Public Information, Private Information and the Multiplicity of Equilibria in Co-ordination Games

By

Christian Hellwig

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Non-technical Summary:

Recent advances in game theory have shown that in coordination games with multiple equilibria, a unique equilibrium is selected, when the assumption of common knowledge is replaced by an assumption whereby players receive conditionally independent private signals about the game's payoffs. This uniqueness result has given rise to a variety of applications, in which this selection methodology has been used, for example to analyse the role of monetary flows ('hot money') in speculative attacks on a currency, or to discuss the role of an optimal policy of a lender of last resort to prevent bank runs.

In this paper, I analyse coordination games, in which players have access to both public and private information. I find that the equilibrium selection based on private information is not robust in the sense that public information reestablishes multiplicity of equilibria in the limit, as the level of noise goes to zero. Moreover, these equilibria converge exactly to the ones known from the common knowledge benchmark. In addition, I show that multiplicity is negatively related to the relative value of the private information and to the overall level of noise in the information. The nature of these results, as well as those obtained without public information can be explained by the fact that with public information (and only then), so-called higher-order uncertainty, i.e. uncertainty about other players' beliefs vanishes, as the level of noise goes to zero.

Besides their game-theoretical implications, these results have important implications for economic applications of coordination games. First, they question the validity of the analysis of bank runs or speculative attacks, based on the selection approach using private information. Second, they highlight the importance of clearly spelling out the informational environment of such market games. In particular, public information, which may come from various sources, such as market prices, public disclosures, or simply strategic revelation of private information, may have a destabilizing effect on markets, since players' strategies are much more sensitive to information in the common domain than to their private information.

Public Information, Private Information and the Multiplicity of Equilibria in Coordination Games

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Abstract

I study an example of a coordination game, and examine the robustness of equilibrium predictions with respect to changes in the information structure. I find two main results: First, the critique of Morris and Shin (1998) is not robust in the sense that if perfect common knowledge is viewed as the limit of imperfect information structures, multiple equilibria are maintained, as long as there exists some valuable public information. I also find that in general, the possibility of coordination is more likely to arise when the overall level of noise is low and when the public information is relatively informative. These results can be related to the structure of higher-order uncertainty: With a public signal, higher-order uncertainty vanishes, as the noise in the signals disappears.

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1 Introduction

Over the past couple of decades, models with multiple equilibria have had a prominent place in explaining possible coordination failures. Arguably the best known example are the bank runs in Diamond and Dybvig (1982). In the closely related context of speculative attacks against a currency peg, multiple equilibria are at the heart of so-called second generation models, following Obstfeld (1986).

All these models have a series of features in common. Most importantly, there are strategic complementarities, that lead to multiplicity of equilibria for some "critical region" of a state parameter. In the Diamond and Dybvig model of bank runs, it is optimal for a consumer to withdraw his savings early, if he observes that a sufficiently large fraction of investors withdraw as well, so that the bank will fail. Similarly, in the crudest versions of the multiple equilibrium model of speculative attacks, if all investors run on the currency, the central bank will abandon the peg. If investors anticipate this, they will run... On the other hand, if investors anticipate that the peg will be maintained, they will not run on the currency, and the central bank has no reason to devalue. The strength of an attack that will induce a single investor to join the run depends in general on an economic state variable, such as the bank's profitability, or, in the context of currency crises, the fundamental economic conditions of the country.

In their simplest form, these models also share some characteristics in their information structure: It is usually assumed that the realized state is commonly observed, i.e. is "common knowledge". Under this assumption, everyone knows the state, everyone knows that everyone else knows the state, everyone knows that everyone else knows that everyone else knows the state, and so on. As a result, all equilibria are rationalizable ex post, once the economic fundamental moves into the critical region. This indeterminacy is inconvenient for dynamic analysis or comparative statics, since the game does not deliver determinate predictions. For a long time therefore, qualitative conclusions drawn from this class of models were restricted to informal predictions as to how policies may alter anticipations, or act as a coordination device towards one equilibrium.

In recent years, this model of coordination failures has been criticized for the lack of realism in the assumption of common knowledge. Carlsson and van Damme (1993) argue that common knowledge should be viewed as a limit of imperfect information structures, in which the noise in the information goes to zero. They show for a general class of 2×2 coordination games that replacing the common knowledge assumption by an incomplete information structure, in which each player observes a private signal about the fundamental, leads to a unique equilibrium. In this equilibrium, there exists a cut-off state such that, if the fundamental falls below this cut-off, in the limit, one action pair will be played in equilibrium, while for higher fundamentals, the other equilibrium will be played. The intuition for the uniqueness result is straight-forward: Whereas the private signal may provide relatively precise information about the fundamental, it provides less information about what other players have observed, and it provides no information at all about one player's signal relative to all other players. Carlsson and van Damme (1993) use this higher-order uncertainty to show by iterated elimination of strictly dominated strategies that under this information structure, there exists a unique equilibrium.

The uniqueness result of Carlsson and van Damme has been used by a series of other authors, in order to determine a unique equilibrium in a coordination game, and to perform a comparative or dynamic analysis. Morris and Shin (1998, henceforth MS) apply this model to speculative attacks (see also the note by Heinemann, 2000 in this respect). In a related paper, Morris and Shin (1999) apply their result to coordination failures in the context of debt pricing. Goldstein and Pauzner (2000), in turn, apply the MS-methodology to the Diamond and Dybvig model of bank runs, and analyze how the stability of a bank depends on interest rates and debt contracts. Rochet and Vives (2000) use the MS-methodology to discuss the effects of a policy of a lender of last resort. Morris and Shin (2000a) provide an excellent survey of this literature.

In this paper, I reconsider the critique of coordination games raised by Carlsson and van Damme. As a starting point, if common knowledge is viewed as a limit of incomplete information structures, it is important to specify, how this limit is taken. As an alternative to MS, one could view common knowledge as a limit of an incomplete information structure, in which all players have access to a common noisy signal about the fundamental. It is straight-forward to show that the analysis of the perfect information game goes through with minor alterations. In this case, multiple equilibria exist, if the common signal falls inside the critical region.

This observation raises two questions, which I shall discuss in this paper: First, if agents have access both to public and private information, when will the information structure allow for multiple equilibria, and when will the equilibrium be unique? Second, if we take the view that common knowledge is the limit of an imperfect information structure, is this limit path-dependent?

In the context of the model discussed here, which is a direct extension of the model considered by MS, both answers are surprisingly clear.[1] With respect to the first question, I show that multiplicity of equilibria depends negatively on the level of noise and the relative importance of the private signal. These results are motivated by the role of higher-order uncertainty: If the level of noise is lower, or if the private signal is uninformative relative to the common signal, a single player has better information not only concerning the fundamental, but also concerning the beliefs other agents have about the fundamental.

With respect to the second question, the answer I provide is even more clear-cut, but less intuitive than the previous. I show that three equilibria are maintained whenever in the limit, the public signal has some value relative to the private signal. Thus, the case studied by MS must be considered a boundary case. Moreover, and more surprisingly, the three equilibria converge exactly to the three equilibria of the game with common knowledge, regardless of the relative importance of the public signal. In the limit, multiple equilibria exist, if the common signal falls into the critical region. The existing public signal provides some information, on which agents can coordinate their actions, and this is sufficient to generate multiplicity of equilibria.

We can also consider the role of the public signal, if the information structure does not allow for multiple equilibria and show that even if the public signal is very uninformative, it may have a disproportionate impact on equilibrium strategies. This result reflects a simple intuition: A player's strategy will be much more sensitive to public than to private information, since the

[1] The results presented here can easily be extended to the general class of coordination games studied by Carlsson and van Damme (1993).

public signal provides him with information not only about the state, but also about other players' beliefs, and therefore other players' actions. If this information is sufficiently precise, relative to private signals, the coordinating effect of public information becomes so large that it induces multiple equilibria. On the basis of this analysis, one would conclude that improved public information may be detrimental to market stability, as this increases the volatility of strategies, and possibly moves the economy into an area with multiple equilibria. Whereas for high levels of noise and higher-order uncertainty, a dramatic shift in strategies occurs at the cut-off fundamental, this cut-off can be perfectly predicted from the economic environment. If higher-order uncertainty is small, abrupt changes in strategies may occur, but insofar as they are due to changes from one equilibrium to another, they remain unpredictable within this game. Morris and Shin (2000b) make a similar point by analyzing the welfare effects of public disclosures in a coordination problem between a principal and two agents.

In explaining the nature of these results, it is useful to formalize the concept of higher-order uncertainty. For this purpose, it is useful to define "common beliefs"[2]. A common belief of a set of players S about an event E is the highest probability p, such that whenever E has occurred, all players in S believe with probability p or larger that E has occurred, believe with probability p or larger that all other agents in S believe with probability p that E has occurred... and so on. If for some event and some set of players, the common belief is 1 or 0, or converges to 1 or 0, as the noise in the signals vanishes, the event is or becomes common knowledge among the set of players. We can ask the following question, which is related to the multiplicity of equilibria: Under what conditions do the common beliefs resulting from a sequence of imperfect information structures converge to common knowledge? Again, we find that, as long as public information retains some value in the limit, as the noise vanishes, common beliefs converge continuously to 0 and 1. This result highlights the strong effect of the public signal on equilibrium structure and strategies. We also show that in the environment considered by MS, common beliefs are independent of the overall level of noise, and only depend on the measure of the set of players who have the common belief.

[2]Hierarchies of beliefs and the idea of common beliefs were first intoduced by Monderer and Samet (1989).

The remainder of this paper is divided into four parts, followed by a short conclusion, which sums up the main points. In part 2, I introduce an example of a coordination game which follows the MS methodology very closely, but allows for more general information structures. In part 3, I use this game to provide answers to the two main questions outlined in this introduction. In part 4, I examine how higher-order uncertainty depends on the information structure, and explore the link between common beliefs and the multiplicity of equilibria. In part 5, I briefly relate the results presented here to Harsanyi's Purification Theorem.

2 The Model

Consider a game played by a $[0,1]$-continuum of risk-neutral players, who all choose between two actions a and b. For any player, the pay-off to playing a is always r. The pay-off to playing b depends on a state parameter θ (economic "fundamental"), and on the proportion of players playing b. We assume that the pay-off to b is given by $f(\theta) > 0$, if the proportion of players who choose to play b exceeds $c(\theta)$, and 0, if the proportion of players who play b falls short of $c(\theta)$. If exactly a proportion $c(\theta)$ of players chooses to play b, playing b yields a pay-off $f(\theta)$ with probability $\frac{r}{f(\theta)}$. The timing of this economy is as follows: Initially, nature draws a fundamental $\theta \in \Re$. I assume that no information exists on the prior distribution of θ.[3] Each player then observes a possibly noisy signal about θ. After observing their signals and forming their beliefs concerning θ, players simultaneously decide whether to play a or b. We assume that $c(\theta)$ is increasing and continuously differentiable, and there exist $\underline{\theta}$ and θ', such that $c(\theta) = 0$ for $\theta \leq \underline{\theta}$ and $c(\theta) \geq 1$ for $\theta \geq \theta'$. We assume that $f(\theta)$ is strictly decreasing and continuously differentiable in θ, and there exists $\theta'' > \underline{\theta}$, such that $f(\theta'') = r$. Finally, let $\bar{\theta} = \min\{\theta', \theta''\}$.

Several observations can be made concerning this framework, which replicates the reduced-form model of speculative attacks against a currency in MS, or existing models of bank runs. First, this game exhibits multiple equilibria,

[3] Loosely speaking, one could say that nature draws the fundamental from a "uniform distribution" over the entire real line. Such an "improper prior" is not essential for the results obtained here. One could easily adapt the model to include a proper prior distribution of θ.

if θ is common knowledge and falls inside a "critical region" $\left[\underline{\theta}, \overline{\theta}\right]$. In one equilibrium, all players play a, in a second equilibrium, all players play b, and finally, there exists a mixed strategy equilibrium, in which players randomize with probability $c(\theta)$. If $\theta > \overline{\theta}$, all players play α in the unique equilibrium, while if $\theta < \overline{\theta}$, all agents attack in the unique equilibrium. Finally, the framework using a continuum of players has some analytical advantages. Assuming that the law of large numbers holds,[4] we can characterize the proportion of agents attacking the currency under a given strategy profile by the value of a cumulative distribution function of the signals across a population.

Within this framework, I shall examine several information structures. Generically, I assume that each agent observes a private signal (observed only by him) and a common signal (observed by all players). In the limiting case, where the variance of the common signal is infinite, we return to the structure studied by MS, in which only private information exists. Similarly, as the variance of the private signal becomes infinite, the economy converges to a situation in which there exists only public information. The situation of common knowledge can be considered as the limiting case of any of these three cases, as the variance of one or both signals converges to zero.

More explicitly, I assume that private signals are independent and identically normally distributed. Given a state θ, player j observes $x_j = \theta + v_j$, where $v_j \sim N\left(0, \varepsilon^2\right)$. Given θ, the common signal y is drawn from a normal distribution independent of all private signals. We have $y = \theta + v$, where $v \sim N\left(0, \eta^2\varepsilon^2\right)$. Let $x_j^1 = \alpha^2 x_j + \left(1-\alpha^2\right) y$, where $\alpha = \frac{\eta}{\sqrt{1+\eta^2}}$, be the posterior expectation of player j about θ. Standard statistical results imply that ex post, player j believes that θ, conditional on y and x_j is normally distributed with mean x_j^1 and variance $\alpha^2\varepsilon^2$.

$$\theta \mid x_j, y \sim N\left(x_j^1, \alpha^2\varepsilon^2\right) \tag{1}$$

By virtue of the law of large numbers, for a given realization of θ and y, posterior expectations are also normally distributed and given by

$$x_j^1 \mid \theta \sim N\left(\alpha^2\theta + \left(1-\alpha^2\right) y, \alpha^4\varepsilon^2\right) \tag{2}$$

[4]See Judd (1985) for a discussion of the role of the law of large numbers for a continuum of random variables.

The various cases mentioned above can be interpreted as special cases of this formulation. MS are concerned with the case, where $\alpha = 1$. In that case, $x_j^1 = x_j$, and agents draw no information from the public signal. If $\alpha = 0$, then $x_j^1 = y$ and all players have identical posterior beliefs. For $\alpha \in (0,1)$, players draw information from both signals.

3 Equilibrium analysis

3.1 Set-up

In this section, I analyze equilibria of the game outlined in the previous section. For simplicity, I restrict attention to symmetric "cut-off" equilibria, i.e. pairs of cut-offs (x^*, θ^*) such that a player plays b, whenever his posterior expectation falls below x, and receives a pay-off of $f(\theta)$, whenever $\theta < \theta^*$. These cut-offs will depend negatively on the public signal y, unless y has no informational value: Suppose there was a cut-off equilibrium with (x^*, θ^*) independent of y. Then a lower y would indicate a higher probability that other players play b, so that θ^* increases. As a consequence, the pay-off to b increases, and so will the cut-off expectation x^*.

For the moment, I shall take y as given. If the cut-off state, below which the pay-off to b is $f(\theta)$, is θ^*, a player will choose to play b, as long as its expected pay-off, given his posterior belief, satisfies

$$r \leq E\left(f(\theta) \mid \theta \leq \theta^*, x_j^1\right) \cdot \Pr\left(\theta \leq \theta^* \mid x_j^1\right) \tag{3}$$

One observes that the right-hand side of (3) is strictly decreasing in x_j^1. It follows that there exists a unique cut-off expectation x^*, given by the *pay-off indifference condition* (PI)

$$r = E\left(f(\theta) \mid \theta \leq \theta^*, x^*\right) \cdot \Pr\left(\theta \leq \theta^* \mid x^*\right) \tag{4}$$

at which a player is indifferent between a and b, while all agents with expectations below x strictly prefer β, and all agents with expectations higher than x strictly prefer a. (4) can be rewritten as

$$r = \frac{1}{\alpha\varepsilon} \int_{-\infty}^{\theta^*} f(\theta)\, \varphi\left(\frac{\theta - x^*}{\alpha\varepsilon}\right) d\theta \tag{5}$$

where $\Phi(\cdot)$ and $\varphi(\cdot)$ denote the cdf. and pdf. of a standard normal distribution, respectively. Let $x_{PI}(\theta^*)$ be implicitly defined by (5).

Next, suppose all agents follow a cut-off strategy x^*. For a given state θ, the fraction of players playing b is derived from (2) and is given by $\Phi\left(\frac{x^*-\alpha^2\theta-(1-\alpha^2)y}{\alpha^2\varepsilon}\right)$. The return to b is $f(\theta)$, iff

$$c(\theta) \leq \Phi\left(\frac{x^* - \alpha^2\theta - (1-\alpha^2)y}{\alpha^2\varepsilon}\right)$$

From the monotonicity in θ and x^*, it follows that the return to b is $f(\theta)$, if $\theta \leq \theta^*$, where θ^* is given by the *critical mass condition* (CM)

$$c(\theta^*) = \Phi\left(\frac{x^* - \alpha^2\theta^* - (1-\alpha^2)y}{\alpha^2\varepsilon}\right) \tag{6}$$

This can be rewritten as

$$x_{CM}(\theta^*, y) = \left(1-\alpha^2\right)y + \alpha^2\theta^* + \alpha^2\varepsilon\Phi^{-1}(c(\theta^*)) \tag{7}$$

$\Phi^{-1}(\cdot)$ indicates the inverse of the cumulative normal distribution function and is defined on $(0,1)$, so that x_{CM} is well-defined and strictly increasing for values of $\theta^* \in (\underline{\theta}, \theta')$. Furthermore, since x_{CM} converges to $+\infty$ or $-\infty$ as it approaches $\underline{\theta}$ and θ', respectively, and x_{PI} is continuous for any $\theta^* \in \Re$, $x_{PI}(\theta^*) = x_{CM}(\theta^*, y)$ for some $\theta^* \in (\underline{\theta}, \theta')$.

3.2 Characterization of equilibria

Using this set-up, I now turn the characterization of equilibria for generic information structures. *For given y, strategies in any symmetric cut-off equilibrium are implicitly defined by the intersection of the PI-condition with the CM-condition.* By examining (PI) and (CM) more in detail, we can give conditions for uniqueness or multiplicity. First, by integrating (5) by parts, and using the Implicit Function Theorem, we have

$$\frac{dx_{PI}(\theta^*)}{d\theta^*} = \frac{f(\theta^*)\frac{1}{\alpha\varepsilon}\varphi\left(\frac{\theta^*-x^*}{\alpha\varepsilon}\right)}{f(\theta^*)\frac{1}{\alpha\varepsilon}\varphi\left(\frac{\theta^*-x^*}{\alpha\varepsilon}\right) - \frac{\partial}{\partial x}\int_{-\infty}^{\theta^*} f'(\theta)\,\Phi\left(\frac{\theta-x}{\alpha\varepsilon}\right)d\theta\Big|_{x=x^*}} \leq 1 \tag{8}$$

Differentiating (CM) yields

$$\frac{\partial x_{CM}(\theta^*, y)}{\partial \theta^*} = \alpha^2 \left(1 + \frac{\varepsilon c'(\theta^*)}{\varphi\left(\Phi^{-1}\left(c(\theta^*)\right)\right)}\right) \geq \alpha^2 \tag{9}$$

For further analysis, we parametrize information structures by the ex post level of noise $\alpha\varepsilon$, and the informativeness of the common signal relative to the private signal, α. I shall first analyze the equilibria of limiting cases, where only public or only private information exists. The results are summarized in proposition 1:

Proposition 1 *(i) If $\alpha = 1$, there exists a unique, symmetric cut-off equilibrium. In the limit, as $\varepsilon \to 0$, $\theta^* - x^* \to 0$, and θ^* is given by $f(\theta^*)(1 - c(\theta^*)) = r$.*

(ii) For any ex post level of noise $\alpha\varepsilon$, as $\alpha \to 0$, there exist multiple cut-off equilibria, if θ falls inside a critical region. In the limit, as $\alpha\varepsilon \to 0$, the critical region converges to $\left(\underline{\theta}, \overline{\theta}\right)$, and $\theta^* - x^* \to 0$ for each equilibrium. As $\alpha\varepsilon \to 0$, there are three equilibria, which converge to $\theta^* = \underline{\theta}$, $\theta^* = \overline{\theta}$, and $\theta^* = y$.

Proposition 1 highlights the uniqueness result found in MS for the case where agents only observe a private signal. In addition, MS show by interim elimination of weakly dominated strategies that this cut-off equilibrium is the unique equilibrium of the game. Convergence to these cut-offs is such that (PI) and (CM) hold with equality. (ii) considers the case of public information only, and shows that under this structure, there exist multiple equilibria, if the common signal falls inside a critical region. From the characterization in the limit, it is easy to see that these three equilibria correspond to the equilibria known from the perfect information game, with the difference that in this case, y performs the same role as the perfectly observed fundamental in the previous. The first and the second equilibrium converge to the two pure strategy equilibria: In the limit, no agent will play β in the first equilibrium, if y falls inside the critical region, while in the second, everyone will play β in the limit. The third equilibrium corresponds to the mixed strategy equilibrium: In this case, θ^* and x^* converge to y in such a way that $\Pr(\theta \leq \theta^* \mid x^*)$ converges to $\frac{r}{f(y)}$ and $\Phi\left(\frac{x^* - \alpha^2\theta^* - (1-\alpha^2)y}{\alpha^2\varepsilon}\right)$ converges to $c(y)$.

Proof of Proposition 1: (i) follows directly from (8) and (9) for $\alpha = 1$. In the limit, the cut-offs can be calculated directly from the (PI) and (CM) conditions. For (ii), observe that (PI) only depends on $\alpha\varepsilon$, so it does not change, as we alter α only. (7) and (9) further imply that changes in y only shift the level of $x_{CM}(\theta^*, y)$ without affecting $\frac{\partial x_{CM}(\theta^*,y)}{\partial \theta^*}$ at any given value of θ^*. It also follows from (9) that as $\alpha \to 0$, for any θ, $\frac{\partial x_{CM}(\theta^*,y)}{\partial \theta^*}$ becomes arbitrarily small in absolute value. Therefore, for any θ, there exists y and α, such that $x_{CM}(\theta, y) = x_{PI}(\theta)$ and $\frac{\partial x_{CM}(\theta^*,y)}{\partial \theta^*} < \frac{dx_{PI}(\theta^*)}{d\theta^*}$. Along with the previously established limit properties of $x_{CM}(\theta^*, y)$, this is sufficient to show the existence of multiple equilibria. For $\alpha\varepsilon$ sufficiently low, $\frac{dx_{PI}(\theta^*)}{d\theta^*}$ is close to 1 and $\frac{\partial x_{CM}(\theta^*,y)}{\partial \theta^*}$ close to α^2 for any $\theta^* \in (\underline{\theta}, \theta')$. Hence, there can be at most three equilibria in the limit, whose properties can be directly calculated from the limits of $x_{CM}(\theta^*, y)$ and $x_{PI}(\theta^*)$.□

I now turn to the general case, where $\alpha \in (0,1)$. For this case, it is useful to characterize the limiting behavior of the (PI) and (CM) conditions, as the ex post level of noise $\alpha\varepsilon$ goes to 0, for a fixed value of α. This is done in the following lemma:

Lemma 2 *(i) As $\alpha\varepsilon \to 0$, $x_{PI}(\theta^*)$ converges to $\overline{x_{PI}}(\theta^*) = \min\{\theta^*, \theta''\}$.*
(ii) As $\alpha\varepsilon \to 0$, the pairs of (θ^*, x^*) satisfying (CM) converge to

$$\theta^* = \begin{cases} \theta' \text{ if } y + \frac{1}{\alpha^2}(x^* - y) \geq \theta' \\ y + \frac{1}{\alpha^2}(x^* - y) \text{ if } y + \frac{1}{\alpha^2}(x^* - y) \in (\underline{\theta}, \theta') \\ \underline{\theta} \text{ if } y + \frac{1}{\alpha^2}(x^* - y) \leq \underline{\theta} \end{cases}$$

Proof of lemma 2: (i) suppose $\theta^* > \theta''$. In the limit, $E(f(\theta) \mid \theta \leq \theta^*, x) \to f(\theta^*)$, if $\theta^* \leq x$, which implies that $x^* < \theta^*$. But then $\Pr(\theta \leq \theta^* \mid x^*) \to 1$, and x^* must satisfy $E(f(\theta) \mid \theta \leq \theta^*, x^*) \to r$, which implies $x^* \to \theta''$. On the other hand, if $\theta^* \leq \theta''$, in the limit, $x^* \leq \theta^*$, since otherwise $\Pr(\theta \leq \theta^* \mid x^*) \to 0$. If, in the limit, $x^* < \theta^* \leq \theta''$, then $\Pr(\theta \leq \theta^* \mid x^*) \to 1$ and (PI) must be violated. Thus, $x_{PI}(\theta^*) \to \min\{\theta^*, \theta''\}$, as $\alpha\varepsilon \to 0$.

(ii) (CM) is defined only on $(\underline{\theta}, \theta')$. Thus, if $y + \frac{1}{\alpha^2}(x^* - y) \geq \theta'$, it follows that $\Phi\left(\frac{x^* - \alpha^2\theta^* - (1-\alpha^2)y}{\alpha^2\varepsilon}\right) \to 1$ for any $\theta^* \in (\underline{\theta}, \theta')$, and $\theta^* = \theta'$

in the limit. similarly, if $y + \frac{1}{\alpha^2}(x^* - y) < \underline{\theta}$, $\Phi\left(\frac{x^*-\alpha^2\theta^*-(1-\alpha^2)y}{\alpha^2\varepsilon}\right) \to 0$ for any $\theta^* \in (\underline{\theta}, \theta')$ and $\theta^* = \underline{\theta}$ in the limit. If $y + \frac{1}{\alpha^2}(x^* - y) \in (\underline{\theta}, \theta')$, any θ^* such that $\Phi\left(\frac{x^*-\alpha^2\theta^*-(1-\alpha^2)y}{\alpha^2\varepsilon}\right) \to 1$ or $\Phi\left(\frac{x^*-\alpha^2\theta^*-(1-\alpha^2)y}{\alpha^2\varepsilon}\right) \to 0$ is contradicted since it would imply $\theta^* = \theta'$ or $\theta^* = \underline{\theta}$ as a solution. It follows that $\Phi\left(\frac{x^*-\alpha^2\theta^*-(1-\alpha^2)y}{\alpha^2\varepsilon}\right) \to c \in (0,1)$, which in turn implies that $\theta^* \to y + \frac{1}{\alpha^2}(x^* - y)$.□

Proposition 3 follows directly from lemma 2 and describes the limiting behavior of equilibria of this economy, as $\alpha\varepsilon \to 0$, holding $\alpha < 1$ fixed.

Proposition 3 *For any $\alpha < 1$, if $\alpha\varepsilon$ is sufficiently small, this economy has three equilibria, if y falls into a critical region. In the limit, as $\alpha\varepsilon \to 0$, the critical region is $y \in (\underline{\theta}, \overline{\theta})$, and equilibria converge to $\theta^* = x^* = \underline{\theta}$, $\theta^* = x^* = y$, and $\theta^* = x^* = \overline{\theta}$,. if $\theta'' \geq \theta'$, or $(\theta^*, x^*) = (\theta', \theta'')$, if $\theta' > \theta'' \geq (1-\alpha^2)y + \alpha^2\theta'$, or $(\theta^*, x^*) = \left(y + \frac{1}{\alpha^2}(\theta'' - y), \theta''\right)$, if $\theta' > (1-\alpha^2)y + \alpha^2\theta' > \theta''$.*

Proof of proposition 3: For any $\alpha < 1$, if $\alpha\varepsilon$ is sufficiently small, then $\frac{\partial x_{CM}(\theta^*,y)}{\partial\theta^*} < \frac{dx_{PI}(\theta^*)}{d\theta^*}$ at some θ^*, yielding multiple equilibria by the same argument as above. In the limit, equilibria and the critical region of y can directly be calculated using the linearity of (PI) and (CM).□

Proposition 3 contains the first important result. It states that the multiple equilibria obtained in the benchmark model with common knowledge continues to hold, if we view common knowledge as the limit of a structure with imperfect information. This result is true, as long as some common information exists and has some value in the limit.[5] Proposition 3 also shows that the equilibria qualitatively converge to the ones obtained under common knowledge. In the limit, multiple equilibria exist, if y falls into the same

[5] It is important to note that y cannot be interpreted as pure sunspots. If y has no informative value at all, then the MS-result continues to hold. Rather, in the case where the existence of sufficiently precise common information leads to multiple equilibria, traditional sunspots might be viewed as one way of selecting among them.

critical region as θ under common knowledge. In addition, the three equilibria have the same properties: In the first equilibrium, no player plays β in the limit, as $\alpha\varepsilon \to 0$. In the third equilibrium, everyone plays β if y falls inside the critical region (abstracting from the problems that arise, when the upper bound on the critical region is θ''). Finally, the second equilibrium converges to the mixed strategy equilibrium under common knowledge. In this equilibrium, the incomplete information structures with private information approximate the mixed strategy equilibrium in pure strategies, similar to Harsanyi's purification theorem.

One should note that proposition 3 holds for every value of $\alpha < 1$. In particular, the equilibrium strategies in the limit do not depend on α. To conclude, common knowledge can be viewed as the limit of a sequence of incomplete information structures in which public information retains some value.

While the previous paragraphs were concerned with results in the limit, as the posterior variance goes to 0, I now turn to the analysis of information structures with finite, positive levels of noise. For this purpose, I shall make the simplifying assumption that $f(\theta) = f$, i.e. the state θ affects the pay-off to β only through $c(\theta)$.[6] This simplifies the (PI)-condition to $\frac{r}{f} = \Phi\left(\frac{\theta^* - x^*}{\alpha\varepsilon}\right)$, or

$$\theta^* = x^* + \Phi^{-1}\left(\frac{r}{f}\right)$$

From proposition 1, we know that if the public signal is completely uninformative, there always exists a unique equilibrium, and if private signals are completely uninformative, there always exist multiple equilibria. Generically, there exist multiple equilibria, if $\frac{\partial x_{CM}(\theta^*, y)}{\partial \theta^*} < 1$, or if, for some $\theta \in (\underline{\theta}, \theta')$,

$$\frac{1-\alpha^2}{\alpha^2 \varepsilon} \geq \frac{c'(\theta)}{\varphi\left(\Phi^{-1}\left(c(\theta)\right)\right)}$$

Returning to the initial parametrization, $\frac{1-\alpha^2}{\alpha^2\varepsilon} = \frac{\varepsilon}{\eta^2\varepsilon^2}$. This condition is formally identical to a uniqueness condition in Morris and Shin (1999), who

[6]It becomes clear from lemma 2, that for low levels of noise, this provides an accurate description of the general case.

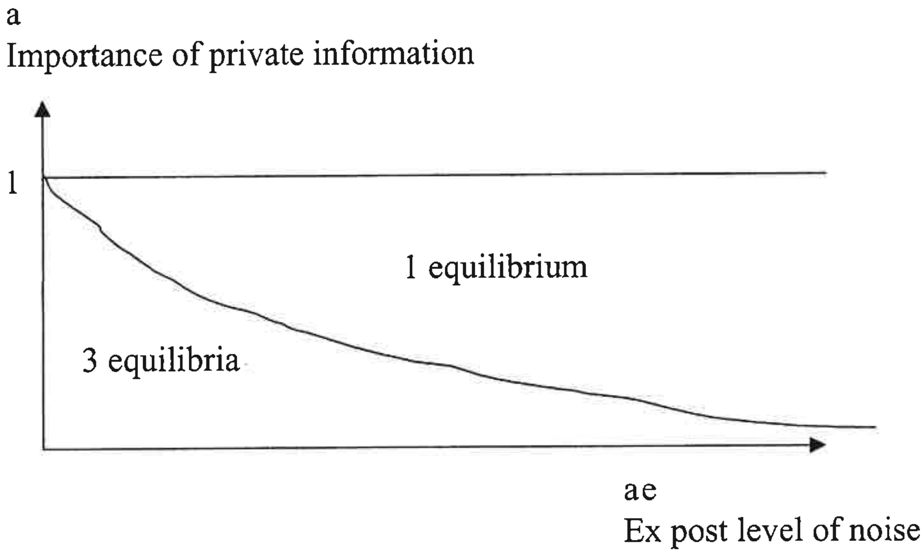

Figure 1:

consider a common normal prior along with private, normally distributed signals.

Lemma 4 *The zone of multiple equilibria is strictly decreasing in $\alpha\varepsilon$ and strictly decreasing in α*

The results of the previous paragraph and lemma 4 are summarized in figure 1.

We can also examine the effect of the common signal on the cut-off state and strategies, in order to examine how sensitive the equilibrium is to public information. Substituting (PI) into (CM), one obtains

$$\theta^* = y + \frac{\alpha^2\varepsilon}{1-\alpha^2}\Phi^{-1}\left(c\left(\theta^*\right)\right) - \frac{\alpha\varepsilon}{1-\alpha^2}\Phi^{-1}\left(\frac{r}{f}\right) \tag{10}$$

and

$$\frac{d\theta^*}{dy} = -\frac{1-\alpha^2}{\alpha^2\left(1+\frac{\varepsilon c'(\theta^*)}{\varphi(\Phi^{-1}(c(\theta^*)))}\right)-1} \tag{11}$$

Several observations follow immediately from (10) and (11). First, in the limit, as $\alpha\varepsilon \to 0$ and $\alpha \to 1$, the cut-off is not uniquely determined, but depends on the rate of convergence of $\frac{\alpha^2\varepsilon}{1-\alpha^2}$. Second, the effect of y on θ^* is potentially much stronger than the relative importance of the public signal. To be precise, the relative importance of the public signal $1-\alpha^2$ is multiplied by a publicity multiplier, which is strictly larger than 1, if ε is sufficiently small. This publicity multiplier goes to infinity, as the game approaches the zone of multiple equilibria. In other words, even if there is a unique equilibrium, public information may well have a disproportionately large impact on equilibrium strategies.

Overall, multiple equilibria are more likely to exist, when the ex post level of noise is low, and when the relative importance of the public signal is high. As the level of noise goes to zero, the economy always exhibits multiple equilibria, unless the public signal becomes perfectly uninformative. Both statements can be explained by the role of information in explicit coordination. In order to be able to explicitly coordinate ex post on one of multiple equilibria, players need to have a sufficient amount of information concerning other agents' beliefs, since their actions depend on other players' actions. Moreover, since other players' actions depend on their beliefs, each agent is also concerned about the beliefs of other players about others... and so on. In a game without public information, higher-order uncertainty is invariably large, even if the private signals themselves are very precise. Public information, on the other hand, not only provides information about the state, but also about what other agents observe. As information becomes more and more precise, the players ultimately have better and better information about other players' beliefs. This logic also explains the disproportionate impact that public information has on the equilibrium, since a change in public information not only alters a player's belief about the state, but also, about other agents' beliefs, hence the multiplier effect on equilibrium strategies.

4 Higher-order beliefs

In this section, I provide an explanation of the previous results based on higher order beliefs. Related to the question of uniqueness vs. multiplicity of equilibria, one may ask whether higher-order uncertainty vanishes, as signals become more and more informative. In order to be precise about higher-order uncertainty in this context, it is useful to start with some definitions.

Following Monderer and Samet (1989), we say that a set of agents S has a common belief p about an event E, if (i) each agent in S believes with at least probability p (henceforth: p-believes) that E has occurred, (ii) each agent in S also p-believes that all agents in S p-believe that E has occurred, (iii) each agent in S also p-believes that all agents in S p-believe that all agents in S p-believe that E has occurred... and so on.

This definition provides a continuous transition from incomplete information to common knowledge, where the latter corresponds to the case where the common belief of all players about any event is either 1 or 0. Now consider an event $E : \theta \leq \theta^*$. If, whenever $\theta \leq \theta^*$, a proportion m of agents has a common belief p about E, and (i) $m \geq c(\theta^*)$ and (ii) $p \geq \frac{\tau}{f}$, there exists an equilibrium, in which agents play β (at least) whenever they p-believe that E has occurred. To see this, it suffices to recongnize that each player knows that whenever E occurs, at least a proportion $m \geq c(\theta^*)$ of players will play β, given the strategy profile. He also knows that given the strategy profile, it is optimal for them to play β, and that they know that whenever E occurs, at least a proportion m of players knows that it is optimal to play β whenever they p-believe E, and so on. In other words, it is common knowledge that, whenever E occurs and (i) and (ii) are satisfied, this strategy profile is a best response strategy profile for at least m players.

We are therefore concerned with finding common beliefs of groups of agents for generic information structures. In a game with a continuum of players, two observations considerably simplify this task. First, the belief operator preserves the relative order of players within the distribution of beliefs. For any two players i and j, if i attaches higher probability to an event $E : \theta \leq \theta^*$ than j, he will also attach a higher probability to the event that (at least) a mass m of players p-believe E (for any m or p). Second, once again by virtue of the law of large numbers, θ is a sufficient statistic for the posterior distribution of expectations, conditional on θ. The second point implies that for any first-order event $E_0 : \theta \leq \theta_0^*$, the second-order event "a proportion m of players p-believes E_0" is equivalent to some suitably defined other first-order event $E_1 : \theta \leq \theta_1^*$. By iteration, we can therefore represent the entire sequence of higher-order events by first-order events and consider their intersection in the state space. The first observation then guarantees that if $\theta \in E_k \cap E_{k+1}$, the set of players who p-believe E_{k-1} and E_k are identical. Thus, at the intersection of all higher-order events, a proportion m of players has a common belief p about E_0.

In the remainder of this paragraph, I use this procedure to derive common beliefs about cut-off events of the type $\theta \leq \theta_0^*$ and $\theta \geq \theta_0^*$. Consider an arbitray event $\theta \leq \theta_0^*$. A player p-believes that $\theta \leq \theta^*$, whenever his posterior belief x_j^1 satisfies

$$\Phi\left(\frac{\theta_0^* - x_j^1}{\alpha\varepsilon}\right) \geq p$$

which defines the first-order cut-off expectation x_0 as

$$x_0 = \theta_0^* - \alpha\varepsilon\Phi^{-1}(p)$$

A mass m of players p-believes that $\theta \leq \theta_0^*$, then

$$\Phi\left(\frac{x_0 - \alpha^2\theta - (1-\alpha^2)\, y}{\alpha^2\varepsilon}\right) \geq m$$

which defines a new first-order event $E_1 : \theta \leq \theta_1^*$, where θ_1^* is given by

$$\theta_1^* = \frac{1}{\alpha^2}\left(x_0 - \left(1-\alpha^2\right) y\right) - \varepsilon\Phi^{-1}(m)$$

A player p-believes that a critical mass of players p-believes that $\theta \leq \theta_0^*$, whenever

$$\Phi\left(\frac{\theta_1^* - x_j^1}{\alpha\varepsilon}\right) \geq p$$

Proceeding in this way, we can recursively define the sequence of cut-off expectations and fundamentals by the following two equations:

$$x_k = \theta_k^* - \alpha\varepsilon\Phi^{-1}(p) \tag{12}$$

$$\theta_k^* = \frac{1}{\alpha^2}\left(x_{k-1} - \left(1-\alpha^2\right) y\right) - \varepsilon\Phi^{-1}(m) \tag{13}$$

Substituting (12) into (13), and rearranging, we find

$$\theta_{k+1}^{*} - y = \frac{1}{\alpha^2}\left(\theta_k^{*} - y\right) - \frac{\alpha\varepsilon}{\alpha^2}\left(\Phi^{-1}(p) + \alpha\Phi^{-1}(m)\right) \tag{14}$$

This is a standard difference equation and has as a solution (c.f. Chiang 1985, p.555f)

$$\theta_k^{*} - y = \frac{\alpha\varepsilon}{1-\alpha^2}\left(\Phi^{-1}(p) + \alpha\Phi^{-1}(m)\right)$$
$$+\frac{1}{\alpha^{2k}}\left(\theta_0^{*} - y - \frac{\alpha\varepsilon}{1-\alpha^2}\left(\Phi^{-1}(p) + \alpha\Phi^{-1}(m)\right)\right)$$

This sequence of cut-offs diverges monotonically, as long as

$$\theta_0^{*} - y - \frac{\alpha\varepsilon}{1-\alpha^2}\Phi^{-1}(p) - \frac{\alpha^2\varepsilon}{1-\alpha^2}\Phi^{-1}(m) \neq 0$$

In order to sustain a common belief p with respect to $\theta \leq \theta_0^{*}$, it is necessary that the sequence of cut-offs is monotonically increasing or stationary, since otherwise the intersection of all higher-order events would be empty.

$$\theta_0^{*} - y - \frac{\alpha\varepsilon}{1-\alpha^2}\left(\Phi^{-1}(p) + \alpha\Phi^{-1}(m)\right)$$

is decreasing in p and m. Thus, the commn belief p of a proportion m of players about an event $\theta \leq \theta_0^{*}$ must satisfy

$$\theta_0^{*} - y - \frac{\alpha\varepsilon}{1-\alpha^2}\Phi^{-1}(p) - \frac{\alpha^2\varepsilon}{1-\alpha^2}\Phi^{-1}(m) = 0$$

which yields

$$\Phi^{-1}(p) = \left(\theta_0^{*} - y\right)\frac{1-\alpha^2}{\alpha\varepsilon} - \alpha\Phi^{-1}(m) \tag{15}$$

The corresponding cut-off expectation is given by

$$x^{*}\left(\theta_0^{*}, p\right) = \theta_0^{*} - \alpha\varepsilon\Phi^{-1}(p) = \alpha^2\theta_0^{*} + \left(1-\alpha^2\right)y + \alpha^2\varepsilon\Phi^{-1}(m) \tag{16}$$

Let $p\left(\theta \leq \theta_0^{*}, m\right)$ be the common belief of a mass m of players, as defined by (15). The following proposition is immediate.

Proposition 5 *Suppose $\alpha < 1$:*

(i) For any $\theta_0^* > y$, $p(\theta \leq \theta_0^*, m) \to 1$, as $\alpha\varepsilon \to 0$. For finite $\alpha\varepsilon$, $p(\theta \leq \theta_0^*, m)$ is decreasing in $\alpha\varepsilon$.

(ii) For any $\theta_0^* < y$, $p(\theta \leq \theta_0^*, m) \to 0$, as $\alpha\varepsilon \to 0$. For finite $\alpha\varepsilon$, $p(\theta \leq \theta_0^*, m)$ is increasing in $\alpha\varepsilon$.

(iii) If $\theta_0^* = y$, $p(\theta \leq \theta_0^*, m)$ does not depend on $\alpha\varepsilon$.

(iv) $p(\theta \leq \theta_0^*, m) \to \Phi\left(\frac{\theta_0^* - y}{\alpha\varepsilon}\right)$, as $\alpha \to 0$, holding $\alpha\varepsilon$ constant. $p(\theta \leq \theta_0^*, m)$ is increasing in α (holding $\alpha\varepsilon$ constant).

If $\alpha = 1$, $p(\theta \leq \theta_0^*, m) = 1 - m$

This proposition has various implications. First, it states that the common signal generates a continuous transition to common knowledge, as long as it retains some value in the limit. The proposition also highlights the very different nature of common beliefs in an environment with only private information. Further inspection shows that the discussion of common beliefs leads to results that are analogous to those on the multiplicity of equilibria in the previous section. We observe that a lower level of ex post noise leads to lower higher-order uncertainty (common beliefs are closer to 0 or 1), and that a higher relative informativeness of the private signal results in more higher-order uncertainty. In line with the previous section on multiplicity, whether the common belief about an event converges to 0 or 1 is uniquely determined by the location of the common signal.

We can also reverse the arguments of this section, and apply them to events of the type $\theta > \theta_0^*$. In this case the dynamic system is characterized by

$$p = 1 - \Phi\left(\frac{\theta_k^* - x_k}{\alpha\varepsilon}\right)$$

$$m = 1 - \Phi\left(\frac{x_k - \alpha^2\theta_{k+1}^* - (1-\alpha^2)\, y}{\alpha^2\varepsilon}\right)$$

which implies as higher-order beliefs

$$\Phi^{-1}(p) = (y - \theta_0^*)\,\frac{1-\alpha^2}{\alpha\varepsilon} - \alpha\Phi^{-1}(m)$$

and conclusions summarized in the following proposition:

Proposition 6 *Suppose* $\alpha < 1$*:*

(i) For any $\theta_0^* > y$, $p(\theta > \theta_0^*, m) \to 0$, *as* $\alpha\varepsilon \to 0$. *For finite* $\alpha\varepsilon$, $p(\theta > \theta_0^*, m)$ *is increasing in* $\alpha\varepsilon$.

(ii) For any $\theta_0^* < y$, $p(\theta > \theta_0^*, m) \to 1$, *as* $\alpha\varepsilon \to 0$. *For finite* $\alpha\varepsilon$, $p(\theta > \theta_0^*, m)$ *is decreasing in* $\alpha\varepsilon$.

(iii) If $\theta_0^* = y$, $p(\theta > \theta_0^*, m)$ *does not depend on* $\alpha\varepsilon$.

(iv) $p(\theta > \theta_0^*, m) \to \Phi\left(\frac{\theta_0^* - y}{\alpha\varepsilon}\right)$, *as* $\alpha \to 0$, *holding* $\alpha\varepsilon$ *constant.* $p(\theta \leq \theta_0^*, m)$ *is increasing in* α *(holding* $\alpha\varepsilon$ *constant).*

If $\alpha = 1$, $p(\theta > \theta_0^*, m) = 1 - m$.

Using the expressions for $p(\theta \leq \theta_0^*, m)$ and $p(\theta > \theta_0^*, m)$, we can use the fact that $\Phi(-x) = 1 - \Phi(x)$ to show the following proposition about complementary events and common beliefs.

Proposition 7 $p(\theta \leq \theta_0^*, m) + p(\theta > \theta_0^*, 1 - m) = 1$

The common belief of a proportion of players about a cut-off event is the complementary probability to the common belief that the complement of players has about the complementary cut-off event. Using this property, we can link higher-order beliefs to equilibrium strategies. For this purpose, consider the higher-order beliefs about events $\theta \leq \theta_0^*$ and $\theta > \theta_0^*$ sustained exactly be the critical masses of players $c(\theta_0^*)$ and $1 - c(\theta_0^*)$. We already know that a strategy profile, in which players play β, (at least) whenever they $p(\theta \leq \theta_0^*, c(\theta_0^*))$-believe that $\theta \leq \theta_0^*$, can be sustained as an equilibrium. It immediately follows, that, for any y, θ', θ'', s.t. $\underline{\theta} < \theta' < y < \theta'' < \overline{\theta}$, there exists $\overline{\alpha\varepsilon}$, s.t. if $\alpha\varepsilon \leq \overline{\alpha\varepsilon}$, the game has multiple equilibria, one in which players play b, whenever they $p\left(\theta \leq \theta'', c\left(\theta''\right)\right)$-believe that $\theta \leq \theta''$, and one, in which they play a, whenever they $p\left(\theta > \theta', 1 - c\left(\theta'\right)\right)$-believe that $\theta \leq \theta'$.

Taking this analysis a step further, we can give a reinterpretation of the initial conditions for a Nash equilibrium in terms of common beliefs. Expressed in terms of common beliefs, a Nash equilibrium is a triplet (θ_0^*, m, p) such that:

$$\begin{aligned} (i)\ p &= p\,(\theta \le \theta_0^*, m) \\ (ii)\ m &= c\,(\theta_0^*) \\ (iii)\ \frac{r}{f} &= p \end{aligned}$$

(i) simply states that p must be a common belief of a proportion m of players. (ii) states that m must equal the critical mass, and (iii) is an indifference condition for the marginal player whose belief about $\theta \le \theta_0^*$ is exactly p. By (12), $p = \Phi\left(\frac{\theta_0^* - x_0}{\alpha\epsilon}\right)$, and substituting into (iii) yields the (PI)-condition, for a given cut-off expectation. The cut-off expectation for a common belief $p\,(\theta \le \theta_0^*, m)$ is given by (16), and substituting (ii) yields the critical mass condition (CM).

5 A Note on Purification

The convergence of common beliefs to common knowledge with valuable public information leads to a reconsideration of Harsanyi's Purification Theorem. Harsanyi (1973) states that every pure and mixed strategy equilibrium of a normal-form game can be approximated as a limit of pure strategy equilibria in a sequence of incomplete information games, in which the noise vanishes in the limit. To prove this theorem, Harsanyi considers perturbations of the initial game, in which pay-offs of players are randomly perturbed, perturbations are independent across players, and are privately observed.

Here, we have considered perturbations from the normal-form, common knowledge game, in which the players receive private, conditionally independent information, as well as public information about the global pay-off perturbations of the game. In the context of the class of coordination games considered here, we can make the following, stronger statement: For any Nash equilibrium of the normal form, common knowledge game at a given state θ^*, any sequence of incomplete information games in which the noise in the signals vanishes in the limit, and in which the public information about the perturbations retains some value, has a sequence of pure strategy Nash equilibria, in which actions almost surely converge to the common knowledge equilibrium. [7]

In order to obtain this strengthening of Harsanyi's Purification Theorem, both public and private information are essential: Some element of private information is needed to sustain pure strategies, i.e. uncertainty about strategies comes from uncertainty about the other players' information, not from mixing. Public information is needed to sustain all equilibria in the limit: As the previous sections have shown, a valuable public signal is sufficient and almost necessary for maintaining the multiplicity of equilibria in the limit.

6 Conclusion

In this paper, I have examined a simple coordination game similar to Morris and Shin (1998) and extended it to more complex incomplete information structures, which incorporate public information. Two results emerge from the analysis. First, a unique equilibrium a la Morris and Shin is less likely to occur if public information is informative relative to private information, and/or if the overall level of noise is small. Second, if common knowledge is viewed as a limit of incomplete information structures, I show that this limit is path independent for as long as public information retains some value. These results can be explained by reinterpreting the equilibrium in terms of common beliefs. In particular, the existence of some valuable common information in the limit is both sufficient and almost necessary for a continuous transition of common beliefs to common knowledge, as noise vanishes.

The results presented here have various implications. They cast doubt on recent papers in which the Morris-Shin methodology is used to derive a unique equilibrium in a coordination model, to perform comparative static or dynamic analysis. If the Morris-Shin methodology is to be used, this would imply that the economic environment allows for no common information. Without engaging in a discussion as to whether such an assumption that

[7] Preliminary calculations seem to indicate that the transition from common beliefs to common knowledge with public information can be extended to arbitrary, measurable, non-null subsets of the state space. This would imply an even stronger statement, that any measurable Bayesian equilibrium of the global game can be approximated by a sequence of pure strategy equilibria in incomplete information games.

no common information can exist is reasonable, I would like to make a few comments.

First, it is common to think that markets serve to aggregate information. In fact, a variety of related models with coordination failures (for instance bank run models) assume that uninformed or partially informed lenders observe how many other lenders run on a bank and can base their decision on this. In these instances, the strength of a bank run aggregates private information and provides a common, usually noisy signal about the true state. Such considerations seem particularly relevant in environments, in which private information is gradually revealed over time and aggregated in the market.

Second, the MS-methodology must implicitly assume that not a single player has the possibility to credibly announce his private information. While there may be arguments against or in favor of this view, it would be interesting to relate this to recent work by Corsetti, Dasgupta, Morris and Shin (2000), in which a single investor has substantial financial wealth, so that by his decision, he can influence other investors and ultimately the likelihood of devaluation. Translated into a dynamic context, one would have to ask whether public information can be generated by strategic revelation of private information.

Alternatively, one may argue that the central bank can publicize its information to the markets. Companies and banks are legally obliged to publish regular data on cash flows and profitability. If the present model is anything to judge on, one would conclude that increased transparency of either the markets or the central bank leads to an increase of indeterminacy in the critical region - a conclusion rather opposite to standard wisdom, and which follows from the reduction of higher-order uncertainty with more precise public information.[8]

Finally, one may argue that no such thing as "common information" can exist, since, even if common public information exists, this information may be interpreted in different ways by players, and therefore lead to different appreciations ex post. Formally, this would amount to transferring uncertainty

[8]See Morris and Shin (2000b) for a formalization of disclosures in a principal-agent environment similar to this.

from pay-off parameters to a higher-level, say, the information structure, or the actual form of the game, and amounts to abandoning the common-prior assumption on the global game that is played. However, as long as the global information structure at some level can be summarized in a common prior assumption, the criticisms presented here would presumably apply in the same way.

Summing up these comments, we can conclude that, if the MS-methodology is to be used to select unique equilibrium in a coordination game, this will require an argument as to why in the existing information structure, markets or individuals are unable to credibly generate public information, or why existing common signals are so noisy that coordination becomes impossible.

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