On Bounded Rationality and Risk Aversion

By

Markus K. Brunnermeier

DISCUSSION PAPER NO 255

FINANCIAL MARKETS GROUP
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ISSN 0956-8549-255

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LSE FINANCIAL MARKETS GROUP

DISCUSSION PAPER SERIES

January 1997

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On Bounded Rationality and Risk Aversion - A Non-technical Summary -

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December 9, 1996

Abstract

Experimental evidence suggests that agents who consume at a familiar income level are reluctant to accept income lotteries. However, these agents after facing an unexpected loss are willing to take on income lotteries they earlier rejected. Experiments show that after an unanticipated income drop they may even become risk loving. This paper provides a theoretical explanation for why we observe such risk attitudes. We consider a decision maker who does not always find her optimal consumption bundle with certainty. Such a boundedly rational decision maker is likely to err when choosing from thousands of different commodities. As she faces this problem repeatedly at her familiar income level, she is able to figure out more precisely her optimal

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bundle of goods. This, in turn, alters her attitude towards income lotteries. The paper shows in two different ways how bounded rationality as described above can explain the mentioned experimental findings.

On Bounded Rationality and Risk Aversion

Markus K. Brunnermeier *
The London School of Economics †

This Version: October 1997

Abstract

Experimental evidence suggests that agents who consume at their usual income level are very risk averse, whereas at lower income levels they often become risk loving. This paper provides a theoretical rationale for these experimental results. It shows that bounded rationality increases risk aversion at the reference income level. However, there is a range of lower income levels at which bounded rationality reduces risk aversion. A decision maker is boundedly rational if he, facing a new income, does not find his new optimal consumption bundle with certainty. This alters his indirect utility function and thus his attitude towards income lotteries. Bounded rationality is modelled in two ways. In the random choice approach the decision maker errs in choosing the new consumption bundle, while in the random utility approach he does not know precisely which bundle is his optimal one.
JEL Classification: D11

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* Special thanks are due to Ariel Rubinstein for his encouragement to write this idea down. I would like to thank in particular Margaret Bray and Kevin Roberts for their guidance. I am grateful for comments from Smita Bhatnagar, Bart Lipman, Volker Nocke, Rohini Pande, Ben Pollak, Sven Rady, Sönje Reiche, Klaus M. Schmidt, Patrick Schmitz, Mimi Tan and David E. Wildasin. Any errors are my own. Support from the University of Bonn is gratefully acknowledged.

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1 Introduction

Choosing the optimal consumption bundle out of thousands of commodities for a given income is a difficult task. This model describes a boundedly rational decision maker who does not always find his optimal consumption bundle with certainty. By facing the problem repeatedly, he can figure out more precisely which is his optimal bundle. It is therefore more likely that he will choose the optimal one at an income level he is used to, called the reference income level. This alters the decision maker's attitude towards income lotteries. Extending the standard model in this way helps to explain experimental findings like loss aversion, status quo bias etc. It also provides a theoretical reasoning as to why people become less risk averse after they faced an unexpected loss.

Aim of this study is to provide a theoretical explanation for experimental findings summarized by Tversky and Kahneman (1986). For these experiments the certainty equivalence method was applied. They show that there are some common reaction patterns in choices involving risk. The following value function (developed by Tversky and Kahneman) captures these patterns.

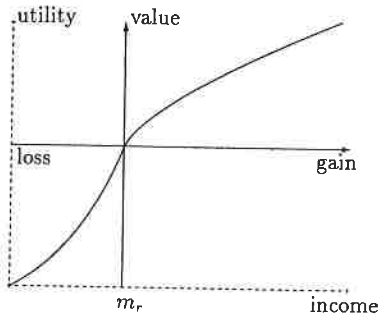

Figure 1: Value function of Tversky and Kahneman

One of the reaction patterns is that preferences are quite insensitive to small changes of wealth, but highly sensitive to corresponding changes in the reference income level, m_r. The reference income plays an important role in determining whether a lottery will be accepted or not. It is therefore useful to consider losses and gains with respect to this reference income level. A significant property is that individuals exhibit loss aversion, i.e. they are much more responsive to losses than to gains. In terms of a utility function, this means that a certain income decrease results in a much higher loss of utility than the utility gain associated with the same income increase. In other words the value function is steeper in the loss region than in the gain region. This leads to risk aversion at the reference income level. Another stylised fact states that the corresponding utility function becomes convex in the loss region. This convexity implies local risk loving behaviour.

In order to explain these findings we will base our theoretical analysis on the classical model framework, where the decision maker has a complete, reflexive and transitive preference ordering over the space of commodity bundles. We show that the Tversky-Kahneman value function can be explained without relaxing the von Neumann-Morgenstern utility axioms. These axioms allow us to represent both the preferences over lotteries and commodity bundles by an affine transformable utility function over the commodity space. Keeping the analysis within these axioms highlights the impact of our approach on the risk attitude of decision makers. Although our analysis can explain the value function developed by Tversky and Kahneman, it cannot explain all experimental results. To achieve this a departure of the von Neumann-Morgenstern axioms and/or a different approach (e.g. Prospect Theory Kahneman and Tversky (1979)) are necessary. Nice summaries of this fast growing literature are given in Camerer (1995) and Hey (1997). In our analysis we will relax the rationality assumptions only slightly in order to derive the relationship between risk aversion and bounded rationality. In the random choice approach, as described in Section 3, the decision maker has to learn "how to choose the most preferred consumption bundle". Since he has chosen his consumption bundle at his reference income level many times before, he is very familiar with this choice. Furthermore, it was worthwhile for him to put a lot of thought into choosing the right consumption bundle at this income level, since he had expected to consume at this reference income level many times over. It is therefore reasonable to assume that he will not make errors (or at least he will not make "larger" errors) at the reference income level. At a different income level, there is scope for the decision maker to err. In Section 4 we employ a different approach. The decision maker has not processed all the information perfectly to figure out his 'true' preference ordering for the entire commodity space. Using reasoning similar to that in Section 3, he knows his optimal consumption bundle at his reference income level. Therefore he can exclude all utility functions which do not lead to this optimal consumption bundle at this reference income level. Both approaches result in reference-dependent preference models, which were introduced by Tversky and Kahneman (1991). We show that in both approaches bounded rationality increases risk aversion at the reference income level and there exists a range of income levels below the reference income where bounded rationality reduces risk aversion. In summary, all three aforementioned properties of the Tversky-Kahneman value function can be explained by our analysis. In Section 5 we draw some conclusions and give some general applications and implications of these results in other economic fields. Before we focus on the impact of bounded rationality on attitudes towards risk, we briefly explain in Section 2 the concept of bounded rationality used in this paper.

2 Bounded Rationality

At any point in time the different elements that constitute reality are related to each other in various complex ways. There is no hope for human beings to understand such a complex system of relations, i.e. to completely grasp reality. Theoretical models, which are a simplified picture of reality, do not claim to describe the true relations, but rather deliver a less complex "as if" relation, which mimics the true unknown relation (Friedman 1953). Models are the biggest concession towards bounded rationality. The following relations are often used in economic models:

$\mathbf{x_n}$	$\overset{\text{relation}}{\longleftrightarrow}$	$\mathbf{x_m}$
input	$\overset{\textit{production function}}{\longleftrightarrow}$	output
action	$\overset{\textit{game}}{\longleftrightarrow}$	payoff new action space
commodity bundle A	$\overset{\textit{preference ordering}}{\longleftrightarrow}$	commodity bundle B
lottery C	$\overset{\textit{preference ordering}}{\longleftrightarrow}$	lottery D

The focus of our analysis is on the preference ordering over commodity bundles and lotteries. Two factors are integral to determining the decision maker's accuracy and speed of grasping the true relationship. The first factor is the information available. The second is the degree of bounded rationality, i.e. the decision makers' ability to process the available information.

Rational decision makers do not have cognitive limitations. They are only restricted by the availability of data. They do not have to bear any calculation costs or exert any extra effort in thinking and do not have to worry about time constraints. All calculations are executed without any error. Moreover, they will not forget anything, since they do not have any memory costs. Rationality can be characterized as instantaneous, costless learning by using all available information. Models with rational decision makers are useful if the bounded rationality effects are small or "average out" by aggregation.

Boundedly rational decision makers, on the other hand, are not only restricted by the availability of information but also in their ability to learn. One question which arises is whether not processing information is equivalent to a lack of information (or similarily, not processing information or data correctly is equivalent to having only imprecise data). In order to save information processing costs or time, boundedly rational decision makers apply simplified thinking and calculation procedures. Since boundedly rational decision makers act within the modeller's framework, they in turn will form a simpler model of this model's framework, i.e. they form heuristics. The answer to the question which heuristic (simplified learning rule) decision makers use goes beyond the scope of this paper.

Irrational decision makers are also restricted in their learning abilities, but unlike boundedly rational individuals they do not take these constraints into account. They just randomise without reasoning. They can be viewed as decision makers with infinitely high information processing costs.

The application of a heuristic, e.g. a cut off learning rule, in choosing the consumption bundle affects individual's risk aversion behaviour. A fully dynamic model should capture the whole learning process. As long as the optimal consumption bundle cannot be derived without cost/effort, our results hold for any possible learning process. Rather than modelling a certain learning process, which does not provide any additional insight, we restrict our analysis to the static consumption problem after the learning process is completed and the reference point is determined. This highlights the ingredients necessary to explain the experimental findings.

3 Random Choice Approach

In the random choice approach we relax the rationality assumptions slightly by assuming that the decision maker has to learn how to choose the optimal commodity bundle. Since he is boundedly rational he will apply a heuristic. This heuristic allows him to choose a consumption bundle faster and with less effort, but at a cost of possible deviations from the optimal bundle. He is willing to put much more effort into thinking and is willing to invest more time in finding the optimal consumption bundle, since the decision maker expects to frequently face the same maximisation problem at the reference income level. Therefore, the deviations from the optimal consumption bundle at the reference income level are much smaller. For simplicity we assume that he chooses the optimal bundle, $x^*(m_r, p_r)$ at the reference income level, m_r, given reference prices, p_r.[1] A decision maker is aware of the fact that if he accepts a lottery over income he has to choose a new consumption bundle at a possibly different income level. At this new income level he will choose the optimal consumption bundle with less than probability one. Therefore if a boundedly rational decision maker accepts a lottery he actually faces a compound lottery. At the first stage a lottery outcome over income is drawn and at the second stage he faces another lottery caused by the error he makes. The latter one has always negative expected value, because making errors can only worsen his situation.

[1] As we keep prices constant in our analysis, we will drop p as argument in the functions.

The effect of applying heuristics on risk aversion is formalized in the following way.

Assumptions

A 1 $u(x) : \mathbb{R}^k_+ \mapsto \mathbb{R}$, the utility function represents a complete, reflexive and transitive preference ordering over the commodity space $\mathbb{R}^k_+$

A 2 $u(x)$ is a von Neumann Morgenstern utility function,
i.e. $\tilde{x} \succeq \tilde{x}' \iff E\, u(\tilde{x}) \geq E\, u(\tilde{x}')$,
where $\tilde{x}$ and $\tilde{x}'$ are random commodity bundles

A 3 $u(x)$ is weakly increasing, and strictly increasing in at least one of its arguments

A 4 $u(x) \in C^2(x)$ such that all resulting indirect utility functions are also well defined and twice continuously differentiable

A 5 The actual consumption bundle chosen at reference income m_r is
$x^E(\hat{x}, m_r) := \hat{x} + \underline{\tilde{e}}(\hat{x}, m_r)$
where
$\hat{x}$ is the target bundle the decision maker tries to achieve
$\underline{\tilde{e}}$ is the error captured by a k-dimensional random variable (function) in the state space $\mathbb{S} = \{1, ..., S\}$ with a subjective probability distribution Π

An optimal target bundle to aim for is given by
$\hat{x}^*(m, m_r) \in \arg\max\{E_s u(\hat{x} + \underline{\tilde{e}}(\hat{x}, m_r))$ s.t. $p\hat{x} \leq m\}$
whereas an optimal bundle is given by
$x^*(m) \in X^*(m) := \arg\max\{u(x)$ s.t. $px \leq m\}$
To simplify notation let $\tilde{e}_x(m, m_r) := \underline{\tilde{e}}(\hat{x}^*(m, m_r), m_r)$,
$\tilde{e}_x(\cdot)$ is such that
(i) $x^E \subset \mathbb{R}^k_+$ *(no negative consumption)*
(ii) $px^E \leq m\ \forall x^E$ *(affordability)*
(iii) there exists $\hat{x}^*(m_r, m_r) \in X^*(m_r)$
s.t. $\tilde{e}(m_r, m_r) = 0$ *(no error at m_r)*
(iv) if possible under (i) - (iii)
for given m_r, there exists for each $\hat{x}$
at least one s' with $\pi_{s'} > 0$ s.t.
$x^E_{s'}(\hat{x}, m_r) \notin X^*(m)$ *(error possibility)*

(v) $\exists\ \hat{x}^*(m, m_r)$
s.t. $e_{x,s}(m, m_r) \in C^2(m)\ \forall s$ *(smoothness)*
(vi) decision maker can choose for all $\underline{m} < m$
$x^E(\underline{m}, m_r) + x_i \frac{m-\underline{m}}{p_i}$, where for x_i, $\frac{\partial u}{\partial x_j} > 0$.
This implies that he strictly *prefers higher income.*

A1 - A4 are standard utility assumptions. We suppose that the decision maker is still fairly rational since he has a transitive preference ordering which satisfies the von Neumann Morgenstern axioms. The assumption that the indirect utility function is twice continuously differentiable together with A5(v) allows us to apply the Arrow-Pratt risk aversion measure. The difference from standard microeconomic models lies in A5, which states that the decision maker makes errors if he is not consuming at his reference income level. He tries to consume target bundle $\hat{x}$ but ends up consuming x^E.[2] Note that the decision maker need not necessarily aim for an optimal bundle, since he knows the distribution of the error term. This can especially be the case when the error term is biased. If e.g. the decision maker will always end up buying accidently too much chocolate, it is probably useful for him to aim at a consumption bundle with less chocolate than in the optimal consumption bundle. A5(i) rules out negative consumption for any commodity. A5(ii) guarantees that the errors are such that the decision maker spends not more than his income. A5(iii) assumes that the decision maker knows how to choose the optimal consumption bundle at the reference income level. In other words he makes no errors at the reference income level, m_r. This can be easily relaxed to the case where the decision maker makes only errors at the reference income level, which have no utility impact. A5(iv) states that the boundedly rational decision maker can make significant errors if he is not at his reference income level. In the case of strictly quasiconcave utility functions any possible error reduces the expected utility. A5(vi) rules out that an increase in income makes the decision maker worse off. In other words, the increase of the error due to higher income has a lower impact on the

[2]There are several possible explanations why the decision maker might not consume his target consumption bundle $\hat{x}$, but x^E. Consider for example a setting where the decision maker does not buy all commodities at once but sequentially. Due to some miscalculation he might buy to many of those goods he purchases in the beginning. He then has not enough money left for the remaining commodities. In the random utility approach (Section 4) the error $x^E - \hat{x}$ results from the ignorance of the optimal consumption bundle.

expected utility than the enlargement of the budget set. This assumption is plausible since the decision maker does not need to spend all of his income. Consequently higher income does no harm. As he has by A3 the opportunity to spend the remaining income for the commodity which leads to a strict increase in his utility, he will always strictly prefer higher income and spend all his money.

To clarify the analysis we define some indirect utility functions and the standard Arrow-Pratt risk aversion measure.

Definitions

D 1 Indirect utility function when decision maker makes no errors

$$v(m) := u(x^*(m))$$
$$\text{where } x^*(m) \in \arg\max\{u(x) \text{ s.t. } px \leq m\}$$

D 2 'Best' indirect utility function when decision maker is boundedly rational, i.e. he knows that he will make some errors

$$Ev^E(m, m_r) := E_s u(\underbrace{\hat{x}^*(m, m_r) + \tilde{e}_x(m, m_r)}_{x^E(\hat{x}^*, m_r)})$$
$$\text{where } \hat{x}^*(\cdot) \text{ and } \tilde{e}_x(\cdot) \text{ are defined in A5}$$

Since it is not sure whether a boundedly rational decision maker can derive his optimal target consumption bundle, $\hat{x}^*$, $Ev^E(\cdot)$ is only an upper bound for his indirect utilility function. Our analysis can be applied to any possible indirect utility function of the boundedly rational decision maker as long as it is twice continuously differentiable and strictly increasing. Note, by A5(vi) $Ev^E(m)$ is strictly increasing.

To simplify notation we drop m_r as arguments in all of the indirect utility functions, since m_r is constant in our analysis.

D 3 The functional $f(v)$ relates the indirect utility function $v(m)$ of an identical *rational* decision maker to the indirect utility function $Ev^E(m)$ of a *boundedly rational* decision maker.
By A3 $v(m)$ is strictly increasing in m. Therefore there exists

the inverse of $v(m)$, $h(v(m)) = m$.
Let

$$f(v) := Ev^E(h(v))$$

Since $v(m)$ is twice continuously differentiable in m, so is $h(v)$; and since $Ev^E(m) \in C^2(m)$, $f(v) \in C^2(v)$.

D 4 Arrow-Pratt measure of (absolute) risk aversion for the indirect utility function of the *rational* and of the *boundedly rational* decision maker

(i) for v: $RA^v(m) := -\frac{\partial^2 v/\partial m^2}{\partial v/\partial m}$

(ii) for Ev^E: $RA^{Ev^E}(m) := -\frac{\partial^2 Ev^E/\partial m^2}{\partial Ev^E/\partial m}$

D 5 Risk aversion contribution of bounded rationality

$$RAC(m) := RA^{Ev^E}(m) - RA^v(m)$$

This definition allows us to separate risk aversion into two parts, one being the actual risk aversion given by the concavity of the utility function and the other being the risk aversion contribution of bounded rationality induced by the optimal consumption bundle not being chosen. By Lemma 1 it is clear that the 'additive' definition of the risk aversion contribution in D5 is reasonable. Lemma 1 has the same flavour as part of Pratt's theorem (Pratt 1964). All proofs are presented in the appendix.

Lemma 1

$$RAC(m) = -\frac{\partial^2 f/\partial v^2}{\partial f/\partial v}\frac{\partial v}{\partial m}$$

Lemma 1 relates the risk aversion contribution term to the functional $f(v)$, which facilitates proving the following propositions.

Lemma 2 shows that an error reduces the decision maker's expected utility, since a non-optimal commodity bundle is consumed with positive probability. This is not the case at a zero income level, since no consumption takes place, and at the reference income level. At the reference income level, m_r the decision maker has learnt how to choose the optimal consumption bundle.

Lemma 2

A 'focal point' is an isolated income level where bounded rationality has no impact on the utility level. At all other income levels bounded rationality strictly reduces the indirect utility function.
'Focal points' are $c_1 = 0$ and $c_2 = m_r$, i.e. e.g. for $Ev^E(m)$
(i) $Ev^E(m) = v(m)$ *for* $m \in \{0, m_r\}$
(ii) $Ev^E(m) < v(m)$ *for* $m \in \mathbb{R}_+ \setminus \{0, m_r\}$.

Lemma 2 illustrates that the indirect utility function of an identical rational decision maker (who makes no errors) is an upper envelope for the indirect utility function of the boundedly rational agent. For the two focal points are 0 and m_r an example is illustrated in figure 2. The focal point $c_1 = 0$ depends on the assumption A5(i) $x^E \subset \mathbb{R}^k_+$, which states that consumption of any commodity cannot be negative. This binds the space for the error term. With decreasing income this space decreases and at zero income the possible consumption set is the single point 0, i.e. the error term vanishes. In other words, since at a zero income level only zero consumption is possible, there is no possibility to err. In the unrealistic case with negative consumption the space for the possible errors need not shrink with decreasing income. Therefore Lemma 2 does not hold in this case. Assumption A5(i) turns out to be important for Proposition 2.

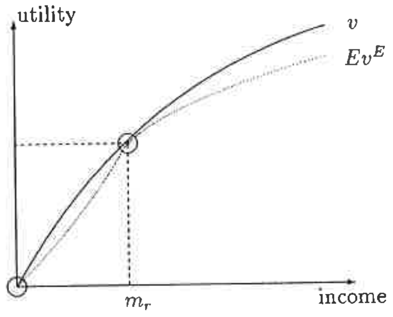

Figure 2: Indirect utility functions of a rational and boundedly rational decision maker

Using Lemmas 1 and 2 we can show that bounded rationality (defined as making small errors in choosing the optimal commodity bundle) increases risk aversion at the reference income level. This is consistent with experimental results.

Proposition 1

Bounded rationality increases absolute risk aversion at the reference income level,

(i) *i.e.* $RAC(m_r) > 0$;

(ii) *a boundedly rational decision maker strictly prefers the reference income level,* m_r, *to* **any** *lottery whose certainty equivalence for an identical rational decision maker is* m_r.

Lotteries are much less attractive for a boundedly rational decision maker because for each outcome of the lottery, leading to an income different from 0 or m_r, he cannot be sure to consume an optimal consumption bundle. The indirect utility function $Ev^E(m)$ can be thought of as resulting from maximisation behaviour subject to an additional constraint, as given by A5.

Given that the indirect utility function of an identically rational individual is the envelope of $Ev^E(m)$ with tangent point at the reference income level, Proposition 1 seems obvious. Proposition 1 is in the same vein as the Le Châtelier principle, especially if one assumes that the decision maker learns to choose the new optimal consumption bundle over time. The Le Châtelier Principle states that the response of optimised variables to a small structural change to the system is reduced, the more constraints are added to how the variables can be changed. In our case $x^*(m)$ maximises the utility function for a given income, m. The additional constraints how x^* can change as m departs from m_r are given by Assumption A5. The Le Châtelier principle fails globally since the constraint summarised by Lemma 2(ii) is not binding at two distinct income levels (Roberts 1996).[3] This Proposition also shows that a decision maker is more risk averse if he has spent a huge amount of his money on durable goods. A sudden income change constrains him from adjusting to the new optimal consumption bundle. He has still to consume the durable commodities, which he bought in previous periods. It is interesting that the expected riskiness of the income stream together with his risk aversion determines the amount he is willing to spend on durable commodities, which, in turn, influences his risk aversion.

Proposition 1 emphasises the importance of considering the reference point for the analysis of risk behaviour and shows why responses in utility to losses are more extreme than responses to gains. It also gives a possible explanation as to why we observe a so-called "status quo bias" (Samuelson and Zeckhauser 1988) or "endowment effect" (Thaler 1980, Kahneman, Knetsch, and Thaler 1991) in experiments. In our model these effects are due to a change in income which results in costs incurred (effort exerted), since the decision maker has to think about choosing a new optimal consumption bundle. A decision maker must be compensated for the additional costs of thinking arising from a lottery over income, since this induces him to find a new commodity bundle. This was the idea behind Proposition 1.

One might argue that lotteries over income can be diversified by borrowing and lending. Consequently, Proposition 1 will then only apply to uncertainty over someone's wealth level. In a world with many independent uncertainties, they may average out. Nonetheless, Proposition 1 refers only

[3] I am indebted to Kevin Roberts for pointing out this similarity.

to a (instantenous) per period utility function. A concave per period utility function causes then diversification over time. In addition, a perfect capital market does not exist in reality, and the decision maker also has to find out (think about) the wealth impact of a lottery outcome.

Since the risk aversion contribution term (a result of bounded rationality) is a real number, we can consider it as a measure of bounded rationality.

Whereas bounded rationality increases risk aversion at the reference income level, Proposition 2 claims that bounded rationality decreases risk aversion or even leads to risk loving behaviour at a lower income level. Our theoretical proposition can explain the experimental findings of Tversky and Kahneman (1986).

Proposition 2

There exists a range of incomes $(\underline{m}, \overline{m})$ *between two 'focal points' where bounded rationality reduces risk aversion or leads to risk loving behaviour,*
i.e. $RAC(m) < 0$.
For the 'focal points' $c_1 = 0$ *and* $c_2 = m_r$ *the income range* $(\underline{m}, \overline{m})$ *is within* $(0, m_r)$.

As mentioned above the result is driven by the fact that at a lower income level only smaller errors are possible. Lower income reduces the budget set within the actually chosen consumption bundle, $x^E = \hat{x} + e$ has to lie.

The range of income levels where risk aversion decreases because of bounded rationality is determined by both the error term and the utility function. There are three factors determining the size and locale of that range of income levels. First, since indifference curves at lower income levels are generally more curved the same error causes a higher disutility at a lower income level. Second, with lowering the income the distance to the reference income level increases. It is plausible that the variance of the error term increases with this distance. A larger variance in turn leads to a lower utility level for strictly quasiconcave utility functions. The third factor is that with lower income the error possibility space shrinks. At the extreme, for zero income there is no 'space' left for any error. One can show that the degree to which the error possiblity space shrinks depends on the number of available commodities. All three factors influence the size and locale of this income range. Whereas the first two suggest that the relevant range should be farther away from the reference income, the third pushes $(\underline{m}, \overline{m})$ closer to m_r.

By Lemma 2 it is obvious that both Propositions are not only true for $Ev^E(m)$ but for any strictly increasing indirect utility function which is twice continuously differentiable.

These results can be generalised to the case where the indirect utility functions are not twice differentiable. Since the traditional Arrow-Pratt risk aversion measure is not defined anymore in such a setting, local risk aversion can be measured by using the preference ordering over ϵ-income lotteries, by comparing their certainty equivalence. It is easy to see that Proposition 1 still holds, and so does a slightly modified Proposition 2.

This approach assumes that the decision maker knows his 'true' preference ordering but is not able to pick his optimal commodity bundle. However, in reality he has to learn his 'true' preference ordering. The implication of not knowing the exact true preference ordering will be the focus of the following section.

4 Random Utility Approach

As pointed out in Section 1, the decision maker has to learn the relationship between different commodity bundles, i.e. his preference ordering. An alternative interpretation for learning the preference ordering directly can be delivered if one considers a household production function. In this case the decision maker buys commodities for producing goods from which he derives his utility. It is plausible that the decision maker has to learn the household production function (e.g. from a cookbook), which is equivalent to saying that he has to learn his preference orderings over the commodity space. This interpretation shows that the same analysis can be applied to the production functions of firms.

Since there are time and effort costs to finding out the 'true' preference ordering for the boundedly rational decision maker, he will focus his learning primarily upon his relevant income level. Apart from focusing upon this reference income level he will also try to find the true preference ordering, but he is not willing to spend too much effort and time learning his 'true' preference ordering over the whole commodity space. In our analysis we assume that the decision maker exerts enough effort to find his most preferred commodity bundle at his reference income level, whereas at some different income he has a certain distribution over possible preference orderings. Each possible preference ordering is represented by a utility function $u_s(x)$. The difference from standard information economics lies in the fact that the boundedly rational decision maker does not know his true utility function, not because he is lacking information, but because he is not able to process all his information in time. In a situation where all the information has been perfectly processed and a rational decision maker still cannot figure out his 'true' utility function, we can consider each $u_s(x)$ as being the true expected utility function over the utility functions of the finest partition of the rational decision maker, $I_s \in \Im$. Note that since utility is linear, if one accepts the axioms of von Neumann Morgenstern, the decision maker still has a complete preference ordering over the whole commodity space despite whether or not any information has been processed.

The above described model is formalised in the following way:

Assumptions

A 1' The decision maker knows a (strictly positive) probability distribution $\Pi = \{\pi_1, \ldots, \pi_S\} > 0$ over the set of possible utility functions $\mathbb{U} = \cup_s u_s(x)$

A 2' All $u_s(x)$ satisfy the assumptions **A1 - A4**

A 3' All $u_s(x)$ are such that

(i) $x_s^*(m_r) = x_{m_r}^* \; \forall s,$
where $x_s^*(m_r) = \arg\max\{u_s(x) \text{ s.t. } px \le m_r\}$

(ii) $\cap_s x_s^*(m) = \emptyset \; \forall m \in \mathbb{R}_+ \setminus \{0, m_r\},$
where $x_s^*(m) = \arg\max\{u_s(x) \text{ s.t. } px \le m\}$

Assumption 3'(i) states that the decision maker knows his optimal consumption bundle at his reference income level, m_r. In other words he can rule out utility functions which do not lead to $x_{m_r}^*$ at m_r. We do not assume that he knows the utility level of this consumption bundle, let alone his utility function at his reference income level. The second part of A 3' rules out the case where at a certain income level all possible utility functions lead to the same optimal consumption bundle.

Proposition 3 illustrates the analogy between the effects on risk aversion in the random choice and the random utility approach. The fact that the 'true' optimal consumption bundle is not chosen with certainty drives these effects in both approaches. The proof in the appendix shows that the random utility approach can be reinterpreted in such a way that the assumptions of the random choice approach are satisfied.

Proposition 3

The random utility approach leads to the same effects on risk aversion as the random choice approach.

While these effects on risk aversion due to not choosing the optimal consumption bundle with probability one are the same for both approaches, in the random utility approach the actual 'true' risk aversion also depends on which of the possible utility functions $u_s(x)$ is the 'true' one. The boundedly rational decision maker does not know his true utility function and hence his true risk aversion. This is not the case if all possible utility functions $u_s(x)$ exhibit the same indirect utility function $v_s(m)$ $\forall s$. However, it is important to notice that a slight change in the relative prices will immediately destroy the property of this special case. But also for the case where the possible utility functions $u_s(x)$ lead to different indirect utility functions the decision maker will take the explained effects into account.

Boundedly rational decision makers probably do not know their exact 'true' preference ordering precisely *and* moreover they err. We do not need any further analysis to see that these results still hold if one combines the random utility approach and the random choice approach.

5 Possible Extensions and Conclusion

Our model can also explain the large discrepancy between one's willingness to buy/pay (WTP) and the willingness to sell/accept (WTA). This is e.g. one of the main problems in evaluating environmental projects (Cropper and Oates 1992). Although standard Hicksian analysis allows for some differene between WTA and WTP (Randall and Stoll 1980, Hanemann 1991), experimental evidence suggests a much larger discrepancy (Bateman, Munro, Rhodes, Starmer, and Sudgen 1997). Purchasing (selling) an additional unit of a certain good results in a reduction (increase) of available money for the remaining goods. This makes a reallocation amoung the remaining goods necessary. Since a boundedly rational decision maker makes errors in this reallocation, bounded rationality increases the difference between WTA and WTP.

Our analysis shows that one factor contributing to risk aversion is the fact that the decision maker must find his new consumption bundle after the outcome of the lottery has been realised. This requires that he incurs thinking costs *in* the realised state of the world. Evaluating a lottery is a much more difficult task because one does not only incur thinking costs in the realised state but in all possible states. Therefore the decision maker will apply a simpler heuristic in evaluating a lottery. It is then plausible that one will observe more misjudgements in decisions made about the acceptance of a lottery.

A related area of research examines the question of finding the *optimal planning horizon* in a world with uncertainty. Planning for distant future increases the number of states exponentially, which makes the maximisation problem much more complicated. Therefore, boundedly rational decision makers will apply a heuristic which is much more precise for short sighted problems. It remains to be shown that the optimal heuristic provides a fairly exact prediction for the near future and a rougher prediction for the distant future. It also seems plausible that increasing uncertainty levels makes people more short sighted, which can explain why high volatility in inflation rate, i.e. price uncertainty hurts the economy. The optimal planning horizon solution also provides an explanation for why we observe incomplete contracts, and the demand for flexibility or liquidity.

The model can be extended to include uncertainty in both income and *prices*. It is a well known fact that in traditional microeconomic models where the decision maker's utility function is quasiconcave and exhibits constant marginal utility of income, the decision maker is risk loving with respect to price uncertainty. This is due to the fact that he chooses his optimal consumption bundle after the prices are realised. In an analysis with error possibilities similar to ours, this risk loving behaviour need not be true. More insight might be gained if we separate the income from the substitution effect by means of a Slutsky decomposition. Endogenous substitution costs can then be derived.

As we pointed out in Section 4 the results of the paper are also applicable to the production sector. Each firm has to "learn" its production function by gathering know-how. But gathering know-how incurs costs, so the firm's production function is only well-understood locally. If the output or input prices were to change suddenly, new information would have to be gathered and processed. A single shock in a one industry sector can change all relative prices and hence affect the whole economy, i.e. all firms have to adjust to their new production plan. If these adjustments cause costs (e.g. the new relevant part of the production function must be learnt) then it is very likely that adjustments do not take place for every single shock.

We have mentioned examples where bounded rationality effects do not average out, thereby affecting the aggregate economy. In these cases it is important to incorporate these boundedly rational aspects of agent behaviour into economic theory in trying to attain a better understanding of the real underlying economic relationships.

Appendix

Proof of Lemma 1:

$$\begin{aligned}
\frac{\partial Ev^E}{\partial m} &= \frac{\partial f}{\partial v}\frac{\partial v}{\partial m} \\
\frac{\partial^2 Ev^E}{\partial m^2} &= \left(\frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial m}\right)\frac{\partial v}{\partial m} + \frac{\partial f}{\partial v}\frac{\partial^2 v}{\partial m^2} \\
-\frac{\partial^2 Ev^E/\partial m^2}{\partial Ev^E/\partial m} &= -\frac{\partial^2 f/\partial v^2}{\partial f/\partial v}\frac{\partial v}{\partial m} - \frac{\partial^2 v/\partial m^2}{\partial v/\partial m} \\
RA^{Ev^E} &= RAC + RA^v \qquad \mathcal{Q.E.D.}
\end{aligned}$$

Proof of Lemma 2:

(i) (1) for $m = 0$

Since $x_s^E \in \mathbb{R}_+^k$ and $px_s^E = m = 0\ \forall s\ \forall x_s^E$

$x_s^E(0, m_r) = x^*(0, m_r)\ \forall s$

(2) for $m = m_r$

By A5(iii) $\underline{e}_s(x_i^{a*}(m_r, m_r), m_r) = 0\ \forall s$.

Therefore $x^E(\hat{x}_i^*(m_r, m_r), m_r) \in X^*(m_r)$.

(ii) for each $m \in \mathbb{R}_+ \setminus \{0, m_r\}$

By definition D1, $v(m) \geq Ev^E(m)$.

By A5(iv) $\exists$ for each $\hat{x}$ at least one s' with $\pi_{s'} > 0$.

such that $x_{s'}^E(\hat{x}, m_r) \notin X^*(m)$. Therefore $\exists$ for each $m \in \mathbb{R}_+ \setminus \{0, m_r\}$ at least one s' with $\pi_{s'}$ such that $u(\hat{x}^* + e_{x,s'}) < u(x^*)$.

(iii) It follows immediately that (i) and (ii) is true not only for the upper bound $Ev^E(m)$, but also for any indirect utility function of the boundedly rational decision maker. $\mathcal{Q.E.D.}$

Proof of Proposition 1:

(i) By Lemma 1 $RAC(m) = -\frac{\partial^2 f/\partial v^2}{\partial f/\partial v}\frac{\partial v}{\partial m}$

It is sufficient to derive the signs for the three factors.

(1) $\frac{\partial v}{\partial m} > 0$,

since $u(x)$ is strictly increasing in at least one argument.

Let $g(v(m)) := f(v(m)) - v(m)$. Since $f(\cdot)$ and $v(m) \in C^2$, $g(\cdot) \in C^2$. By Lemma 2 $g(\cdot)$ has a local maximum at $v(m_r)$, Therefore $\frac{\partial g}{\partial v}\mid_{v(m_r)} = 0$ and $\frac{\partial^2 g}{\partial v^2}\mid_{v(m_r)} < 0$, which yields

(2) $\frac{\partial f}{\partial v}\mid_{v(m_r)} = 1 > 0$,

(3) $\frac{\partial^2 f}{\partial v^2}\mid_{v(m_r)} < 0$.

(ii) Take any lottery $\tilde{m}$ (with distribution F) whose certainty equivalence for an rational decision maker is m_r, i.e. $E_F[v(\tilde{m})] = v(m_r)$. By Lemma 2 for any realisation m_j of $\tilde{m}$, $Ev^E(m_j) < v(m_j)$. Thus, $E_F[Ev^E(\tilde{m})] < E_q[v(\tilde{m}] = v(m_r) = Ev^E(m_r)$.

$\mathcal{Q.E.D.}$

Proof of Proposition 2:

By Lemma 1 $RAC(m) = -\frac{\partial^2 f/\partial v^2}{\partial f/\partial v}\frac{\partial v}{\partial m}$

(1) $\frac{\partial v}{\partial m} > 0$ (see Proposition 1)

(2) $\frac{\partial f}{\partial v} > 0$,
since $\frac{\partial E v^E}{\partial m} = \frac{\partial f}{\partial v}\frac{\partial v}{\partial m}$ and v and Ev^E are strictly increasing in m.

(3) $\exists\ (\underline{m}, \overline{m}) \subset (c_1, c_2)$, s.t. $\frac{\partial^2 f}{\partial v^2} > 0$ on $(v(\underline{m}), v(\overline{m}))$.
This is shown in the following three steps:

(3.1) $\exists\ a \subset (v(c_1), v(c_2))$ s.t. $\frac{\partial^2 f}{\partial v^2}\mid_a > 0$.
By Lemma 2 $f(v(c_1)) = v(c_1)$ and
$f(v(c_1+\epsilon)) < v(c_1+\epsilon)$ for sufficiently small $\epsilon > 0$.
From this we can conclude that $\frac{\partial f}{\partial v}\mid_{v(c_1+\epsilon/2)} < 1$.
We also know from Proposition 1 that $\frac{\partial f}{\partial v}\mid_{v(c_2)} = 1$.
Applying the mean value theorem on $\frac{\partial f}{\partial v}(\cdot)$,
$\exists\ a \ \in (v(c_1+\epsilon/2), v(c_2))$ such that

$$\frac{\partial^2 f}{\partial v^2}\mid_a = \frac{\overbrace{\partial f/\partial v\mid_{v(c_2)}}^{=1} - \overbrace{\partial f/\partial v\mid_{v(c_1+\epsilon/2)}}^{<1}}{\underbrace{v(c_2) - v(c_1+\epsilon/2)}_{>0}} > 0.$$

(3.2) Since $f \in C^2$ and $\frac{\partial^2 f}{\partial v^2} > 0$ at a, this must be also true at $(a-\epsilon', a+\epsilon')$ for small $\epsilon' > 0$.

(3.3) Since $v(\cdot)$ is strictly increasing and continuous in m there exists for each $\vartheta \in (a-\epsilon', a+\epsilon')$ a corresponding m such that $\vartheta = v(m)$. $\mathcal{Q.E.D.}$

Proof of Proposition 3:

It is sufficient to show that the random utility approach can be reinterpreted such that assumptions A1-A5 of the random choice approach are satisfied.
For this purpose let

$e_x(m, m_r) := x^{Eu*}(m) - x_s^*(m)$
where
$x^{Eu*}(m) \quad \in X^{Eu*}(m) \quad := \arg\max\{E_s u_s(x) \text{ s.t. } px \leq m\}$
$x_s^*(m) \quad \in X_s^*(m) \quad := \arg\max\{u_s(x) \text{ s.t. } px \leq m\}.$

(1) A1-A4 are satisfied for the true $u_{s'}(x)$ by A2'.

(2) It remains to show that $\tilde{e}_x(\cdot)$ satisfies all restrictions of A5. (Prices are kept constant.)

(i) $x^{Eu*} \subset \mathbb{R}_+^k$ and $x_s^* \subset \mathbb{R}_+^k\ \forall s$
by A2' and A1.

(ii) $px^{Eu*} \leq m\ \forall x^{Eu*} \in X^{Eu*}$,
which is satisfied by definition of $X^{Eu*}(m)$.

(iii) $\exists\ x^{Eu*}(m_r) \in X_s^*(m_r)\ \forall s$ s.t. $\tilde{e}_x(m_r, m_r) = 0$
By A3'(i) $x_s^*(m_r) = x_{m_r}^* \forall s$, thus $x^{Eu*}(m_r) = x_{m_r}^*$.

(iv) $\exists$ for each $m \in \mathbb{R}_+ \setminus \{0, m_r\}$ (incorporates Lemma 2)
and for each $x^{Eu*}(m) \in X^{Eu*}(m)$ at least one
s' with $\pi_{s'} > 0$ such that $x_i^{Eu*}(m) \notin X_{s'}^*(m)$
This is satisfied by A3'(ii).

(v) $\exists\ x^{Eu*}(m)$ s.t. $e_{x,s}(m, m_r) \in C^2(m)\ \forall s$
By A2' and A4 all $u_s(x)$, $Eu_s(x)$ and all resulting indirect utility functions are twice continously differentiable. Thus there exist income expansion paths $x_s^*(m)\ \forall s$ and $x^{Eu*}(m)$ satisfying this property.

(vi) The decision maker always strictly prefers higher income since by A2' in conjunction with A3 all $u_s(x)$ are strictly increasing in at least one argument. $\mathcal{Q.E.D.}$

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