin{aligned}
A\Big(\hat{Q}(l_t,t)\Big) - w_R\Big(\pi(\tilde{p}_0,t),\pi(l_t,t)\Big) &\le A\Big(\hat{Q}(1,t)\Big) - w_R\Big(\pi(\tilde{p}_0,t),\pi(l_t,t)\Big) \\
&\le A\Big(\hat{Q}(1,t)\Big) - w_R\Big(\pi(1,t),\pi(l_t,t)\Big) < rV
\end{aligned}$$

for every t $\ge \bar{T}$ and $l_t \in [\underline{p}, 1]$ since beliefs about the any agent in the support converge to 1 for a sufficiently long track record of good performance. This implies that the intermediary prefers to replace every agent by time $\bar{T}$. □

Suppose there exists a different equilibrium and $(k_t)_{t\ge 0}$ is the clients' belief process about the type of the agent that may leave the intermediary at time t. As before, process $l_t = \pi(k_t, -t)$ denotes the initial type of the departing agent. As we focus on pure-strategy equilibria, it is without loss to identify the intermediary's strategy by the deterministic time she fires the agent along the path of good performance.

Lemma A.14. *Suppose the intermediary finds it* strictly *optimal to fire the agent of skill $\tilde{p}_0$ at time $T \le \bar{T}$. Then the intermediary also* strictly *prefers to fire every agent $\tilde{p}_0 < \tilde{p}_0'$ by time T.*

Proof. Strict optimality of the intermediary's stopping decision implies that for every stopping time $\hat{\tau} > 0$

$$\begin{aligned}
V &> \mathrm{E}_{\tilde{p}_0}\left[\int_t^{\hat{\tau}} e^{-r(s-t)}\left[A\Big(\hat{Q}(l_s,s)\Big)ds - d\tilde{C}_s^{\tilde{p}_0} - rV\right] + V\right] \\
&\overset{(i)}{\geq} \mathrm{E}_{\tilde{p}_0}\left[\int_t^{\hat{\tau}} e^{-r(s-t)}\left[A\Big(\hat{Q}(l_s,s)\Big)ds - d\tilde{C}_s^{\tilde{p}_0'} - rV\right] + V\right] \\
&\overset{(ii)}{\geq} \mathrm{E}_{\tilde{p}_0'}\left[\int_t^{\hat{\tau}} e^{-r(s-t)}\left[A\Big(\hat{Q}(l_s,s)\Big)ds - d\tilde{C}_s^{\tilde{p}_0'} - rV\right] + V\right],
\end{aligned}$$

where (i) holds due to Lemma A.10 and (ii) holds as the intermediary's expected payoff is weakly increasing in $\tilde{p}_0$ given identical cash flows due to the possibility of randomized termination. $\square$

Lemma A.15. *Suppose the intermediary is indifferent between letting the agent of skill $\tilde{p}_0$ go at time T and time $T' > T$. Then,*

- *either the intermediary strictly prefers to let go of all agents $\tilde{p}_0' < \tilde{p}_0$ time T;*
- *or the type of the cutoff agent does not change between T and T', i.e., $l_T = l_{T'}$, the profit wedge of the intermediary is constant for $t \in [T, T']$*

$$A\Big(\hat{Q}(l_t,t)\Big) - w_R\Big(\pi(\tilde{p}_0,t),\pi(l_t,t)\Big) = rV$$

  and, as a result, the intermediary is indifferent between letting the agent of skill $\tilde{p}_0'$ go at time T and at time $T' > T$.

Proof. Follows from the uniform ranking of compensation processes if l_t increases between T and T'. $\square$

Denote by z_t the lowest skilled agent still employed by the intermediary at time t. This is known as the cutoff type. While we've characterized the equilibrium using the cutoff type, it is not necessarily the case that $z_t = l_t$.

Lemma A.16. *Denote by $T(\tilde{p}_0)$ the equilibrium time when the agent of ex-ante skill $\tilde{p}_0$ leaves the intermediary. Then, in equilibrium, it must be the case that $T(\tilde{p}_0)$ is weakly increasing in $\tilde{p}_0$.*

Proof. Define

$$\underline{T} = \inf_{\tilde{p}_0} T(\tilde{p}_0).$$

Suppose the contrary and for $\tilde{p}_0 > l_0^c$ it is the case that $\underline{T} = T(\tilde{p}_0) < T(l_0^c)$. Define

$$\bar{T}_1 = \sup_{\hat{p}_0 \in [l_0^c, \tilde{p}_0]} T(\hat{p}_0).$$

Lemma A.15 implies that it is only weakly optimal for the intermediary employing agent $\tilde{p}_0$ to let him go until $\bar{T}_1$. It implies that

$$l_t = l_{T(\tilde{p}_0)} \qquad and \qquad A\Big(\hat{Q}(l_t,t)\Big) - w_R\Big(\pi(\tilde{p}_0,t),\pi(l_t,t)\Big) = rV$$

for $t \in [T(\tilde{p}_0), \bar{T}_1]$. For $t > \bar{T}$ no type of agent $[l_0^c, \tilde{p}_0]$ remains with the intermediary. If $\mathrm{P}(\tau > T) = 0$, then the independence of never a weak best response implies that $l_T = 1$. If $\mathrm{P}(\tau > T) > 0$, then belief consistency requirement implies that $l_{\bar{T}} \geq \tilde{p}_0$. Lemma A.15 implies that $l_t \geq \tilde{p}_0$ for every $t \in [T(\tilde{p}_0), \bar{T}]$.

Define $\bar{l}(T) = sup\{p_0 : \quad T(\tilde{p}_0) \leq T\}$. Then, define

$$\bar{T}_2 = \sup_{\hat{p}_0 \in [l_0^c, \bar{l}(\bar{T}_1)]} T(\hat{p}_0).$$

By construction, $\bar{T}_2 \geq \bar{T}_1$. Continuing this construction by induction define $\bar{T} = \lim_{n\to\infty} \bar{T}_n$. Then for $t \in [T(\tilde{p}_0), \bar{T}]$ it must be the case that

$$l_t = l_{T(\tilde{p}_0)} \qquad and \qquad A\Big(\hat{Q}(l_t, t)\Big) - w_R\Big(\pi(\tilde{p}_0, t), \pi(l_t, t)\Big) = rV$$

Moreover, belief consistency requires that $l_{\bar{T}} > \bar{l}(\bar{T}) \geq \tilde{p}_0$. Suppose that $\bar{l}(\bar{T}) = \tilde{p}_0$. Then this is a violation of belief consistency of clients. If $\bar{l}(\bar{T}) > \tilde{p}_0$, then it implies that there is a positive belief jump occurring at time $\bar{T}$. The global indifference of the intermediary implies that $\tilde{p}_0$ type is better of waiting until $\bar{T}$ and separating then. □

Lemma A.16 implies that there exists a weakly increasing process $z_t \in [\underline{p}, \overline{p}]$ such that the set of types retained by time t are given by $\{\tilde{p}_0 \geq z_t\}$.

Lemma A.17. *[Uniqueness] Suppose $V > 0$. Then process $(k_t)_{t\geq 0}$, and the corresponding k_0, pin down the unique pure-strategy equilibrium.*

Proof. The equilibrium monotonicity requirement implies that $(l_t)_{t\geq 0}$ is weakly increasing in t. A monotone process can be decomposed into the unique sum of an absolutely continuous (l_t^c), a discrete (l_t^d, and a continuous singular (l_t^s) weakly increasing processes

$$l_t = l_t^c + l_t^s + l_t^d. \tag{A.23}$$

The optimal termination time of the intermediary solves

$$\sup_{\hat{\tau}} \mathrm{E}_{p_0}\left[\int_0^{\hat{\tau}} e^{-rt}\Big[A\Big(\hat{Q}(l_t, t)\Big)\,dt - rV\,dt - d\tilde{C}_t\Big]\right]. \tag{A.24}$$

On-path dynamics. The necessary optimality condition to let go of the agent of initial skill p_0 is given by

$$A\Big(\hat{Q}(l_t, t)\Big)\,dt - rV\,dt - d\tilde{C}_t \leq 0 \qquad \forall t \in support(\tau). \tag{A.25}$$

Substituting $d\tilde{C}_t$ from (A.21) into (A.25) it is easy to see that if process l_t has a jump at time t, i.e., $dl_t^d > 0$, then it is sub-optimal to let go of the agent in some neighborhood $(t-\varepsilon, t)$. Similarly, if $dl_t^s > 0$, then (A.25) is not satisfied at time t, implying that it is sub-optimal to let go of the agent then. This disciplines process $(l_t)_{t\geq 0}$ at times $t \in support(\tau)$.

Off-path dynamics. Lemma A.16 shows that the equilibrium features cutoff strategies. Denoting by $(z_t)_{t\geq 0}$ to be the cutoff type, it implies that if agents of skill z_t is let go, then all agents with skill $p_0 < z_t$ is let go as well by time t. As process z_t is determined on- and off-path, if z_t increases discontinuously, then it implies that there is a positive mass of agents leaving the intermediary. Belief consistency implies that $l_{t-} < l_t$, implying an on-path jump in process

l_t, leading to a contradiction with the optimal stopping condition (A.24). This implies that process z_t cannot have jumps in equilibrium, leading to a continuous process of beliefs $(l_t)_{t\geq 0}$.

If process $(z_t)_{t\geq 0}$ does not have jumps, it implies that the equilibrium is separating. Moreover, it implies that along the equilibrium path $z_t = l_t$ for every $t \in \textit{support}(\tau)$, or, in other words, $\tilde{p}_\tau = k_\tau$. The fact that the equilibrium is separating and that process $(l_t)_{t\geq 0}$ is continuous implies that $\tilde{p}_\tau = k_\tau = \pi(l_\tau, \tau)$. The total mass of ex-ante types that separate from the intermediary is given by

$$\begin{aligned}
\int_0^\infty \mathbb{1}\{t \in \tau\}\, dF(l_t) &= \int_0^\infty \int t \in \tau dF(l_t^c + l_t^s) \\
&= \int_0^\infty \mathbb{1}\{t \in \tau\}\, f(l_t^c + l_t^s)\, dl_t^c + \int_0^\infty \mathbb{1}\{t \in \tau\}\, f(l_t^c + l_t^s)\, dl_t^s \\
&= \int_0^\infty \mathbb{1}\{t \in \tau\}\, f(l_t^c + l_t^s)\, dl_t^c \leq \int_0^\infty f(l_t^c + l_t^s)\, dl_t^c \\
&\overset{(i)}{\leq} \int_0^\infty f(l_t^c + l_t^s)\, d(l_t^c + l_t^s) = 1.
\end{aligned}$$

where inequality (i) is strict if $dl_t^s > 0$ for any t such that $f(l_t) > 0$. Intuitively, since no separation occurs when $dl_t^s > 0$ stemming from the optimality of the stopping condition, and implies that in the separating equilibrium there is a positive mass of types that do not leave the intermediary, leading to a contradiction. □

Proof of Lemma 4 (equilibrium value V)

The intermediary's equilibrium expected value satisfies

$$V = \max\left[e^{-r\Delta} \cdot \mathrm{E}\left[\int_0^\tau e^{-rt}\Big(A(Q(k_t,t)) - w(\tilde{p}_t, k_t) - rV\Big)\, dt + V\right] - I, \quad 0\right]. \tag{A.26}$$

Similar to earlier, denote by $G(V)$ the incremental value of the intermediary of retaining the current agent

$$G(V) \overset{def}{=} \mathrm{E}\left[\int_0^\tau e^{-rt}\Big(A(Q(k_t,t)) - w(\tilde{p}_t, k_t) - rV\Big)\, dt\right] > 0,$$

where equilibrium processes k and q, as well as the stopping time τ depend on V.

Lemma A.18. *The intermediary's expected payoff $G(V) + V$ is weakly increasing in V.*

Proof. Hold V fixed. Then k_t^V is increasing in V point-wise. Can prove this by contradiction. Need to derive it from the general nature of the stopping rule. Hence q_t^V is increasing in V pointwise. □

Rewrite (A.26) as

$$V = \max\left[e^{-r\Delta} \cdot (G(V) + V) - I, \quad 0\right].$$

The value of the continuation is

$$\begin{cases} V_n = e^{-r\Delta} \cdot (G(V_{n-1}) + V_{n-1}) - I, \\ V_{n+1} = e^{-r\Delta} \cdot (G(V_n) + V_n) - I. \end{cases} \tag{A.27}$$

The difference is

$$V_{n+1} - V_n = e^{-r\Delta} \cdot (G(V_n) + V_n - G(V_{n-1}) - V_{n-1}) \geq 0.$$

This implies the sequence $(V_n)_{n\in\mathbb{N}}$ is increasing in n and there exists a unique limit

$$V^* = \lim_{n\to\infty} V_n.$$

Taking limits of both sides of (A.27) and noting that $G(V)$ is a bounded, continuous function of V we obtain

$$V^* = e^{-r\Delta} \cdot G(V^*) + e^{-r\Delta} \cdot V^* - I.$$

Lemma A.19. *Employment gain $G(V)$ satisfies $G'(0) = -\left[q_0 + (1-q_0)\frac{r}{r+\lambda}\right]$.*

Proof. For $V = 0$ the optimal stopping time $\tau = \inf\{X_t < \mu t\}$. By Envelope theorem with respect to stopping time τ around $V = 0$ we have

$$\begin{aligned} G'(0) &= \frac{d}{dV}\mathrm{E}\left[\int_0^\tau e^{-rt}\Big(A(q_t) - w(\tilde{p}_t, k_t) - rV\Big)\,dt\right]\Bigg|_{V=0} \\ &= -\int_0^\infty \left(q_0 e^{-rt} + (1-q_0)e^{-(r+\lambda)t}\right) r\,dt = -\frac{r+\lambda q_0}{r+\lambda} < 0. \end{aligned}$$

□

Proof of Lemma 5 (comparative statics)

First, we show that if I or Δ are sufficiently large, then V is increasing the initial quality of the average agent $\mathrm{E}\left[\tilde{p}_0\right]$ and decreasing in the skill of the worst agent initially employed $\underline{p}$. The fixed point equation is given by

$$V = \max\left[e^{-r\Delta} \cdot \mathrm{E}\left[\int_0^\tau e^{-rt}\Big[A(Q(k_t,t)) - w(\tilde{p}_t, k_t) - rV\Big]\,dt + V\right] - I, \quad 0\right].$$

If we have a corner solution, then the continuation value of the intermediary is 0. This implies her expected profit is to employ the agent until he generates a bad return leading to

$$\mathrm{E}\left[\int_0^\infty \left[A\Big(\hat{Q}(\tilde{p}_0, t)\Big) - w_R\Big(\pi(\tilde{p}_0, t), \pi(k_0, t)\Big)\right]\,dt\right].$$

This expression is increasing in $\mathrm{E}\left[\tilde{p}_0\right]$ and weakly decreasing in $\underline{p}$ as the intermediary decides which types to let go. Suppose that $V > 0$. Then the fixed point equation (A.26) rewritten above can be expressed as

$$\begin{aligned} Ie^{r\Delta} + \left(e^{r\Delta} - 1\right)V &= \mathrm{E}\left[\int_0^\tau e^{-rt}\Big[A\Big(Q(k_t,t)\Big) - w\Big(\tilde{p}_t, k_t\Big) - rV\Big]\,dt\right] \\ &= \mathrm{E}\left[\int_0^\tau e^{-rt}\Big[A\Big(Q(k_t,t)\Big) - w\Big(\tilde{p}_t, k_t\Big) - rV\Big] \cdot \mathbb{1}\left\{t \notin \mathbb{T}\right\}\,dt\right] \\ &\quad + \mathrm{E}\left[\int_0^\tau e^{-rt}\Big[rV + w\Big(\tilde{p}_t, k_t\Big) - A\Big(Q(k_t,t)\Big)\Big]\frac{\partial_2 U(\tilde{p}_t, k_t) - \partial_2 U(k_t, k_t)}{\partial_2 U(k_t, k_t)}\mathbb{1}\left\{t \in \mathbb{T}\right\}\,dt\right] \end{aligned} \tag{A.28}$$

$$
\begin{aligned}
&= \mathrm{E}\left[\int_0^{t^*\wedge\eta} e^{-rt}\Big[A\Big((\pi(q_0,t)\Big) - w\Big(\tilde{p}_t, \pi(k_0,t)\Big) - rV\Big]\, dt\right] \\
&+ \mathrm{E}\left[\int_{t^*\wedge\eta}^{\tau} e^{-rt}\Big[A\Big(Q(k_t,t)\Big) - w\Big(\tilde{p}_t, k_t\Big) - rV\Big] \cdot \mathbb{1}\left\{t \notin \mathbb{T}\right\}\, dt\right] \\
&+ \mathrm{E}\left[\int_{t^*\wedge\eta}^{\tau} e^{-rt}\Big[rV + w\Big(\tilde{p}_t, k_t\Big) - A\Big(Q(k_t,t)\Big)\Big] \frac{\partial_2 U(\tilde{p}_t,k_t) - \partial_2 U(k_t,k_t)}{\partial_2 U(k_t,k_t)} \mathbb{1}\left\{t \in \mathbb{T}\right\}\, dt\right]
\end{aligned}
$$

First, we show that the right hand side of (A.28) is negative if V is small

$$
\begin{aligned}
&- r\mathrm{E}\left[\int_0^{t^*\wedge\eta} e^{-rt}\, dt\right] + \mathrm{E}\left[\int_0^{t^*\wedge\eta} e^{-rt}\, dt\right]\Big[\partial_1 A\Big((\pi(q_0,t)\Big)\partial_1\pi(q_0,t) - w\Big(\tilde{p}_t, \pi(k_0,t)\Big) - rV\Big] \\
&+ \mathrm{E}\left[\int_{t^*\wedge\eta}^{\tau} e^{-rt}\Big[\partial_1 A\Big(Q(k_t,t)\Big)\partial_1 Q(k_t,t)\partial_V k_t - \partial_2 w\Big(\tilde{p}_t, k_t\Big)\partial_V k_t - r\Big] \cdot \mathbb{1}\left\{t \notin \mathbb{T}\right\}\, dt\right] \\
&+ \mathrm{E}\left[\int_{t^*\wedge\eta}^{\tau} e^{-rt}\Big[r + \partial_2 w\Big(\tilde{p}_t, k_t\Big)\partial_V k_t - \partial_1 A\Big(Q(k_t,t)\Big)\partial_1 Q(k_t,t)\partial_V k_t\Big] \frac{\partial_2 U(\tilde{p}_t,k_t) - \partial_2 U(k_t,k_t)}{\partial_2 U(k_t,k_t)} \mathbb{1}\left\{t \in \mathbb{T}\right\}\, dt\right] \\
&+ \mathrm{E}\left[\int_{t^*\wedge\eta}^{\tau} e^{-rt}\Big[rV + w\Big(\tilde{p}_t, k_t\Big) - A\Big(Q(k_t,t)\Big)\Big] \cdot \partial_V\left[\frac{\partial_2 U(\tilde{p}_t,k_t) - \partial_2 U(k_t,k_t)}{\partial_2 U(k_t,k_t)}\right] \cdot \mathbb{1}\left\{t \in \mathbb{T}\right\}\, dt\right]
\end{aligned}
$$

Note that if V is sufficiently small, then t^* is very large. In this case the only term is $-r\int_0^{t^*\wedge\eta} e^{-rt}\, dt$ of higher order than $o(V)$. This implies that for V sufficiently low, the derivative of the right hand side of (A.28) is negative, while the derivative of the left hand side is clearly positive. This implies that the objective is decreasing in V.
Consider a Fréchét derivative as the distribution of private information $F \to \hat{F}$ such that $k_0 = \hat{k}_0$ and $q_0 < \hat{q}_0$. Such a change affects all of the endogenous belief process dynamics. Then, by the similar argument as before,[53]

$$
\begin{aligned}
&\mathrm{E}\left[\int_0^{\tau} e^{-rt}\Big[A\Big(Q(k_t,t)\Big) - w\Big(\tilde{p}_t, k_t\Big) - rV\Big]\, dt\right] \\
-&\mathrm{E}\left[\int_0^{\tau} e^{-rt}\Big[A\Big(Q(k_t,t)\Big) - w\Big(\tilde{p}_t, k_t\Big) - rV\Big]\, dt\right] \\
\approx&\mathrm{E}\left[\int_0^{t^*\wedge\eta} e^{-rt}\Big[A\Big((\pi(q_0,t)\Big) - w\Big(\tilde{p}_t, \pi(k_0,t)\Big) - rV\Big]\, dt\right] \\
-&\mathrm{E}\left[\int_0^{t^*\wedge\eta} e^{-rt}\Big[A\Big((\pi(q_0,t)\Big) - w\Big(\tilde{p}_t, \pi(k_0,t)\Big) - rV\Big]\, dt\right] \\
=&\mathrm{E}\left[\int_0^{t^*\wedge\eta} e^{-rt}\Big[A\Big((\pi(\hat{q}_0,t)\Big) - w\Big(\tilde{p}_t, \pi(k_0,t)\Big) - rV\Big]\, dt\right] \\
-&\mathrm{E}\left[\int_0^{t^*\wedge\eta} e^{-rt}\Big[A\Big((\pi(q_0,t)\Big) - w\Big(\tilde{p}_t, \pi(k_0,t)\Big) - rV\Big]\, dt\right] \\
=&\mathrm{E}\left[\int_0^{t^*\wedge\eta} e^{-rt}\Big[A\Big((\pi(\hat{q}_0,t)\Big) - A\Big((\pi(q_0,t)\Big)\Big]\, dt\right] > 0.
\end{aligned}
$$

This implies that if the Fréchét derivative in the direction of $\hat{F}$ is positive. This implies that the right hand side of

[53] All $\approx$ signs in the proof stand in for an equality with a lower order $\overline{o}(V)$ term in V.

(A.28) is increasing in the mean of $\mathrm{E}\left[\tilde{p}_0\right]$, implying that a greater V is necessary to satisfy (A.28). Similar argument carries over for a decrease in k_0.

Note that

$$\begin{aligned}
\pi(\tilde{p}_0,t) &= \frac{\tilde{p}_0}{\tilde{p}_0+(1-\tilde{p}_0)e^{-\lambda t}}\\
\partial_2\pi(\tilde{p}_0,t) &= \frac{\lambda\tilde{p}_0(1-\tilde{p}_0)e^{-\lambda t}}{\tilde{p}_0+(1-\tilde{p}_0)e^{-\lambda t}}\\
\frac{\partial}{\partial\lambda}\pi(\tilde{p}_0,t) &= \frac{t\tilde{p}_0(1-\tilde{p}_0)e^{-\lambda t}}{\tilde{p}_0+(1-\tilde{p}_0)e^{-\lambda t}} = \frac{t}{\lambda}\partial_2(\tilde{p}_0,t).
\end{aligned}$$

If t^* is sufficiently high, then the derivative of the right hand side of (A.28) with respect to λ is approximately

$$\begin{aligned}
&\frac{\partial}{\partial\lambda}\mathrm{E}\left[\int_0^\tau e^{-rt}\Big[A\Big(Q(k_t,t)\Big)-w\Big(\tilde{p}_t,k_t\Big)-rV\Big]\,dt\right]\\
\approx&\frac{\partial}{\partial\lambda}\mathrm{E}\left[\int_0^{t^*\wedge\eta} e^{-rt}\Big[A\Big((\pi(q_0,t)\Big)-w\Big(\tilde{p}_t,\pi(k_0,t)\Big)-rV\Big]\,dt\right]\\
=&\mathrm{E}\left[\int_0^{t^*\wedge\eta} e^{-rt}\frac{\partial}{\partial\lambda}\Big[A\Big((\pi(q_0,t)\Big)-w\Big(\tilde{p}_t,\pi(k_0,t)\Big)-rV\Big]\,dt\right]0\\
=&\mathrm{E}\left[\int_0^{t^*\wedge\eta} e^{-rt}\frac{t}{\lambda}\frac{\partial}{\partial t}\Big[A\Big((\pi(q_0,t)\Big)-w\Big(\tilde{p}_t,\pi(k_0,t)\Big)-rV\Big]\,dt\right]\\
=&\mathrm{E}\left[\int_0^{t^*\wedge\eta} e^{-rt}\frac{t}{\lambda}\,d\Big[A\Big((\pi(q_0,t)\Big)-w\Big(\tilde{p}_t,\pi(k_0,t)\Big)-rV\Big]\right]\\
\approx&-\mathrm{E}\left[\int_0^{t^*\wedge\eta}\Big[A\Big((\pi(q_0,t)\Big)-w\Big(\tilde{p}_t,\pi(k_0,t)\Big)-rV\Big]d\left(e^{-rt}\frac{t}{\lambda}\right)\right]\\
=&\frac{1}{\lambda}\mathrm{E}\left[\int_0^{t^*\wedge\eta} e^{-rt}\Big[A\Big((\pi(q_0,t)\Big)-w\Big(\tilde{p}_t,\pi(k_0,t)\Big)-rV\Big]\,(rt-1)\;dt\right]. \qquad \text{(A.29)}\\
=&\underbrace{\frac{1}{\lambda}\mathrm{E}\left[\int_0^{t^*\wedge 1/r\wedge\eta} e^{-rt}\Big[A\Big((\pi(q_0,t)\Big)-w\Big(\tilde{p}_t,\pi(k_0,t)\Big)-rV\Big]\,(rt-1)\;dt\right]}_{(i)}\\
&\underbrace{+\frac{1}{\lambda}\mathrm{E}\left[\int_{t^*\wedge 1/r\wedge\eta}^{t^*\wedge\eta} e^{-rt}\Big[A\Big((\pi(q_0,t)\Big)-w\Big(\tilde{p}_t,\pi(k_0,t)\Big)-rV\Big]\,(rt-1)\;dt\right]}_{(ii)}.
\end{aligned}$$

If r is sufficiently low of λ is sufficiently high, then (A.29) is negative as the negative term in (i) is significantly greater than the negative term in (ii) as in that case the profit wedge of the intermediary is substantively lower.

Proof of Lemma 6 (imperfect competition among intermediaries)

As before, denote the intermediary's expected value of employing type $\tilde{p}$ while the belief is k as

$$V(\tilde{p},k) = \mathrm{E}_{\tilde{p}}\left[\int_0^{\tau} e^{-rt}(A(q_t) - w(\tilde{p}_t,k_t))\,dt + e^{-r\tau}\cdot V\right]. \tag{A.30}$$

Then upon finding multiple potential employers, the agent obtains his equilibrium value $U(\tilde{p},k)$, but also extracts the value added from the family, $V(\tilde{p},k) - V$, in the form of the sign-on bonus. This implies that, in equilibrium, the agent's expected value in each period must satisfy

$$U^{\zeta}(\tilde{p}_t,k_t) = \zeta_0 \cdot U(\tilde{p}_t,k_t) + \zeta_1 \cdot U^{\zeta}(\tilde{p}_t,k_t) + \zeta_2 \cdot \big(U^{\zeta}(\tilde{p}_t,k_t) + V(\tilde{p}_t,k_t) - V\big). \tag{A.31}$$

since the current employer makes take-it-or-leave-it offers to match the agent's outside option. As before, denote $\zeta = \frac{\zeta_2}{1-\zeta_1}$ to be the probability of the agent meeting multiple firms conditional on not meeting just one. We can rewrite the relation between the agent's and the intermediary's equilibrium expected values as

$$\begin{aligned}(1-\zeta_1)\cdot U^{\zeta}(\tilde{p}_t,k_t) - \zeta_0 \cdot U(\tilde{p}_t,k_t) &= \zeta_2 \cdot \big(U^{\zeta}(\tilde{p}_t,k_t) + V(\tilde{p}_t,k_t) - V\big),\\ (1-\zeta)\cdot\Big(U^{\zeta}(\tilde{p}_t,k_t) - U(\tilde{p}_t,k_t)\Big) &= \zeta\cdot\Big(V(\tilde{p}_t,k_t) - V\Big).\end{aligned} \tag{A.32}$$

Lemma A.20. *The necessary retention wage for the agent is*

$$\tilde{w}_{\zeta}(\tilde{p}_t,k_t) \stackrel{def}{=} \zeta\Big((\rho - r)V(\tilde{p}_t,k_t) + A(Q(k_t,t)) - \rho V\Big) + (1-\zeta)\Big(w_R(\tilde{p}_t,k_t) - \gamma(k_t,t)\partial_2 U(\tilde{p}_t,k_t)\Big). \tag{A.33}$$

Proof. Given belief processes $(k_t,q_t)_{t\geq 0}$ the value of the intermediary from employing the agent of skill $\tilde{p}_t$ is

$$\begin{aligned} rV(\tilde{p}_t,k_t) = A(Q(k_t,t)) - \tilde{w}_t^{\zeta} + \big(\lambda k_t(1-k_t) + \gamma(k_t,t)\big)\cdot \partial_2 V(\tilde{p}_t,k_t)\\ +\lambda\tilde{p}_t(1-\tilde{p}_t)\cdot\partial_1 V(\tilde{p}_t,k_t) + \lambda(1-\tilde{p}_t)\big(V - V(\tilde{p}_t,k_t)\big).\end{aligned} \tag{A.34}$$

Given wage process $\tilde{w}_t$ the equilibrium payoff of the agent is given by $U_1(\tilde{p}_t,k_t)$

$$\begin{aligned}\rho U^{\zeta}(\tilde{p}_t,k_t) = \tilde{w}_t^{\zeta} + \big(\lambda k_t(1-k_t) + \gamma_t\big)\cdot\partial_2 U^{\zeta}(\tilde{p}_t,k_t)\\ + \lambda\tilde{p}_t(1-\tilde{p}_t)\cdot\partial_1 U^{\zeta}(\tilde{p}_t,k_t) + \lambda(1-\tilde{p}_t)\big(L - U^{\zeta}(\tilde{p}_t,k_t)\big).\end{aligned} \tag{A.35}$$

The payoff to the agent from opening his own firm is given by $U(\tilde{p}_t,k_t)$ and is described by Lemma 1. Together with (A.35) this implies

$$\begin{aligned}\rho\Big(U^{\zeta}(\tilde{p}_t,k_t) - U(\tilde{p}_t,k_t)\Big) = \tilde{w}_t^{\zeta} - w_R(\tilde{p}_t,k_t) + \big(\lambda k_t(1-k_t) + \gamma(k_t,t)\big)\Big(\partial_2 U^{\zeta}(\tilde{p}_t,k_t) - \partial_2 U(\tilde{p}_t,k_t)\Big)\\ + \gamma(k_t,t)\partial_2 U(\tilde{p}_t,k_t) + \lambda\tilde{p}_t(1-\tilde{p}_t)\Big(\partial_1 U^{\zeta}(\tilde{p}_t,k_t) - \partial_1 U(\tilde{p}_t,k_t)\Big)\\ - \lambda(1-\tilde{p}_t)\Big(U^{\zeta}(\tilde{p}_t,k_t) - U(\tilde{p}_t,k_t)\Big).\end{aligned}$$

Substituting (A.32) into the equation above obtain

$$\begin{aligned}\frac{\zeta}{1-\zeta}\cdot\rho\Big(V(\tilde{p}_t,k_t)-V\Big) &= \tilde{w}_t^{\zeta}-w_R(\tilde{p}_t,k_t)+\frac{\zeta}{1-\zeta}\big(\lambda k_t(1-k_t)+\gamma(k_t,t)\big)\cdot\partial_2 V(\tilde{p}_t,k_t)\\ &\quad+\gamma(k_t,t)\cdot\partial_2 U(\tilde{p}_t,k_t)+\frac{\zeta}{1-\zeta}\lambda\tilde{p}_t(1-\tilde{p}_t)\cdot\partial_1 V(\tilde{p}_t,k_t)\\ &\quad+\frac{\zeta}{1-\zeta}\lambda(1-\tilde{p}_t)\big(V-V(\tilde{p}_t,k_t)\big).\end{aligned}\tag{A.36}$$

Multiply (A.34) by $\frac{\zeta}{1-\zeta}$ and subtract from (A.36)

$$\begin{aligned}\frac{\zeta}{1-\zeta}\cdot\Big(\rho\big(V(\tilde{p}_t,k_t)-V\big)-rV(\tilde{p}_t,k_t)\Big) &= \tilde{w}_t^{\zeta}-w_R(\tilde{p}_t,k_t)+\gamma(k_t,t)\cdot\partial_2 U(\tilde{p}_t,k_t)\\ &\quad-\frac{\zeta}{1-\zeta}\left(A(Q(k_t,t))-\tilde{w}_t^{\zeta}\right).\end{aligned}$$

Simplifying terms obtain

$$\begin{aligned}\frac{\zeta}{1-\zeta}\Big((\rho-r)V(\tilde{p}_t,k_t)-\rho V\Big) &= \frac{1}{1-\zeta}\tilde{w}_t^{\zeta}-w_R(\tilde{p}_t,k_t)+\gamma(k_t,t)\cdot\partial_2 U(\tilde{p}_t,k_t)-\frac{\zeta}{1-\zeta}A(Q(k_t,t))\\ \zeta\Big((\rho-r)V(\tilde{p}_t,k_t)-\rho V\Big) &= \tilde{w}_t^{\zeta}-(1-\zeta)\Big(w_R(\tilde{p}_t,k_t)+\gamma(k_t,t)\cdot\partial_2 U(\tilde{p}_t,k_t)\Big)-\zeta A(Q(k_t,t)).\end{aligned}$$

This results in

$$\tilde{w}_t^{\zeta}=\zeta\cdot\Big((\rho-r)V(\tilde{p}_t,k_t)+A(Q(k_t,t))-\rho V\Big)+(1-\zeta)\cdot\Big(w_R(\tilde{p}_t,k_t)-\gamma(k_t,t)\cdot\partial_2 U(\tilde{p}_t,k_t)\Big).\tag{A.37}$$

The wage is the convex combination of the profit flow of the family and the opportunity cost of the agent in staying with the mutual fund. □

The intermediary's profit wedge is equal to her opportunity cost when she lets go of the agent at time τ. This implies

$$\begin{aligned}A(Q(k_\tau,\tau))-w^{\zeta}(k_\tau,k_\tau) &= rV,\\ A(Q(k_\tau,\tau))-\zeta\Big(A(Q(k_\tau,\tau))-rV\Big)-(1-\zeta)\Big(w_R(\tilde{p}_t,k_t)-\gamma(k_t,t)\partial_2 U(\tilde{p}_t,k_t)\Big) &= rV,\\ A(Q(k_\tau,\tau))-rV-w_R(\tilde{p}_\tau,k_\tau)+\gamma(k_t,t)\partial_2 U(\tilde{p}_\tau,k_\tau) &= 0.\end{aligned}\tag{A.38}$$

The equilibrium churning rate $\gamma(k_t,t)$ is, then, given by

$$\gamma(k_t,t)=\frac{rV+w_R(k_t,k_t)-A(Q(k_t,t))}{\partial_2 U(\tilde{p}_t,k_t)}$$

for $t\in\mathbb{T}$. For a given equilibrium value of V, coincides with the rate of separation obtained in (17). Belief process $k=(k_t)_{t\geq 0}$ does not depend on the agent's bargaining power α given the intermediary's expected value V. Can express

$$\begin{aligned}U^{\zeta}(\tilde{p}_0,k_0) &= \mathrm{E}_{\tilde{p}_0}\left[\int_0^{\tau}e^{-\rho t}\tilde{w}_t^{\zeta}\,dt+e^{-\rho\tau}\cdot U(k_\tau,k_\tau)\right],\\ U(\tilde{p}_0,k_0) &= \mathrm{E}_{\tilde{p}_0}\left[\int_0^{\tau}e^{-\rho t}\tilde{w}_t^{0}\,dt+e^{-\rho\tau}\cdot U(k_\tau,k_\tau)\right].\end{aligned}$$

Subtracting the two obtain

$$U^{\zeta}\left(\tilde{p}_0, k_0\right) - U\left(\tilde{p}_0, k_0\right) = \mathrm{E}_{\tilde{p}_0}\left[\int_0^{\tau} e^{-\rho t}\left(\tilde{w}_t^{\zeta} - \tilde{w}_t^{0}\right) dt\right].$$

Suppose that $\rho = r$. Substituting the above and (A.30) into (A.32) obtain

$$\begin{aligned}(1-\zeta)\cdot \mathrm{E}_{\tilde{p}_0}\left[\int_0^{\tau} e^{-\rho t}\Big(\tilde{w}_t^{\zeta} - \tilde{w}_t^{0}\Big)\, dt\right] &= \zeta \cdot \mathrm{E}_{\tilde{p}_0}\left[\int_0^{\tau} e^{-rt}\Big(A(Q(k_t,t)) - \tilde{w}_t^{\zeta} - rV\Big)\, dt\right],\\ \mathrm{E}_{\tilde{p}_0}\left[\int_0^{\tau} e^{-\rho t}\Big(\tilde{w}_t^{\zeta} - \tilde{w}_t^{0}\Big)\, dt\right] &= \zeta \cdot \mathrm{E}_{\tilde{p}_0}\left[\int_0^{\tau} e^{-rt}\Big(A(Q(k_t,t)) - \tilde{w}_t^{0} - rV\Big)\, dt\right], \end{aligned} \tag{A.39}$$

The intermediary's expected value is pinned down by

$$\begin{aligned} V &= e^{-r\Delta}\mathrm{E}\left[\int_0^{\tau} e^{-rt}\Big(A(Q(k_t,t)) - rV - \tilde{w}_t^{\zeta}\Big)\, dt\right] + e^{-r\Delta}V - I\\ V &= e^{-r\Delta}\cdot(1-\zeta)\cdot \mathrm{E}\left[\int_0^{\tau} e^{-rt}\Big(A(Q(k_t,t)) - rV - \tilde{w}_t^{0}\Big)\, dt\right] + e^{-r\Delta}V - I \end{aligned}$$

Rewrite the above expression as

$$\frac{1}{1-\zeta}\cdot\Big(e^{r\Delta}\cdot I + \left(e^{r\Delta}-1\right)\cdot V^{\zeta}\Big) = \mathrm{E}\left[\int_0^{\tau} e^{-rt}\Big(A(Q(k_t,t) - rV^{\zeta} - \tilde{w}_t^{0}\Big)\, dt\right]. \tag{A.40}$$

As we see, ζ magnifies the replacement cost faced by the intermediary.

Proof of Lemma 7 (signaling outside option $U_{\beta}(p,k)$)

Define $\beta(p,k)$ the fee chosen by the agent of type p, given that the lowest agent in the support of the distribution is of type k. Denote by $U(p,\hat{p},\beta)$ the expected value to the agent who has expected skill p but is perceived as skill $\hat{p}$

$$U(p,\hat{p},\beta) = \mathrm{E}_p\left[\int_0^{\eta} e^{-\rho t}\beta A(\pi(\hat{p},t))\, dt + e^{-\rho\eta}\cdot L\right] = \beta U(p,\hat{p}) + \frac{\lambda}{\rho+\lambda}(1-\beta)(1-p)L.$$

Denote by $\hat{p}(\beta,k)$ the inference made by clients about the agent's ability when they observe discount β and conjecture that the lowest type is k. Because higher skilled agents value reputation more, they may be willing to accept a lower fee in exchange for higher reputation:

$$\begin{aligned}\beta(p,k) &\in \arg\max_{\hat{\beta}}\; U\Big(p,\hat{p}(\hat{\beta},k),\hat{\beta}\Big)\\ &= \arg\max_{\hat{\beta}}\left[\hat{\beta}\cdot U\left(p,\hat{p}(\hat{\beta},k)\right) + \frac{\lambda}{\rho+\lambda}(1-\hat{\beta})(1-p)L\right]. \end{aligned} \tag{A.41}$$

Taking the first order condition of (A.41) it must be the case that $\beta(p,k)$ must satisfy

$$U(p,\hat{p}(\beta(p,k),k)) - \frac{\lambda}{r+\lambda}(1-p)L + \beta(p,k)\cdot\partial_2 U\left(p,\hat{p}(\beta(p,k),k)\right)\cdot\partial_1\hat{p}(\beta(p,k),k) = 0. \tag{A.42}$$

In a separating equilibrium belief consistency requires that $\hat{p}(\beta(p,k),k) = p$. Differentiating this identify with respect to p obtain $\partial_1\hat{p}(\beta(p,k),k))\cdot\partial_1\beta(p,k) = 1$. Substituting into (A.43), obtain a first order differential equation characterizing the equilibrium discount

$$\partial_1\beta(p,k)\cdot\left(U(p,p)-\frac{\lambda}{\rho+\lambda}(1-p)L\right)+\beta(p,k)\cdot\partial_2 U(p,p)=0, \tag{A.43}$$

$$\frac{\partial_1\beta(p,k)}{\beta(p,k)}=-\frac{\partial_2 U(p,p)}{U(p,p)-\frac{\lambda}{\rho+\lambda}(1-p)L}.$$

The lowest type $p=k$ maximizes revenues, implying that $\beta(k,k)=1$, for every k. Solving (A.43) for $\beta(p,k)$ obtain the result of Lemma 7 given by

$$\beta(p,k)=exp\left[-\int_k^p\frac{\partial_2 U(x,x)}{U(x,x)-\frac{\lambda}{\rho+\lambda}(1-p)L}\,dx\right].$$

Proof of Lemma 8 (training incentives)

By direct computation obtain

$$\frac{\partial}{\partial p}\left(\frac{\tilde{p}_t}{\pi(\tilde{p}_s,s-t)}\right)=\frac{\partial}{\partial p}\left(\frac{\tilde{p}}{\frac{\tilde{p}}{\tilde{p}+(1-\tilde{p})e^{-\lambda(s-t)}}}\right)$$

$$=\frac{\partial}{\partial p}\left(\tilde{p}+(1-\tilde{p})e^{-\lambda(s-t)}\right)=1-e^{-\lambda(s-t)}.$$

Applying Envelope theorem with respect to stopping time τ,

$$\frac{\partial}{\partial\tilde{p}_t}V(\tilde{p}_t,k_t)=\frac{\partial}{\partial p}\mathrm{E}_{\tilde{p}}\left[\int_t^\tau e^{-r(s-t)}\cdot\Big[A(q_s)-w_R(\tilde{p}_s,k_s)-rV\Big]\,ds\right]=\frac{\partial}{\partial p}\mathrm{E}_{p=1}\left[\int_t^\tau e^{-r(s-t)}\cdot\Big[A(q_s)-w_R(\tilde{p}_s,k_s)-rV\Big]\frac{\tilde{p}_t}{\tilde{p}_s}\,ds\right]$$

This implies that $\frac{\partial}{\partial\tilde{p}_t}V(k_t,k_t)=0$ since the $\tau=t$.

Proof of Proposition 3 (equilibrium given signaling)

The agent's expected value can be written as

$$\hat{U}(p,k)=\beta(p,k)\cdot\hat{U}(p,p)+\frac{\lambda}{\rho+\lambda}(1-\beta(p,k))(1-p)L.$$

Define the wage necessary for the agent to delay signaling $w(p,k)$ as

$$\rho\hat{U}(p,k)=w_S(p,k)+\lambda p(1-p)\cdot\partial_1\hat{U}(p,k)+\lambda(1-p)(L-\hat{U}(p,k))+(\lambda k(1-k)+\gamma)\cdot\partial_2\hat{U}(p,k).$$

Rearranging terms obtain

$$\begin{aligned}w_S(p,k)&=\rho\hat{U}(p,k)-\lambda p(1-p)\partial_1\hat{U}(p,k)+\lambda(1-p)(\hat{U}(p,k)-L)-(\lambda k(1-k)+\gamma)\partial_2\hat{U}(p,k)\\&=\rho\left(\beta(p,k)U(p,p)+\frac{\lambda}{\rho+\lambda}(1-\beta(p,k))(1-p)L\right)\\&-\lambda p(1-p)\left(\beta(p,k)\partial_1 U(p,p)-\frac{\lambda}{\rho+\lambda}(1-\beta(p,k))L\right)\end{aligned}$$

$$+\lambda(1-p)\left(\beta(p,k)U(p,p)+\frac{\lambda}{\rho+\lambda}(1-\beta(p,k))(1-p)L-L\right)$$
$$-(\lambda k(1-k)+\gamma)\partial_2\beta(p,k)\left(U(p,p)-\frac{\lambda}{\rho+\lambda}(1-p)L\right).$$

Note that

$$\partial_2\beta(p,k)=\beta(p,k)\cdot\frac{\partial_2 U(k,k)}{U(k,k)-\frac{\lambda}{\rho+\lambda}(1-k)L}.$$

Simplifying terms obtain

$$\begin{aligned}w_S(p,k)=&\ \rho\left(\beta(p,k)U(p,p)+\frac{\lambda}{\rho+\lambda}(1-\beta(p,k))(1-p)L\right)\\&-\lambda p(1-p)\left(\beta(p,k)\partial_1 U(p,p)-\frac{\lambda}{\rho+\lambda}(1-\beta(p,k))L\right)\\&+\lambda(1-p)\left(\beta(p,k)U(p,p)+\frac{\lambda}{\rho+\lambda}(1-\beta(p,k))(1-p)L-L\right)\\&-(\lambda k(1-k)+\gamma)\beta(p,k)\left(U(p,p)-\frac{\lambda}{\rho+\lambda}(1-p)L\right)\frac{\partial_2 U(k,k)}{U(k,k)-\frac{\lambda}{\rho+\lambda}(1-k)L}\end{aligned}$$

Rearranging terms obtain

$$\begin{aligned}w_S(p,k)=&\ (\rho+\lambda(1-p))\left(\beta(p,k)U(p,p)+\frac{\lambda}{\rho+\lambda}(1-\beta(p,k))(1-p)L\right)\\&-\lambda p(1-p)\left(\beta(p,k)\partial_1 U(p,p)-\frac{\lambda}{\rho+\lambda}(1-\beta(p,k))L\right)\\&-(\lambda k(1-k)+\gamma)\left(\beta(p,k)U(p,p)+(1-\beta(p,k))\frac{\lambda}{\rho+\lambda}(1-p)L\right)\frac{\partial_2 U(k,k)}{U(k,k)-\frac{\lambda}{\rho+\lambda}(1-k)L}\\&-\lambda(1-p)L+(\lambda k(1-k)+\gamma)\cdot\frac{\lambda}{\rho+\lambda}(1-p)L\cdot\frac{\partial_2 U(k,k)}{U(k,k)-\frac{\lambda}{\rho+\lambda}(1-k)L}\end{aligned}$$

Define

$$\hat{u}_1(p)\overset{def}{=}u_1(p),\qquad \hat{u}_0(p)\overset{def}{=}u_0(p)-\frac{\lambda}{\rho+\lambda}L.$$

Then can rewrite

$$\begin{aligned}w_S(p,k)=&\ (\rho+\lambda(1-p))\left(\beta(p,k)\Big(p\hat{u}_1(p)+(1-p)\hat{u}_0(p)\Big)+\frac{\lambda}{\rho+\lambda}(1-p)L\right)\\&-\lambda p(1-p)\left(\beta(p,k)\Big(\hat{u}_1(p)-\hat{u}_0(p)\Big)-\frac{\lambda}{\rho+\lambda}L\right)\\&-(\lambda k(1-k)+\gamma)\beta(p,k)\Big(p\hat{u}_1(p)+(1-p)\hat{u}_0(p)\Big)\frac{k\hat{u}_1'(k)+(1-k)\hat{u}_0'(k)}{k\hat{u}_1(k)+(1-k)\hat{u}_0(k)}\\&-\lambda(1-p)L\end{aligned}$$

The terms containing L add up to

$$(\rho+\lambda(1-p))\frac{(1-p)\lambda L}{\rho+\lambda}+\lambda p(1-p)\frac{\lambda L}{\rho+\lambda}-\lambda(1-p)L=0.$$

This implies

$$\begin{aligned}w_S(p,k)=&(\rho+\lambda(1-p))\beta(p,k)\Big(p\hat{u}_1(p)+(1-p)\hat{u}_0(p)\Big)-\lambda p(1-p)\beta(p,k)\Big(\hat{u}_1(p)-\hat{u}_0(p)\Big)\\&-(\lambda k(1-k)+\gamma)\beta(p,k)\Big(p\hat{u}_1(p)+(1-p)\hat{u}_0(p)\Big)\frac{k\hat{u}_1'(k)+(1-k)\hat{u}_0'(k)}{k\hat{u}_1(k)+(1-k)\hat{u}_0(k)}\end{aligned}$$

Simplifying terms obtain

$$\begin{aligned}w_S(p,k)=&\rho\beta(p,k)\Big(p\hat{u}_1(p)+(1-p)\hat{u}_0(p)\Big)\\&+\lambda\Big(p(1-p)-p(1-p)\Big)\beta(p,k)\hat{u}_1(p)+\lambda\Big((1-p)^2+p(1-p)\Big)\beta(p,k)\hat{u}_0(p)\\&-(\lambda k(1-k)+\gamma)\beta(p,k)\Big(p\hat{u}_1(p)+(1-p)\hat{u}_0(p)\Big)\frac{k\hat{u}_1'(k)+(1-k)\hat{u}_0'(k)}{k\hat{u}_1(k)+(1-k)\hat{u}_0(k)}\end{aligned}$$

Simplifying terms obtain

$$\begin{aligned}w_S(p,k)=&\rho\beta(p,k)\Big(p\hat{u}_1(p)+(1-p)\hat{u}_0(p)\Big)+\lambda(1-p)\beta(p,k)\hat{u}_0(p)\\&-(\lambda k(1-k)+\gamma)\beta(p,k)\Big(p\hat{u}_1(p)+(1-p)\hat{u}_0(p)\Big)\frac{\partial_2 U(k,k)}{k\hat{u}_1(k)+(1-k)\hat{u}_0(k)}\end{aligned}$$

Lemma A.21. *Suppose for every $p\in[\underline{p},1]$*

$$A'(p)\geq\frac{A(1)-A(p)}{1-p}\cdot\frac{\frac{\rho}{\lambda}\left(\frac{\rho}{\lambda}+1\right)}{\frac{\rho}{\lambda}+p}. \tag{A.44}$$

This condition satisfied if $\frac{\lambda}{\rho}$ is sufficiently large. Then, the signaling reservation wage $w_S(p,k)$ is decreasing in p.

Proof. Under the new notation can write

$$\beta(p,k)=exp\left(-\int_k^p\frac{p\hat{u}_1'(p)+(1-p)\hat{u}_0'(p)}{p\hat{u}_1(p)+(1-p)\hat{u}_0(p)}\right).$$

By the envelope theorem

$$\frac{\partial}{\partial p}\Big[\beta(p,k)(p\hat{u}_1(p)+(1-p)\hat{u}_0(p))\Big]=\beta(p,k)(\hat{u}_1'(p)-\hat{u}_0'(p))$$

implying

$$\partial_1\beta(p,k)=-\beta(p,k)\frac{p\hat{u}_1'(p)+(1-p)\hat{u}_0'(p)}{p\hat{u}_1(p)+(1-p)\hat{u}_0(p)}.$$

Then

$$\begin{aligned}\partial_1 w_S(p,k)=&\rho\beta(p,k)(\hat{u}_1(p)-\hat{u}_0(p))\\&+\lambda(1-p)\partial_1\beta(p,k)\hat{u}_0(p)+\lambda(1-p)\beta(p,k)\hat{u}_0'(p)-\lambda\beta(p,k)\hat{u}_0(p)\end{aligned}$$

$$- (\lambda k(1-k)+\gamma)\beta(p,k)(\hat{u}_1(p)-\hat{u}_0(p))\frac{\partial_2 U(k,k)}{k\hat{u}_1(k)+(1-k)\hat{u}_0(k)}$$

Dividing both sides by $\beta(p,k)$ obtain

$$\begin{aligned}\frac{\partial_1 w_S(p,k)}{\beta(p,k)} &= \rho(\hat{u}_1(p)-\hat{u}_0(p)) \\ &\quad - \lambda(1-p)\beta(p,k)\frac{p\hat{u}_1'(p)+(1-p)\hat{u}_0'(p)}{p\hat{u}_1(p)+(1-p)\hat{u}_0(p)}\hat{u}_0(p)+\lambda(1-p)\hat{u}_0'(p)-\lambda\hat{u}_0(p) \\ &\quad - (\lambda k(1-k)+\gamma)(\hat{u}_1(p)-\hat{u}_0(p))\frac{k\hat{u}_1'(k)+(1-k)\hat{u}_0'(k)}{k\hat{u}_1(k)+(1-k)\hat{u}_0(k)}\end{aligned}$$

Simplifying terms obtain

$$\begin{aligned}\frac{\partial_1 w_S(p,k)}{\beta(p,k)} &= \rho\hat{u}_1(p)-(\rho+\lambda)\hat{u}_0(p) \\ &\quad - \lambda(1-p)\frac{p\hat{u}_1'(p)+(1-p)\hat{u}_0'(p)}{p\hat{u}_1(p)+(1-p)\hat{u}_0(p)}\hat{u}_0(p)+\lambda(1-p)\hat{u}_0'(p) \\ &\quad - (\lambda k(1-k)+\gamma)(\hat{u}_1(p)-\hat{u}_0(p))\frac{k\hat{u}_1'(k)+(1-k)\hat{u}_0'(k)}{k\hat{u}_1(k)+(1-k)\hat{u}_0(k)}\end{aligned}$$

Simplifying terms obtain

$$\begin{aligned}\frac{\partial_1 w_S(p,k)}{\beta(p,k)} &= \rho\hat{u}_1(p)-(\rho+\lambda)\hat{u}_0(p)+\lambda p(1-p)\frac{\hat{u}_1(p)\hat{u}_0'(p)-\hat{u}_1'(p)\hat{u}_0(p)}{p\hat{u}_1(p)+(1-p)\hat{u}_0(p)} \\ &\quad - (\lambda k(1-k)+\gamma)(\hat{u}_1(p)-\hat{u}_0(p))\frac{k\hat{u}_1'(k)+(1-k)\hat{u}_0'(k)}{k\hat{u}_1(k)+(1-k)\hat{u}_0(k)}\end{aligned} \tag{A.45}$$

It is useful to express

$$\begin{aligned}\rho\hat{u}_1(p) &= A(p)+\hat{u}_1'(p)\lambda p(1-p) \\ (\rho+\lambda)\hat{u}_0(p) &= A(p)+\hat{u}_0'(p)\lambda p(1-p)\end{aligned} \tag{A.46}$$

Substituting (A.46) into (A.45) obtain

$$\begin{aligned}\frac{\partial_1 w_S(p,k)}{\beta(p,k)} &= \rho\hat{u}_1(p)-(\rho+\lambda)\hat{u}_0(p)+\frac{\hat{u}_1(p)\Big((\rho+\lambda)\hat{u}_0(p)-A(p)\Big)-\Big(\rho\hat{u}_1(p)-A(p)\Big)\hat{u}_0(p)}{p\hat{u}_1(p)+(1-p)\hat{u}_0(p)} \\ &\quad - (\lambda k(1-k)+\gamma)(\hat{u}_1(p)-\hat{u}_0(p))\frac{k\hat{u}_1'(k)+(1-k)\hat{u}_0'(k)}{k\hat{u}_1(k)+(1-k)\hat{u}_0(k)}\end{aligned}$$

Rearranging terms obtain

$$\begin{aligned}\frac{\partial_1 w_S(p,k)}{\beta(p,k)} &= \rho\hat{u}_1(p)-(\rho+\lambda)\hat{u}_0(p)+\frac{\hat{u}_1(p)(\lambda\hat{u}_0(p)-A(p))+A(p)\hat{u}_0(p)}{p\hat{u}_1(p)+(1-p)\hat{u}_0(p)} \\ &\quad - (\lambda k(1-k)+\gamma)(\hat{u}_1(p)-\hat{u}_0(p))\frac{k\hat{u}_1'(k)+(1-k)\hat{u}_0'(k)}{k\hat{u}_1(k)+(1-k)\hat{u}_0(k)}\end{aligned}$$

Simplifying terms

$$\frac{\partial_1 w_S(p,k)}{\beta(p,k)} = \rho(\hat{u}_1(p)-\hat{u}_0(p))-\lambda\hat{u}_0(p)+\frac{\lambda\hat{u}_1(p)\hat{u}_0(p)-A(p)(\hat{u}_1(p)-\hat{u}_0(p))}{p\hat{u}_1(p)+(1-p)\hat{u}_0(p)}$$

$$-(\lambda k(1-k)+\gamma)(\hat{u}_1(p)-\hat{u}_0(p))\frac{k\hat{u}_1'(k)+(1-k)\hat{u}_0'(k)}{k\hat{u}_1(k)+(1-k)\hat{u}_0(k)}$$

Simplifying terms

$$\begin{aligned}\frac{\partial_1 w_S(p,k)}{\beta(p,k)} &= \rho(\hat{u}_1(p)-\hat{u}_0(p))+\frac{\lambda(1-p)\hat{u}_0(p)(\hat{u}_1(p)-\hat{u}_0(p))-A(p)(\hat{u}_1(p)-\hat{u}_0(p))}{p\hat{u}_1(p)+(1-p)\hat{u}_0(p)}\\ &\quad -(\lambda k(1-k)+\gamma)(\hat{u}_1(p)-\hat{u}_0(p))\frac{k\hat{u}_1'(k)+(1-k)\hat{u}_0'(k)}{k\hat{u}_1(k)+(1-k)\hat{u}_0(k)}\end{aligned}$$

Dividing both sides by $\hat{u}_1(p)-\hat{u}_0(p)>0$ obtain

$$\frac{\partial_1 w_S(p,k)}{\beta(p,k)(\hat{u}_1(p)-\hat{u}_0(p))}=\rho+\frac{\lambda(1-p)\hat{u}_0(p)-A(p)}{p\hat{u}_1(p)+(1-p)\hat{u}_0(p)}-(\lambda k(1-k)+\gamma)\frac{k\hat{u}_1'(k)+(1-k)\hat{u}_0'(k)}{k\hat{u}_1(k)+(1-k)\hat{u}_0(k)}$$

Simplifying terms obtain

$$\begin{aligned}\frac{\partial_1 w_S(p,k)}{\beta(p,k)(\hat{u}_1(p)-\hat{u}_0(p))} &= \frac{p\rho\hat{u}_1(p)+(1-p)(\rho+\lambda)\hat{u}_0(p)-A(p)}{p\hat{u}_1(p)+(1-p)\hat{u}_0(p)}\\ &\quad -(\lambda k(1-k)+\gamma)\frac{k\hat{u}_1'(k)+(1-k)\hat{u}_0'(k)}{k\hat{u}_1(k)+(1-k)\hat{u}_0(k)}\end{aligned}$$

Set $\gamma=0$. Then use a similar calculation as before to obtain

$$\begin{aligned}\frac{\partial_1 w_S(p,k)}{\beta(p,k)(\hat{u}_1(p)-\hat{u}_0(p))} &\le \frac{p\cdot\rho\hat{u}_1(p)+(1-p)\cdot(\rho+\lambda)\hat{u}_0(p)-A(p)}{p\hat{u}_1(p)+(1-p)\hat{u}_0(p)}\\ &\quad -\frac{k\cdot\rho\hat{u}_1(k)+(1-k)\cdot(\rho+\lambda)\hat{u}_0(k)-A(k)}{k\hat{u}_1(k)+(1-k)\hat{u}_0(k)}.\end{aligned}$$

Define an auxiliary function

$$g(p)\overset{def}{=}\frac{p\cdot\rho\hat{u}_1(p)+(1-p)\cdot(\rho+\lambda)\hat{u}_0(p)-A(p)}{p\hat{u}_1(p)+(1-p)\hat{u}_0(p)}. \tag{A.47}$$

The goal is to show that $g'(p)\le 0$. Then

$$\begin{aligned}g'(p) &= \frac{\rho\hat{u}_1(p)+p\rho\hat{u}_1'(p)-(\rho+\lambda)\hat{u}_0(p)+(\rho+\lambda)(1-p)\hat{u}_0'(p)-A'(p)}{p\hat{u}_1(p)+(1-p)\hat{u}_0(p)}\\ &\quad -\frac{\Big(p\cdot\rho\hat{u}_1(p)+(1-p)\cdot(\rho+\lambda)\hat{u}_0(p)-A(p)\Big)\Big(\hat{u}_1(p)-\hat{u}_0(p)+p\hat{u}_1'(p)+(1-p)\hat{u}_0'(p)\Big)}{(p\hat{u}_1(p)+(1-p)\hat{u}_0(p))^2}.\end{aligned}$$

Multiplying by the common denominator

$$\begin{aligned}&\Big(\rho\hat{u}_1(p)+p\rho\hat{u}_1'(p)-(\rho+\lambda)\hat{u}_0(p)+(\rho+\lambda)(1-p)\hat{u}_0'(p)-A'(p)\Big)\Big(p\hat{u}_1(p)+(1-p)\hat{u}_0(p)\Big)\\ &-\Big(p\rho\hat{u}_1(p)+(1-p)(\rho+\lambda)\hat{u}_0(p)-A(p)\Big)\Big(\hat{u}_1(p)-\hat{u}_0(p)+p\hat{u}_1'(p)+(1-p)\hat{u}_0'(p)\Big)\\ =&\Big(\rho\hat{u}_1(p)+p\rho\hat{u}_1'(p)-(\rho+\lambda)\hat{u}_0(p)+(\rho+\lambda)(1-p)\hat{u}_0'(p)-A'(p)\Big)\Big(p\hat{u}_1(p)+(1-p)\hat{u}_0(p)\Big)\\ &-\rho\Big(p\hat{u}_1(p)+(1-p)\hat{u}_0(p)\Big)\Big(\hat{u}_1(p)-\hat{u}_0(p)+p\hat{u}_1'(p)+(1-p)\hat{u}_0'(p)\Big)\\ &-\Big((1-p)\lambda\hat{u}_0(p)-A(p)\Big)\Big(\hat{u}_1(p)-\hat{u}_0(p)+p\hat{u}_1'(p)+(1-p)\hat{u}_0'(p)\Big)\end{aligned}$$

$$
\begin{aligned}
=&\Big(-\lambda\hat{u}_0(p)+\lambda(1-p)\hat{u}_0'(p)-A'(p)\Big)\Big(p\hat{u}_1(p)+(1-p)\hat{u}_0(p)\Big)\\
-&\Big(\lambda(1-p)\hat{u}_0(p)-A(p)\Big)\Big(\hat{u}_1(p)-\hat{u}_0(p)+p\hat{u}_1'(p)+(1-p)\hat{u}_0'(p)\Big)
\end{aligned}
$$

Splitting terms obtain

$$
\begin{aligned}
&\Big(-\lambda\hat{u}_0(p)+\lambda(1-p)\hat{u}_0'(p)-A'(p)\Big)p\hat{u}_1(p)\\
+&\Big(-\lambda\hat{u}_0(p)+\lambda(1-p)\hat{u}_0'(p)-A'(p)\Big)(1-p)\hat{u}_0(p)\\
-&\lambda(1-p)\hat{u}_0(p)\Big(\hat{u}_1(p)-\hat{u}_0(p)+p\hat{u}_1'(p)+(1-p)\hat{u}_0'(p)\Big)\\
+&A(p)\Big(\hat{u}_1(p)-\hat{u}_0(p)+p\hat{u}_1'(p)+(1-p)\hat{u}_0'(p)\Big)
\end{aligned}
$$

Simplifying terms obtain

$$
\begin{aligned}
&\Big(-\lambda\hat{u}_0(p)+\lambda(1-p)\hat{u}_0'(p)-A'(p)\Big)p\hat{u}_1(p)-A'(p)(1-p)\hat{u}_0(p)\\
-&\lambda(1-p)\hat{u}_0(p)\Big(\hat{u}_1(p)+p\hat{u}_1'(p)\Big)+A(p)\Big(\hat{u}_1(p)-\hat{u}_0(p)+p\hat{u}_1'(p)+(1-p)\hat{u}_0'(p)\Big)
\end{aligned}
$$

Simplifying terms further obtain

$$
\begin{aligned}
&-\lambda\hat{u}_1(p)\hat{u}_0(p)-A'(p)\Big(p\hat{u}_1(p)+(1-p)\hat{u}_0(p)\Big)\\
&+\lambda p(1-p)\Big(\hat{u}_1(p)\hat{u}_0'(p)-\hat{u}_1'(p)\hat{u}_0(p)\Big)\\
&+A(p)\Big(\hat{u}_1(p)-\hat{u}_0(p)+p\hat{u}_1'(p)+(1-p)\hat{u}_0'(p)\Big)
\end{aligned}
$$

Substituting (A.46) into the above equation obtain

$$
\begin{aligned}
&-\lambda\hat{u}_1(p)\hat{u}_0(p)-A'(p)\Big(p\hat{u}_1(p)+(1-p)\hat{u}_0(p)\Big)\\
&+\hat{u}_1(p)\Big((\rho+\lambda)\hat{u}_0(p)-A(p)\Big)-\Big(\rho\hat{u}_1(p)-A(p)\Big)\hat{u}_0(p)\\
&+A(p)\Big(\hat{u}_1(p)-\hat{u}_0(p)+p\hat{u}_1'(p)+(1-p)\hat{u}_0'(p)\Big)
\end{aligned}
$$

Simplifying terms obtain

$$
\begin{aligned}
&-A'(p)\Big(p\hat{u}_1(p)+(1-p)\hat{u}_0(p)\Big)+A(p)\Big(p\hat{u}_1'(p)+(1-p)\hat{u}_0'(p)\Big)\\
=&p\Big(A(p)\hat{u}_1'(p)-A'(p)\hat{u}_1(p)\Big)+(1-p)\Big(A(p)\hat{u}_0'(p)-A'(p)\hat{u}_0(p)\Big)
\end{aligned}
$$

Using (A.46), can write

$$
\begin{aligned}
A(p)\hat{u}_1'(p)-A'(p)\hat{u}_1(p)&=A(p)\frac{\rho\hat{u}_1(p)-A(p)}{\lambda p(1-p)}-A'(p)\hat{u}_1(p)\le A(p)\frac{A(1)-A(p)}{\lambda p(1-p)}-A'(p)\frac{A(p)}{\rho}\\
A(p)\hat{u}_0'(p)-A'(p)\hat{u}_0(p)&=A(p)\frac{(\rho+\lambda)\hat{u}_0(p)-A(p)}{\lambda p(1-p)}-A'(p)\hat{u}_0(p)\le A(p)\frac{A(1)-A(p)}{\lambda p(1-p)}-A'(p)\frac{A(p)}{\rho+\lambda}
\end{aligned}
$$

This implies

$$
\begin{aligned}
&p\Big(A(p)\hat{u}_1'(p) - A'(p)\hat{u}_1(p)\Big) + (1-p)\Big(A(p)\hat{u}_0'(p) - A'(p)\hat{u}_0(p)\Big) \\
\leq &A(p)p\left(\frac{A(1)-A(p)}{\lambda p(1-p)} - \frac{A'(p)}{\rho}\right) + A(p)(1-p)\left(\frac{A(1)-A(p)}{\lambda p(1-p)} - \frac{A'(p)}{\rho+\lambda}\right) \\
= &\frac{A(1)-A(p)}{\lambda p(1-p)} - A'(p)\frac{\rho+\lambda p}{\rho(\rho+\lambda)}
\end{aligned}
$$

The sufficient condition for this to be negative is

$$
A'(p) \geq \frac{A(1)-A(p)}{1-p} \cdot \frac{\rho(\rho+\lambda)}{\lambda(\rho+\lambda p)} = \frac{A(1)-A(p)}{1-p} \cdot \frac{\frac{\rho}{\lambda}\left(\frac{\rho}{\lambda}+1\right)}{\frac{\rho}{\lambda}+p}.
$$

□

Remark 1. *Note that $w_S(p,k)$ is decreasing in p. This implies that, even though there is no churning, higher skilled agents are paid less than lower skilled agents. Specifically, it implies that $w_S(p,k) < w_R(p,k)$ for $t \notin \mathbb{T}$. This may seem like a contradiction with the fact that additional bargaining power increases the equilibrium value of the agent. What occurs is that the higher skilled agent is willing to pay more for information from the performance process X, before signaling his ability, but less for reputation building γ. The net effect makes the agent better of in equilibrium.*

Proof of Lemma 9 (agent's outside option under Brownian signals)

Suppose the leaves the family and investors assume that $\mu = k_t$. Based on (A.51), the posterior of investors for $s \geq t$ is given by (26) rewritten here as

$$
k_{t,s} = \mathrm{E}\left[\theta \,|\, X_s - X_t, k_t\right] = (\phi t + 1)\left(\frac{\phi}{\phi s + 1} \cdot (X_s - X_t) + \frac{1}{\phi s + 1} \cdot k_t\right).
$$

Suppose the agent leaves the intermediary at time t. Suppose the agent's private belief is p_t at the time he leaves. We can write the agent's expected utility at time t as

$$
U(\tilde{p}_t, k_t, t) = \sup_{\hat{\eta}} \mathrm{E}_{\tilde{p}_t}\left[\int_t^{\hat{\eta}} e^{-\rho(s-t)} A\left(\frac{\phi(\phi t+1)}{\phi s+1} \cdot (X_s - X_t) + \frac{\phi t + 1}{\phi s + 1} \cdot k_t\right) ds + e^{-\rho(\hat{\eta}-t)} L\right] \tag{A.48}
$$

where $\hat{\eta}$ is the time when he leaves the industry. Denote by η^* the optimal stopping time of the agent. The agent's continuation value at time s is a function of states $(\tilde{p}_s, k_{t,s}, s)$. Also, note that the value of t does not matter as the forward dynamics of $k_{t,s}$ do not depend on the initial t and are given by

$$
dk_{t,s} = \frac{\phi(\phi t + 1)}{\phi s + 1}(dX_s - k_{t,s}\, ds).
$$

The agent's continuation value increases in $\tilde{p}_s$, holding other variables constant. It implies that there exists a boundary $b(k_{t,s}, s)$ such that

$$
\eta^* = \inf\{s \geq 0 : \tilde{p}_s \leq b(k_{t,s}, s)\}.
$$

Moreover, since the agent's expected value is increasing in $k_{t,s}$, then $b(k,s)$ is decreasing in k for a given s. By envelope theorem, the derivative (A.48) with respect to k_t is

$$\frac{\partial}{\partial k_t}U(\tilde{p}_t,k_t,t)=\mathrm{E}_{\tilde{p}_t}\left[\int_t^{\eta^*}e^{-\rho(s-t)}\cdot\frac{\phi t+1}{\phi s+1}\cdot A'\left(\frac{\phi(\phi t+1)}{\phi s+1}\cdot(X_s-X_t)+\frac{\phi t+1}{\phi s+1}\cdot k_t\right)ds\right].$$

For $\hat{p}_t>\tilde{p}_t$ it is the case that $\hat{\eta}^*>\eta^*$ since X_t takes higher values with higher probability for $\hat{p}>\tilde{p}$. This implies

$$\frac{\partial}{\partial k_t}U(\hat{p}_t,k_t,t)-\frac{\partial}{\partial k_t}U(\tilde{p}_t,k_t,t)=\mathrm{E}_{\tilde{p}_t}\left[\int_{\eta^*}^{\hat{\eta}^*}e^{-\rho(s-t)}\frac{\phi t+1}{\phi s+1}A'\left(\frac{\phi(\phi t+1)}{\phi s+1}(X_s-X_t)+\frac{\phi t+1}{\phi s+1}k_t\right)ds\right]>0.$$

Proof of Lemma 10 (boundary $B(p,t)$)

Consider the following optimal stopping problem of the intermediary

$$\sup_{\hat{\tau}}\mathrm{E}_{\tilde{p}_0}\left[\int_0^{\hat{\tau}}e^{-rt}\Big(A\big(Q\left(\tilde{p}_t,t,X_t\right)\big)-w_R\big(\tilde{p}_t,\tilde{p}_t,t\big)-rV\Big)\,dt+V\right].$$

It is convenient to rewrite the above optimal stopping problem using explicit dependency on past performance

$$\sup_{\hat{\tau}}\mathrm{E}_{\tilde{p}_0}\left[\int_0^{\hat{\tau}}e^{-rt}\Big[A\Big(\hat{Q}(\tilde{p}_0,t,X_t)\Big)-w_R\Big(\Pi(\tilde{p}_0,t,X_t),\Pi(\tilde{p}_0,t,X_t),t\Big)\Big]\,dt+e^{-r\hat{\tau}}\cdot V\right].$$

Holding $\tilde{p}_0$ fixed, this is an optimal stopping problem with (X_t,t) being the relevant state variables. It is not immediately clear that an optimal stopping time τ exists and, if it does, how it is characterized.

Lemma A.22. *The optimal stopping rule is given by a boundary $\hat{B}(\tilde{p}_0,s)$ such that*

$$\tau=\inf\{s:X_s\le\hat{B}(\tilde{p}_0,s)\}.$$

Proof. Fix $\tilde{p}_0$. Define $\mathcal{T}(T,n)$ take values in $\left\{\frac{1}{n},\frac{2}{n},\dots,\frac{T\cdot n}{n}\right\}$. Consider the finite horizon problem

$$V_n^T(\tilde{p}_0,0,X_0)=\sup_{\hat{\tau}}\mathrm{E}_{\mu}\left[\int_0^{\hat{\tau}}e^{-rt}\Big[A\Big(Q\Big(\Pi(\tilde{p}_0,t,X_t),t,X_t\Big)\Big)-w_R\Big(\Pi(\tilde{p}_0,t,X_t),\Pi(\tilde{p}_0,t,X_t),t\Big)\Big]\,dt+e^{-r\hat{\tau}}V\right]$$

This is a discrete time problem and, by backward induction, there exists τ_n^T such that

$$V_n^T(\tilde{p}_0,0,X_0)=\mathrm{E}\left[\int_0^{\tau_n^T}e^{-rt}\Big[A\Big(Q\Big(\Pi(\tilde{p}_0,t,X_t),t,X_t\Big)\Big)-w_R\Big(\Pi(\tilde{p}_0,t,X_t),\Pi(\tilde{p}_0,t,X_t),t\Big)\Big]\,dt+e^{-r\tau_n^T}V\right].$$

Stopping rule τ_n^T is Markov in (X_t,t). Moreover, it is monotone in X_t, i.e., there exists boundary $\hat{B}_n^T(\tilde{p}_0,t)$ such that

$$\tau_n^T=\inf\left\{t:X_t\le\hat{B}_n^T(\tilde{p}_0,t)\right\}.$$

The sequence $V_{2^n}^T(\tilde{p}_0,0,X_0)$ is increasing in n. This implies that there exists a limit

$$\hat{B}_{2^n}^T(\tilde{p}_0,t)\overset{n\to\infty}{\searrow}\hat{B}^T(\tilde{p}_0,t).$$

Boundary $\hat{B}^T(\tilde{p}_0, t)$ and the corresponding stopping time τ^T satisfy

$$\begin{aligned} V^T(\tilde{p}_0, 0, X_0) &\stackrel{def}{=} \sup_{\hat{\tau}\leq T} \mathrm{E}_\mu \left[\int_0^{\hat{\tau}} e^{-rt}\Big[A\Big(Q\Big(\Pi(\tilde{p}_0, t, X_t), t, X_t\Big)\Big) - w_R\Big(\Pi(\tilde{p}_0, t, X_t), \Pi(\tilde{p}_0, t, X_t), t\Big)\Big]\, dt + e^{-r\hat{\tau}}V\right] \\ &= \mathrm{E}\left[\int_0^{\tau^T} e^{-rt}\Big[A\Big(Q\Big(\Pi(\tilde{p}_0, t, X_t), t, X_t\Big)\Big) - w_R\Big(\Pi(\tilde{p}_0, t, X_t), \Pi(\tilde{p}_0, t, X_t), t\Big)\Big]\, dt + e^{-r\tau^T}V\right]. \end{aligned}$$

Functions $V^T(\tilde{p}_0, t, k)$ are increasing in T. This implies there exists a limit

$$\hat{B}^T(\tilde{p}_0, t) \overset{T\to\infty}{\searrow} \hat{B}(\tilde{p}_0, t).$$

The stopping time τ corresponding to $\hat{B}(\tilde{p}_0, t)$ satisfies

$$\begin{aligned} V(\tilde{p}_0, 0, X) &\stackrel{def}{=} \sup_{\hat{\tau}} \mathrm{E}_{\tilde{p}_0} \left[\int_0^{\hat{\tau}} e^{-rt}\Big[A\Big(\hat{Q}(\tilde{p}_0, t, X_t)\Big) - w_R\Big(\Pi(\tilde{p}_0, t, X_t), \Pi(\tilde{p}_0, t, X_t), t\Big)\Big]\, dt + e^{-r\hat{\tau}}V\right] \\ &= \mathrm{E}_{\tilde{p}_0} \left[\int_0^{\tau} e^{-rt}\Big[A\Big(\hat{Q}(\tilde{p}_0, t, X_t)\Big) - w_R\Big(\Pi(\tilde{p}_0, t, X_t), \Pi(\tilde{p}_0, t, X_t), t\Big)\Big]\, dt + e^{-r\tau}V\right]. \end{aligned}$$

□

Lemma A.23. *Suppose $F(\cdot)$ has a log-concave density function $f(\cdot)$. Then there exists a boundary $B(k_t, t)$ such that*

$$X_t < \hat{B}(\tilde{p}_0, t) \quad \Leftrightarrow \quad X_t < B(\Pi(\tilde{p}_0, t, X_t), t). \tag{A.49}$$

Proof. By definition, $\Pi(\tilde{p}_0, t, X_t) = \frac{\phi}{\phi t+1}X_t + \frac{1}{\phi t+1}\tilde{p}_0$. Inverting this mapping,

$$\tilde{p}_0 = (\phi t + 1)\Pi(\tilde{p}_0, t, X_t) - \phi X_t.$$

Then need to show that

$$X_t < \hat{B}\Big((\phi t + 1)\Pi(\tilde{p}_0, t, X_t) - \phi X_t, t\Big).$$

In order to establish (A.49), it is sufficient to prove that $\partial_1 \hat{B}(\tilde{p}_0, t) \geq -\frac{1}{\phi}$. Consider $\varepsilon > 0$. Log-concavity of $f(\cdot)$ ensures that

$$\frac{\partial}{\partial k_0}\Big[\mathrm{E}\left[\tilde{p}_0 | \tilde{p}_0 - k_0\right] - k_0\Big] < 0.$$

Moreover, this property is preserved conditional on learning from the Brownian performance process X_t. It, further, implies that

$$\begin{aligned} Q\left(\Pi\left(\tilde{p}_0 + \varepsilon, t, X_t - \frac{\varepsilon}{\phi}\right), t, X_t - \frac{\varepsilon}{\phi}\right) &= \mathrm{E}\left[\theta \,|\, \tilde{p}_0 > \tilde{p}_0 + \varepsilon, X_t - \frac{\varepsilon}{\phi}\right] \\ &= \frac{\phi}{\phi t + 1}\left(X_t - \frac{\varepsilon}{\phi}\right) + \frac{1}{\phi t + 1}\mathrm{E}\left[\tilde{p}_0 | \tilde{p}_0 > \tilde{p}_0 + \varepsilon\right] \\ &= \frac{\phi}{\phi t + 1}X_t + \frac{1}{\phi t + 1}\Big(\mathrm{E}\left[\tilde{p}_0 | \tilde{p}_0 > \tilde{p}_0 + \varepsilon\right] - \varepsilon\Big) \\ &< Q\left(\Pi\left(\tilde{p}_0, t, X_t\right), t, X_t\right). \end{aligned}$$

If we rewrite it via $\hat{Q}(l,t,X_t) \overset{def}{=} Q(\Pi(l,t,X_t),t,X_t)$ then

$$\hat{Q}(\tilde{p}_0+\varepsilon,t,X_t-\varepsilon/\phi) < \hat{Q}(\tilde{p}_0,t,X_t). \tag{A.50}$$

This implies that

$$\begin{aligned}
&V\left(\tilde{p}_0+\varepsilon,0,X_0-\frac{\varepsilon}{\phi}\right)\\
&=\sup_{\hat{\tau}}\mathrm{E}_{\tilde{p}_0}\left[\int_0^{\hat{\tau}} e^{-rt}\left[A\left(\hat{Q}\left(\tilde{p}_0+\varepsilon,t,X_t-\frac{\varepsilon}{\phi}\right)\right)-w_R\left(\Pi\left(\tilde{p}_0+\varepsilon,t,X_t-\frac{\varepsilon}{\phi}\right),\Pi\left(\tilde{p}_0+\varepsilon,t,X_t-\frac{\varepsilon}{\phi}\right),t\right)\right]dt+e^{-r\hat{\tau}}V\right]\\
&=\sup_{\hat{\tau}}\mathrm{E}_{\tilde{p}_0}\left[\int_0^{\hat{\tau}} e^{-rt}\left[A\left(\hat{Q}\left(\tilde{p}_0+\varepsilon,t,X_t-\frac{\varepsilon}{\phi}\right)\right)-w_R\Big(\Pi(\tilde{p}_0,t,X_t),\Pi(\tilde{p}_0,t,X_t),t\Big)\right]dt+e^{-r\hat{\tau}}V\right]\\
&\leq\sup_{\hat{\tau}}\mathrm{E}_{\tilde{p}_0}\left[\int_0^{\hat{\tau}} e^{-rt}\left[A\Big(\hat{Q}(\tilde{p}_0,t,X_t)\Big)-w_R\Big(\Pi(\tilde{p}_0,t,X_t),\Pi(\tilde{p}_0,t,X_t),t\Big)\right]dt+e^{-r\hat{\tau}}V\right]\\
&\leq V(\tilde{p}_0,0,X_0).
\end{aligned}$$

This implies that $V(\tilde{p}_0,0,X_0) \geq V(\tilde{p}_0+\varepsilon,0,X_0-\varepsilon/\phi)$. It implies that

$$\begin{aligned}
\hat{B}(\tilde{p}_0+\varepsilon,0) &> \hat{B}(\tilde{p}_0,0)-\frac{\varepsilon}{\phi},\\
\frac{\hat{B}(\tilde{p}_0+\varepsilon,0)-\hat{B}(\tilde{p}_0,0)}{\varepsilon} &> -\frac{1}{\phi} \qquad \Rightarrow \qquad \partial_1\hat{B}(\tilde{p}_0,0) > -\frac{1}{\phi}.
\end{aligned}$$

The same logic follows through unchanged for $t>0$. This implies that function $x-\hat{B}((\phi t+1)y-x,t)$ is an increasing function in x. Thus, there exists a root $B(y,t)$ to the equation

$$\begin{aligned}
&B(y,t) = \hat{B}\big((\phi t+1)y-B(y,t),t\big)\\
\Rightarrow \qquad &\big\{x \leq \hat{B}\big((\phi t+1)y-x,t\big)\big\} = \big\{x \leq B(y,t)\big\}.
\end{aligned}$$

This concludes the proof. □

Lemma A.24. *The posterior belief conditional on $\mu=\mu_0$ and the performance history $(X_s)_{s\leq t}$ is given by*

$$\tilde{p}_t = \mathrm{E}_t\left[\theta\,|\,(X_s)_{s\leq t},\tilde{p}_0\right] = \frac{\phi}{\phi t+1}\cdot X_t + \frac{1}{\phi t+1}\cdot\tilde{p}_0. \tag{A.51}$$

The posterior variance is given by

$$\Sigma_t = \mathrm{E}_t\left[(\theta-\tilde{p}_t)^2\right] = \frac{\sigma_\theta^2\sigma^2}{\sigma_\theta^2 t+\sigma^2} = \frac{\sigma^2\phi}{\phi t+1}. \tag{A.52}$$

Proof. The proof if an application of the Kalman filter

$$\begin{aligned}
\mathrm{E}_t\left[\theta\,|\,(X_s)_{s\leq t},\tilde{p}_0\right] &= \mathrm{E}_t\left[\tilde{p}_0+\theta-\tilde{p}_0\,|\,(X_s)_{s\leq t}\right] = \mathrm{E}_t\left[\tilde{p}_0+\theta-\tilde{p}_0\,|\,X_t\right]\\
&= \tilde{p}_0 + \frac{\mathrm{cov}(\tilde{p}_0+\theta-\tilde{p}_0,X_t)}{\mathrm{var}(X_t)}\cdot(X_t-\tilde{p}_0 t) = \tilde{p}_0 + \frac{\sigma_\theta^2\cdot t}{\sigma_\theta t^2+\sigma^2 t}\cdot(X_t-\tilde{p}_0 t)
\end{aligned}$$

$$= \tilde{p}_0 + \frac{\phi}{\phi t + 1} \cdot (X_t - \tilde{p}_0 t) = \frac{1}{\phi t + 1} \cdot \tilde{p}_0 + \frac{\phi}{\phi t + 1} \cdot X_t.$$

Rewrite (A.52) as

$$\begin{aligned}
\mathrm{E}\left[(\theta - \tilde{p}_t)^2\right] &= \mathrm{E}\left[\left(\theta - \frac{\phi}{\phi t + 1}\Big(\tilde{p}_0 t + (\theta - \tilde{p}_0)t + \sigma B_t\Big) - \frac{1}{\phi t + 1}\tilde{p}_0\right)^2\right] \\
&= \mathrm{E}\left[\left(\theta - \tilde{p}_0 - \frac{\phi}{\phi t + 1}\Big((\theta - \tilde{p}_0)t + \sigma B_t\Big)\right)^2\right] \\
&= \mathrm{E}\left[\left(\frac{1}{\phi t + 1}(\theta - \tilde{p}_0) - \frac{\phi}{\phi t + 1}\sigma B_t\right)^2\right] \\
&= \frac{\phi \sigma^2}{(\phi t + 1)^2} + \frac{\phi^2 t \sigma^2}{(\phi t + 1)^2} = \frac{\sigma^2 \phi}{\phi t + 1}.
\end{aligned}$$

□

Lemma A.25. *The posterior average conditional on $\tilde{p}_0 > l$ after a performance history $(X_s)_{s \le t}$ is given by a function $\hat{Q}(l, t, X_t)$*

$$\begin{aligned}
\hat{Q}(l, t, X_t) &\overset{def}{=} \mathrm{E}_t\left[\theta \,|\, (X_s)_{s \le t}, \mu \ge k\right] = \mathrm{E}_t\left[\theta \,|\, X_t, \tilde{p}_0 \ge l\right] \\
&= \frac{\phi}{\phi t + 1} X_t + \frac{1}{\phi t + 1} \mathrm{E}_t\left[\tilde{p}_0 \,|\, X_t, \tilde{p}_0 \ge l\right]
\end{aligned} \tag{A.53}$$

Moreover, $\frac{\partial}{\partial x}\hat{Q}(l, t, x) > \frac{\phi}{\phi t + 1}$.

Proof. The sufficient condition result is given by

$$\begin{aligned}
\mathrm{E}\left[\theta \,\middle|\, (X_s)_{s \le t}, \tilde{p}_0 \ge l\right] &= \mathrm{E}\left[\ \mathrm{E}\left[\theta \,\middle|\, (X_s)_{s \le t}, \tilde{p}_0\right] \ \middle|\, (X_s)_{s \le t}, \tilde{p}_0 \ge l\right] \\
&= \mathrm{E}\left[\ \mathrm{E}\left[\mu + \xi \,\middle|\, (X_s)_{s \le t}, \tilde{p}_0\right] \ \middle|\, (X_s)_{s \le t}, \tilde{p}_0 \ge l\right] \\
&= \mathrm{E}\left[\ \mathrm{E}\left[\tilde{p}_0 + \theta - \tilde{p}_0 \,\middle|\, X_t, \tilde{p}_0\right] \ \middle|\, (X_s)_{s \le t}, \tilde{p}_0 \ge l\right] \\
&= \mathrm{E}\left[\ \frac{1}{\phi t + 1}\tilde{p}_0 + \frac{\phi}{\phi t + 1} X_t \,\middle|\, (X_s)_{s \le t}, \tilde{p}_0 \ge l\right] \\
&= \frac{\phi}{\phi t + 1} X_t + \frac{1}{\phi t + 1} \cdot \mathrm{E}\left[\tilde{p}_0 \,\middle|\, (X_s)_{s \le t}, \tilde{p}_0 \ge l\right] \\
&= \frac{\phi}{\phi t + 1} X_t + \frac{1}{\phi t + 1} \mathrm{E}\left[\tilde{p}_0 \,\middle|\, X_t, \left(X_s - \frac{s}{t} X_t\right)_{s \le t}, \tilde{p}_0 \ge l\right] \\
&= \frac{\phi}{\phi t + 1} X_t + \frac{1}{\phi t + 1} \mathrm{E}\left[\tilde{p}_0 \,\middle|\, X_t, \left(B_s - \frac{s}{t} B_t\right)_{s \le t}, \tilde{p}_0 \ge l\right] \\
&\overset{(i)}{=} \frac{\phi}{\phi t + 1} X_t + \frac{1}{\phi t + 1} \mathrm{E}\left[\tilde{p}_0 \,\middle|\, X_t, \tilde{p}_0 \ge l\right]
\end{aligned}$$

where (i) follows from X_t being independent from $B_s - \frac{s}{t} \cdot B_t$. This holds because

– Conditional on μ, X_t is uncorrelated with $B_s - \frac{s}{t} \cdot B_t$

$$\operatorname{cov}\left[X_t, B_s - \frac{s}{t}B_t \,\middle|\, \tilde{p}_0\right] = \mathrm{E}\left[\sigma B_t \left(B_s - \frac{s}{t}B_t\right)\right] = \sigma s - \sigma \frac{s}{t} t = 0.$$

– Both X_t and $B_s - \frac{s}{t}B_t$ are normal, conditional on $\tilde{p}_0$. This implies that X_t and $B_s - \frac{s}{t}B_t$ are independent, conditional on $\tilde{p}_0$.

– Since $\tilde{p}_0$ is independent from $(B_s)_{s \le t}$ we can write

$$\begin{aligned}
\mathrm{P}\Big(X_t \le x_1, B_s - \frac{s}{t}B_t \le x_2\Big) &= \mathrm{E}\left[\mathrm{P}\Big(X_t \le x_1, B_s - \frac{s}{t}B_t \le x_2 \,\Big|\, \mu\Big)\right] \\
&= \mathrm{E}\left[\mathrm{P}(X_t \le x_1 \,|\, \mu) \cdot \mathrm{P}\Big(B_s - \frac{s}{t}B_t \le x_2 \,\Big|\, \mu\Big)\right] \\
&= \mathrm{E}\left[\mathrm{P}(X_t \le x_1 \,|\, \mu) \cdot \mathrm{P}\Big(B_s - \frac{s}{t}B_t \le x_2\Big)\right] \\
&= \mathrm{E}\left[\mathrm{P}(X_t \le x_1 \,|\, \mu)\right] \cdot \mathrm{P}\Big(B_s - \frac{s}{t}B_t \le x_2\Big) \\
&= \mathrm{P}(X_t \le x_1) \cdot \mathrm{P}\Big(B_s - \frac{s}{t}B_t \le x_2\Big).
\end{aligned}$$

This proves that X_t is a sufficient statistic of returns. Following the definition of $q(l,t,X_t)$, we have

$$q(l,t,X_t) = \frac{\phi}{\phi t + 1} X_t + \frac{1}{\phi t + 1} \mathrm{E}\left[\tilde{p}_0 \,\middle|\, X_t, \tilde{p}_0 \ge l\right].$$

The second part of the Lemma is to show that $\mathrm{E}\left[\tilde{p}_0 \,\middle|\, X_t = x, \tilde{p}_0 \ge l\right]$ is increasing in x for fixed t and k. The conditional density of $\tilde{p}_0$ is given by

$$\begin{aligned}
\mathrm{P}(\tilde{p}_0 = y \,|\, X_t = x, \tilde{p}_0 \ge l) &= \frac{\mathrm{P}(\tilde{p}_0 = y, X_t = x, \tilde{p}_0 \ge l)}{\mathrm{P}(X_t = x, \tilde{p}_0 \ge l)} = \frac{\mathrm{P}(\tilde{p}_0 = y, \sigma B_t = (x - yt))}{\int_l^\infty \mathrm{P}(\tilde{p}_0 = z, \sigma B_t = (x - zt))\, dz} \\
&= \frac{\frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-yt)^2}{2t}} \cdot \mathrm{P}(\mu = y)}{\int_l^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-yt)^2}{2t}} \cdot f(z)\, dz} = \frac{e^{\frac{2xyt-(yt)^2}{2\sigma^2 t}} \cdot f(y)}{\int_l^\infty e^{\frac{2xzt-(zt)^2}{2\sigma^2 t}} \cdot f(z)\, dz}.
\end{aligned}$$

The cumulative distribution of $\tilde{p}_0$, conditional on $X_t = x$ and $\tilde{p}_0 > l$ is given by

$$\mathrm{P}(\tilde{p}_0 \le y \,|\, X_t = x, \tilde{p}_0 \ge l) = \frac{\int_l^y e^{\frac{2xzt-(yt)^2}{2\sigma^2 t}} \cdot f(z)\, dz}{\int_l^\infty e^{\frac{2xzt-(zt)^2}{2\sigma^2 t}} \cdot f(z)\, dz}.$$

The derivative with respect to x is given by

$$\begin{aligned}
&\frac{\partial}{\partial x} \mathrm{P}\Big(\tilde{p}_0 \le y \,\Big|\, X_t = x, \tilde{p}_0 \ge l\Big) = \\
&\frac{\int_k^y \frac{z_1}{\sigma^2} e^{\frac{2xz_1t-(yt)^2}{2\sigma^2 t}} f(z_1) dz_1}{\int_k^\infty e^{\frac{2xz_2t-(z_2t)^2}{2\sigma^2 t}} f(z_2) dz_2} - \frac{\left(\int_k^y e^{\frac{2xz_1t-(yt)^2}{2\sigma^2 t}} f(z_1) dz_1\right)\left(\int_k^\infty \frac{z_2}{\sigma^2} e^{\frac{2xz_2t-(z_2t)^2}{2\sigma^2 t}} f(z_2) dz_2\right)}{\left(\int_k^\infty e^{\frac{2xz_2t-(z_2t)^2}{2\sigma^2 t}} f(z_2) dz_2\right)^2} =
\end{aligned}$$

$$\frac{\left(\int_k^y \frac{z_1}{\sigma^2} e^{\frac{2xz_1t-(yt)^2}{2\sigma^2 t}} f(z_1)dz_1\right)\left(\int_k^\infty e^{\frac{2xz_2t-(z_2t)^2}{2\sigma^2 t}} f(z_2)dz_2\right)-\left(\int_k^y e^{\frac{2xz_1t-(yt)^2}{2\sigma^2 t}} f(z_1)dz_1\right)\left(\int_k^\infty \frac{z_2}{\sigma^2} e^{\frac{2xz_2t-(z_2t)^2}{2\sigma^2 t}} f(z_2)dz_2\right)}{\left(\int_k^\infty e^{\frac{2xz_2t-(z_2t)^2}{2\sigma^2 t}} f(z_2)dz_2\right)^2} =$$

$$\frac{\left(\int_k^\infty e^{\frac{2xz_1t-(yt)^2}{2\sigma^2 t}} f(z_1)dz_1\right)\left(\int_k^y \frac{z_2}{\sigma^2} e^{\frac{2xz_2t-(z_2t)^2}{2\sigma^2 t}} f(z_2)dz_2\right)-\left(\int_k^y e^{\frac{2xz_1t-(yt)^2}{2\sigma^2 t}} f(z_1)dz_1\right)\left(\int_k^\infty \frac{z_2}{\sigma^2} e^{\frac{2xz_2t-(z_2t)^2}{2\sigma^2 t}} f(z_2)dz_2\right)}{\left(\int_k^\infty e^{\frac{2xz_2t-(z_2t)^2}{2\sigma^2 t}} f(z_2)dz_2\right)^2} =$$

$$\frac{\left(\int_y^\infty e^{\frac{2xz_1t-(yt)^2}{2\sigma^2 t}} f(z_1)dz_1\right)\left(\int_k^y \frac{z_2}{\sigma^2} e^{\frac{2xz_2t-(z_2t)^2}{2\sigma^2 t}} f(z_2)dz_2\right)-\left(\int_k^y e^{\frac{2xz_1t-(yt)^2}{2\sigma^2 t}} f(z_1)dz_1\right)\left(\int_y^\infty \frac{z_2}{\sigma^2} e^{\frac{2xz_2t-(z_2t)^2}{2\sigma^2 t}} f(z_2)dz_2\right)}{\left(\int_k^\infty e^{\frac{2xz_2t-(z_2t)^2}{2\sigma^2 t}} f(z_2)dz_2\right)^2} =$$

$$\frac{\int_y^\infty \int_k^y \frac{z_2-z_1}{2} e^{\frac{2xz_1t-(yt)^2}{2\sigma^2 t}} e^{\frac{2xz_2t-(yt)^2}{2\sigma^2 t}} f(z_1)f(z_2)dz_1dz_2}{\left(\int_k^\infty e^{\frac{2xz_2t-(z_2t)^2}{2\sigma^2 t}} f(z_2)dz_2\right)^2} < 0$$

where the last inequality holds since $z_2 \leq y \leq z_1$. This implies

$$\text{Law}(\tilde{p}_0 \,|\, X_t = x_1, \tilde{p}_0 \geq l) \overset{FOSD}{<} \text{Law}(\tilde{p}_0 \,|\, X_t = x_2, \tilde{p}_0 \geq l)$$

for any $x_1 < x_2$. This implies that

$$\frac{\partial}{\partial x}\text{E}_t\left[\tilde{p}_0 \,|\, X_t = x, \tilde{p}_0 \geq l\right] > 0$$

and it follows that

$$\frac{\partial}{\partial x}\hat{Q}(l,t,x) > \frac{\phi}{\phi t + 1}.$$

□

Proof of Proposition 4 (equilibrium verification given Brownian signals)

Define by $\mathbb{T}(\tilde{p}_0, l, t)$ to be the boundary at which the intermediary lets go of agent p_0 given the market perception of l. Formally, it is the optimal non-stationary boundary of the optimal stopping problem

$$V(\tilde{p}_0, l, t, X_t) = \sup_{\hat{\tau}} \text{E}_{\tilde{p}_0}\left[\int_t^{\hat{\tau}-t} e^{-r(s-t)}\left[A\Big(\hat{Q}\left(l,s,X_s\right)\Big) - w_R\Big(\Pi(\tilde{p}_0,s,X_s),\Pi(l,s,X_s),s\Big) - rV\right]ds + V\right]$$

Lemma A.26. *Boundary $\mathbb{T}(\tilde{p}_0, l, t)$ is weakly decreasing in p_0 and $\mathbb{T}(l,l,t) \equiv \hat{B}(l,t)$.*

Proof. Application of the Envelope theorem with respect to p_0 shows that $V(\tilde{p}_0, l, t, X_t)$ is weakly increasing in p_0 leading to the optimal stopping boundary $\mathbb{T}(\tilde{p}_0, l, t)$ to be weakly decreasing. □

Lemma A.27. *Suppose that process $(l_t)_{t\geq 0}$ is a weakly increasing. Moreover, suppose that $dl_t > 0$ if and only if*

$X_t \le \mathbb{T}(\tilde{p}_0, l_t, t)$[54] *Then*

$$V(\tilde{p}_0,l_t,t,X_t)=\mathrm{E}_{\tilde{p}_0}\left[\int_t^{\infty} e^{-r(s-t)}\left[A\Big(\hat{Q}(l_s,s,X_s)\Big)-w_R\Big(\Pi(\tilde{p}_0,s,X_s),\Pi(l_s,s,X_s),s\Big)-rV\right]\mathbb{1}\{X_s\geq\mathbb{T}(\tilde{p}_0,l_s,s)\}\,ds+V\right]$$

Proof. Using Ito's lemma for $X_t > \mathbb{T}(\tilde{p}_0, l_t, t)$ since $dl_t = 0$ we have

$$\begin{aligned} rV(\tilde{p}_0, l_t, t, X_t) &= A\Big(\hat{Q}(l_t, t, X_t)\Big) - w_R\Big(\Pi(\tilde{p}_0, t, X_t), \Pi(l_t, t, X_t), t\Big) \\ &\quad + \frac{\partial}{\partial t}V(\tilde{p}_0, l_t, t, X_t) + \Pi(\tilde{p}_0, t, X_t) \cdot \frac{\partial}{\partial x}V(\tilde{p}_0, l_t, t, X_t) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}V(\tilde{p}_0, l_t, t, X_t). \end{aligned}$$

For $X_t \le \mathbb{T}(\tilde{p}_0, l_t, t)$ we have $V(\tilde{p}_0, l_t, t, X_t) = V$, which can also be written (in differential form in case dl_t is non-differentiable due to a discrete or singular component)

$$\begin{aligned} rV(\tilde{p}_0, l_t, t, X_t)\, dt &= rV\, dt + \frac{\partial}{\partial t}V(\tilde{p}_0, l_t, t, X_t)\, dt + \Pi(\tilde{p}_0, l_t, t, X_t) \cdot \frac{\partial}{\partial x}V(\tilde{p}_0, l_t, t, X_t)\, d_t \\ &\quad + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}V(\tilde{p}_0, l_t, t, X_t) dt + \frac{\partial}{\partial l}V(\tilde{p}_0, l_t, t, X_t)\, dl_t. \end{aligned}$$

where it does not matter if dl_t has a discrete jump or not as long as it is self-contained in the churning set. It is without loss to prove the result for $t = 0$. Define process $(Z_t)_{t\geq 0}$ as

$$\begin{aligned} Z_t &\overset{def}{=} \int_0^t e^{-rs}\left[A\Big(\hat{Q}\Big(l_s, s, X_s\Big)\Big) - w_R\Big(\Pi(\tilde{p}_0, s, X_s), \Pi(l_s, s, X_s), s\Big)\right]\mathbb{1}\{X_s \geq \mathbb{T}(\tilde{p}_0, l_s, s)\}\; ds \\ &+ \int_0^t e^{-rs} rV \cdot \mathbb{1}\{X_s < \mathbb{T}(\tilde{p}_0, l_s, s)\}\; ds + e^{-rt}V(\tilde{p}_0, l_t, t, X_t). \end{aligned}$$

Using the Ito decomposition for $V(\tilde{p}_0, l_t, t, X_t)$ obtained earlier, we see that process $(Z_t)_{t\geq 0}$ is a martingale. Thus

$$V(\tilde{p}_0, l_0, 0, X_0) = Z_0 = \mathrm{E}\left[Z_\infty\right],$$

which proves that the intermediary is indifferent between stopping and continuing in her stopping region as long as she obtains her opportunity cost rV in that region. This argument follows through almost unchanged for every starting value $t > 0$. □

Lemma A.27 establishes that the principal is willing to retain the agent of skill p_0 in period t given performance X_t as long as the stream of revenues weakly exceeds

$$s(\tilde{p}_0, l_t, t, X_t) \overset{def}{=} \begin{cases} A\Big(\hat{Q}(l_t, t, X_t)\Big) - w_R\Big(\Pi(\tilde{p}_0, t, X_t), \Pi(l_t, t, X_t), t\Big) & \text{if } X_t \geq \mathbb{T}(\tilde{p}_0, l_t, t), \\ rV & \text{if } X_t < \mathbb{T}(\tilde{p}_0, l_t, t). \end{cases} \tag{A.54}$$

Lemma A.28. *Denote by S_t to be the flow profit of the intermediary employing agent p_0. Suppose there exists a time τ such that*

$$S_t - \int_0^s s(\tilde{p}_0, l_s, s, X_s)\, ds$$

is weakly increasing for $t \le \tau$ and weakly decreasing for $t > \tau$. Then the intermediary (weakly) prefers to let go of

[54] In particular, it implies that $X_t \le \mathbb{T}(\tilde{p}_0, l, t)$ for any $l \in [l_{t-}, l_t]$.

the agent and collect her outside option at time τ.

Proof. It is without loss to prove the result for $t = 0$. Define process $(\hat{V}_t)_{t\geq 0}$ as

$$\begin{aligned} Z_t =& \int_0^t e^{-rs}\Big[A\Big(\hat{Q}\big(l_s, s, X_s\big)\Big) - w_R\Big(\Pi(\tilde{p}_0, s, X_s), \Pi(l_s, s, X_s), s\Big)\Big]\mathbb{1}\left\{X_s \geq \mathbb{T}(\tilde{p}_0, l_s, s)\right\} ds \\ &+ \int_0^t e^{-rs} rV \cdot \mathbb{1}\left\{X_s < \mathbb{B}(\tilde{p}_0, l_s, s)\right\} ds + e^{-rt}V(l_t, \tilde{p}_0, t, X_t). \end{aligned}$$

Given a stopping time τ, both $t \vee \tau$ and $t \wedge \tau$ are also well-defined stopping times. By Ito's lemma it implies that for any stopping time $\hat{\tau}$

- process $Z_{(t\vee\tau)\wedge\hat{\tau}}$ is a super-martingale;
- process $Z_{t\wedge\tau\wedge\hat{\tau}}$ is a sub-martingale.

This implies

$$\begin{aligned} \mathrm{E}\left[Z_{\hat{\tau}}\right] &= \mathrm{E}\left[Z_{\hat{\tau}} \cdot \mathbb{1}\left\{\hat{\tau} \geq \tau\right\} + Z_{\hat{\tau}} \cdot \mathbb{1}\left\{\hat{\tau} < \tau\right\}\right] \\ &= \mathrm{E}\left[\Big(Z_{\hat{\tau}} \cdot \mathbb{1}\left\{\hat{\tau} \geq \tau\right\} + Z_\tau \mathbb{1}\left\{\hat{\tau} < \tau\right\}\Big) + \Big(Z_{\hat{\tau}} \cdot \mathbb{1}\left\{\hat{\tau} < \tau\right\} + Z_\tau \mathbb{1}\left\{\hat{\tau} \geq \tau\right\}\Big) - Z_\tau\right] \\ &= \mathrm{E}\left[Z_{\hat{\tau}\vee\tau} + Z_{\hat{\tau}\wedge\tau} - Z_\tau\right] = \mathrm{E}\left[\mathrm{E}_\tau\left[Z_{\hat{\tau}\vee\tau}\right] + Z_{\hat{\tau}\wedge\tau} - Z_\tau\right] \\ &\overset{(i)}{\leq} \mathrm{E}\left[Z_\tau + Z_{\hat{\tau}\wedge\tau} - Z_\tau\right] = \mathrm{E}\left[Z_{\hat{\tau}\wedge\tau}\right] \overset{(ii)}{\leq} \mathrm{E}\left[Z_\tau\right]. \end{aligned}$$

where (i) holds because process $Z_{(t\vee\tau)\wedge\hat{\tau}}$ is a super-martingale and (ii) holds because process $Z_{t\wedge\tau\wedge\hat{\tau}}$ is a sub-martingale. The principal, thus, finds it strictly optimal to take her outside option at time τ. □

Lemma A.29. *Consider an increasing continuous process $l = (l_t)_{t\geq 0}$. The cumulative compensation necessary to retain the agent of skill $(\pi(\tilde{p}_0, t))_{t\geq 0}$ denoted by $(\tilde{C}_t^{\tilde{p}_0})_{t\geq 0}$ is given by*

$$\begin{aligned} d\tilde{C}_t^{\tilde{p}_0} =& w_R\Big(\Pi(\tilde{p}_0, t, X_t), \Pi(l_t, t, X_t)\Big) dt - \partial_2 U\Big(\Pi(\tilde{p}_0, t, X_t), \Pi(l_t, t, X_t)\Big)\partial_1\Pi(l_t, t, X_t)\, dl_t \mathbb{1}\left\{l_t - l_{t-} = 0\right\} \\ &- \Big[\partial_2 U\Big(\Pi(\tilde{p}_0, t, X_t), \Pi(l_t, t, X_t)\Big) - \partial_2 U\Big(\Pi(\tilde{p}_0, t, X_t), \Pi(l_{t-}, t, X_t)\Big)\Big]. \end{aligned} \tag{A.55}$$

Proof. Proof is identical to Lemma A.10 of the Poisson case and relies on the Martingale property of the resulting payoff to the agent if he accepts current compensation and then collects his outside option. Because we focus on equilibria which are separating at the time the agent leaves the industry, it implies that $(l_t)_{t\geq 0}$ is continuous for $t > 0$. □

Lemma A.30. *Consider process $(k_t)_{t\geq 0}$ defined in (32). Then it is optimal for the intermediary to retain the agent for $\tilde{p}_t > k_t$ and let go of the agent the first time when $\tilde{p}_t < k_t$.*

Proof. Given process k_t, define the corresponding initial cutoff of the agent given by l_t given by the unique solution to

$$k_t = \Pi(l_t, t, X_t) \qquad \Rightarrow \qquad l_t = (\phi t + 1)k_t - \phi X_t.$$

The dynamics for process $(l_t)_{t\geq 0}$ are implied by the dynamics of process $(k_t)_{t\geq 0}$ given by (32)

$$\begin{aligned}
dl_t &= \phi k_t + (\phi t+1)dk_t - \phi dX_t \\
&= \phi k_t + (\phi t+1)\frac{\phi}{\phi t+1}(dX_t - k_t\,dt) - \phi\,dX_t + (\phi t+1)\gamma(k_t,t,X_t)\cdot \mathbb{1}\left\{X_t < B(k_t,t)\right\}\,dt \\
&= \phi k_t + \phi(dX_t - k_t\,dt) - \phi\,dX_t + (\phi t+1)\gamma(k_t,t,X_t)\cdot \mathbb{1}\left\{X_t < B(k_t,t)\right\}\,dt \\
&= (\phi t+1)\gamma(k_t,t,X_t)\cdot \mathbb{1}\left\{X_t < B(k_t,t)\right\}\,dt \\
&= (\phi t+1)\gamma(\Pi(l_t,t,X_t),t,X_t)\cdot \mathbb{1}\left\{X_t < \hat{B}(l_t,t)\right\}\,dt.
\end{aligned}$$

Denote by S_t the cumulative profit flow accruing to the intermediary from retaining the agent. We show that for

$$dS_t \geq s(l_t,\tilde{p}_0,t,X_t)\,dt \qquad \Leftrightarrow \qquad \tilde{p}_0 \geq l_t. \tag{A.56}$$

The retention compensation needed to retain the agent is given by (A.55) leading to a compensation flow given by

$$\begin{aligned}
\tilde{w}(\tilde{p}_t,k_t,t,X_t) &= \tilde{w}\big(\Pi(\tilde{p}_0,t,X_t),\Pi(l_t,t,X_t)\big) \\
&= w_R\big(\tilde{p}_t,k_t,t\big) - \partial_2 U\big(\tilde{p}_t,k_t,t\big)\gamma(k_t,t,X_t)\mathbb{1}\left\{X_t < B(k_t,t)\right\} \\
&= w_R\big(\tilde{p}_t,k_t,t\big) - \frac{\partial_2 U\big(\tilde{p}_t,k_t,t\big)}{\partial_2 U(k_t,k_t,t)}\cdot\Big[rV - A(Q(k_t,t,X_t)) + w_R(\tilde{p}_t,k_t,X_t)\Big]\cdot \mathbb{1}\left\{X_t < B(k_t,t)\right\}
\end{aligned}$$

Note that due to the agent's single-crossing condition $\frac{\partial^2}{\partial p \partial k}U(\tilde{p},k,t) > 0$ we have

$$\frac{\partial_2 U\big(\tilde{p}_t,k_t,t\big)}{\partial_2 U(k_t,k_t,t)} \geq 1 \qquad \Leftrightarrow \qquad \tilde{p}_t \geq k_t.$$

Suppose $\tilde{p}_t \leq k_t$ or, equivalently, $\tilde{p}_0 \leq l_t$, the flow profit of the intermediary satisfies

$$\begin{aligned}
&A(Q(k_t,t,X_t)) - \tilde{w}(\tilde{p}_t,k_t,t,X_t) \\
=&A(Q(k_t,t,X_t)) - A(k_t) + \frac{\partial_2 U\big(\tilde{p}_t,k_t,t\big)}{\partial_2 U(k_t,k_t,t)}\Big[rV - A(Q(k_t,t,X_t)) + w_R(\tilde{p}_t,k_t,X_t)\Big]\mathbb{1}\left\{X_t < B(k_t,t)\right\} \\
\leq&A(Q(k_t,t,X_t)) - A(k_t) + \frac{\partial_2 U\big(k_t,k_t,t\big)}{\partial_2 U(k_t,k_t,t)}\Big[rV - A(Q(k_t,t,X_t)) + w_R(\tilde{p}_t,k_t,X_t)\Big]\mathbb{1}\left\{X_t < B(k_t,t)\right\} \\
=&A(Q(k_t,t,X_t)) - A(k_t) + \Big[rV - A(Q(k_t,t,X_t)) + w_R(\tilde{p}_t,k_t,t)\Big]\mathbb{1}\left\{X_t < B(k_t,t)\right\} \\
=&\Big[A(Q(k_t,t,X_t)) - A(k_t)\Big]\cdot \mathbb{1}\left\{X_t \geq B(k_t,t)\right\} + rV\cdot \mathbb{1}\left\{X_t < B(k_t,t)\right\} \\
=&\Big[A\Big(\hat{Q}(l_t,t,X_t)\Big) - A\Big(\Pi(l_t,t,X_t)\Big)\Big]\cdot \mathbb{1}\left\{X_t \geq \hat{B}(l_t,t)\right\} + rV\cdot \mathbb{1}\left\{X_t < \hat{B}(l_t,t)\right\} \\
\overset{(i)}{\leq}&\Big[A\Big(\hat{Q}(l_t,t,X_t)\Big) - A\Big(\Pi(l_t,t,X_t)\Big)\Big]\cdot \mathbb{1}\left\{X_t \geq \mathbb{T}\left(\tilde{p}_0,l_t,t\right)\right\} + rV\cdot \mathbb{1}\left\{X_t < \mathbb{T}\left(\tilde{p}_0,l_t,t\right)\right\}
\end{aligned}$$

where inequality (i) follows from $\mathbb{T}\left(l_t,l_t,t\right) < \mathbb{T}\left(\tilde{p}_0,l_t,t\right)$ and

$$A\Big(\hat{Q}(l_t,t,X_t)\Big) - A\Big(\Pi(l_t,t,X_t)\Big) < rV \qquad \forall \quad X_t \in \left[\mathbb{T}\left(l_t,l_t,t\right),\mathbb{T}\left(\tilde{p}_0,l_t,t\right)\right] \quad \textit{if} \quad \tilde{p}_0 < l_t.$$

Similarly, for $\tilde{p}_t > k_t$ or, equivalently, $\tilde{p}_0 > l_t$, the flow profit of the intermediary satisfies

$$
\begin{aligned}
& A(Q(k_t,t,X_t)) - \tilde{w}(\tilde{p}_t,k_t,t,X_t) \\
=& A(Q(k_t,t,X_t)) - A(k_t) + \frac{\partial_2 U\left(\tilde{p}_t,k_t,t\right)}{\partial_2 U(k_t,k_t,t)}\Big[rV - A(Q(k_t,t,X_t)) + w_R(\tilde{p}_t,k_t,X_t)\Big]\mathbb{1}\left\{X_t < B(k_t,t)\right\} \\
\geq& A(Q(k_t,t,X_t)) - A(k_t) + \frac{\partial_2 U\left(k_t,k_t,t\right)}{\partial_2 U(k_t,k_t,t)}\Big[rV - A(Q(k_t,t,X_t)) + w_R(\tilde{p}_t,k_t,X_t)\Big]\mathbb{1}\left\{X_t < B(k_t,t)\right\} \\
=& A(Q(k_t,t,X_t)) - A(k_t) + \Big[rV - A(Q(k_t,t,X_t)) + w_R(\tilde{p}_t,k_t,t)\Big]\mathbb{1}\left\{X_t < B(k_t,t)\right\} \\
=& \Big[A(Q(k_t,t,X_t)) - A(k_t)\Big]\cdot\mathbb{1}\left\{X_t \geq B(k_t,t)\right\} + rV\cdot\mathbb{1}\left\{X_t < B(k_t,t)\right\} \\
=& \Big[A\Big(\hat{Q}(l_t,t,X_t)\Big) - A\Big(\Pi(l_t,t,X_t)\Big)\Big]\cdot\mathbb{1}\Big\{X_t \geq \hat{B}(l_t,t)\Big\} + rV\cdot\mathbb{1}\Big\{X_t < \hat{B}(l_t,t)\Big\} \\
\overset{(ii)}{\geq}& \Big[A\Big(\hat{Q}(l_t,t,X_t)\Big) - A\Big(\Pi(l_t,t,X_t)\Big)\Big]\cdot\mathbb{1}\left\{X_t \geq \mathbb{T}\left(\tilde{p}_0,l_t,t\right)\right\} + rV\cdot\mathbb{1}\left\{X_t < \mathbb{T}\left(\tilde{p}_0,l_t,t\right)\right\}
\end{aligned}
$$

where inequality (ii) follows from $\mathbb{T}\left(l_t,l_t,t\right) > \mathbb{T}\left(\tilde{p}_0,l_t,t\right)$

$$A\Big(\hat{Q}(l_t,t,X_t)\Big) - A\Big(\Pi(l_t,t,X_t)\Big) < rV \qquad \forall \quad X_t \in \left[\mathbb{T}\left(\tilde{p}_0,l_t,t\right), \mathbb{T}\left(l_t,l_t,t\right)\right] \quad \textit{if} \quad \tilde{p}_0 > l_t.$$

This proves (A.56) and, by Lemma llows that it is optimal to stop the first time when $\tilde{p}_0 = l_t$ or, equivalently, $\tilde{p}_t = k_t$. Now, suppose that $k_t < b(k_t,t)$. Then for $p_0 < l_t$ the cutoff agent does not value reputation implying that the flow profit of the intermediary is

$$
\begin{aligned}
& A(Q(k_t,t,X_t)) - \tilde{w}(\tilde{p}_t,k_t,t,X_t) \\
=& A(Q(k_t,t,X_t)) - \tilde{w}(\tilde{p}_t,k_t,t,X_t) + \partial_2 U\big(\tilde{p}_t,k_t,t\big)\gamma(k_t,t,X_t)\mathbb{1}\left\{X_t < B(k_t,t)\right\} \\
=& A(Q(k_t,t,X_t)) - \tilde{w}(\tilde{p}_t,k_t,t,X_t) \\
=& A\Big(\hat{Q}(l_t,t,X_t)\Big) - \tilde{w}(\tilde{p}_t,k_t,t,X_t) \\
\overset{(i')}{\leq}& \Big[A\Big(\hat{Q}(l_t,t,X_t)\Big) - \tilde{w}(\tilde{p}_t,k_t,t,X_t)\Big]\cdot\mathbb{1}\left\{X_t \geq \mathbb{T}\left(\tilde{p}_0,l_t,t\right)\right\} + rV\cdot\mathbb{1}\left\{X_t < \mathbb{T}\left(\tilde{p}_0,l_t,t\right)\right\}.
\end{aligned}
$$

Similarly, if $p_0 > l_t$ then the cumulative compensation is given by

$$
\begin{aligned}
dS_t =& A(Q(k_t,t,X_t))\,dt - w_R(\tilde{p}_t,k_t,t)\,dt + \partial_2 U\big(\tilde{p}_t,k_t,t\big)\gamma(k_t,t,X_t)\mathbb{1}\left\{X_t < B(k_t,t), k_t > b(k_t,t)\right\}\,dt \\
&+ \Big(U(\tilde{p}_t,k_t,t) - U(\tilde{p}_t,k_{t-},t)\Big)\cdot\epsilon(k_{t-},t,X_t)\cdot\mathbb{1}\left\{X_t < B(k_t,t), k_t \leq b(k_t,t)\right\} \\
&\geq A(Q(k_t,t,X_t))\,dt - w_R(\tilde{p}_t,k_t,t)\,dt + \partial_2 U\big(\tilde{p}_t,k_t,t\big)\gamma(k_t,t,X_t)\mathbb{1}\left\{X_t < B(k_t,t), k_t > b(k_t,t)\right\}\,dt \\
\overset{(ii')}{\leq}& \Big[A\Big(\hat{Q}(l_t,t,X_t)\Big) - \tilde{w}(\tilde{p}_t,k_t,t,X_t)\Big]\cdot\mathbb{1}\left\{X_t \geq \mathbb{T}\left(\tilde{p}_0,l_t,t\right)\right\} + rV\cdot\mathbb{1}\left\{X_t < \mathbb{T}\left(\tilde{p}_0,l_t,t\right)\right\}.
\end{aligned}
$$

Thus, dS_t exceeds the necessary payment flow the intermediary would agree to to delay letting go of the agent. This proves that it is optimal to retain the agent as long as $\tilde{p}_t > k_t$ and let him go the moment $\tilde{p}_t < k_t$. □

B Online Appendix (Not for Printed Publication)

The expected value to the intermediary is given by

$$\begin{aligned}
G(V) =& \mathrm{E}\left[\int_0^\tau e^{-rt}\Big(A\big(Q(k_t,t,X_t)\big) - w(\tilde{p}_t,k_t,t) - rV\Big)\,dt\right] \\
=& \mathrm{E}\left[\int_0^\tau e^{-rt}e^{\sigma B_t - \frac{\theta^2\sigma^2}{2}t}\Big[A\Big(\hat{Q}(l_t,t,\sigma B_t\Big) - w\Big(\Pi(l_t,t,\sigma B_t),\Pi(l_t,t,\sigma B_t),t\Big) - rV\Big]\,dt\right] \\
=& \mathrm{E}\left[\int_0^\tau e^{-rt}\mathrm{E}_t\left[e^{\sigma B_t - \frac{\theta^2\sigma^2}{2}t}\,\Big|\,\tilde{p}_0 \geq l_t, X_t\right]\Big[A\Big(\hat{Q}(l_t,t,\sigma B_t\Big) - w\Big(\Pi(l_t,t,\sigma B_t),\Pi(l_t,t,\sigma B_t),t\Big) - rV\Big]\,dt\right] \\
=& \mathrm{E}\left[\int_0^\tau e^{-rt}\overline{G}(l_t,t,X_t)\Big[A\Big(\hat{Q}(l_t,t,\sigma B_t\Big) - w\Big(\Pi(l_t,t,\sigma B_t),\Pi(l_t,t,\sigma B_t),t\Big) - rV\Big]\,dt\right] \qquad \text{(B.1)}
\end{aligned}$$

where we obtain the closed form solution for $\overline{G}(l_t,t,X_t)$ for the case of $F(\cdot)$ being a truncated normal distribution in (B.16). This results in evaluating (B.1) via a dynamic program in three state variables (l_t,t,X_t). See below for details.

B.1 Truncated Normal Distributions

Lemma B.1. *Suppose that $p \sim \mathcal{N}(p_0,\sigma_p^2,\underline{p},\overline{p})$. Then the posterior distribution is given by*

$$\mathrm{Law}(\tilde{p}_0|X_t) = \mathcal{N}\left(\frac{\sigma_\theta^2 t + \sigma^2}{(\sigma_p^2+\sigma_\theta^2)\cdot t + \sigma^2}\cdot\tilde{p}_0 + \frac{\sigma_p^2}{(\sigma_p^2+\sigma_\theta^2)\cdot t+\sigma^2}\cdot X_t, \quad \frac{\sigma_p^2(\sigma_\theta^2 t+\sigma^2)}{(\sigma_p^2+\sigma_\theta^2)t+\sigma^2}, \quad [\underline{p},\overline{p}]\right).$$

Proof. Note that process X is given by

$$X_t = \tilde{p}_0 t + (\theta - \tilde{p}_0)t + \sigma B_t.$$

Absent truncation the normal distribution stays a normal distribution. This implies

$$\begin{aligned}
\mathrm{E}\left[\theta|X_t\right] &= \tilde{p}_0 + \frac{\mathrm{cov}(\tilde{p}_0,X_t)}{\mathrm{var}(X_t)}\cdot(X_t - \tilde{p}_0 t) \\
&= \tilde{p}_0 + \frac{\sigma_p^2\cdot t}{(\sigma_p^2+\sigma_\theta^2)\cdot t^2 + \sigma^2\cdot t}\cdot(X_t - \tilde{p}_0 t) \\
&= \tilde{p}_0 + \frac{\sigma_p^2}{(\sigma_p^2+\sigma_\theta^2)\cdot t + \sigma^2}\cdot(X_t - \tilde{p}_0 t) \\
&= \frac{\sigma_\theta^2\cdot t + \sigma^2}{(\sigma_p^2+\sigma_\theta^2)\cdot t + \sigma^2}\cdot\tilde{p}_0 + \frac{\sigma_p^2}{(\sigma_p^2+\sigma_\theta^2)\cdot t+\sigma^2}\cdot X_t
\end{aligned}$$

Then

$$\begin{aligned}
&\mathrm{E}\left[\left(\theta - \frac{\sigma_\theta^2\cdot t+\sigma^2}{(\sigma_p^2+\sigma_\theta^2)\cdot t+\sigma^2}\cdot\tilde{p}_0 + \frac{\sigma_p^2}{(\sigma_p^2+\sigma_\theta^2)\cdot t+\sigma^2}\cdot X_t\right)^2\right] \\
=&\mathrm{E}\left[\left(\theta - \frac{\sigma_\theta^2\cdot t+\sigma^2}{(\sigma_p^2+\sigma_\theta^2)\cdot t+\sigma^2}\cdot\tilde{p}_0 + \frac{\sigma_p^2}{(\sigma_p^2+\sigma_\theta^2)\cdot t+\sigma^2}\cdot(\theta t + \theta t + \sigma B_t)\right)^2\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{E}\left[\left(\frac{\sigma_\theta^2 \cdot t + \sigma^2}{(\sigma_p^2+\sigma_\theta^2)\cdot t + \sigma^2}\cdot(\theta - \tilde{p}_0) + \frac{\sigma_p^2}{(\sigma_p^2+\sigma_\theta^2)\cdot t+\sigma^2}\cdot(\theta t + \sigma B_t)\right)^2\right] \\
&= \left(\frac{\sigma_\theta^2 \cdot t + \sigma^2}{(\sigma_p^2+\sigma_\theta^2)\cdot t + \sigma^2}\right)^2 \sigma_p^2 + \left(\frac{\sigma_p^2}{(\sigma_p^2+\sigma_\theta^2)\cdot t+\sigma^2}\right)^2 (\sigma_\theta^2 t^2 + \sigma^2 t) \\
&= \frac{\sigma_p^2(\sigma_\theta^2 \cdot t + \sigma^2)}{(\sigma_p^2+\sigma_\theta^2)\cdot t+\sigma^2}\left(\frac{\sigma_\theta^2 \cdot t + \sigma^2}{(\sigma_p^2+\sigma_\theta^2)\cdot t+\sigma^2} + \frac{\sigma_p^2 \cdot t}{(\sigma_p^2+\sigma_\theta^2)\cdot t+\sigma^2}\right) \\
&= \frac{\sigma_p^2(\sigma_\theta^2 t + \sigma^2)}{(\sigma_p^2+\sigma_\theta^2)t+\sigma^2}.
\end{aligned}
$$

□

Lemma B.2. *The expected value of a truncated normal distribution $p \sim \mathcal{N}(p_0, \sigma^2, [\underline{p}, \overline{p}])$ is given by*

$$
\mathrm{E}\left[\,\tilde{p}\,\middle|\,\tilde{p} \in [\underline{p}, \overline{p}]\right] = p_0 + \sigma \cdot \frac{\phi\left(\frac{\underline{p}-p_0}{\sigma}\right) - \phi\left(\frac{\overline{p}-p_0}{\sigma}\right)}{\Phi\left(\frac{\overline{p}-p_0}{\sigma}\right) - \Phi\left(\frac{\underline{p}-p_0}{\sigma}\right)}
$$

Proof. Integration by parts. □

B.1.1 Agent's Subgame if $A(p) = e^{\alpha p}$ and $L = 0$

Suppose the agent leaves at time t. The posterior variance about θ is given by

$$
\Sigma_t \overset{def}{=} \frac{\sigma_\theta^2 \sigma^2}{\sigma_\theta^2 t + \sigma^2} \tag{B.2}
$$

At time t the agent's posterior belief about θ is given by $\mathcal{N}(\tilde{p}_t, \Sigma_t)$, and we can write

$$
\theta = \tilde{p}_t + \xi_t.
$$

The investors' posterior belief about θ is given by $\mathcal{N}(k, \Sigma_t)$.[55]

Investors' posterior beliefs at time $s > t$ are given by

$$
\begin{aligned}
k_{t,s} &= \frac{\Sigma_t}{\Sigma_t(s-t)+\sigma^2}\cdot(X_s - X_t) + \frac{\sigma^2}{\Sigma_t(s-t)+\sigma^2}\cdot k_t \\
&\overset{(i)}{=} \frac{\Sigma_t}{\Sigma_t(s-t)+\sigma^2}\cdot(\sigma(B_s - B_t) + k_t(s-t) + \xi_t(s-t)) + \frac{\sigma^2}{\Sigma_t(s-t)+\sigma^2}\cdot k_t. \\
&= \frac{\Sigma_t}{\Sigma_t(s-t)+\sigma^2}\cdot(\sigma(B_s - B_t) + \xi_t(s-t)) + k_t
\end{aligned}
$$

where (i) holds along the equilibrium path, i.e., from the perspective of investors, and $\tilde{\theta}_t \sim \mathcal{N}(0, \Sigma_t)$. From the perspective of the agent

$$
X_s - X_t = \sigma(B_s - B_t) + \tilde{p}_t(s-t) + \xi_t(s-t).
$$

[55] Note that the equilibrium is separating so there is no uncertainty in p here.

Hence belief process evolves according to

$$\begin{aligned}
k_{t,s} &\overset{(ii)}{=} \frac{\Sigma_t}{\Sigma_t(s-t)+\sigma^2}\cdot(\sigma(B_s-B_t)+\tilde{p}_t(s-t)+\xi_t(s-t))+\frac{\sigma^2}{\Sigma_t(s-t)+\sigma^2}\cdot k_t \\
&= \frac{\Sigma_t}{\Sigma_t(s-t)+\sigma^2}\cdot(\sigma(B_s-B_t)+\xi_t(s-t))+\frac{\Sigma_t(t-s)}{\Sigma_t(s-t)+\sigma^2}\cdot\tilde{p}_t+\frac{\sigma^2}{\Sigma_t(s-t)+\sigma^2}\cdot k_t.
\end{aligned}$$

From the perspective of the agent, beliefs have an upward drift since $p \geq k$. In this normalization, $\tilde{\theta}_t \sim \mathcal{N}(0, \Sigma_t)$ from the perspective of both the agent and the market.

Suppose $L = 0$ and $g(x) = \log\left(\frac{x}{\alpha}\right)$. Then $A(x) = e^{\alpha x}$. Moreover, the agent never finds it profitable to exit the industry. It implies that his value function is given by

$$\begin{aligned}
U(p,k,t) &= \mathrm{E}\left[\int_t^\infty e^{-r(s-t)}\cdot e^{\alpha k_s}\,ds\right] \\
&= \mathrm{E}\left[\int_t^\infty e^{-r(s-t)}\cdot e^{\alpha\left(\frac{\Sigma_t}{\Sigma_t(s-t)+\sigma^2}\cdot(\sigma(B_s-B_t)+\tilde{\theta}_t(s-t))+\frac{\Sigma_t(s-t)}{\Sigma_t(s-t)+\sigma^2}p+\frac{\sigma^2}{\Sigma_t(s-t)+\sigma^2}k\right)}\,ds\right] \\
&= \int_t^\infty e^{-r(s-t)}\mathrm{E}\left[e^{\alpha\left(\frac{\Sigma_t}{\Sigma_t(s-t)+\sigma^2}\cdot(\sigma(B_s-B_t)+\tilde{\theta}_t(s-t))+\frac{\Sigma_t(s-t)}{\Sigma_t(s-t)+\sigma^2}p+\frac{\sigma^2}{\Sigma_t(s-t)+\sigma^2}k\right)}\right]ds \\
&= \int_t^\infty e^{-r(s-t)}\cdot e^{\alpha\left(\frac{\Sigma_t(s-t)}{\Sigma_t(s-t)+\sigma^2}p+\frac{\sigma^2}{\Sigma_t(s-t)+\sigma^2}k\right)}\mathrm{E}\left[e^{\alpha\left(\frac{\Sigma_t}{\Sigma_t(s-t)+\sigma^2}\cdot(\sigma(B_s-B_t)+\tilde{\theta}_t(s-t))\right)}\right]ds \\
&= \int_t^\infty e^{-r(s-t)}\cdot e^{\alpha\left(\frac{\Sigma_t(s-t)}{\Sigma_t(s-t)+\sigma^2}p+\frac{\sigma^2}{\Sigma_t(s-t)+\sigma^2}k\right)}\cdot e^{\frac{\alpha^2}{2}\left(\frac{\Sigma_t}{\Sigma_t(s-t)+\sigma^2}\right)^2\cdot(\sigma^2(s-t)+\Sigma_t(s-t)^2)}\,ds \\
&= \int_t^\infty e^{-r(s-t)}\cdot e^{\alpha\left(\frac{\Sigma_t(s-t)}{\Sigma_t(s-t)+\sigma^2}p+\frac{\sigma^2}{\Sigma_t(s-t)+\sigma^2}k\right)}\cdot e^{\frac{\alpha^2}{2}\frac{\Sigma_t^2(s-t)}{\Sigma_t(s-t)+\sigma^2}}\,ds \qquad \text{(B.3)}
\end{aligned}$$

At $t = 0$ we have $\Sigma_0 = \sigma_\theta^2$ and (B.3) as

$$U(p,k,0) = \int_0^\infty e^{-rs}\cdot e^{\alpha\left(\frac{\sigma_\theta^2 s}{\sigma_\theta^2 s+\sigma^2}p+\frac{\sigma^2}{\sigma_\theta^2 s+\sigma^2}k\right)}\cdot e^{\frac{\alpha^2}{2}\frac{\sigma_\theta^4 s}{\sigma_\theta^2 s+\sigma^2}}\,ds$$

We can simplify (B.3) by showing that

$$\begin{aligned}
\frac{\Sigma_t^2(s-t)}{\Sigma_t(s-t)+\sigma^2} &= \Sigma_t(s-t)\frac{\Sigma_t}{\Sigma_t(s-t)+\sigma^2} \\
&= \frac{\sigma_\theta^2\sigma^2(s-t)}{\sigma_\theta^2 t+\sigma^2}\frac{\frac{\sigma_\theta^2\sigma^2}{\sigma_\theta^2 t+\sigma^2}}{\frac{\sigma_\theta^2\sigma^2}{\sigma_\theta^2 t+\sigma^2}(s-t)+\sigma^2} \\
&= \frac{\sigma_\theta^2\sigma^2(s-t)}{\sigma_\theta^2 t+\sigma^2}\frac{\sigma_\theta^2}{\sigma_\theta^2 s+\sigma^2} \\
&= \frac{\sigma_\theta^4\sigma^2(s-t)}{(\sigma_\theta^2 t+\sigma^2)(\sigma_\theta^2 s+\sigma^2)} \\
&= \frac{\sigma_\theta^2\sigma^2(\sigma_\theta^2 s+\sigma^2-\sigma_\theta^2 t-\sigma^2)}{(\sigma_\theta^2 t+\sigma^2)(\sigma_\theta^2 s+\sigma^2)} \\
&= \frac{\sigma_\theta^2\sigma^2}{\sigma_\theta^2 t+\sigma^2}-\frac{\sigma_\theta^2\sigma^2}{\sigma_\theta^2 s+\sigma^2}
\end{aligned}$$

$$= \Sigma_t - \Sigma_s.$$

This implies that we can rewrite (B.3) as

$$U(p,k,t) = \int_t^\infty e^{-r(s-t)} \cdot e^{\alpha\left(\frac{\Sigma_t(s-t)}{\Sigma_t(s-t)+\sigma^2}p + \frac{\sigma^2}{\Sigma_t(s-t)+\sigma^2}k\right)} \cdot e^{\frac{\alpha^2}{2}(\Sigma_t - \Sigma_s)}\, ds.$$

Along the equilibrium path $p = k$ and we can simplify the second exponent in the integral

$$\begin{aligned} U(k,k,t) &= \int_t^\infty e^{-r(s-t)} \cdot e^{\alpha\left(\frac{\Sigma_t(s-t)}{\Sigma_t(s-t)+\sigma^2}p + \frac{\sigma^2}{\Sigma_t(s-t)+\sigma^2}k\right)} \cdot e^{\frac{\alpha^2}{2}(\Sigma_t - \Sigma_s)}\, ds \\ &= e^{\alpha k} \cdot \int_t^\infty e^{-r(s-t)} \cdot e^{\frac{\alpha^2}{2}(\Sigma_t - \Sigma_s)}\, ds. \end{aligned}$$

Define

$$\hat{u}_1(t) \stackrel{def}{=} e^{rt + \frac{\alpha^2}{2}\Sigma_t} \cdot \int_t^\infty e^{-rs - \frac{\alpha^2}{2}\Sigma_s}\, ds. \tag{B.4}$$

Then

$$U(k,k,t) = e^{\alpha k} \cdot \hat{u}_1(t). \tag{B.5}$$

The expected payoff if perceived correctly is given by

$$\mathrm{E}\left[U(p,p,0)\right] = \mathrm{E}\left[e^{\alpha p}\right] \cdot \hat{u}_1(0).$$

To characterize the forward dynamics we must consider the partial derivative of the agent. It is given by

$$\frac{\partial}{\partial k}U(p,k,t) = \int_t^\infty e^{-r(s-t)} \cdot \frac{\alpha\sigma^2}{\Sigma_t(s-t)+\sigma^2} \cdot e^{\alpha\left(\frac{\Sigma_t(s-t)}{\Sigma_t(s-t)+\sigma^2}p + \frac{\sigma^2}{\Sigma_t(s-t)+\sigma^2}k\right)} \cdot e^{\frac{\alpha^2}{2}(\Sigma_t - \Sigma_s)}\, ds.$$

This is difficult to evaluate off-path, but we only need to compute it for $p = k$. This, again, allows us to separate k out and write

$$\begin{aligned} \frac{\partial U}{\partial k}(k,k,t) &= \int_t^\infty e^{-r(s-t)} \cdot \frac{\alpha\sigma^2}{\Sigma_t(s-t)+\sigma^2} \cdot e^{\alpha\left(\frac{\Sigma_t(s-t)}{\Sigma_t(s-t)+\sigma^2}k + \frac{\sigma^2}{\Sigma_t(s-t)+\sigma^2}k\right)} \cdot e^{\frac{\alpha^2}{2}(\Sigma_t - \Sigma_s)}\, ds \\ &= \int_t^\infty e^{-r(s-t)} \cdot \frac{\alpha\sigma^2}{\Sigma_t(s-t)+\sigma^2} \cdot e^{\alpha k} \cdot e^{\frac{\alpha^2}{2}(\Sigma_t - \Sigma_s)}\, ds \\ &= \alpha e^{\alpha k} \cdot \int_t^\infty e^{-r(s-t)} \cdot \frac{\sigma^2}{\frac{\sigma_\theta^2\sigma^2}{\sigma_\theta^2 t + \sigma^2}(s-t) + \sigma^2} \cdot e^{\frac{\alpha^2}{2}(\Sigma_t - \Sigma_s)}\, ds \\ &= \alpha e^{\alpha k} \cdot \int_t^\infty e^{-r(s-t)} \cdot \frac{\sigma_\theta^2 t + \sigma^2}{\sigma_\theta^2 s + \sigma^2} \cdot e^{\frac{\alpha^2}{2}(\Sigma_t - \Sigma_s)}\, ds \\ &= \alpha e^{\alpha k} \cdot \int_t^\infty e^{-r(s-t)} \cdot \frac{\Sigma_s}{\Sigma_t} \cdot e^{\frac{\alpha^2}{2}(\Sigma_t - \Sigma_s)}\, ds \\ &= e^{\alpha k} \cdot \alpha \cdot \frac{e^{rt + \frac{\alpha^2}{2}\Sigma_t}}{\Sigma_t} \cdot \int_t^\infty e^{-rs - \frac{\alpha^2}{2}\Sigma_s} \cdot \Sigma_s\, ds \end{aligned}$$

Define function $\hat{u}_2(t)$ as

$$\hat{u}_2(t) \stackrel{def}{=} \alpha \cdot \frac{e^{rt+\frac{\alpha^2}{2}\Sigma_t}}{\Sigma_t} \cdot \int_t^\infty e^{-rs-\frac{\alpha^2}{2}\Sigma_s} \cdot \Sigma_s \, ds. \tag{B.6}$$

Then

$$\frac{\partial}{\partial k} U(k,k,t) = e^{\alpha t}\hat{u}_2(t). \tag{B.7}$$

Agent's Reservation Utility. The principal must compensate the agent for his ex-ante reservation utility. The expected value of the agent perceived as the lowesst type at $t=0$ is given by

$$\begin{aligned} U(p,\underline{p},0) &= \int_0^\infty e^{-rs} \cdot e^{\alpha\left(\frac{\sigma_\theta^2 s}{\sigma_\theta^2 s+\sigma^2}p + \frac{\sigma^2}{\sigma_\theta^2 s+\sigma^2}\underline{p}\right)} \cdot e^{\frac{\alpha^2}{2}(\Sigma_0-\Sigma_s)}\, ds \\ &= \int_0^\infty e^{-rs} \cdot e^{\alpha\left(\frac{\sigma_\theta^2 s}{\sigma_\theta^2 s+\sigma^2}p + \frac{\sigma^2}{\sigma_\theta^2 s+\sigma^2}\underline{p}\right)} \cdot e^{\frac{\alpha^2}{2}\frac{\sigma_\theta^4 s}{\sigma_\theta^2 s+\sigma^2}}\, ds. \end{aligned}$$

Considering the "expected" reservation value of the agent, we must take the expected value with respect to p. It is given by

$$\begin{aligned} \mathrm{E}\left[U(p,\underline{p},0)\right] &= \mathrm{E}\left[\int_0^\infty e^{-rs} \cdot e^{\alpha\left(\frac{\sigma_\theta^2 s}{\sigma_\theta^2 s+\sigma^2}p + \frac{\sigma^2}{\sigma_\theta^2 s+\sigma^2}\underline{p}\right)} \cdot e^{\frac{\alpha^2}{2}\frac{\sigma_\theta^4 s}{\sigma_\theta^2 s+\sigma^2}}\, ds\right] \\ &= \int_0^\infty e^{-rs} \cdot \mathrm{E}\left[e^{\frac{\alpha\sigma_\theta^2 s}{\sigma_\theta^2 s+\sigma^2}p}\right] \cdot e^{\frac{\alpha\sigma^2}{\sigma_\theta^2 s+\sigma^2}\underline{p} + \frac{\alpha^2}{2}\cdot\frac{\sigma_\theta^4 s}{\sigma_\theta^2 s+\sigma^2}}\, ds. \end{aligned}$$

Define $b(s) \stackrel{def}{=} \frac{\alpha\sigma_\theta^2 s}{\sigma_\theta^2 s+\sigma^2}$. Define the moment generating function of the truncated normal distribution as

$$\begin{aligned} MGF_{TR}(b) &\stackrel{def}{=} \mathrm{E}\left[e^{bp}\right] \\ &= \frac{1}{\Phi\left(\frac{\bar{p}-p_0}{\sigma_p}\right) - \Phi\left(\frac{\underline{p}-p_0}{\sigma_p}\right)} \cdot \frac{1}{\sqrt{2\pi\sigma_p^2}} \cdot \int_{\underline{p}}^{\overline{p}} e^{bz} e^{-\frac{(z-p_0)^2}{2\sigma_p^2}}\, dz \end{aligned}$$

Then have

$$\begin{aligned} \left[\Phi\left(\frac{\bar{p}-p_0}{\sigma_p}\right) - \Phi\left(\frac{\underline{p}-p_0}{\sigma_p}\right)\right] MGF_{TR}(b) &= \frac{1}{\sqrt{2\pi\sigma_p^2}} \cdot \int_{\underline{p}}^{\overline{p}} e^{bz-\frac{(z-p_0)^2}{2\sigma_p^2}}\, dz \\ &= \frac{1}{\sqrt{2\pi\sigma_p^2}} \cdot \int_{\underline{p}}^{\overline{p}} e^{-\frac{1}{2}\left(\frac{z}{\sigma_p}-b\sigma_p-\frac{p_0}{\sigma_p}\right)^2 + \frac{1}{2}\left(b\sigma_p+\frac{p_0}{\sigma_p}\right)^2 - \frac{1}{2}\frac{p_0^2}{\sigma_p^2}}\, dz \\ &= \frac{1}{\sqrt{2\pi\sigma_p^2}} \cdot \int_{\underline{p}}^{\overline{p}} e^{-\frac{1}{2}\left(\frac{z}{\sigma_p}-b\sigma_p-\frac{p_0}{\sigma_p}\right)^2 + \frac{b^2\sigma_p^2}{2}+bp_0}\, dz \\ &= \frac{1}{\sqrt{2\pi\sigma_p^2}} \cdot e^{\frac{b^2\sigma_p^2}{2}+bp_0} \cdot \int_{\underline{p}}^{\overline{p}} e^{-\frac{1}{2}\left(\frac{z}{\sigma_p}-b\sigma_p-\frac{p_0}{\sigma_p}\right)^2}\, dz \end{aligned}$$

$$= e^{\frac{b^2\sigma_p^2}{2}+bp_0} \cdot \frac{1}{\sqrt{2\pi}} \int_{\frac{\underline{p}}{\sigma_p}-b\sigma_p-\frac{p_0}{\sigma_p}}^{\frac{\overline{p}}{\sigma_p}-b\sigma_p-\frac{p_0}{\sigma_p}} e^{-\frac{y^2}{2}}\, dy$$

$$= e^{\frac{b^2\sigma_p^2}{2}+bp_0} \cdot \left[\Phi\left(\frac{\overline{p}-p_0}{\sigma_p} - b\sigma_p\right) - \Phi\left(\frac{\underline{p}-p_0}{\sigma_p} - b\sigma_p\right)\right].$$

This implies that the moment generating function of the truncated normal distribution is equal to

$$MGF_{TR}(b) = e^{\frac{b^2\sigma_p^2}{2}+bp_0} \cdot \frac{\Phi\left(\frac{\overline{p}-p_0}{\sigma_p} - b\sigma_p\right) - \Phi\left(\frac{\underline{p}-p_0}{\sigma_p} - b\sigma_p\right)}{\Phi\left(\frac{\overline{p}-p_0}{\sigma_p}\right) - \Phi\left(\frac{\underline{p}-p_0}{\sigma_p}\right)}.$$

Substitute the value of $b(s) = \frac{\alpha\sigma_\theta^2 s}{\sigma_\theta^2 s+\sigma^2}$ to obtain

$$\mathrm{E}\left[U\left(\tilde{p}, \underline{p}, 0\right)\right] = \int_0^\infty e^{-rs} \cdot e^{\frac{\alpha\sigma^2 \underline{p}}{\sigma_\theta^2 s+\sigma^2} + \frac{\alpha^2}{2}\frac{\sigma_\theta^4 s}{\sigma_\theta^2 s+\sigma^2}} \cdot MGF_{TR}\left(\frac{\alpha\sigma_\theta^2 s}{\sigma_\theta^2 s+\sigma^2}\right) ds. \tag{B.8}$$

This can be computed once and is independent of the fixed point calculation of M.

Agent's first-best payoff. We have

$$\mathrm{E}\left[U\left(\tilde{p}, \tilde{p}, 0\right)\right] = \mathrm{E}\left[e^{\alpha p}\right] \cdot \hat{u}_1(0) = MGF_{TR}(\alpha) \cdot \hat{u}_1(0). \tag{B.9}$$

B.1.2 Evaluating Expected Payoff under Girsanov Change of Measure

Suppose X_t is the path of returns and l_t is the corresponding cutoff agent type process. The realized expected payoff to the intermediary is

$$\int_0^\tau e^{-rt} A\Big(Q(\Pi(l_t, t, X_t), t, X_t)\Big)\, dt + e^{-r\tau}\Big[U\Big(\Pi(l_\tau, \tau, X_\tau), \Pi(l_\tau, \tau, X_\tau), \tau\Big) + V\Big] - U(\tilde{p}, \underline{p}, 0).$$

Define

$$V_1 \stackrel{def}{=} \mathrm{E}\left[\int_0^\tau e^{-rt} A\Big(Q(\Pi(l_t, t, X_t), t, X_t)\Big)\, dt\right], \tag{B.10}$$

$$V_2 \stackrel{def}{=} \mathrm{E}\left[e^{-r\tau}\left(U\Big(\Pi(l_\tau, \tau, X_\tau), \Pi(l_\tau, \tau, X_\tau), \tau\Big) + V\right)\right]. \tag{B.11}$$

The agent's reservation utility is already computed in (B.8).

Integrating flow profits. Process X_t is given by

$$X_t = \sigma B_t + \theta t.$$

Then

$$\mathrm{E}_P\left[\int_0^\tau e^{-rt} A\Big(Q(\Pi(l_t,t,X_t),t,X_t)\Big)\, dt\right] = \mathrm{E}_{\hat{P}}\left[e^{\frac{\theta}{\sigma}\hat{B}_T - \frac{T}{2}\left(\frac{\theta}{\sigma}\right)^2} \int_0^\tau e^{-rt} A\Big(Q(\Pi(l_t,t,\sigma\hat{B}_t),t,X_t)\Big)\, dt\right] \tag{B.12}$$

where $(\hat{B}_t)_{t\geq 0}$ is a standard Brownian motion under the risk-neutral measure $\hat{P}$. This is true since, under the change of measure density

$$e^{\frac{\theta}{\sigma}\hat{B}_T-\frac{T}{2}\left(\frac{\theta}{\sigma}\right)^2},$$

Brownian motion $\hat{B}_t$ has a drift $\frac{\theta}{\sigma}$ under the initial measure. This implies

$$\begin{aligned}V_1 &= \mathrm{E}\left[e^{\frac{\theta}{\sigma}B_\tau-\frac{\tau}{2}\left(\frac{\theta}{\sigma}\right)^2}\cdot\int_0^\tau e^{-rt}A\Big(Q(\Pi(l_t,t,X_t),t,X_t)\Big)\,dt\right]\\ &\overset{(i)}{=} \mathrm{E}\left[\int_0^\tau e^{-rt}\cdot e^{\frac{\theta}{\sigma}B_t-\frac{t}{2}\left(\frac{\theta}{\sigma}\right)^2}\cdot A\Big(Q(\Pi(l_t,t,X_t),t,X_t)\Big)\,dt\right]\end{aligned}$$

where (i) holds by the law of iterated expectation. Note that path by path $X_t=\sigma B_t$, which has drift $(p+\theta)t$ under the new change of measure. Thus, we can write

$$V_1 = \mathrm{E}\left[\int_0^\tau e^{-rt}\cdot e^{\frac{\theta}{\sigma^2}X_t-\frac{t}{2}\left(\frac{\theta}{\sigma}\right)^2}\cdot A\Big(Q(\Pi(l_t,t,X_t),t,X_t)\Big)\,dt\right].$$

Define the Girsanov density

$$G(t,x,p,\theta)\overset{def}{=}e^{\frac{\theta}{\sigma^2}X_t-\frac{t}{2}\left(\frac{\theta}{\sigma}\right)^2} \tag{B.13}$$

Then

$$V_1 = \mathrm{E}\left[\int_0^\tau e^{-rt}\cdot G(t,X_t,p,\theta)\cdot A(Q(t,X_t,l_t))\,dt\right].$$

Note that X_t is independent of θ since we perform the change of measure conditional on θ.[56] We can compute the expected value of $G(t,x,p,\theta)$ with respect to θ as

$$\begin{aligned}G(p,t,x) &\overset{def}{=} \mathrm{E}_{X_t,p}\left[G(t,x,\theta)\,|\,\tilde{p}_0=p\right]=\mathrm{E}_{X_t,p}\left[e^{\frac{\theta}{\sigma^2}X_t-\frac{t}{2}\left(\frac{\theta}{\sigma}\right)^2}\right]\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{\frac{p+\sigma_\theta z}{\sigma^2}X_t-\frac{t}{2}\left(\frac{p+\sigma_\theta z}{\sigma}\right)^2}e^{-\frac{z^2}{2}}\,dz\\ &= \frac{1}{\sqrt{2\pi}}e^{\frac{p}{\sigma^2}X_t-\frac{t}{2}\left(\frac{p}{\sigma}\right)^2}\cdot\int_{-\infty}^{\infty}e^{\frac{\sigma_\theta}{\sigma^2}X_t z-\frac{(t\sigma_\theta^2+\sigma^2)z^2+2p\sigma_\theta z}{2\sigma^2}}\,dz\\ &= \frac{1}{\sqrt{2\pi}}e^{\frac{p}{\sigma^2}X_t-\frac{t}{2}\left(\frac{p}{\sigma}\right)^2}\cdot\int_{-\infty}^{\infty}e^{\left(\frac{\sigma_\theta}{\sigma^2}X_t-p\frac{\sigma_\theta}{\sigma^2}t\right)z-\frac{(\sigma_\theta^2t+\sigma^2)\cdot z^2}{2\sigma^2}}\,dz\\ &\overset{z=\sqrt{\frac{\sigma^2}{\sigma_\theta^2t+\sigma^2}}\hat{z}}{=} \frac{\sigma}{\sqrt{2\pi(\sigma_\theta^2t+\sigma^2)}}e^{\frac{p}{\sigma^2}X_t-\frac{t}{2}\left(\frac{p}{\sigma}\right)^2}\int_{-\infty}^{\infty}e^{\frac{\sigma_\theta}{\sigma}\frac{X_t-pt}{\sqrt{\sigma_\theta^2t+\sigma^2}}\hat{z}-\frac{\hat{z}^2}{2}}\,d\hat{z}\\ &= \frac{\sigma}{\sqrt{\sigma_\theta^2t+\sigma^2}}\cdot e^{\frac{p}{\sigma^2}X_t-\frac{t}{2}\left(\frac{p}{\sigma}\right)^2+\frac{1}{2}\frac{\sigma_\theta^2(X_t-pt)^2}{\sigma^2(\sigma_\theta^2t+\sigma^2)}}\\ &= \frac{\sigma}{\sqrt{\sigma_\theta^2t+\sigma^2}}\cdot e^{\frac{p}{\sigma^2}X_t-\frac{1}{2}\frac{p^2t}{\sigma^2}+\frac{1}{2}\frac{\sigma_\theta^2(X_t^2-2pX_tt+p^2t^2)}{\sigma^2(\sigma_\theta^2t+\sigma^2)}}\end{aligned}$$

[56] This can be verified by direct integration.

$$= \frac{\sigma}{\sqrt{\sigma_\theta^2 t + \sigma^2}} \cdot e^{\frac{p}{\sigma^2} X_t - \frac{1}{2} \frac{p^2 t(\sigma_\theta^2 t + \sigma^2)}{\sigma^2(\sigma_\theta^2 t+\sigma^2)} + \frac{1}{2} \frac{\sigma_\theta^2 (X_t^2 - 2pX_t t + p^2 t^2)}{\sigma^2(\sigma_\theta^2 t + \sigma^2)}}$$

$$= \frac{\sigma}{\sqrt{\sigma_\theta^2 t + \sigma^2}} \cdot e^{\frac{p}{\sigma^2} X_t - \frac{1}{2} \frac{p^2 t}{\sigma_\theta^2 t+\sigma^2} + \frac{1}{2} \frac{\sigma_\theta^2 (X_t^2 - 2pX_t t)}{\sigma^2(\sigma_\theta^2 t + \sigma^2)}}$$

$$= \frac{\sigma}{\sqrt{\sigma_\theta^2 t + \sigma^2}} \cdot e^{\frac{pX_t}{\sigma_\theta^2 t + \sigma^2} - \frac{1}{2} \frac{p^2 t}{\sigma_\theta^2 t+\sigma^2} + \frac{1}{2} \frac{\sigma_\theta^2 X_t^2}{\sigma^2(\sigma_\theta^2 t + \sigma^2)}} \tag{B.14}$$

Using Law of Iterated Expectation, we can write

$$\begin{aligned} V_1 &= \mathrm{E}\left[\int_0^\tau e^{-rt} \cdot G(t, X_t, p, \theta) \cdot A\Big(Q(\Pi(l_t, t, X_t), t, X_t)\Big)\, dt\right] \\ &= \mathrm{E}\left[\int_0^\tau e^{-rt} \cdot \mathrm{E}\left[G(t, X_t, p, \theta) \mid t, X_t, p\right] \cdot A\Big(Q(\Pi(l_t, t, X_t), t, X_t)\Big)\, dt\right] \\ &= \mathrm{E}\left[\int_0^\tau e^{-rt} \cdot G(t, X_t, p) \cdot A\Big(Q(\Pi(l_t, t, X_t), t, X_t)\Big)\, dt\right]. \end{aligned} \tag{B.15}$$

The next step is to compute

$$\begin{aligned} \bar{G}(l, t, X_t) &\stackrel{def}{=} \mathrm{E}\left[G(\tilde{p}, t, X_t) \cdot \mathbb{1}\{\tilde{p}_0 > l\}\right] \\ &= \mathrm{E}\left[\frac{\sigma}{\sqrt{\sigma_\theta^2 t + \sigma^2}} \cdot e^{\frac{pX_t}{\sigma_\theta^2 t + \sigma^2} - \frac{1}{2} \frac{p^2 t}{\sigma_\theta^2 t+\sigma^2} + \frac{1}{2} \frac{\sigma_\theta^2 X_t^2}{\sigma^2(\sigma_\theta^2 t + \sigma^2)}} \cdot \mathbb{1}\{\tilde{p}_0 > l\}\right]. \end{aligned}$$

Denote $b = \frac{1}{\sigma_\theta^2 t + \sigma^2}$. Then

$$\begin{aligned} &\int_l^{\bar{p}} e^{b\left(X_t y - \frac{t}{2} y^2\right)} d\Phi(y, p_0, \sigma_p) \\ =& \frac{1}{\sqrt{2\pi\sigma_p}} \int_l^{\bar{p}} e^{b\left(X_t y - \frac{t}{2} y^2\right)} e^{-\frac{(y-p_0)^2}{2\sigma_p^2}}\, dy \\ \stackrel{y = \sigma_p \hat{y} + p_0}{=}& \frac{1}{\sqrt{2\pi}} \int_{\frac{l-p_0}{\sigma_p}}^{\frac{\bar{p}-p_0}{\sigma_p}} e^{b\left(X_t(\sigma_p \hat{y} + p_0) - \frac{t}{2}(\sigma_p \hat{y} + p_0)^2\right)} e^{-\frac{\hat{y}^2}{2}}\, d\hat{y} \\ =& \frac{1}{\sqrt{2\pi}} e^{bX_t p_0 - \frac{t}{2} b p_0^2} \int_{\frac{l-p_0}{\sigma_p}}^{\frac{\bar{p}-p_0}{\sigma_p}} e^{b\sigma_p(X_t - tp_0)\hat{y} - \frac{1}{2}\left(b\sigma_p^2 t + 1\right)\hat{y}^2}\, d\hat{y} \\ \stackrel{\hat{y} = \frac{1}{\sqrt{b\sigma_p^2 t+1}} y'}{=}& \frac{1}{\sqrt{2\pi(b\sigma_p^2 t + 1)}} e^{bX_t p_0 - \frac{t}{2} b p_0^2} \int_{\frac{l-p_0}{\sigma_p}\sqrt{b\sigma_p^2 t+1}}^{\frac{\bar{p}-p_0}{\sigma_p}\sqrt{b\sigma_p^2 t+1}} e^{\frac{b\sigma_p(X_t - tp_0)}{\sqrt{b\sigma_p^2 t+1}} y' - \frac{1}{2}(y')^2}\, dy' \\ =& \frac{1}{\sqrt{2\pi(b\sigma_p^2 t + 1)}} e^{bX_t p_0 - \frac{t}{2} b p_0^2 + \frac{1}{2} \frac{b^2 \sigma_p^2 (X_t - p_0 t)^2}{b\sigma_p^2 t + 1}} \int_{\frac{l-p_0}{\sigma_p}\sqrt{b\sigma_p^2 t+1} - \frac{b\sigma_p(X_t - tp_0)}{\sqrt{b\sigma_p^2 t+1}}}^{\frac{\bar{p}-p_0}{\sigma_p}\sqrt{b\sigma_p^2 t+1} - \frac{b\sigma_p(X_t - tp_0)}{\sqrt{b\sigma_p^2 t+1}}} e^{-\frac{1}{2}(y')^2}\, dy' \\ =& \frac{1}{\sqrt{b\sigma_p^2 t + 1}} e^{bX_t p_0 - \frac{t}{2} b p_0^2 + \frac{1}{2} \frac{b^2 \sigma_p^2 (X_t - p_0 t)^2}{b\sigma_p^2 t + 1}} \end{aligned}$$

$$\times\left[\Phi\left(\frac{\bar{p}-p_0}{\sigma_p}\sqrt{b\sigma_p^2 t+1}-\frac{b\sigma_p\left(X_t-tp_0\right)}{\sqrt{b\sigma_p^2 t+1}}\right)-\Phi\left(\frac{l-p_0}{\sigma_p}\sqrt{b\sigma_p^2 t+1}-\frac{b\sigma_p\left(X_t-tp_0\right)}{\sqrt{b\sigma_p^2 t+1}}\right)\right]$$

We can now write

$$\begin{aligned}\bar{G}(l,t,x) &= \frac{\sigma\sqrt{b}}{\sqrt{b\sigma_p^2 t+1}} e^{bX_t p_0-\frac{t}{2}bp_0^2+\frac{1}{2}\frac{b^2\sigma_p^2(X_t-p_0 t)^2}{b\sigma_p^2 t+1}+\frac{1}{2}\frac{b\sigma_\theta^2 X_t^2}{\sigma^2}} \\ &\times \frac{\Phi\left(\frac{\bar{p}-p_0}{\sigma_p}\sqrt{b\sigma_p^2 t+1}-\frac{b\sigma_p(X_t-tp_0)}{\sqrt{b\sigma_p^2 t+1}}\right)-\Phi\left(\frac{l-p_0}{\sigma_p}\sqrt{b\sigma_p^2 t+1}-\frac{b\sigma_p(X_t-tp_0)}{\sqrt{b\sigma_p^2 t+1}}\right)}{\Phi\left(\frac{\bar{p}-p_0}{\sigma_p}\right)-\Phi\left(\frac{\underline{p}-p_0}{\sigma_p}\right)}\end{aligned} \tag{B.16}$$

For computational reasons, we can define $c = b\sigma_p^2 t+1$. Then we have[57]

$$\begin{aligned}\bar{G}(l,t,x) &= \frac{\sigma\sqrt{b}}{\sqrt{c}}\cdot e^{bX_t p_0-\frac{t}{2}bp_0^2+\frac{1}{2}\frac{b^2\sigma_p^2(X_t-p_0 t)^2}{c}+\frac{1}{2}\frac{b\sigma_\theta^2 X_t^2}{\sigma^2}} \\ &\times \frac{\Phi\left(\frac{\bar{p}-p_0}{\sigma_p}\sqrt{c}-\frac{b\sigma_p(X_t-tp_0)}{\sqrt{c}}\right)-\Phi\left(\frac{l-p_0}{\sigma_p}\sqrt{c}-\frac{b\sigma_p(X_t-tp_0)}{\sqrt{c}}\right)}{\Phi\left(\frac{\bar{p}-p_0}{\sigma_p}\right)-\Phi\left(\frac{\underline{p}-p_0}{\sigma_p}\right)}.\end{aligned}$$

We can now express

$$V_1(\underline{p},0,0) \stackrel{def}{=} \mathrm{E}\left[\int_0^\tau e^{-rt}\cdot\bar{G}(t,X_t,l_t)\cdot A\Big(Q(\Pi(l_t,t,X_t),t,X_t)\Big)\,dt\right]. \tag{B.17}$$

We can, thus, write

$$V_1(l_t,t,X_t) \stackrel{def}{=} \mathrm{E}\left[\int_t^\tau e^{-r(s-t)}\bar{G}(s,X_s,l_s)A\Big(Q(\Pi(l_s,s,X_s),s,X_s)\Big)\,ds\right]. \tag{B.18}$$

Note that $V_1(l,t,x)$ is not a continuation value of the principal since we are not conditioning the change of measure. However, $V_1 = V_1(\underline{p},0,0)$ since at $t=0$ no such conditioning needs to be done. We can solve for $V_1(\underline{p},0,0)$ via a dynamic program by solving the "fictitious" value function $V_1(l,t,x)$

$$rV_1(l,t,x) = \bar{G}(l,t,x)\cdot e^{\alpha Q(t,x,l)}+\frac{\partial}{\partial t}V_1(l,t,x)+\frac{\sigma^2}{2}\cdot\frac{\partial^2}{\partial x^2}V_1(l,t,x)+\gamma(t,x,l)\cdot\frac{\partial}{\partial l}V_1(l,t,x), \tag{B.19}$$

where policy function $\gamma(t,x,l)$ is pinned down by the optimal stopping problem of the principal given by

$$\gamma(\Pi(l,t,x),t,x) = \frac{rV+A(Q(l,t,x))-A(\Pi(l,t,x))}{\frac{\sigma^2}{\sigma_\theta^2 t+\sigma^2}\partial_k U(\Pi(l,t,x),\Pi(l,t,x),t)}\cdot\mathbb{1}\left\{x<hatB(l,t)\right\} \tag{B.20}$$

$$= \frac{\sigma_\theta^2 t+\sigma^2}{\sigma^2}\cdot\frac{rV+A(\hat{Q}(l,t,x))-A(\Pi(l,t,x))}{\partial_k U(\Pi(l,t,x),\Pi(l,t,x),t)}\cdot\mathbb{1}\left\{x<\hat{B}(l,t)\right\} \tag{B.21}$$

$$= \frac{\sigma_\theta^2 t+\sigma^2}{\sigma^2}\cdot\frac{rV+A(\hat{Q}(l,t,x))-A(\Pi(l,t,x))}{e^{\alpha\Pi(l,t,x)}\cdot\hat{u}_2(t)}\cdot\mathbb{1}\left\{x<\hat{B}(l,t)\right\}. \tag{B.22}$$

[57]This expression has been checked numerically.

We can see here that over time there needs to be more turnover to compensate for the fact that information becomes more stale over time. Note that $\gamma(t, x, l)$ is written in terms of ex-ante types, not ex-post types as in the Poisson model. This helps with tractability significantly.

Integrating sale profits. We can write

$$V_2 \overset{def}{=} \mathrm{E}\left[e^{-r\tau}\Big(U(P(l_\tau, \tau, X_\tau), \Pi(l_\tau, \tau, X_\tau), \tau) + V\Big)\right] \tag{B.23}$$

$$= \mathrm{E}\left[e^{\theta X_\tau - \frac{\sigma^2\theta^2\tau}{2}} e^{-r\tau}\Big(U(\Pi(p, \tau, X_\tau), \Pi(p, \tau, X_\tau), \tau) + V\Big)\right] \tag{B.24}$$

$$= \mathrm{E}\left[G(p, \tau, X_\tau) e^{-r\tau}\Big(U(\Pi(p, \tau, X_\tau), \Pi(p, \tau, X_\tau), \tau) + V\Big)\right] \tag{B.25}$$

$$= \mathrm{E}\left[e^{-r\tau} \cdot G(p, \tau, X_\tau) \cdot \Big(e^{\alpha \Pi(p,\tau,X_\tau)} \cdot \hat{u}_1(\tau) + V\Big)\right] \tag{B.26}$$

$$= \frac{\mathrm{E}\left[\int_0^\infty e^{-rt} G(l_t, t, X_t)\left(e^{\alpha\Pi(l_t,t,X_t)}\hat{u}_1(t) + V\right)\, d\Phi\left(\frac{l_t - p_0}{\sigma_p}\right)\right]}{\Phi\left(\frac{\overline{p}-p}{\sigma_p}\right) - \Phi\left(\frac{\underline{p}-p}{\sigma_p}\right)} \tag{B.27}$$

$$= \frac{\mathrm{E}\left[\frac{1}{\sqrt{2\pi\sigma_p^2}}\int_0^\infty e^{-rt} G(l_t, t, X_t)\left(e^{\alpha\Pi(l_t,t,X_t)}\hat{u}_1(t) + V\right) e^{-\frac{1}{2}\left(\frac{l_t - p}{\sigma_p}\right)^2} \cdot \gamma_t\, dt\right]}{\Phi\left(\frac{\overline{p}-\mathrm{E}[\tilde{p}_0]}{\sigma_p}\right) - \Phi\left(\frac{\underline{p}-\mathrm{E}[\tilde{p}_0]}{\sigma_p}\right)}. \tag{B.28}$$

where γ_t is given by (B.22). We can write

$$V_2(l_t, t, X_t) \overset{def}{=} \frac{\mathrm{E}\left[\frac{1}{\sqrt{2\pi\sigma_p^2}}\int_t^\infty e^{-r(s-t)} \cdot G(l_s, s, X_s) \cdot \left(e^{\alpha\Pi(l_s,s,X_s)} \cdot \hat{u}_1(s) + V\right) \cdot e^{-\frac{1}{2}\left(\frac{l_s - \mathrm{E}[\tilde{p}_0]}{\sigma_p}\right)^2} \cdot \gamma_s\, ds\right]}{\Phi\left(\frac{\overline{p}-\mathrm{E}[\tilde{p}_0]}{\sigma_p}\right) - \Phi\left(\frac{\underline{p}-\mathrm{E}[\tilde{p}_0]}{\sigma_p}\right)} \tag{B.29}$$

We can then apply a dynamic programming approach

$$\begin{aligned} rV_2(l, t, x) = {} & \frac{1}{\sqrt{2\pi\sigma_p^2}} \cdot \frac{G(t, x, l) \cdot \left(e^{\alpha\Pi(()l,t,x)} \cdot \hat{u}_1(t) + V\right) \cdot e^{-\frac{1}{2}\left(\frac{l - \mathrm{E}[\tilde{p}_0]}{\sigma_p}\right)^2} \cdot \gamma(t, x, l)}{\Phi\left(\frac{\overline{p}-\mathrm{E}[\tilde{p}_0]}{\sigma_p}\right) - \Phi\left(\frac{\underline{p}-\mathrm{E}[\tilde{p}_0]}{\sigma_p}\right)} \\ & + \frac{\partial}{\partial t} V_2(l, t, x) + \frac{\sigma^2}{2} \cdot \frac{\partial^2}{\partial x^2} V_2(l, t, x) + \gamma(t, x, l) \cdot \frac{\partial}{\partial l} V_2(l, t, x), \end{aligned} \tag{B.30}$$

where $\gamma(t, x, l)$ is taken from the principal's value function. Note that, since $V_1(t, x, l)$ and $V_2(t, x, l)$ are both integrals of the state variables, can compute the backward induction for the sum

$$V(l, t, x) = V_1(l, t, x) + V_2(l, t, x). \tag{B.31}$$

Can solve for $V(l, t, x)$ as a dynamic program in three state variables (l, t, x). In our computations, we assumed the time step was 0.05, the size of the performance jumps (modeled via a binomial tree) is 0.05, and l takes 30 values on $[\underline{p}, \overline{p}]$. Including the computation of the stopping frontier, the expected payoff can be evaluated in about 5 hours for a given V. We then rely on parallel computation to obtain the fixed point V, as well as the comparative statics presented in the paper.

C.1 Online Appendix. Derivations for Numerical Analysis

C.1.1 Poisson Model

Assume the initial distribution of types is uniform: $\tilde{p}_0 \sim U[\underline{p}_0, \bar{p}_0]$ with $\bar{p}_0 \leq 1$. Denote $k_0 = \underline{p}_0$. The posterior mean is given by

$$q_t = \mathrm{E}\left[\tilde{p}_t \,|\, \tilde{p}_t > k_t\right] = \frac{\frac{k_{0,t}+\bar{p}_0}{2}}{\frac{k_{0,t}+\bar{p}_0}{2} + \left(1 - \frac{k_{0,t}+\bar{p}_0}{2}\right) e^{-\lambda t}} \tag{C.1}$$

where $k_{0,t}$ is the ex-ante type corresponding to posterior k_t at time t. It solves

$$k_t = \frac{k_{0,t}}{k_{0,t} + (1-k_{0,t})e^{-\lambda t}} \qquad \Rightarrow \qquad k_{0,t} = \frac{k_t}{k_t + (1-k_t)e^{\lambda t}}.$$

This implies that q_t can be computed as

$$q_t = \frac{\frac{\frac{k_t}{k_t+(1-k_t)e^{\lambda t}}+\bar{p}_0}{2}}{\frac{\frac{k_t}{k_t+(1-k_t)e^{\lambda t}}+\bar{p}_0}{2} + \left(1 - \frac{\frac{k_t}{k_t+(1-k_t)e^{\lambda t}}+\bar{p}_0}{2}\right) \cdot e^{-\lambda t}} \tag{C.2}$$

For a given p_0 the fund family's value function is given by

$$\begin{aligned}
V_F(p_0) &= \mathrm{E}_{p_0}\left[\int_0^\tau e^{-rt} A(q_t) dt + e^{-r\tau} \cdot V\right] \\
&\quad + \mathrm{E}_{p_0}\left[e^{-r\tau}\Big(\mathbb{1}\left\{N_\tau = 0\right\} U\Big(\pi(\tau, p_0), \pi(\tau, p_0)\Big) + \mathbb{1}\left\{N_\tau > 0\right\} L\Big) - U(p_0, k_0)\right] \\
&= \mathrm{E}_{p_0}\left[\int_0^\tau e^{-rt} \cdot (A(q_t) - rV) dt\right] \\
&\quad + \mathrm{E}_{p_0}\left[e^{-r\tau}\Big(\mathbb{1}\left\{N_\tau = 0\right\} U\Big(\pi(\tau, p_0), \pi(\tau, p_0)\Big) + \mathbb{1}\left\{N_\tau > 0\right\} L\Big)\right] + V - U(p_0, k_0)
\end{aligned}$$

We can write

$$V_F(p_0) = V_{F,1}(p_0) + V_{F,2}(p_0)$$

where

$$\begin{aligned}
V_{F,1}(p_0) &= \mathrm{E}_{p_0}\left[\int_0^\tau e^{-rt} A(q_t)\, dt + e^{-r\tau} \cdot V\right], \\
V_{F,2}(p_0) &= \mathrm{E}_{p_0}\left[e^{-r\tau}\Big(\mathbb{1}\left\{N_\tau = 0\right\} U\Big(\pi(\tau, p_0), \pi(\tau, p_0)\Big) + \mathbb{1}\left\{N_\tau > 0\right\} L\Big) - U(p_0, k_0)\right].
\end{aligned}$$

By the martingale property

$$p_0 = p_t \cdot \mathrm{P}(\tau > t) + 0 \cdot \mathrm{P}(\tau \leq t) \qquad \Rightarrow \qquad \mathrm{P}(\tau > t) = \frac{p_0}{p_t}.$$

Define

$$T(p_0) = \inf\{t : k_t \geq \pi(t, p_0)\}.$$

The first part of the family's payoff can be written as

$$
\begin{aligned}
V_{F,1}(p_0) &= \int_0^{T(p_0)} e^{-rt} \cdot (A(q_t) - rV) \cdot \frac{p_0}{p_t}\, dt + V \\
&= \int_0^{T(p_0)} e^{-rt} \cdot (A(q_t) - rV) \cdot \left(p_0 + (1-p_0)e^{-\lambda t}\right) dt + V \\
&= \int_0^{T(\bar{p})} e^{-rt} \cdot (A(q_t) - rV) \cdot \left(p_0 + (1-p_0)e^{-\lambda t}\right) \cdot \mathbb{1}\left\{t \le T(p_0)\right\} dt + V \\
&= \int_0^{T(\bar{p})} e^{-rt} \cdot (A(q_t) - rV) \cdot \left(p_0 + (1-p_0)e^{-\lambda t}\right) \cdot \mathbb{1}\left\{p_0 \ge T^{-1}(t)\right\} dt + V.
\end{aligned}
$$

The second part of the family's payoff can be written as

$$
\begin{aligned}
V_{F,2}(p_0) &= e^{-rT(p_0)} U\Big(\pi(T(p_0), p_0),\ \pi(T(p_0), p_0)\Big) \cdot \frac{p_0}{p_{T(p_0)}} - U(p_0, k_0) \\
&= e^{-rT(p_0)} U\Big(\pi(T(p_0), p_0),\ \pi(T(p_0), p_0)\Big) \cdot \left(p_0 + (1-p_0)e^{-\lambda T(p_0)}\right) - U(p_0, k_0).
\end{aligned}
$$

The ex-ante value to the family is given by

$$
\mathrm{E}\left[V_F(\tilde{p}_0)\right] = \mathrm{E}\left[V_{F,1}(\tilde{p}_0) + V_{F,2}(\tilde{p}_0)\right]. \tag{C.3}
$$

The second term of (C.3) is given b

$$
\mathrm{E}\left[V_{F,2}(\tilde{p}_0)\right] = \mathrm{E}\left[e^{-rT(\tilde{p}_0)} U\Big(\pi(T(\tilde{p}_0), \tilde{p}_0),\ \pi(T(\tilde{p}_0), \tilde{p}_0)\Big) \left(\tilde{p}_0 + (1-\tilde{p}_0)e^{-\lambda T(\tilde{p}_0)}\right) - U(\tilde{p}_0, k_0)\right].
$$

The first term of (C.3) can be written as

$$
\begin{aligned}
\mathrm{E}[V_{F,1}(\tilde{p}_0)] &= \mathrm{E}\left[\int_0^{T(\bar{p})} e^{-rt}(A(q_t) - rV)\left(\tilde{p}_0 + (1-\tilde{p}_0)e^{-\lambda t}\right)\mathbb{1}\left\{\tilde{p}_0 \ge T^{-1}(t)\right\} dt\right] + V \\
&= \int_0^{T(\bar{p})} e^{-rt}(A(q_t) - rV)\left(\mathrm{E}\left[\tilde{p}_0 \mathbb{1}\left\{\tilde{p}_0 \ge T^{-1}(t)\right\}\right]\left(1 - e^{-\lambda t}\right) + \mathrm{P}\big(\tilde{p}_0 \ge T^{-1}(t)\big)\, e^{-\lambda t}\right) dt + V \\
&= \int_0^{T(\bar{p})} e^{-rt}(A(q_t) - rV)\left(\frac{\bar{p}^2 - (T^{-1}(t))^2}{2(\bar{p} - \underline{p})}\left(1 - e^{-\lambda t}\right) + \frac{\bar{p} - T^{-1}(t)}{\bar{p} - \underline{p}} e^{-\lambda t}\right) dt + V.
\end{aligned}
$$

By equilibrium construction

$$
\pi(T(p_0), p_0) = k_{T(p_0)} \quad \Rightarrow \quad T^{-1}(t) = k_{0,t} = \frac{k_t}{k_t + (1-k_t)e^{\lambda t}} = \pi(-t, k_t).
$$

Substituting this identity into the expression for $\mathrm{E}\left[V_{F,1}(\tilde{p}_0)\right]$ we obtain

$$
\begin{aligned}
\mathrm{E}\left[V_{F,1}(\tilde{p}_0)\right] &= \int_0^{T(\bar{p})} e^{-rt} \cdot (A(q_t) - rV)\left(\frac{\bar{p}^2 - k_{0,t}^2}{2(\bar{p} - \underline{p})}\left(1 - e^{-\lambda t}\right) + \frac{\bar{p} - k_{0,t}}{\bar{p} - \underline{p}} e^{-\lambda t}\right) dt + V \\
&= \int_0^{T(\bar{p})} e^{-rt} \cdot (A(q_t) - rV)\left(\frac{\bar{p} - k_{0,t}}{\bar{p} - \underline{p}}\right)\left(\frac{\bar{p} + k_{0,t}}{2}\left(1 - e^{-\lambda t}\right) + e^{-\lambda t}\right) dt + V
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{T(\bar{p})} e^{-rt} \cdot (A(q_t) - rV) \left(\frac{\overline{p} - k_{0,t}}{\overline{p} - \underline{p}} \right) \left(\frac{q_t}{q_t + (1-q_t)e^{\lambda t}} \left(1 - e^{-\lambda t}\right) + e^{-\lambda t} \right) dt + V \\
&= \int_0^{T(\bar{p})} e^{-rt} \cdot (A(q_t) - rM) \left(\frac{\overline{p} - k_{0,t}}{\overline{p} - \underline{p}} \right) \left(\frac{1}{q_t + (1-q_t)e^{\lambda t}} \right) dt + M \\
&= \frac{1}{\overline{p} - \underline{p}} \cdot \int_0^{T(\bar{p})} e^{-rt} \cdot (A(q_t) - rM) \cdot \frac{\overline{p} - \frac{k_t}{k_t + (1-k_t)e^{\lambda t}}}{q_t + (1-q_t)e^{\lambda t}} \, dt + M \\
&= \int_0^{T(\bar{p})} e^{-rt} \cdot (A(q_t) - rV) \cdot \underbrace{\frac{\overline{p} - \frac{k_t}{k_t + (1-k_t)e^{\lambda t}}}{\overline{p} - \underline{p}}}_{(i)} \cdot \underbrace{\frac{1}{q_t + (1-q_t)e^{\lambda t}}}_{(ii)} \, dt + V
\end{aligned}
$$

where (i) is the ex-ante measure of types not being let go endogenously at time t and (ii) is the average probability of these types having good returns up to time t.

Comparative Static Profit Decomposition

We can decompose the profit of the principal as

$$
\begin{aligned}
V &= \frac{\delta}{r+\delta} \mathrm{E}\left[\int_0^{\tau} e^{-rt} (A(q_t) - w_t) \, dt + e^{-r\tau} V \right] - I \\
&= \frac{\frac{\delta}{r+\delta} \mathrm{E}\left[\int_0^{\tau} e^{-rt} (A(q_t) - w_t) \, dt \right] - I}{1 - \frac{\delta}{r+\delta} \mathrm{E}\left[e^{-r\tau} \right]} \\
&= \frac{\frac{\delta}{r+\delta} \mathrm{E}\left[\int_0^{\tau \wedge t^*} e^{-rt} (A(q_t) - w_t) \, dt \right] + \int_{\tau \wedge t^*}^{\tau} -I}{1 - \frac{\delta}{r+\delta} \mathrm{E}\left[e^{-r\tau} \right]} \\
&= \frac{\frac{\delta}{r+\delta} \mathrm{E}\left[\int_0^{\tau \wedge t^*} e^{-rt} A(q_t) \, dt + e^{-r\tau \wedge t^*} U(\tilde{p}_{\tau \wedge t^*}, k_{\tau \wedge t^*}) - U(\tilde{p}_0, k_0) \right]}{1 - \frac{\delta}{r+\delta} \mathrm{E}\left[e^{-r\tau} \right]} \\
&\quad + \frac{\frac{\delta}{r+\delta} \mathrm{E}\left[\int_{\tau \wedge t^*}^{\tau} e^{-rt} A(q_t) \, dt + e^{-r\tau} U(\tilde{p}_\tau, \tilde{p}_\tau) - e^{-r\tau \wedge t^*} U(\tilde{p}_{\tau \wedge t^*}, k_{\tau \wedge t^*}) \right]}{1 - \frac{\delta}{r+\delta} \mathrm{E}\left[e^{-r\tau} \right]} - \frac{I}{1 - \frac{\delta}{r+\delta} \mathrm{E}\left[e^{-r\tau} \right]}.
\end{aligned}
$$

The fraction of profits obtained by the manager in the reputation building phase is given by

$$
\frac{\mathrm{E}\left[\int_{\tau \wedge t^*}^{\tau} e^{-rt} A(q_t) \, dt + e^{-r\tau} U(\tilde{p}_\tau, \tilde{p}_\tau) - e^{-r\tau \wedge t^*} U(\tilde{p}_{\tau \wedge t^*}, k_{\tau \wedge t^*}) \right]}{\int_0^{\tau} e^{-rt} A(q_t) \, dt + e^{-r\tau} U(\tilde{p}_\tau, k_\tau) - U(\tilde{p}_0, k_0)}.
$$

We can also decompose

$$
V_{ch}^{\%} = \frac{\mathrm{E}\left[\int_0^{\tau} e^{-rt} A(k_t) \, dt + e^{-r\tau} U(p_\tau, p_\tau) - U(\tilde{p}_\tau, k_0) \right]}{\mathrm{E}\left[\int_0^{\tau} e^{-rt} A(q_t) \, dt + e^{-r\tau} U(p_\tau, p_\tau) - U(\tilde{p}_\tau, k_0) \right]}
$$

C.1.1.1 Pay for Training Plot

The agent's marginal value of ability is

$$
\partial_1 U(\tilde{p}, k) = u_1(k) - u_0(k).
$$

Take the equilibrium value V. Denote $T(p_0)$ the time when the agent leaves the principal along the path of good performance. Then

$$\begin{aligned}
V(p,t) = \max_{\tau} \Bigg[& \int_t^{\tau} e^{-r(s-t)} \cdot (A(q_s) - rV) \cdot \frac{p}{\pi(p, s-t)}\, ds + V \\
& + e^{-r(\tau-t)} \Big[\mathrm{P}(N_\tau - N_t = 0) \cdot U(\pi(p, \tau - t), k_\tau) + \mathrm{P}(N_\tau - N_t = 1) \cdot L \Big] - U(p, k_t) \Bigg] \\
= \max_{\tau} \Bigg[& \int_t^{\tau} e^{-r(s-t)} \cdot (A(q_s) - rV) \cdot \frac{p}{\pi(p, s-t)}\, ds + V \\
& + e^{-r(\tau-t)} \left[\frac{p}{\pi(p, \tau - t)} \cdot U(\pi(p, \tau - t), k_\tau) + \left(1 - \frac{p}{\pi(p, s-t)}\right) \cdot L \right] - U(p, k_t) \Bigg]
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{\partial}{\partial p}\left(\frac{p}{\pi(p, s-t)}\right) &= \frac{\partial}{\partial p}\left(\frac{p}{\frac{p}{p+(1-p)e^{-\lambda(s-t)}}}\right) \\
&= \frac{\partial}{\partial p}\left(p + (1-p)e^{-\lambda(s-t)}\right) \\
&= 1 - e^{-\lambda(s-t)}.
\end{aligned}$$

Using Envelope Theorem with respect to time τ, the derivative with respect to p is given b

$$\partial_1 U(p, k_t) + \partial_1 V(p,t) = \int_t^{\tau} e^{-r(s-t)} \cdot (A(q_s) - rV) \cdot \left(1 - e^{-\lambda(s-t)}\right) ds \tag{C.4}$$

$$+ e^{-r(\tau-t)} \left(1 - e^{-\lambda(\tau-t)}\right) \cdot (U(k_\tau, k_\tau) - L) \tag{C.5}$$

$$+ e^{-r(\tau-t)} \cdot \left(p + (1-p)e^{-\lambda(\tau-t)}\right) \cdot \frac{e^{-\lambda(\tau-t)}}{\left(p + (1-p)e^{-\lambda(\tau-t)}\right)^2} \cdot \partial_1 U(k_\tau, k_\tau). \tag{C.6}$$

Rearranging terms obtain

$$\partial_1 U(p, k_t) + \partial_1 V(p,t) = \int_t^{\tau} e^{-r(s-t)} \cdot (A(q_s) - rV) \cdot \left(1 - e^{-\lambda(s-t)}\right) ds \tag{C.7}$$

$$+ e^{-r(\tau-t)} \left(1 - e^{-\lambda(\tau-t)}\right) \cdot (U(k_\tau, k_\tau) - L) \tag{C.8}$$

$$+ e^{-r(\tau-t)} \cdot \frac{e^{-\lambda(\tau-t)}}{p + (1-p)e^{-\lambda(\tau-t)}} \cdot \partial_1 U(k_\tau, k_\tau). \tag{C.9}$$

The total investment in training for a starting belief of p_0 is given by

$$\begin{aligned}
\partial_1 U(\pi(p_0, t), k_t) + \partial_1 V(\pi(p_0, t), k_t) = & \int_t^{T(p_0)} e^{-r(s-t)} (A(q_s) - rV) \left(1 - e^{-\lambda(s-t)}\right) ds \\
& + e^{-r(T(p_0)-t)} \left(1 - e^{-\lambda(T(p_0)-t)}\right) \left(U\left(k_{T(p_0)}, k_{T(p_0)}\right) - L\right) \\
& + e^{-r(T(p_0)-t)} \frac{e^{-\lambda(T(p_0)-t)}}{\pi(p_0, t) + (1 - \pi(p_0, t))e^{-\lambda(T(p_0)-t)}} \partial_1 U(k_{T(p_0)}, k_{T(p_0)}).
\end{aligned}$$

Can rewrite the second term

$$\begin{aligned}\partial_1 U(\pi(p_0,t),k_t) + \partial_1 V(\pi(p_0,t),k_t) = &\int_t^{T(p_0)} e^{-r(s-t)}(A(q_s) - rV)\left(1 - e^{-\lambda(s-t)}\right) ds \\ &+ e^{-r(T(p_0)-t)}\left(1 - e^{-\lambda(T(p_0)-t)}\right)\left(U\left(k_{T(p_0)}, k_{T(p_0)}\right) - L\right) \\ &+ e^{-(r+\lambda)(T(p_0)-t)} \cdot \frac{k_{T(p_0)}}{\pi(p_0,t)} \cdot \partial_1 U(k_{T(p_0)}, k_{T(p_0)}).\end{aligned}$$

Tracking the agent's incentive to invest is easy

$$\partial_1 U(\pi(p_0,t),k_t) = u_1(k_t) - u_0(k_t).$$

Given p_0 and $T(p_0)$ the second and third terms are known. The first term is easily integrated numerically. It is useful to compute integrals

$$\int_t^T e^{-rs}(A(q_s) - rV)\,ds, \qquad \int_t^T e^{-(r+\lambda)s}(A(q_s) - rV)\,ds \tag{C.10}$$

Can then express

$$\begin{aligned}v(t) &\overset{def}{=} \partial_1 V(\pi(p_0,t),k_t) \\ &= e^{rt} \cdot \int_t^T e^{-rs}(A(q_s) - rV)\,ds - e^{(r+\lambda)t} \cdot \int_t^T e^{-(r+\lambda)s}(A(q_s) - rV)\,ds \\ &+ \left(e^{-r(T-t)} - e^{-(r+\lambda)(T-t)}\right)(k_T u_1(k_T) + (1-k_T)u_0(k_T) - L) \\ &+ e^{-(r+\lambda)(T-t)} \cdot \frac{k_T}{\pi(p_0,t)} \cdot (u_1(k_T) - u_0(k_T)).\end{aligned}$$

C.1.2 Brownian Model

Optimal Stopping Boundary. Numerical Solution

Denote by k the lower bound of the ex-ante distribution of μ. Define

$$k(X_t,t) = \frac{\sigma^2}{\sigma_\xi^2 \cdot t + \sigma^2} \cdot \mu_0 + \frac{\sigma_\xi^2}{\sigma_\xi^2 \cdot t + \sigma^2} \cdot X_t.$$

Discrete grid. Denote by dt to be the time step, and Δ to be the Brownian diffusion step X. In each period t increases by ϵ. The per-period variance of $X_{t+dt} - X_t$ is equal to $\sigma^2 dt$. The drift of X_t is equal to $k(X_t,t)$.

The discrete scheme is

$$X_{t+dt} = \begin{cases} X_t + (n_d + n_\sigma)\cdot\Delta & \text{with probability} \quad \lambda_+(X_t,t) \\ X_t + n_d\cdot\Delta & \text{with probability} \quad 1 - \lambda_+(X_t,t) - \lambda_-(X_t,t) \\ X_t + (n_d - n_\sigma)\cdot\Delta & \text{with probability} \quad \lambda_-(X_t,t) \end{cases}$$

The drift of X_t is given by

$$\left(\lambda_+ - \lambda_-\right) \cdot n_\sigma \Delta = k(X_t, t)dt - n_d \Delta$$
$$\left(\lambda_+ - \lambda_-\right) \cdot n_\sigma = k(X_t, t)\frac{dt}{\Delta} - n_d$$

Ideally, set $n_d = k(X_t, t) \cdot \frac{dt}{\Delta}$ to limit the drift term impact on the probabilities. Since $n_d \in \mathbb{Z}$, set

$$n_d = \left\lfloor k(X_t, t) \cdot \frac{dt}{\Delta} \right\rfloor = \text{np.floor}\left(k(X_t, t) \cdot \frac{dt}{\Delta}\right).$$

This implies

$$\lambda_+ - \lambda_- = \frac{1}{n_\sigma} \cdot \left(k(X_t, t)\frac{dt}{\Delta} - n_d\right) \in \left[0, \frac{1}{n_\sigma}\right]$$

An imprecise way to capture the volatility term is

$$(\lambda_+ + \lambda_-) \cdot n_\sigma^2 \Delta^2 \approx \sigma^2 dt$$
$$\lambda_+ + \lambda_- \approx \frac{\sigma^2 dt}{n_\sigma^2 \Delta^2}$$

The resulting probabilities are given by

$$\begin{cases} \lambda_- = \frac{1}{2}\left(\frac{\sigma^2 dt}{n_\sigma^2 \Delta^2} - \frac{1}{n_\sigma} \cdot \left(k(X_t, t)\frac{dt}{\Delta} - n_d\right)\right) \\ \lambda_+ = \frac{1}{2}\left(\frac{\sigma^2 dt}{n_\sigma^2 \Delta^2} + \frac{1}{n_\sigma} \cdot \left(k(X_t, t)\frac{dt}{\Delta} - n_d\right)\right) \end{cases} \tag{C.11}$$

The above imply that $\lambda_- < \lambda_+$. The sufficient conditions for existence is such an n_σ that

$$\begin{cases} \frac{\sigma^2 dt}{n_\sigma^2 \Delta^2} < 1 & \Rightarrow \quad \lambda_- + \lambda_+ < 1 \\ \frac{1}{2}\left(\frac{\sigma^2 dt}{n_\sigma^2 \Delta^2} - \frac{1}{n_\sigma}\right) > 0 & \Rightarrow \quad \lambda_- > 0 \\ \frac{1}{2}\left(\frac{\sigma^2 dt}{n_\sigma^2 \Delta^2} + \frac{1}{n_\sigma}\right) < 1 & \Rightarrow \quad \lambda_+ < 1 \end{cases}$$

If $n_\sigma \geq 1$, then the third constraint is implied by the first constraint. The combination of the first two constraints jointly imply

$$\frac{1}{n_\sigma} < \frac{\sigma^2 dt}{n_\sigma^2 \Delta^2} < 1$$
$$n_\sigma < \frac{\sigma^2 dt}{\Delta^2} < n_\sigma^2$$

Then $n_\sigma = 2$ and $\frac{\sigma^2 dt}{\Delta^2} \in [2, 4]$ satisfies this approximate contraint. The vector $(\lambda_-, 1 - \lambda_- - \lambda_+, \lambda_+)$ is, then, guarranteed to be a probability vector. We can rewrite the above constraint as

$$\frac{1}{n_\sigma^2} < \frac{\Delta^2}{\sigma^2 dt} < \frac{1}{n_\sigma}$$

$$\frac{\sigma\sqrt{dt}}{n_\sigma} < \Delta < \frac{\sigma\sqrt{dt}}{\sqrt{n_\sigma}}$$
$$\frac{\sigma\sqrt{dt}}{n_\sigma} < \frac{\overline{x}-\underline{x}}{N_x} < \frac{\sigma\sqrt{dt}}{\sqrt{n_\sigma}}$$
$$\frac{\overline{x}-\underline{x}}{\sigma\sqrt{dt}} \cdot \sqrt{n_\sigma} < N_x < \frac{\overline{x}-\underline{x}}{\sigma\sqrt{dt}} \cdot n_\sigma$$

Setting $n_\sigma = 1$ this can be satisfied precisely, but then it leads to a discreteness problem. Instead, set

$$N_x = \left[\frac{\sqrt{n_\sigma}+n_\sigma}{2}\frac{\overline{x}-\underline{x}}{\sigma\sqrt{dt}}\right].$$

Setting $n_\sigma = 2$ becomes

$$N_x = \left[\frac{\sqrt{2}+2}{2} \cdot \frac{\overline{x}-\underline{x}}{\sigma\sqrt{dt}}\right].$$

Backward induction. Denote by $s \in \{0, \epsilon, 2\epsilon, \dots\}$ the number of periods remaining until period T which is sufficiently large. Write the principal's value function as $V(\mu, s)$ and we are interested in forward dynamics

$$V(\mu, s) \to V(\mu, s+\epsilon).$$

1. In the final period the fund family is forced to stop

$$V(\mu, 0) = M$$

pinning down the initial condition.

2. In the period before last the fund family chooses whether to stop immediately or wait for an additional period.

$$V(X,t) = \max\Big[M, \big(fA(q(X, t+dt) - f) - hA(k(X_t, t+dt) - h)\big) \cdot dt$$
$$+ e^{-rdt}\Big((1-2\lambda_k(\epsilon))V(\mu, 0) + \lambda_k(\epsilon)V(\mu-\Delta, 0) + \lambda_k(\epsilon)V(\mu+\Delta, 0)\Big)\Big]$$

Determine threshold $\underline{\mu}(\epsilon)$ at which the family chooses to stop.

3. Continue solving it forward

$$V(\mu, s+\epsilon) = \max\Big[M, \big(fA(q(\mu, k, s+\epsilon) - f) - hA(k-h)\big) \cdot \epsilon$$
$$+ e^{-r\epsilon}\Big((1-2\lambda_k(s+\epsilon))V(\mu, s) + \lambda_k(s+\epsilon)V(\mu-\Delta, s) + \lambda_k(s+\epsilon)V(\mu+\Delta, s)\Big)\Big]$$

Determine threshold $\underline{\mu}(s)$ at which the family chooses to stop.

C.1.2.1 Optimal Stopping Surface

Denote by l the lower bound of the ex-ante distribution of μ. Define

$$k(t, X_t, l) \stackrel{def}{=} \frac{\sigma^2}{\sigma_\xi^2 \cdot t + \sigma^2} \cdot l + \frac{\sigma_\xi^2}{\sigma_\xi^2 \cdot t + \sigma^2} \cdot X_t.$$

Denote by dt to be the time step, and Δ to be the Brownian diffusion step X. In each period t increases by ϵ. The per-period variance of $X_{t+dt} - X_t$ is equal to $\sigma^2 dt$. The drift of X_t is equal to $k(X_t, t)$. The discrete scheme is

$$X_{t+dt} = \begin{cases} X_t + (n_d + n_\sigma) \cdot \Delta & \text{with probability} & \lambda_+(X_t, t) \\ X_t + n_d \cdot \Delta & \text{with probability} & 1 - \lambda_+(X_t, t) - \lambda_-(X_t, t) \\ X_t + (n_d - n_\sigma) \cdot \Delta & \text{with probability} & \lambda_-(X_t, t) \end{cases}$$

The drift of X_t is given by

$$\begin{aligned} \left(\lambda_+ - \lambda_-\right) \cdot n_\sigma \Delta + n_d \Delta &= k(X_t, t) dt \\ \left(\lambda_+ - \lambda_-\right) \cdot n_\sigma &= k(X_t, t) \frac{dt}{\Delta} - n_d \end{aligned}$$

Ideally, set $n_d = k(X_t, t) \cdot \frac{dt}{\Delta}$ to limit the drift term impact on the probabilities. Since $n_d \in \mathbb{Z}$, set

$$n_d = \left\lfloor k(X_t, t) \cdot \frac{dt}{\Delta} \right\rfloor = \text{np.floor}\left(k(X_t, t) \cdot \frac{dt}{\Delta}\right).$$

This implies

$$\lambda_+ - \lambda_- = \frac{1}{n_\sigma} \cdot \left(k(X_t, t) \frac{dt}{\Delta} - n_d\right) \in \left[0, \frac{1}{n_\sigma}\right]$$

An imprecise way to capture the volatility term is

$$\begin{aligned} (\lambda_+ + \lambda_-) \cdot n_\sigma^2 \Delta^2 &\approx \sigma^2 dt \\ \lambda_+ + \lambda_- &\approx \frac{\sigma^2 dt}{n_\sigma^2 \Delta^2} \end{aligned}$$

The resulting probabilities are given by

$$\begin{cases} \lambda_- = \frac{1}{2} \left(\frac{\sigma^2 dt}{n_\sigma^2 \Delta^2} - \frac{1}{n_\sigma} \cdot \left(k(X_t, t) \frac{dt}{\Delta} - n_d \right) \right) \\ \lambda_+ = \frac{1}{2} \left(\frac{\sigma^2 dt}{n_\sigma^2 \Delta^2} + \frac{1}{n_\sigma} \cdot \left(k(X_t, t) \frac{dt}{\Delta} - n_d \right) \right) \end{cases} \tag{C.12}$$

The above imply that $\lambda_- < \lambda_+$. The sufficient conditions for existence is such an n_σ that

$$\begin{cases} \dfrac{\sigma^2 dt}{n_\sigma^2 \Delta^2} < 1 & \Rightarrow \quad \lambda_- + \lambda_+ < 1 \\ \dfrac{1}{2}\left(\dfrac{\sigma^2 dt}{n_\sigma^2 \Delta^2} - \dfrac{1}{n_\sigma}\right) > 0 & \Rightarrow \quad \lambda_- > 0 \\ \dfrac{1}{2}\left(\dfrac{\sigma^2 dt}{n_\sigma^2 \Delta^2} + \dfrac{1}{n_\sigma}\right) < 1 & \Rightarrow \quad \lambda_+ < 1 \end{cases}$$

If $n_\sigma \geq 1$, then the third constraint is implied by the first constraint. The combination of the first two constraints jointly imply

$$\frac{1}{n_\sigma} < \frac{\sigma^2 dt}{n_\sigma^2 \Delta^2} < 1$$
$$n_\sigma < \frac{\sigma^2 dt}{\Delta^2} < n_\sigma^2$$

Then $n_\sigma = 2$ and $\frac{\sigma^2 dt}{\Delta^2} \in [2, 4]$ satisfies this approximate constraint. The vector $(\lambda_-, 1 - \lambda_- - \lambda_+, \lambda_+)$ is, then, guaranteed to be a probability vector. We can rewrite the above constraint as

$$\frac{1}{n_\sigma^2} < \frac{\Delta^2}{\sigma^2 dt} < \frac{1}{n_\sigma}$$
$$\frac{\sigma\sqrt{dt}}{n_\sigma} < \Delta < \frac{\sigma\sqrt{dt}}{\sqrt{n_\sigma}}$$
$$\frac{\sigma\sqrt{dt}}{n_\sigma} < \frac{\overline{x} - \underline{x}}{N_x} < \frac{\sigma\sqrt{dt}}{\sqrt{n_\sigma}}$$
$$\frac{\overline{x} - \underline{x}}{\sigma\sqrt{dt}} \cdot \sqrt{n_\sigma} < N_x < \frac{\overline{x} - \underline{x}}{\sigma\sqrt{dt}} \cdot n_\sigma$$

Setting $n_\sigma = 1$ this can be satisfied precisely, but then it leads to a discreteness problem. Instead, set

$$N_x = \left[\frac{\sqrt{n_\sigma} + n_\sigma}{2} \frac{\overline{x} - \underline{x}}{\sigma\sqrt{dt}}\right].$$

Setting $n_\sigma = 2$ becomes

$$N_x = \left[\frac{\sqrt{2} + 2}{2} \cdot \frac{\overline{x} - \underline{x}}{\sigma\sqrt{dt}}\right].$$

Backward induction. Denote by $s \in \{0, \epsilon, 2\epsilon, \dots\}$ the number of periods remaining until period T which is sufficiently large. Write the principal's value function as $V(\mu, s)$ and we are interested in forward dynamics

$$V(s, X, l) \to V(s + \epsilon, X, l)$$

1. In the final period the fund family is forced to stop

$$V(0, X, l) \equiv V$$

pinning down the initial condition.

2. In the period before last the fund family chooses whether to stop immediately or wait for an additional period.

$$V(X,t) = \max\Big[M,\ \big(A(q(X,t+dt)) - A(k(X_t,t+dt))\big)\cdot dt$$
$$+e^{-rdt}\Big((1-2\lambda_k(\epsilon))V(\mu,0) + \lambda_k(\epsilon)V(\mu-\Delta,0) + \lambda_k(\epsilon)V(\mu+\Delta,0)\Big)\Big]$$

Determine threshold $\underline{\mu}(\epsilon)$ at which the family chooses to stop.

3. Continue solving it forward

$$V(\mu,s+\epsilon) = \max\Big[M,\ \big(fA(q(\mu,k,s+\epsilon)-f) - hA(k-h)\big)\cdot\epsilon$$
$$+e^{-r\epsilon}\Big((1-2\lambda_k(s+\epsilon))V(\mu,s) + \lambda_k(s+\epsilon)V(\mu-\Delta,s) + \lambda_k(s+\epsilon)V(\mu+\Delta,s)\Big)\Big]$$

Determine threshold $\underline{\mu}(s)$ at which the family chooses to stop.

C.1.3 Inverting the Stopping Surface

We have

$$k_t = \frac{\phi}{\phi t+1}\cdot X_t + \frac{1}{\phi t+1}\cdot l_t$$
$$X_t = \frac{(\phi t+1)k_t - l_t}{\phi}.$$

The stopping boundary is $X < S(l,t)$. Then

$$\frac{(\phi t+1)k - l}{\phi} < B(l,t)$$
$$(\phi t+1)k - l \leq \phi B(l,t)$$
$$k \leq \frac{\phi}{\phi t+1}\cdot B(l,t) + \frac{1}{\phi t+1}\cdot l.$$

The set of boundaries is

$$k \in \left[\underline{k}_t, \overline{k}_t\right] = \left[\frac{\phi}{\phi t+1}\cdot B\left(\underline{l},t\right) + \frac{1}{\phi t+1}\cdot\underline{l},\ \frac{\phi}{\phi t+1}\cdot B\left(\bar{l},t\right) + \frac{1}{\phi t+1}\cdot\bar{l}\right].$$

Note that $B(\bar{l},t) = +\infty$ for every t. Thus $\overline{k}_t = +\infty$ and we only have the lower constraint $k_t \geq \underline{k}_t$. If $k_t < \underline{k}_t$ then it is optimal to stop immediately. The intuition is that even for the most generous l it is still optimal to stop immediately.

For each k satisfying the above inequality define $l(k,t)$ such that

$$k = \frac{\phi}{\phi t+1}\cdot B(l(k,t),t) + \frac{1}{\phi t+1}\cdot l(k,t).$$

If $k \to \underline{k}_t$, then $l \to \underline{l}$. Also, $l(k,t)$ is increasing in k. Once k drops below $\underline{k}_t$ then it is optimal to stop immediately.

$$\hat{B}(k,t) = B(l(k,t),t) = \frac{(\phi t+1)k - l(k,t)}{\phi}.$$

The optimal stopping rule is

$$X \leq \begin{cases} \dfrac{(\phi t+1)k - l(k,t)}{\phi} & if \quad k \geq \underline{k}, \\ +\infty & if \quad k < \underline{k}. \end{cases}$$

Note that we can smooth out the above solution a bit.

Is it also true that if $k < \underline{k}$, then it necessarily implies that $X < B(\underline{l},t)$? Yes. So this implies I can adjust above to be continuous

$$X \leq \begin{cases} \hat{B}(k,t) = B(l(k,t),t) & if \quad k \geq \underline{k}, \\ B(\underline{l},t) & if \quad k < \underline{k}. \end{cases}$$

Computing $l(k,t)$. Have

$$\frac{\phi}{\phi t+1} \cdot B(l[i-1],t) + \frac{1}{\phi t+1} \cdot l[i-1] < k < \frac{\phi}{\phi t+1} \cdot B(l[i],t) + \frac{1}{\phi t+1} \cdot l[i]$$

Then have

$$l(k,t) = l[i-1] + \frac{k - \frac{\phi}{\phi t+1} \cdot B(l[i-1],t) - \frac{1}{\phi t+1} \cdot l[i-1]}{\frac{\phi}{\phi t+1} \cdot B(l[i],t) + \frac{1}{\phi t+1} \cdot l[i] - \frac{\phi}{\phi t+1} \cdot B(l[i-1],t) - \frac{1}{\phi t+1} \cdot l[i-1]} \cdot \Big(l[i] - l[i-1]\Big)$$

$$l(k,t) = l[i-1] + \frac{(\phi t+1)k - \phi B(l[i-1],t) - l[i-1]}{\phi B(l[i],t) + l[i] - \phi B(l[i-1],t) - l[i-1]} \cdot \Big(l[i] - l[i-1]\Big)$$