Skewness and Kurtosis Implied by Option Prices: A Second Comment*

Emmanuel Jurczenko† Bertrand Maillet‡ Bogdan Negrea§

- July 2002 -

Abstract

Several authors have proposed series expansion methods to price options when the risk-neutral density is asymmetric and leptokurtic. Among these, Corrado and Su (1996) provide an intuitive pricing formula based on a Gram-Charlier Type A series expansion. However, their formula contains a typographic error that can be significant. Brown and Robinson (2002) correct their pricing formula and provide an example of economic significance under plausible market conditions. The purpose of this comment is to slightly modify their pricing formula to provide consistency with a martingale restriction. We also compare the sensitivities of option prices to shifts in skewness and kurtosis using parameter values from Corrado- Su (1996) and Brown-Robinson (2002), and market data from the French options market. We show that differences between the original, corrected, and our modified versions of the Corrado-Su (1996) original model are minor on the whole sample, but could be economically significant in specific cases, namely for long maturity and far-from-the-money options when markets are turbulent.

Keywords: Option Pricing Models, Skewness, Kurtosis.

JEL Classification: G.10, G.12, G.13.

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1 Introduction

The main goal of this note is to correct a common misuse of the martingale restriction within the context of the Corrado-Su (1996) model. We also evaluate some approximations made in the literature in order to obtain tractable pricing and implied risk-neutral density formulae.

To correct the well-documented Black and Scholes (1973) option pricing model biases, several authors have proposed series expansions of a given probability density in order to approximate the "true" underlying risk-neutral implied return distribution. Under this approach, skewness and kurtosis may have significant impact on option prices and correction terms in the Black-Scholes formula might lead to a plausible explanation of strike price and time-to-maturity biases.

Following the seminal paper by Jarrow and Rudd (1982) - who appear to be the first to use a Gram-Charlier series expansion of the lognormal density function - Corrado and Su (1996) find an approximate implied density function by using a Gram-Charlier type A series expansion of the (normal) density function of underlying log-returns. The option price is then an intuitive function of the third and fourth moments of the risk-neutral log-return distribution. The first two moments of the approximating distribution remain the same as that of the normal distribution, but third and fourth moments are introduced as higher order terms of the density expansion. Brown and Robinson (2002) correct two typographic errors in Corrado and Su (1996) and provide examples of how these errors can have economic significance.

This note is organized as follows. Section 2 starts with the martingale restriction to be used in a Multi-moment Approximate Option Pricing Models framework. Section 3 provides the different expressions of option price sensitivities to departures from Gaussianity. Section 4 evaluates the different expressions of sensitivities of option prices corresponding to the original, corrected, and modified Corrado-Su (1996) models. The study case parameter values provided by Corrado and Su (1996) and Brown and Robinson (2002) are presented and, as an illustration, densities of pricing errors are estimated on the French CAC 40 options market. Section 5 summarizes and concludes this note.

2 Option Pricing and the Martingale Restriction

Consider an European call option written on an underlying asset whose price at time t is denoted S_t, with a strike price K, a maturity τ and a date of expiration T. Under the assumptions of a complete market with no-arbitrage opportunity, and further assuming that the risk-free rate of interest, denoted by r, is constant, the price of the option at time t is the present value of the expected payoff at expiry, given by the following pricing kernel (see Harrison and Kreps, 1979):

$$\begin{aligned} C &= e^{-r\tau} E_Q \left[Max \left(S_T - K, 0 \right) \right] \\ &= e^{-r\tau} \int_{S_T=K}^{+\infty} \left(S_T - K \right) f \left(S_T \right) dS_T \end{aligned} \tag{1}$$

where $E_Q[.]$ is the expectation under the risk-neutral probability measure, S_T is the underlying asset terminal price and $f(.)$ is the risk-neutral density.

Let the τ-period log-return of the underlying asset $\log(S_T/S_t)$ have a conditional mean $\mu\tau$ and a standard deviation $\sigma\sqrt{\tau}$, then, introducing the standardized variable z:

$$z = \frac{\log\left(S_T/S_t\right) - \mu\tau}{\sigma\sqrt{\tau}} \tag{2}$$

the fair price of the call becomes:

$$C = e^{-r\tau} \int_{z=\frac{\log(K/S_t)-\mu\tau}{\sigma\sqrt{\tau}}}^{+\infty} \left(S_t e^{\mu\tau + z\sigma\sqrt{\tau}} - K \right) f(z)\, dz \tag{3}$$

Under the no-arbitrage condition, the expected price under the correct probability measure should be equal to the current asset price compounded at the risk-free rate. Accordingly, the probability measure to be considered must satisfy the so-called martingale restriction (see Longstaff, 1995):

$$E_Q[S_T] = e^{r\tau} S_t \tag{4}$$

and then the risk-neutral density $f(z)$ must respect:

$$\mu\tau = r\tau - \ln\left[\int_{-\infty}^{+\infty} e^{\mu\tau + z\sigma\sqrt{\tau}} f(z)\ dz \right] \tag{5}$$

The choice of the density representing the "true" underlying risk-neutral density leads to different expressions for the mean of the log-return. In particular, when assuming a log-normal distribution for the terminal price of the asset, as in Black and Scholes (1973), the "traditional" martingale restriction is (see Baxter and Rennie, 1996, p.85):

$$\mu\tau = r\tau - \frac{1}{2}\sigma^2\tau \tag{6}$$

If instead a statistical series expansion of the density related to the continuously compounded return (as in Corrado and Su, 1996) is used, the martingale restriction implies now (see for instance[1], Kochard, 1999, Jurczenko *et al.*, 2002):

$$\mu\tau = r\tau - \frac{1}{2}\sigma^2\tau - \ln\left[1 + \frac{\gamma_1(f)}{3!}\sigma^3\tau^{3/2} + \frac{\gamma_2(f)}{4!}\sigma^4\tau^2 \right] \tag{7}$$

where f is the "true" density and $\gamma_1(.)$ and $\gamma_2(.)$ are the Fisher parameters for skewness and kurtosis:

$$\gamma_1(.) = \frac{\mu_3(.)}{\mu_2^{3/2}(.)} \quad \text{and} \quad \gamma_2(.) = \frac{\mu_4(.)}{\mu_2^2(.)} - 3$$

[1] See also Brenner and Eom (1997) in the context of option pricing with Laguerre polynomial series.

with $\mu_i(.)$ the corresponding centered moments of order i, for $i = [2, 3, 4]$.

The difference between the two martingale restrictions - the traditional (6) and the Gram-Charlier induced one (7) - depends on terms such as $\gamma_1(.)$, $\gamma_2(.)$, $\sigma^3\tau^{3/2}$ and $\sigma^4\tau^2$.

The purpose of this note is to evaluate the impact of these terms on the pricing of options. Indeed, the implicit martingale restriction used by Corrado-Su (1996) is mispecified. Nevertheless, as underlined by Brown and Robinson (2002), the magnitude of the differences in parameters and option prices is mainly an empirical question. The following section reviews the different versions of Corrado-Su (1996) original model and investigate these differences.

3 The Different Versions of the Corrado-Su (1996) Formula

From the seminal approach of Jarrow and Rudd (1982), Corrado and Su (1996) propose a new option pricing formula that is easily implemented. Using a Gram-Charlier type A series expansion, they begin their option price expression with the Black-Scholes formula, and then add two terms related to a skewed and leptokurtic risk-neutral density.

As shown by Brown and Robinson (2002), the original Corrado-Su (1996) formula contains an error in the term corresponding to the sensitivity of the option price to the excess skewness of the implied risk-neutral density. In addition, we provide a correction to the martingale restriction implicitly assumed by these authors to yield a complete option price expression. Finally, we note that differences among previous formulae involve terms of a "high" degree in volatility and time to maturity, which motivates us, following Backus *et al.* (1997), to give the option price expression neglecting these terms.

3.1 The Original Corrado-Su (1996) Formula

To allow for moments of higher order in the risk-neutral log-return distribution, Corrado and Su (1996) find an approximate risk-neutral probability density function using a Gram-Charlier expansion of the normal density function. The Black and Scholes (1973) model is then adjusted in an intuitive way by introducing third and fourth moments as higher order terms of the expansion. For practical reasons, the series is truncated after the fourth term, noting that the first four moments of the underlying distribution should capture most of the effect on option prices (see Jarrow and Rudd, 1982).

Using previous notation and following Corrado and Su (1996), the price of the European call option C_{CS} can be written as (see Corrado and Su, 1996):

$$C_{CS} = C_{BS} + \gamma_1(f)\, Q_3 + \gamma_2(f)\, Q_4 \tag{8}$$

with:

$$\begin{cases} Q_3 = \frac{1}{3!} S_t\, \sigma\sqrt{\tau}\, \left[P_3\,(d)\, \varphi(d) - \sigma^2\tau\, \Phi(d)\right] \\ Q_4 = \frac{1}{4!} S_t\, \sigma\sqrt{\tau}\, \left[P_4\,(d)\, \varphi(d) + \sigma^3\tau^{3/2}\, \Phi(d)\right] \end{cases}$$

and:

$$\begin{cases} P_3(y) = 2\sigma\sqrt{\tau} - y \\ P_4(y) = y^2 - 3y\sigma\sqrt{\tau} + 3\,\sigma^2\tau - 1 \end{cases}$$

where $\varphi(.)$ and $\Phi(.)$ are respectively the standard normal density and the standard normal distribution, $P_3(.)$ and $P_4(.)$ are two polynomial, $\gamma_1(f)$ and $\gamma_2(f)$ are the Fisher parameters (defined previously), while C_{BS} is the Black-Scholes (1973) price evaluated at d - the "standardized moneyness" - that is:

$$d = \left(\sigma\sqrt{\tau}\right)^{-1} \left[\log\left(S_t/Ke^{-r\tau}\right) + \sigma^2\tau/2\right]$$

Using S&P 500 index options traded on the Chicago Board Option Exchange (CBOE), Corrado and Su (1996) show that expression (8) improves significantly the in-sample option pricing accuracy of the Black-Scholes (1973) model. Corrado and Su (1997) also demonstrate improvements using their formula on an out-of-sample basis, from actively traded individual equity options on the Chicago Board Option Exchange (CBOE).

3.2 The Corrected Corrado-Su (1996) Formula

Noting that the definition of an Hermite Polynomial was incomplete in Corrado-Su (1996)[2], Brown and Robinson (1999 and 2002) correct the previous formula that then becomes (using previous notation):

$$C_{CS}^{'} = C_{BS} + \gamma_1 (f) \; Q_3^{'} + \gamma_2 (f) \; Q_4 \tag{9}$$

with:

$$Q_3^{'} = \frac{1}{3!} S_t \; \sigma \sqrt{\tau} \; \left[P_3 \; (d) \, \varphi (d) + \sigma^2 \tau \; \Phi(d) \right]$$

The correction to be done is proportional to $\sigma^3 \tau^{3/2}$ (see below) and should be very small on real data as underlined by Backus *et al.* (1997). Nevertheless, some specific option prices can be truly biased as shown by Brown and Robinson (2002) when calculated using the original Corrado-Su (1996) formula.

[2] Hermite Polynomials should read:

$$H_i (x) = (-1)^n \, \varphi (x)^{-1} \, d^i \varphi (x) \, / dx^i$$

instead of:

$$H_i (x) = \varphi (x)^{-1} \, d^i \varphi (x) \, / dx^i$$

in the original Corrado-Su (1996) model.

Using SPI index future options traded on the Sydney Futures Exchange (SFE) and the Black-Scholes (1973) model as a benchmark, Brown and Robinson (1999) show that expression (9) seems to improve in-sample option pricing accuracy significantly. Moreover, Brown and Robinson (2002) show the correction to the Corrado and Su (1996) formula is economically significant for some options.

3.3 The Modified Corrado-Su (1996) Formula

When using the "true" martingale restriction recalled in Section 2, the European Call price finally is (see for instance Kochard, 1999, Jurczenko *et al.*, 2002 and Appendix 1 at the referee's attention):

$$C_{CS}^{*} = C_{BS}^{*} + \gamma_1 (f)\ Q_3^{*} + \gamma_2 (f)\ Q_4^{*} \tag{10}$$

with (using previous notation):

$$\begin{cases} Q_3^{*} = [3!\,(1+\omega)]^{-1}\ S_t\ \sigma\sqrt{\tau} P_3 (d^*)\,\varphi (d^*) \\ Q_4^{*} = [4!\,(1+\omega)]^{-1}\ S_t\,\sigma\sqrt{\tau} P_4 (d^*)\,\varphi (d^*) \end{cases}$$

and:

$$\begin{cases} d^* = (\sigma\sqrt{\tau})^{-1} \left[\log\left(S_t/Ke^{-r\tau}\right) + \frac{1}{2}\sigma^2\tau - \ln\left[1+\omega\right]\right] \\ \omega = \frac{\gamma_1(f)}{3!}\,\sigma^3\tau^{3/2} + \frac{\gamma_2(f)}{4!}\,\sigma^4\tau^2 \end{cases}$$

where C_{BS}^{*} is the Black and Scholes (1973) call price evaluated at a "corrected" standardized moneyness level denoted d^*, and ω a constant. Note that parameters Q_3^{*} and Q_4^{*}, no longer represent the true marginal effects of the non normal log-return skewness and kurtosis on the option price, since terms depending on kurtosis (skewness) appear in Q_3^{*} (Q_4^{*}). Nevertheless, when realistic values are considered, the modified parameters are close - even if different - from the corrected $Q_3^{'}$ and Q_4 sensitivities.

Using S&P 500 index futures options traded on the Chicago Mercantile Exchange (CME) and the Black-Scholes (1973) model as a benchmark, Kochard (1999) also obtains a better fit from formula[3] (10) on an in- and out-of-sample basis.

3.4 The Simplified Corrado-Su (1996) Formula

Arguing that some terms are numerically small in real markets, Backus *et al.* (1997) suggest that they can be dropped from the pricing formula. Simplifying formula (10) by deleting terms involving $\sigma^3\tau^{3/2}$ and $\sigma^4\tau^2$ leads to the following simplified expression (using previous notation):

$$C_{CS}^{**} = C_{BS} + \gamma_1(f)\ Q_3^{**} + \gamma_2(f)\ Q_4^{**} \tag{11}$$

with:

$$\begin{cases} Q_3^{**} = \frac{1}{3!} S_t\ \sigma\sqrt{\tau}\, P_3(d)\ \varphi(d) \\ Q_4^{**} = \frac{1}{4!} S_t\, \sigma\sqrt{\tau}\, P_5(d)\ \varphi(d) \end{cases}$$

and:

$$P_5(y) = y^2 - 3y\sigma\sqrt{\tau} - 1$$

Note that when neglecting these terms in the final price expression, the error of Corrado-Su (1996) corrected by Brown and Robinson (2002) has no importance at all. Further simulations done by Backus *et al.* (1997) show that the simplified Corrado-Su (1997) formula constitutes a good approximation of the option price when the underlying asset follows a jump-diffusion process.

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[3] Adopting a very similar approach, Kochard (1999) indeed obtains an equivalent *formula* to equation (10).

4 Corrado-Su (1996) Model Sensitivity Analysis

Extending the Brown and Robinson (2002) static analysis, the next sub-section gives expressions of sensitivity and option price differences related to the original Corrado-Su (1996) model, followed by an illustration and an estimation of such biases.

4.1 Skewness and Kurtosis Sensitivities, Related Option Price and Hedging Differences

The various differences in skewness and kurtosis sensitivities (denoted $\dot{Q}$) thus read (using previous notation):

$$\begin{cases} \dot{Q}_3^{'} = \left|Q_3 - Q_3^{'}\right| = 2\alpha \\ \dot{Q}_3^{*} = \left|Q_3^{*} - Q_3^{'}\right| = \left|S_t\sigma\sqrt{\tau}\left[k^{*}P_3\left(d^{*}\right)\varphi\left(d^{*}\right) - kP_3\left(d\right)\varphi\left(d\right)\right] - \alpha\right| \\ \dot{Q}_3^{**} = \left|Q_3^{**} - Q_3^{'}\right| = \alpha \\ \dot{Q}_4^{*} = \left|Q_4^{*} - Q_4\right| = \left|S_t\sigma\sqrt{\tau}\left[l^{*}P_4\left(d^{*}\right)\varphi\left(d^{*}\right) - lP_4\left(d\right)\varphi\left(d\right)\right] - \beta\right| \\ \dot{Q}_4^{**} = \left|Q_4^{**} - Q_4\right| = \left|S_t\,\sigma\sqrt{\tau}\;l\left(1 - 3\,\sigma^2\tau\right)\varphi\left(d\right) - \beta\right| \end{cases} \tag{12}$$

with $[\alpha, \beta] \in (IR^{+})^2$ and:

$$\begin{cases} \alpha = \frac{1}{3!}\,\sigma^3\tau^{3/2}S_t\,\Phi(d) \\ \beta = \frac{1}{4!}\,\sigma^4\tau^2 S_t\,\Phi(d) \\ k = \frac{1}{3!} \text{ and } k^{*} = \frac{1}{3!(1+\omega)} \\ l = \frac{1}{4!} \text{ and } l^{*} = \frac{1}{4!(1+\omega)} \\ \omega = \frac{\gamma_1(f)}{3!}\,\sigma^3\tau^{3/2} + \frac{\gamma_2(f)}{4!}\,\sigma^4\tau^2 \end{cases}$$

Following Brown and Robinson (2002), one can say that variables of interest α and β can be roughly bounded for short-term options[4] by:

$$\begin{cases} 0 \le \alpha \le .06 \times \sigma^3 S_t \\ 0 \le \beta \le .01 \times \sigma^4 S_t \end{cases} \tag{13}$$

If we now evaluate induced differences in option prices (denoted by $\dot{C}$), one can write that (using previous notation):

$$\begin{cases} \dot{C}_{CS}' = \left| C_{CS} - C_{CS}' \right| = \left| \gamma_1 (f)\ 2\alpha \right| \\ \dot{C}_{CS}^* = \left| C_{CS}^* - C_{CS}' \right| \\ \qquad = \left| C_{BS}^* - C_{BS} + \gamma_1 (f) \left\{ S_t \sigma \sqrt{\tau} \left[k^* P_3 \left(d^* \right) \varphi \left(d^* \right) - k P_3 (d) \varphi (d) \right] - \alpha \right\} \right. \\ \qquad \quad \left. + \gamma_2 (f) \left\{ S_t \sigma \sqrt{\tau} \left[l^* P_4 \left(d^* \right) \varphi \left(d^* \right) - l P_4 (d) \varphi (d) \right] - \beta \right\} \right| \\ \dot{C}_{CS}^{**} = \left| C_{CS}^{**} - C_{CS}' \right| = \left| \gamma_1 (f)\ \alpha + \gamma_2 (f) \left[S_t\, \sigma \sqrt{\tau}\ l \left(1 - 3\, \sigma^2 \tau \right) \varphi (d) - \beta \right] \right| \end{cases} \tag{14}$$

For hedging purposes, one might also highlight differences in delta. Using the previous formulae to derive the sensitivity of option prices to changes in underlying prices - for, respectively, the original, corrected, modified and simplified models - leads to (with previous notation):

$$\begin{cases} \Delta_{CS}^{C} = \Phi (d) + \varphi (d) \left\{ \gamma_1 (f)\, k \left[P_5 (d) - \frac{\sigma^2 \tau}{S_t} \right] + \gamma_2 (f)\, l \left[P_6 (d) + \frac{\sigma^3 \tau^{3/2}}{S_t} \right] \right\} \\ \Delta_{CS'}^{C} = \Phi (d) + \varphi (d) \left\{ \gamma_1 (f)\, k \left[P_5 (d) + \frac{\sigma^2 \tau}{S_t} \right] + \gamma_2 (f)\, l \left[P_6 (d) + \frac{\sigma^3 \tau^{3/2}}{S_t} \right] \right\} \\ \Delta_{CS^*}^{C} = \Phi (d^*) + \varphi (d^*) \left\{ \frac{1}{\sigma \sqrt{\tau}} \left[1 - \frac{1}{(1+\omega)} \right] + \gamma_1 (f)\, k^* P_6 (d^*) + \gamma_2 (f)\, l^* P_7 (d^*) \right\} \\ \Delta_{CS^{**}}^{C} = \Phi (d) + \varphi (d) \left[\gamma_1 (f)\, k P_6 (d) + \gamma_2 (f)\, l P_8 (d) \right] \end{cases} \tag{15}$$

with the following polynomials:

$$\begin{cases} P_6 (y) = y^2 - 3 y \sigma \sqrt{\tau} + 2\, \sigma^2 \tau - 1 \\ P_7 (y) = -y^3 + 4 y^2 \sigma \sqrt{\tau} + 3 y \left(1 - 2 \sigma^2 \tau \right) + 3 \sigma^3 \tau^{3/2} - 4 \sigma \sqrt{\tau} \\ P_8 (y) = -y^3 + 4 y^2 \sigma \sqrt{\tau} + 3 y \left(1 - \sigma^2 \tau \right) - 4 \sigma \sqrt{\tau} \end{cases}$$

[4] Less than 6 month maturity options.

Absolute differences in delta, denoted by $\dot{\Delta}$, then read:

$$\begin{cases} \dot{\Delta}^{C}_{CS'} = \left|\Delta^{C}_{CS} - \Delta^{C}_{CS'}\right| = 2\gamma_1(f)\,k\varphi(d)\,\frac{\sigma^2\tau}{S_t} \\ \dot{\Delta}^{C}_{CS^*} = \left|\Delta^{C}_{CS^*} - \Delta^{C}_{CS'}\right| \\ \qquad = |\Phi(d^*) - \Phi(d) \\ \qquad\quad + \varphi(d^*)\left\{\frac{1}{\sigma\sqrt{\tau}}\left[1 - \frac{1}{(1+\omega)}\right] + \gamma_1(f)\,k^* P_6(d^*) + \gamma_2(f)\,l^* P_7(d^*)\right\} \\ \qquad\quad -\varphi(d)\left\{\gamma_1(f)\,k\left[P_5(d) + \frac{\sigma^2\tau}{S_t}\right] + \gamma_2(f)\,l\left[P_6(d) + \frac{\sigma^3\tau^{3/2}}{S_t}\right]\right\}\Big| \\ \dot{\Delta}^{C}_{CS^{**}} = \left|\Delta^{C}_{CS^{**}} - \Delta^{C}_{CS'}\right| \\ \qquad = \left|\varphi(d)\left\{\gamma_1(f)\,k\left(2\,\sigma^2\tau - \frac{\sigma^2\tau}{S_t}\right) + \gamma_2(f)\,l\left[P_9(d) - \frac{\sigma^3\tau^{3/2}}{S_t}\right]\right\}\right| \end{cases} \tag{16}$$

with:

$$P_9(y) = -y^3 + y^2\left(4\sigma\sqrt{\tau} - 1\right) + 3y(1 + \sigma\sqrt{\tau} - \sigma^2\tau) - 4\sigma\sqrt{\tau} - 2\,\sigma^2\tau + 1$$

Differences in term of sensitivities to departures from Gaussianity, in terms of pricing and hedging are clearly an empirical question. We now investigate the magnitude of these differences.

4.2 The Study Cases of Corrado and Su (1996 and 1997)

In Figure 1, using the same parameter values as Corrado and Su (1996) and Brown and Robinson (2002), we reproduce the adjustment for skewness plotted in Figure I of Corrado and Su (1996)[5]. We compute the various expressions of sensitivities of option price to excess skewness - related to the original, corrected and our modified Corrado-Su (1996) models - for $S_0 = 100$, $\sigma = 15\%$, $\tau = 0.25$, $r = 4\%$ and with K varying from 65 to 135.

The vertical axis measures the adjustments for options that are - on the horizontal axis - from 35% in-the-money to 35% out-the-money, where moneyness is defined as $M = (Ke^{-r\tau} -$

[5]See Corrado and Su (1996), p.180 and Brown and Robinson (2002), p.9.

$S_0)/(Ke^{-r\tau})$. In-the-money call options therefore have negative moneyness in this figure. Clearly, differences in parameters are relatively low - as shown on Figure 2, where absolute differences in option price sensitivity to skewness are reproduced, and magnitudes of the absolute differences in sensitivity vary with the moneyness of the option.

- Please insert Figures 1 and 2 somewhere here -

For example, when skewness is equal to -.7 (and kurtosis is 3.53)[6], the absolute size of differences for the coefficient of skewness are 3.68 10^{-4}, 1.84 10^{-4} and 1.56 10^{-4} when comparing the corrected, the simplified and the modified Corrado-Su (1996) model to the original one. Relative differences between the corrected and modified sensitivities are significant while those between the simplified and modified skewness parameters are clearly negligible.

Figure 3 represents differences in option price sensitivity to kurtosis as a function of moneyness. Results exhibited in this figure show that while the correction of Brown and Robinson (2002) is irrelevant for these parameters, the correction for martingale restriction has a very small impact on the option price sensitivity to kurtosis, especially for at-the-money options. For example, when kurtosis is equal to 3.53 (and skewness is -.7), the absolute size of differences for the coefficient of kurtosis are 5.29 10^{-6} and 5.19 10^{-4} when comparing the simplified and the modified Corrado-Su (1996) model to the original one.

- Please insert Figures 3 somewhere here -

[6] These values are realistic in the sense that they correspond to mean parameter values when backing-out implied moments corresponding to the Jarrow-Rudd (1982) model on the CAC 40 options on the French market on the period 1997-1999 (see Capelle-Blancard *et al.*, 2001-a and 2001-b, for details).

When considering, now, both skewness and kurtosis effects on price changes, displayed in Figure 4, we see that the modification corresponding to the martingale restriction diminishes the impact of the correction by Brown and Robinson (2002) on option prices and that simplifications of the modified version cannot be judged as totally pertinent in this context. When taking the numerical values corresponding to the Corrado-Su (1996) study case - adding $M = -0.25$, a skewness equal to $-.7$ and a kurtosis equal to 3.53, the absolute size of differences for the related option prices are (USD) 7.17 10^{-7}, 3.53 10^{-7} and 2.93 10^{-7} when comparing the corrected, the simplified and the modified Corrado-Su (1996) model to the original one. We see here that the correction, modification and simplification have a very limited effect on option prices in this study case. If now we take the numerical example by Brown and Robinson (2002) - with $S_0 = 100$, $\sigma = 50\%$, $\tau = 0.25$, $r = 4\%$ and with $M = -0.25$, a skewness equal to -.1 and a kurtosis equal to 3 - the absolute size of differences for the related option price are (USD) 0.44, 0.22 and 0.22 when comparing the corrected, the simplified and the modified Corrado-Su (1996) model to the original one. In other words, the correction by Brown and Robinson (2002) overestimates the necessary correction with the martingale restriction. We also remark that the simplification leads to results that are very close to those corresponding to the true martingale correction.

- Please insert Figure 4 somewhere here -

In Table 1, using the same parameter values as Corrado and Su (1997), we reproduce the differences in the number of option contracts needed to delta-hedge a USD 10 million stock portfolio with beta of one. This illustrates how a hedging strategy based on the various models may differ. We compute the different expressions of sensitivities of option price to the excess skewness - related to Black-Scholes (1973) and to the original, corrected, simplified and our modified Corrado-Su (1996) models - for[7] $S_0 = USD\ 700$, $\sigma = 11.62\%$, $\gamma_1(f) = -1.68, \gamma_2(f) = 5.39, \tau = 0.25$, $r = 5\%, q = 2\%$ and with K varying from 660 to 750 with increments of 10. Columns 1 and 7 in Table 1 list strike prices from 660 to 750 in increments of 10. For each of these strike prices, columns 2 to 6 (and 8 to 12) list the number of S&P 500 index option contracts needed to delta-hedge the assumed USD 10 million stock portfolio. Columns 2 and 8 report the number of option contracts needed to delta-hedge this portfolio based on the Black-Scholes model, while columns 3 and 9, 4 and 10, 5 and 11, and 6 and 12 report the number of option contracts needed to delta-hedge this portfolio based on, respectively, the original, corrected, simplified and our modified Corrado-Su (1996) models. In all cases, numbers of contracts required are computed as follows:

$$N = \frac{1}{\Delta}\frac{P}{s} \tag{17}$$

where N is the number of contracts, P is the portfolio value, s is the contract size and Δ is the delta of the option computed from the various option fair price expressions (see Corrado and Su, 1997).

Table 1 reveals that for in-the money options, a delta-hegded strategy based on Black-Scholes model specifies a greater number of contracts than a delta-hedge based on other competitive models[8].

- Please insert Table 1 somewhere here -

But, for out-of-the-money options, the opposite is also true since the various Corrado-Su models require a greater number of contracts. Differences in the number of contracts specified

[7] We use here the realistic values reported by Corrado and Su (1997).

[8] Note that the Corrado-Su original model results reported in columns 3 and 9 in Table 1 are very similar to those of columns 3 and 6 in Exhibit 6 of Corrado and Su (1997) corresponding to the Jarrow-Rudd model.

by each model are also greatest for out-of-the-money options ($1/\Delta$ has a multiplication effect, values of delta being closed to zero for out-of-the-money options), while negligible for at- and in-the-money options ($1/\Delta$ has a division effect, values of delta being close to one for in-the-money options). Moreover, there is no difference between the original and corrected versions, and the differences between the modified and its simplified version are very small (4 contracts is the maximum discrepancy). significant differences however appear between the original and our modified version in terms of the numbers of hedging contracts for out-of-the-money options. For deep out-of-money options, approximately double the number of contracts are needed if we compare our modified model to the Black-Scholes one (1602 *versus* 802) and approximately 60% more contracts have to be added to the portfolio if the reference is the original or the corrected model (1602 *versus* 1008).

Based on simulations, we have seen that differences between model-induced prices are generally low but can be important in some specific situations. In the following section, we compare results obtained using the different versions of the Corrado-Su (1996) model with market data in order to investigate the magnitude of differences in terms of pricing accuracy.

4.3 An Estimation of Induced Approximation Biases

Using French Long Term CAC 40 option prices[9] from 01/97 through 12/98, we compute differences in option pricing relative to the four versions of the original Corrado-Su (1996) model. Figure 5 represents the absolute pricing error probability densities on the whole sample (12,705 data point estimations) relative to Black-Scholes (1973) and the modified Corrado-Su (1996) inspired models, while Figure 6 shows the absolute difference of probability associated with the absolute error terms of the three related versions of the Corrado-Su (1996) model relative to the original model. As indicated in this last figure, the differences

[9] See Capelle-Blancard *et al.,* 2001, for details on the database, filters, optimization *criterion* and routines.

between models are very small in terms of pricing error density (and not significant).

This general conclusion must be interpreted since differences in the skewness and kurtosis parameters can be important for the sample considered as illustrated on Figures 7 and 8. Nevertheless, when comparing the original and modified versions, relative differences in skewness parameters vary from $-.86$ to $.06$ with a mean of $-.01$ and a median equal to $.000$, whilst differences in kurtosis parameters range between $-.03$ to $.02$ with a mean value of $-.002$ and a median of $.000$. In other words, the modified version of the model yields very close estimations of sensitivity parameters to those related to the original model, while this is not the case for the corrected and simplified models as illustrated on Figures 7 and 8.

- Please insert Figures 7 to 8 somewhere here -

Finally, when comparing option prices obtained using the various models, one can say that Corrado-Su (1996) prices are relatively closer to the market prices (as illustrated on Figure 9) whatever the model and that model-dependant relative price differences are generally low, but sometimes economically significant as shown on Figure 10.

- Please insert Figures 9 to 10 somewhere here -

5 Conclusion

Following Brown and Robinson (2002), we focus on the Corrado and Su (1996) model dealing with excess skewness and kurtosis of the "true" implied risk-neutral density. We first present the link between multi-moment approximate models and the martingale restriction, and second provide different versions of the original Corrado-Su (1996) model. We finally give an empirical illustration of these different versions, both with the Corrado-Su study case and with market data.

As highlighted by Brown and Robinson (2002), the correction of the Corrado-Su (1996) model can be numerically important for a range of specific option (namely deep in-the-money and long maturity options when markets are turbulent) even if, on the whole, it can be judged as minor on our sample.

The martingale restriction highlighted by Kochard (1999) - and directly related to the Corrado-Su (1996) model hypotheses - yields a modified Corrado-Su (1996) formula, which is both analytically and numerically significantly different (for some options at least) from the original lead to by the appropriate martingale restriction. The resulting differences in option price estimates are sometimes economically significant. Nevertheless, investigations of the French market of empirical errors induced by the misprint in Corrado-Su (1996) seem to indicate that the general results concerning the better accuracy of this model - claimed in particular in Corrado-Su (1996) and (1997) - are out of question.

Finally, the simplifications of the Corrado-Su (1996) formula advocated in some papers (see Backus *et al.*, 1997, for instance), if they truly yield to more tractable and intuitive formulae very accurate in general, are not always negligible in-sample; they thus should be avoided when pricing and hedging options in a skewed and leptokurtic world.

However, from a theoretical point of view, we have to highlight the point that further restrictions should be imposed when applied to real markets. In particular, for the expanded distribution to correspond to an economically meaningful density of agents' expectations, some constraints should be added to ensure that implied moments backed-out from market prices correspond to a well-defined return density that is uniformly positive and unimodal (see for instance Jondeau and Rockinger, 2001). In the same vein, recovered implied moments should lead to pricing relations that respect the no-arbitrage condition (especially the call-put parity[10]).

We lastly note that if the correct use of the martingale restriction has a proven impact on the option pricing formula in the Corrado-Su (1996) model, it might also have more generally a significant economic effect on estimated implied risk-neutral densities, implicit smile functions and related Greek parameters induced by multi-moment approximate option pricing models (see Jurczenko *et al.*, 2002).

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[10] We thank Charles Corrado for pointing out that the different Corrado-Su (1996) models presented here do not strictly fulfilled the call-put parity relation on a theoretical basis (see Brown and Robinson, 1999). Nevertheless, this relation is also empirically challenged when considering transaction costs and real data (see Bakshi *et al.*, 2000, for other arbitrage violations).

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Appendix 1

(at the referee's attention)

Under the hypotheses of existence of the first non-central moments of the underlying asset log-return density, the choice of a normal as the approximate density of the continuous compound return density, and perfection and completeness of financial markets, the modified Corrado-Su formula European call C_{CS} can be expressed as:

$$C_{CS}^{*} = C_{BS}^{*} + \gamma_1(f)\, Q_3^{*} + \gamma_2(f)\, Q_4^{*} \tag{10}$$

with:

$$\begin{cases} Q_3^{*} = \dfrac{S_t\, \sigma\sqrt{\tau}\left(2\sigma\sqrt{\tau}-d^{*}\right)\varphi(d^{*})}{3!\left(1+\frac{\gamma_1(f)}{3!}\,\sigma^3\tau^{3/2}+\frac{\gamma_2(f)}{4!}\,\sigma^4\tau^2\right)} \\ Q_4^{*} = \dfrac{S_t\, \sigma\sqrt{\tau}\left({d^{*}}^2-3d^{*}\sigma\sqrt{\tau}+3\,\sigma^2\tau-1\right)\varphi(d^{*})}{4!\left(1+\frac{\gamma_1(f)}{3!}\,\sigma^3\tau^{3/2}+\frac{\gamma_2(f)}{4!}\,\sigma^4\tau^2\right)} \end{cases}$$

and:

$$d^{*} = \frac{\log\left(S_t/Ke^{-r\tau}\right) + \frac{1}{2}\sigma^2\tau - \ln\left[1 + \frac{\gamma_1(f)}{3!}\,\sigma^3\tau^{3/2} + \frac{\gamma_2(f)}{4!}\,\sigma^4\tau^2\right]}{\sigma\sqrt{\tau}}$$

Proof. Under a Gram-Charlier Type A series expansion, the risk-neutral price of an European call written on a stock S_t with strike price K is:

$$\begin{aligned} C_{CS} &= e^{-r\tau}\int_{\frac{\ln(K/S_T)-\mu\tau}{\sigma\sqrt{\tau}}}^{+\infty}\left(S_t\, e^{\mu\tau+\sigma\sqrt{t}z} - K\right) f(z)\, dz \qquad \text{(A.1.1.)} \\ &= e^{-rt}\int_{\frac{\ln(K/S_T)-\mu\tau}{\sigma\sqrt{\tau}}}^{+\infty}\left(S_t\, e^{\mu\tau+\sigma\sqrt{t}z} - K\right) \\ &\quad \times\left[1 + \frac{\kappa_3(f)}{3!}H_3(z) + \frac{\kappa_4(f)}{4!}H_4(z)\right]\varphi(z)\; dz \end{aligned}$$

where the change of variable $z = \left[\ln\left(S_T/S_t\right) - \mu\tau\right]/\sigma\sqrt{\tau}$ have been performed on S_T and the Gram-Charlier expansion residual have been dropped.

In order to evaluate expression $(A.1.1.)$, we need to compute the following integral:

$$I_j = \int_{-d+\sigma\sqrt{\tau}}^{+\infty}\left(S_t\, e^{\mu\tau+\sigma\sqrt{\tau}z} - K\right) H_j(z)\; \varphi(z)\; dz \tag{A.1.2.}$$

for $j = [3, 4]$.

Using the definition of Hermite polynomials, we get:

$$\begin{aligned} I_j &= \int_{-d+\sigma\sqrt{\tau}}^{+\infty} \left(S_t\, e^{\mu\tau+\sigma\sqrt{\tau}z} - K\right) (-1)^j \, \frac{d^j \varphi(z)}{dz^j}\, dz \\ &= -\int_{-d+\sigma\sqrt{\tau}}^{+\infty} \left(S_t\, e^{\mu\tau+\sigma\sqrt{\tau}z} - K\right) \frac{d}{dz} \left[(-1)^{j-1} \frac{d^{j-1}\varphi(z)}{dz^{j-1}}\right] dz \\ &= -\int_{-d+\sigma\sqrt{\tau}}^{+\infty} \left(S_t\, e^{\mu\tau+\sigma\sqrt{\tau}z} - K\right) \frac{d}{dz} H_{j-1}(z)\, \varphi(z)\, dz \end{aligned} \tag{A.1.3.}$$

and an integration by parts yields:

$$\begin{aligned} I_j &= -\left[\left(S_t\, e^{\mu\tau+\sigma\sqrt{\tau}z} - K\right) H_{j-1}(z)\, \varphi(z)\right]_{-d+\sigma\sqrt{\tau}}^{+\infty} \\ &\quad +\sigma\sqrt{\tau} S_t \int_{-d+\sigma\sqrt{\tau}}^{+\infty} e^{\mu\tau+\sigma\sqrt{\tau}z}\, H_{j-1}(z)\, \varphi(z)\, dz \end{aligned} \tag{A.1.4.}$$

It is readily verified that the first term in the above expression equals zero. Noting also that $\lim\limits_{z\to\infty} \varphi(z) = 0$, this then leaves the expression:

$$I_j = \sigma\sqrt{\tau} S_t \int_{-d+\sigma\sqrt{\tau}}^{+\infty} e^{\mu\tau+\sigma\sqrt{\tau}z}\, H_{j-1}(z)\, \varphi(z)\, dz \tag{A.1.5.}$$

Using once again the definition of Hermite polynomials, we have:

$$\begin{aligned} I_j &= \sigma\sqrt{\tau} S_t \int_{-d+\sigma\sqrt{\tau}}^{+\infty} e^{\mu\tau+\sigma\sqrt{\tau}z}\, (-1)^{j-1} \frac{d^{j-1}\varphi(z)}{dz^{j-1}}\, dz \\ &= -\sigma\sqrt{\tau} S_t \int_{-d+\sigma\sqrt{\tau}}^{+\infty} e^{\mu\tau+\sigma\sqrt{\tau}z} \frac{d}{dz} H_{j-2}(z)\, \varphi(z)\, dz \end{aligned} \tag{A.1.6.}$$

and integrating by parts, we get:

$$\begin{aligned} I_j &= -\sigma\sqrt{\tau} S_t \left[e^{\mu\tau+\sigma\sqrt{\tau}z} H_{j-2}(z)\, \varphi(z)\right]_{-d+\sigma\sqrt{\tau}}^{+\infty} \\ &\quad + \left(\sigma\sqrt{\tau}\right)^2 S_t \int_{-d+\sigma\sqrt{\tau}}^{+\infty} e^{\mu\tau+\sigma\sqrt{\tau}z}\, H_{j-2}(z)\, \varphi(z)\, dz \\ &= \sigma\sqrt{\tau} K\, H_{j-2}\left(-d+\sigma\sqrt{\tau}\right) \varphi\left(d-\sigma\sqrt{\tau}\right) \\ &\quad + \left(\sigma\sqrt{\tau}\right)^2 S_t \int_{-d+\sigma\sqrt{\tau}}^{+\infty} e^{\mu\tau+\sigma\sqrt{\tau}z}\, H_{j-2}(z)\, \varphi(z)\, dz \end{aligned} \tag{A.1.7.}$$

Then, by induction, we obtain:

$$
\begin{aligned}
I_j &= \sigma\sqrt{\tau}K\ H_{j-2}\left(-d+\sigma\sqrt{\tau}\right)\ \varphi\left(d-\sigma\sqrt{\tau}\right) \\
&\quad +\sigma\sqrt{\tau}\left[\sigma\sqrt{\tau}K\,H_{j-3}\left(-d+\sigma\sqrt{\tau}\right)\ \varphi\left(d-\sigma\sqrt{\tau}\right)\right. \\
&\quad \left.+\left(\sigma\sqrt{\tau}\right)^2 S_t \int_{-d+\sigma\sqrt{\tau}}^{+\infty} e^{\mu\tau+\sigma\sqrt{\tau}z}\,H_{j-3}(z)\ \varphi(z)\ dz\right] \\
&= K\,\varphi\left(d-\sigma\sqrt{\tau}\right)\sum_{k=1}^{j-1}\left(\sigma\sqrt{\tau}\right)^k H_{j-1-k}\left(-d+\sigma\sqrt{\tau}\right) \\
&\quad +\left(\sigma\sqrt{\tau}\right)^j S_t \int_{-d+\sigma\sqrt{\tau}}^{+\infty} e^{\mu\tau+\sigma\sqrt{\tau}z}\ \varphi(z)\ dz \\
&= K\,\varphi\left(d-\sigma\sqrt{\tau}\right)\sum_{k=1}^{j-1}\left(\sigma\sqrt{\tau}\right)^k H_{j-1-k}\left(-d+\sigma\sqrt{\tau}\right) \\
&\quad +\left(\sigma\sqrt{\tau}\right)^j S_t\,e^{\mu\tau+\sigma^2\tau/2}\Phi(d)
\end{aligned}
\tag{A.1.8.}
$$

Using the following equality (Stoll and Whaley, 1993, p.245):

$$
K\,\varphi\left(d-\sigma\sqrt{\tau}\right)=S_t\,e^{\mu\tau+\sigma^2\tau/2}\varphi(d) \tag{A.1.9.}
$$

leads to the following expression for I_j:

$$
\begin{aligned}
I_j^{**} &= S_t\,e^{\mu\tau+\sigma^2\tau/2}\left[\sum_{k=1}^{j-1}\left(\sigma\sqrt{\tau}\right)^k H_{j-1-k}\left(-d+\sigma\sqrt{\tau}\right)\varphi(d)\right. \\
&\quad \left.+\left(\sigma\sqrt{\tau}\right)^j\,\Phi(d)\right]
\end{aligned}
\tag{A.1.10.}
$$

Under the Gram-Charlier series expansion, the martingale restriction also implies that:

$$
\begin{aligned}
\mu\tau &= r\tau-\ln\left[\int_{-\infty}^{+\infty} e^{\mu\tau+z\sigma\sqrt{\tau}} f(z)\ dz\right] \\
&= r\tau-\ln\left\{\int_{-\infty}^{+\infty} e^{\mu\tau+z\sigma\sqrt{\tau}}\left[1+\frac{\gamma_1(f)}{3!}H_3(z)+\frac{\gamma_2(f)}{4!}H_4(z)\right]\varphi(z)\ dz\right\}
\end{aligned}
\tag{A.1.11.}
$$

where the remainder term of the series expansion have been neglected.

In order to evaluate expression (A.1.11.), we need to compute the following integral:

$$
I_j^{*}=\int_{-\infty}^{+\infty} e^{\mu\tau+\sigma\sqrt{\tau}z}H_j(z)\ \varphi(z)\ dz \tag{A.1.12.}
$$

for $j = [3, 4]$.

Note that when the current underlying asset price is unitary, the exercise price is equal to zero and the limit of integration are taken between minus and plus infinity, the integral I_j^* is equivalent to the integral I_j. Thus, integrating expression $(A.1.12.)$ by parts yields for $j = [3, 4]$:

$$I_j^* = \left(\sigma\sqrt{\tau}\right)^j \int_{-\infty}^{+\infty} e^{\mu\tau+\sigma\sqrt{\tau}z}\, \varphi(z)\; dz \tag{A.1.13.}$$

Equation $(A.1.13.)$ then becomes:

$$\begin{aligned}
\mu\tau &= r\tau - \ln\left[\int_{-\infty}^{+\infty}\left(e^{\mu\tau+\sigma\sqrt{\tau}z}\right)\varphi(z)\; dz\right. \\
&\quad +\frac{\gamma_1(f)}{3!}\sigma^3\tau^{3/2}e^{-r\tau}\int_{-\infty}^{+\infty}\left(e^{\mu\tau+\sigma\sqrt{\tau}z}\right)\varphi(z)\; dz \\
&\quad \left.+\frac{\gamma_2(f)}{4!}\sigma^4\tau^2e^{-r\tau}\int_{-\infty}^{+\infty}\left(e^{\mu\tau+\sigma\sqrt{\tau}z}\right)\varphi(z)\; dz\right] \\
&= r\tau - \frac{1}{2}\sigma^2\tau - \ln\left[1+\frac{\gamma_1(f)}{3!}\sigma^3\tau^{3/2}+\frac{\gamma_2(f)}{4!}\sigma^4\tau^2\right]
\end{aligned} \tag{A.1.14.}$$

Substituting this expression into equation $(A.1.11.)$, using Hermite polynomial definitions such as $H_0(z) = 1$, $H_1(z) = z$, $H_2(z) = z^2 - 1$, the value of an European call becomes:

$$\begin{aligned}
C_{CS}^* &= \frac{S_t\Phi(d^*)}{\left(1+\frac{\gamma_1(f)}{3!}\sigma^3\tau^{3/2}+\frac{\gamma_2(f)}{4!}\sigma^4\tau^2\right)} - e^{-r\tau}K\Phi\left(d^*-\sigma\sqrt{\tau}\right) \\
&\quad +\frac{\gamma_1(f)\; S_t}{3!\left(1+\frac{\gamma_1(f)}{3!}\sigma^3\tau^{3/2}+\frac{\gamma_2(f)}{4!}\sigma^4\tau^2\right)}\;\left[\left(2\sigma^2\tau - d^*\,\sigma\sqrt{\tau}\right)\varphi(d^*)\right. \\
&\quad \left.+\sigma^3\tau^{3/2}\Phi(d^*)\right] + \frac{\gamma_2(f)\,S_t\,\varphi(d^*)}{4!\left(1+\frac{\gamma_1(f)}{3!}\sigma^3\tau^{3/2}+\frac{\gamma_2(f)}{4!}\sigma^4\tau^2\right)} \\
&\quad \left[\left(3\,\sigma^3\tau^{3/2} - 3d^*\sigma^2\tau + d^{*^2}\,\sigma\sqrt{\tau} - \sigma\sqrt{\tau}\right)\varphi(d^*) + \sigma^4\tau^2\Phi(d^*)\right]
\end{aligned} \tag{A.1.15.}$$

that yields the final call price proposed expression:

$$
\begin{aligned}
C^{*}_{CS} &= S_t\Phi\left(d^{*}\right) - e^{-r\tau}K\Phi\left(d^{*}-\sigma\sqrt{\tau}\right) \qquad \text{(A.1.16.)} \\
&+\frac{\gamma_1\left(f\right)\ S_t}{3!\left(1+\frac{\gamma_1(f)}{3!}\,\sigma^3\tau^{3/2}+\frac{\gamma_2(f)}{4!}\,\sigma^4\tau^2\right)}\left(-d^{*}\,\sigma\sqrt{\tau}+2\sigma^2\tau\right)\varphi\left(d^{*}\right) \\
&+\frac{\gamma_2\left(f\right)\ S_t}{4!\left(1+\frac{\gamma_1(f)}{3!}\,\sigma^3\tau^{3/2}+\frac{\gamma_2(f)}{4!}\,\sigma^4\tau^2\right)}\left(d^{*^2}\,\sigma\sqrt{\tau}-3d^{*}\sigma^2\tau\right. \\
&\quad\left.+\,3\,\sigma^3\tau^{3/2}-\sigma\sqrt{\tau}\right)\varphi\left(d^{*}\right)
\end{aligned}
$$

with:

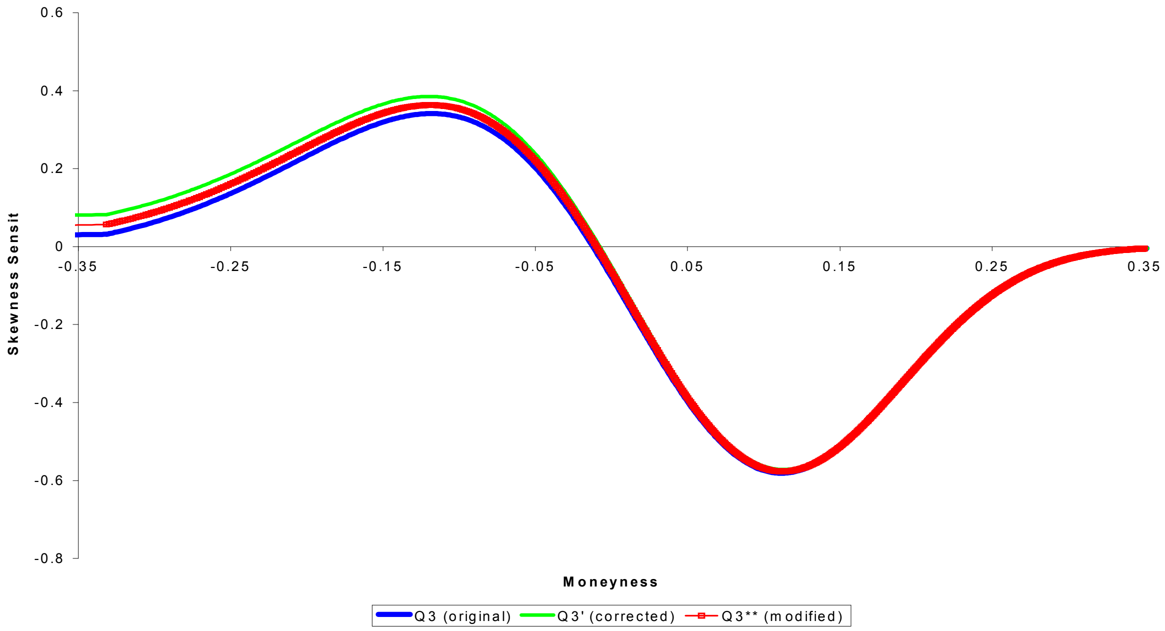
$$
d^{*}=\frac{\log\left(S_t/Ke^{-r\tau}\right)+\frac{1}{2}\sigma^2\tau-\ln\left[1+\frac{\gamma_1(f)}{3!}\,\sigma^3\tau^{3/2}+\frac{\gamma_2(f)}{4!}\,\sigma^4\tau^2\right]}{\sigma\sqrt{\tau}}
$$

Rearranging terms, and posing polynomials P_3 and P_4 and a constant ω, lead finally to the desired results (equation 10) $\odot$

Appendix 2

In the figures below (1 to 4), Skewness and Kurtosis have been fixed at -.7 and 3.53 for the modified Corrado-Su (1996) model parameter representations (which are mean parameter values of implied moments when backed out them from French market data using the Jarrow-Rudd (1982) model - see Capelle-Blancard *et al.*, 2001-a and 2001-b, for details). Other parameters are those of the Corrado-Su (1996) case study, also reported in Brown and Robinson (2002) - see text.

Figure 1: Option Price Adjustment for Skewness of the Underlying Return Risk-neutral Density ($-Q_3$, $-Q_3^{'}$, $-Q_3^{*}$ and $-Q_3^{**}$)



No difference between the simplified and modified parameters can be highlighted on this Figure.

Figure 2: Absolute Differences in Option Price Skewness "Sensitivities" Compared to the Original Corrado-Su (1996) Model

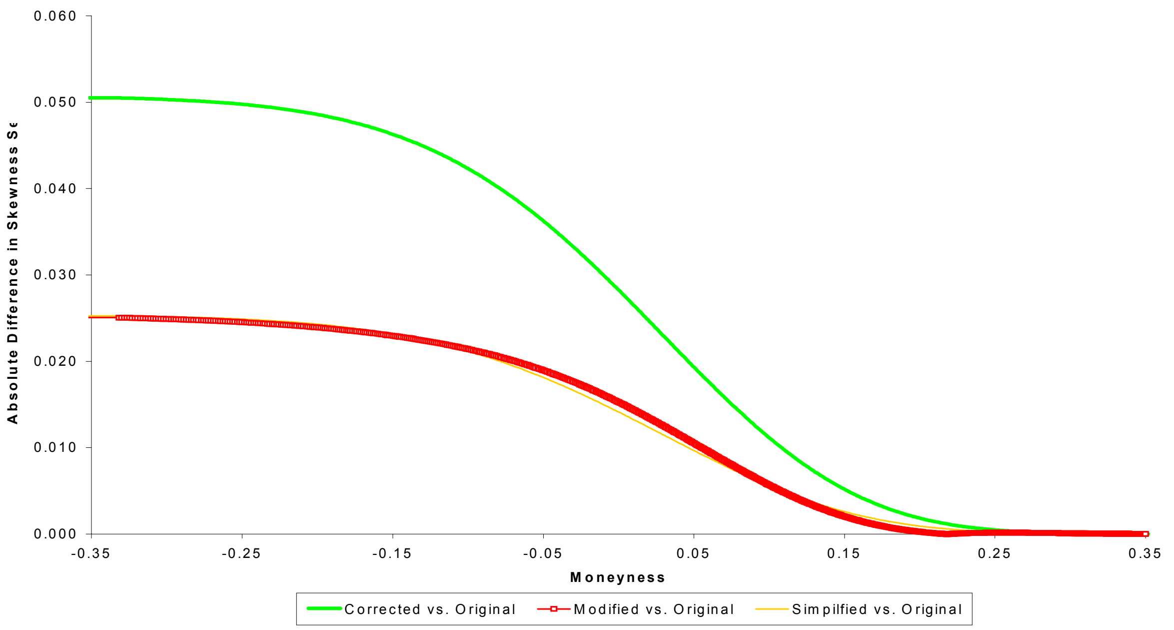
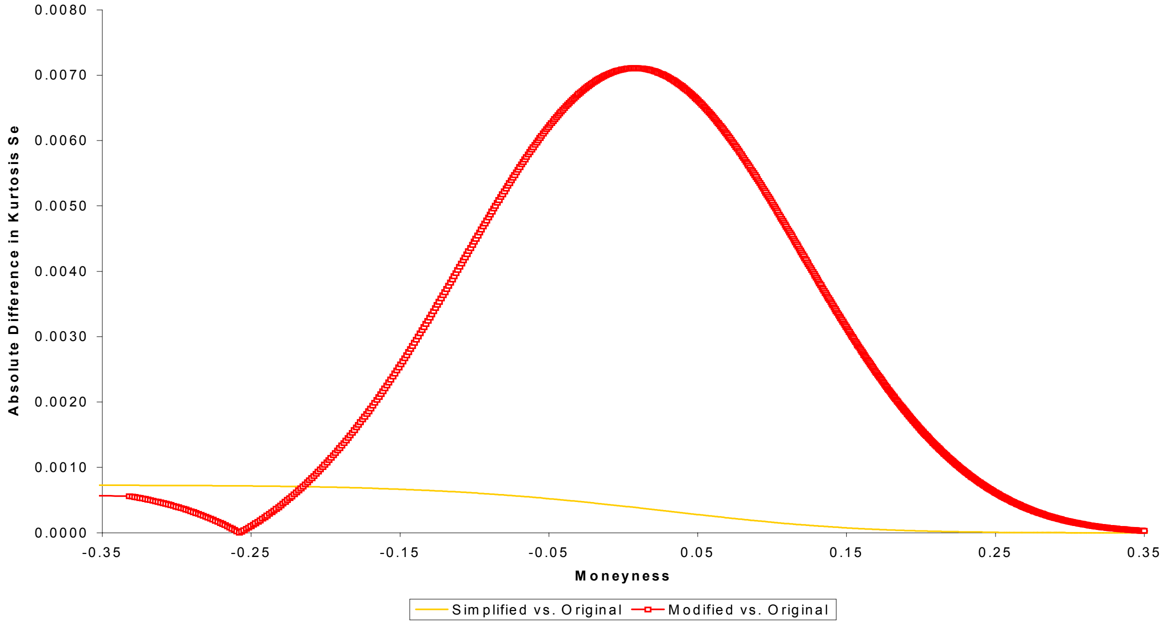
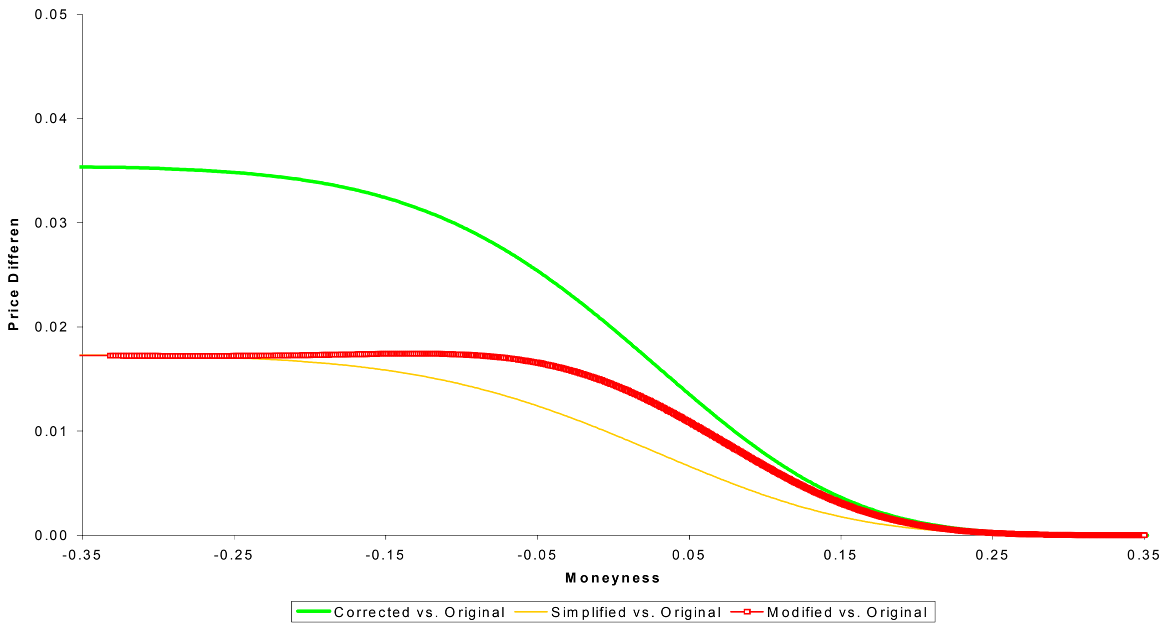


Figure 3: Absolute Differences in Option Price Kurtosis "Sensitivities" Compared to the Original Corrado-Su (1996) Model



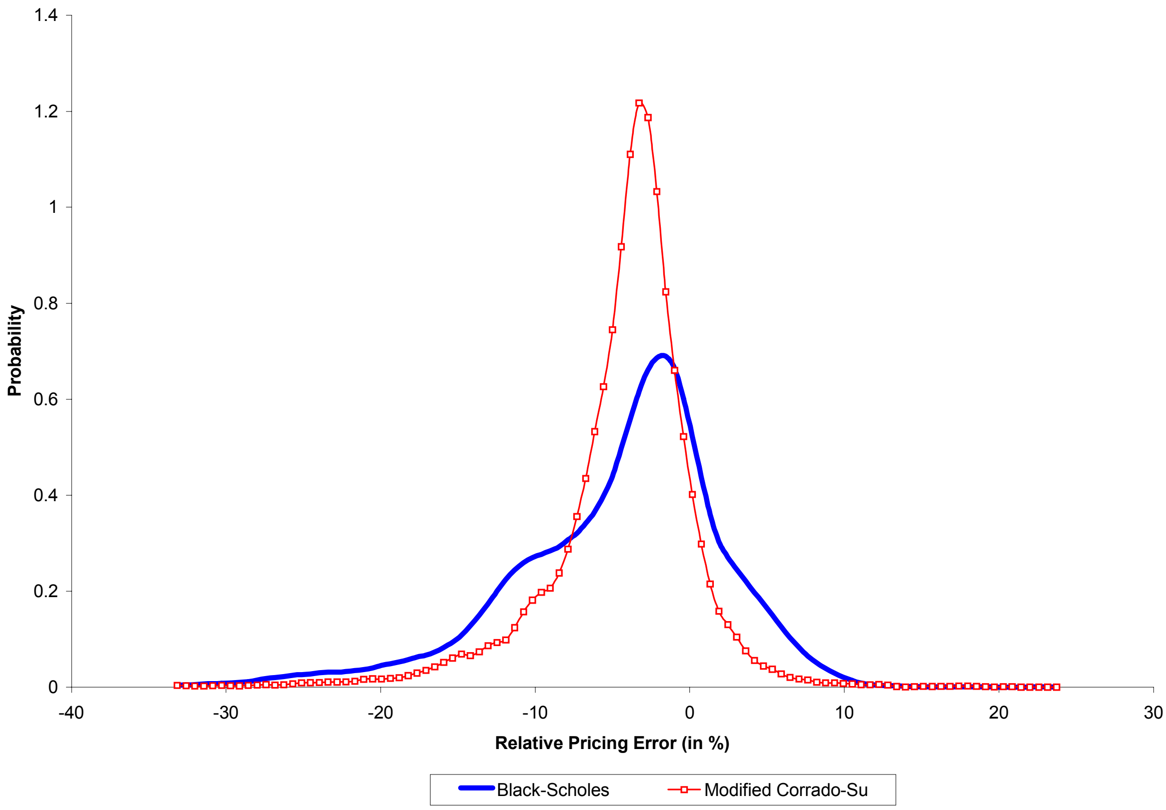
The correction of Brown and Robinson (2002) do not apply (directly) to the kurtosis parameter.

Figure 4: Absolute Induced Option Price Absolute Differences for the Corrected, Modified and Simplified Models Compared to the Original Corrado-Su (1996) Model



In the figures below (5 and 6), French CAC 40 Long Term options on the period 01/97 through 12/98 have been used to estimate the error terms and related density probabilities (see Capelle-Blancard *et al.*, 2001-a and 2001-b, for details on the database, filters, optimization *criterion* and routines). For easy representations, Figures 7 to 11 illustrate estimations on sub-samples.

Figure 5: Estimated Probability Densities of Relative Pricing Errors for the Black-Scholes (1973) and the Modified Corrado-Su (1996) Models



A non-parametric kernel density estimator was used with the Silverman (1986) rule of thumb.

Figure 6: Estimated Probability Density Differences of Relative Pricing Errors for the Corrected, Modified and Simplified Models Compared to the Original Corrado-Su (1996) Model

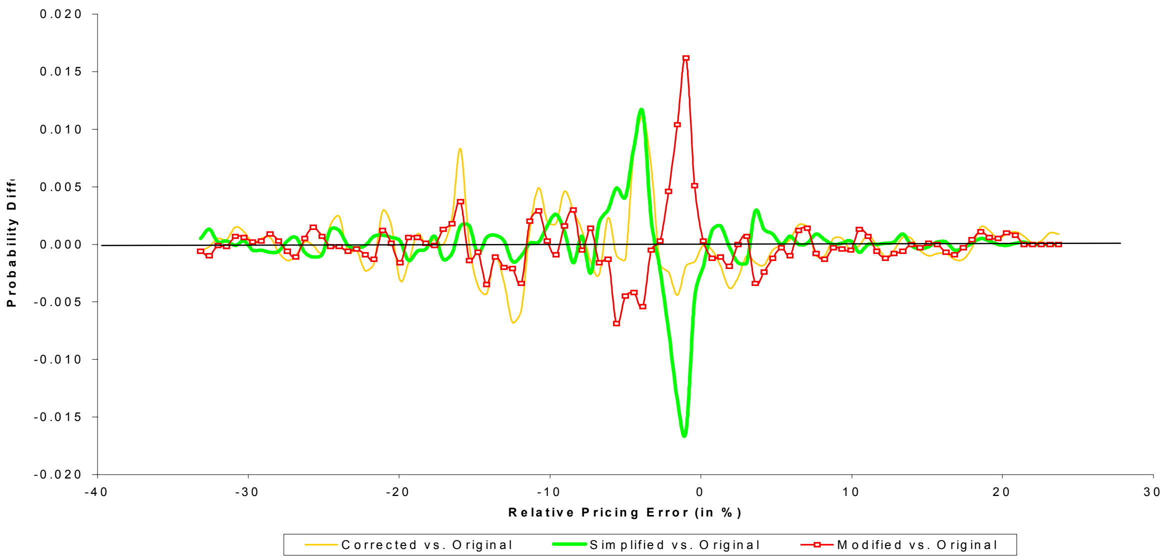


Figure 7: An illustration of Time-variations of Skewness Parameters for the Corrected, Modified and Simplified Models Compared to the Original Corrado-Su (1996) Model - on the period 06/23/97-09/19/97 -

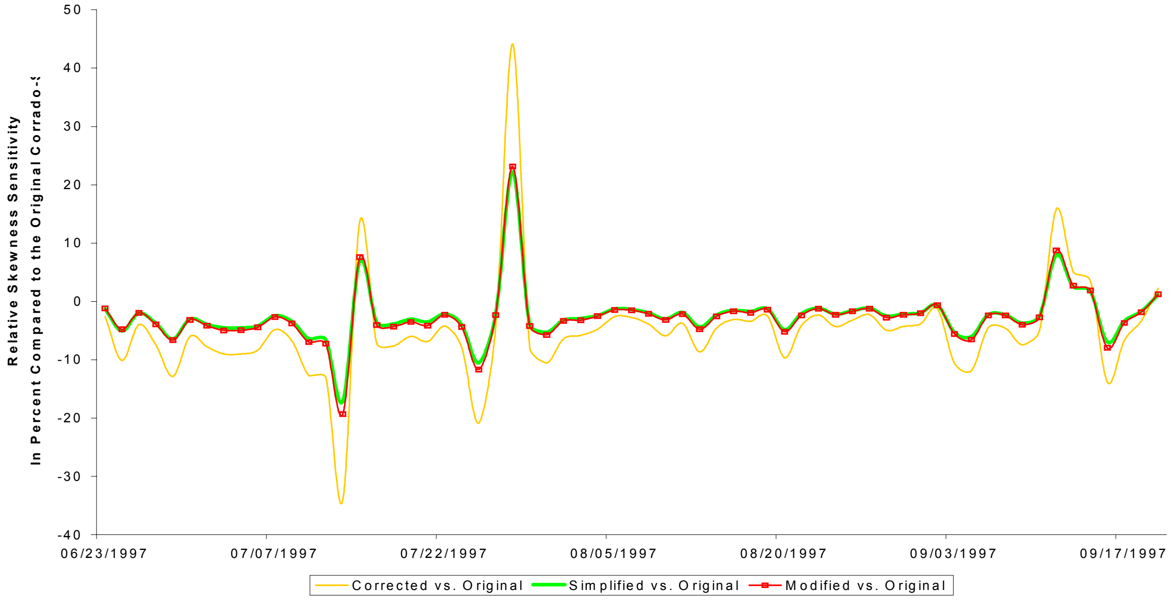
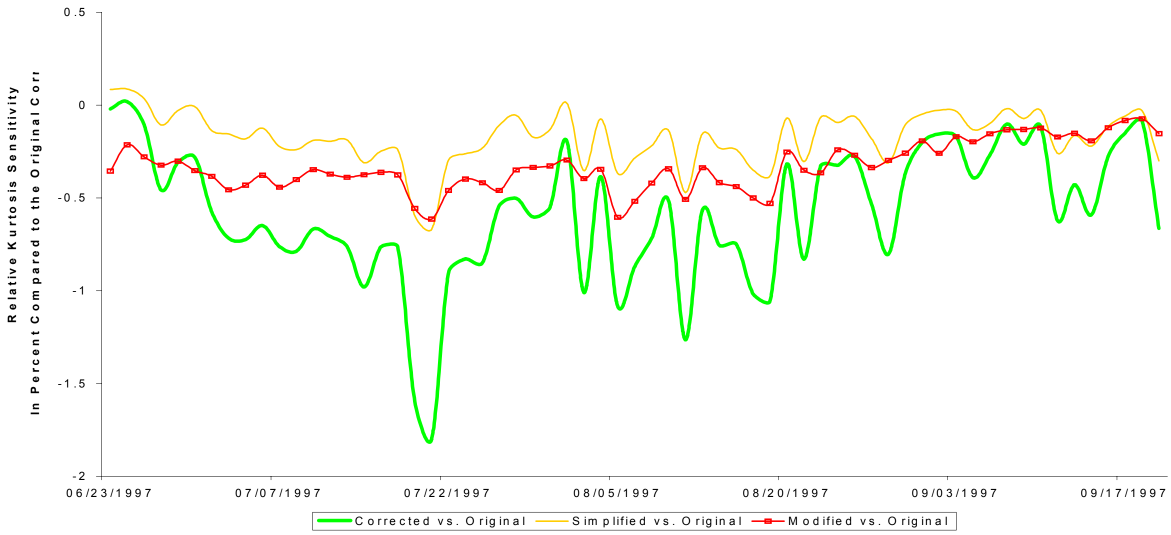


Figure 8: An illustration of Time-variations of Kurtosis Parameters for the Corrected[11], Simplified and Modified Models Compared to the Original Corrado-Su (1996) Model - on the period 06/23/97-09/19/97 -



[11] The absolute difference between the corrected and original kurtosis parameter is empirically non null since the modification of the definition of the skewness parameter in the corrected model has an impact on the in-sample estimation of the kurtosis parameter.

Figure 9: An illustration of Time-variations of Implied Index Prices[12] for the Black-Scholes and the Modified Corrado-Su Models Compared to the "True" Index Prices - on the period 06/23/97-09/19/97 -

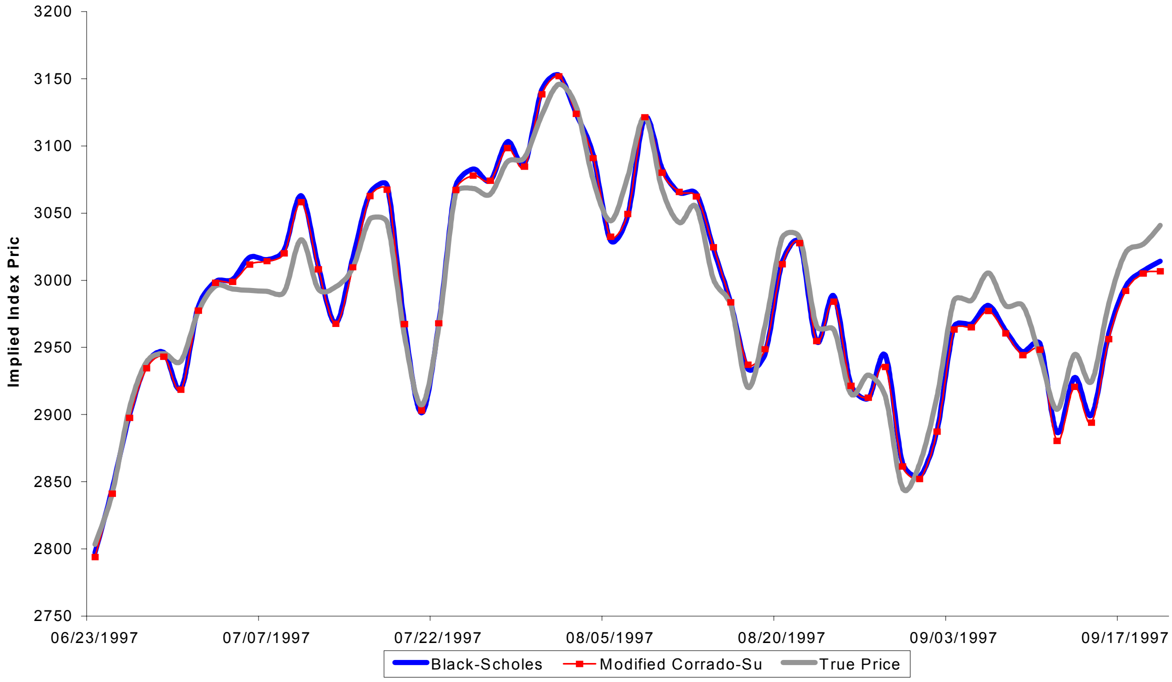
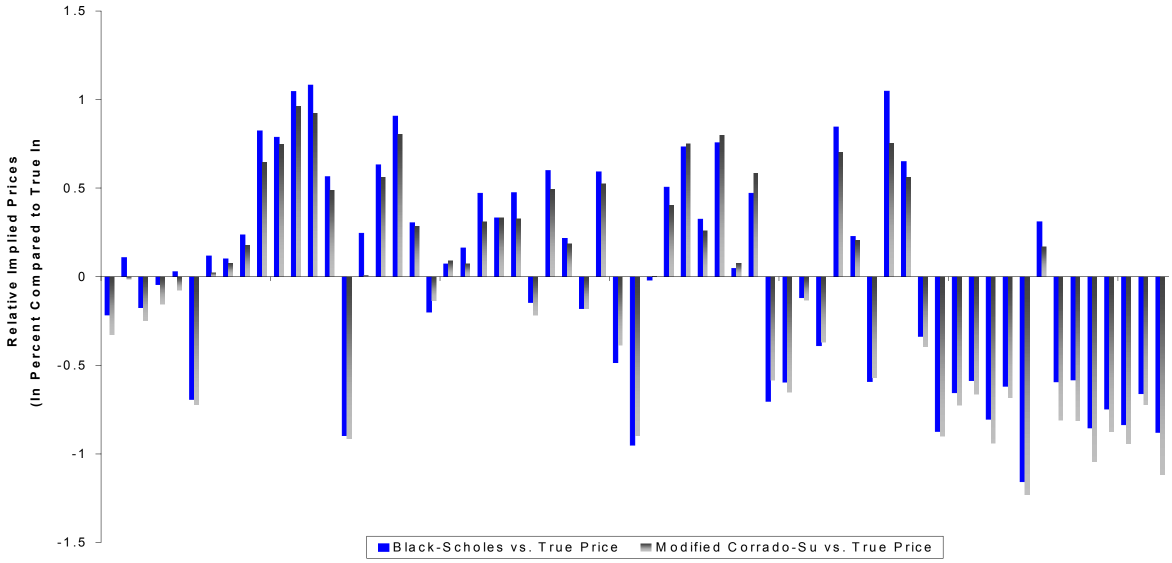


Figure 10: An illustration of Time-variations of Differences in Implied Index Prices[12] for the Modified Corrado-Su and Black-Scholes (1973) Models Compared to the True Index Price - on the period 06/23/97-09/19/97 -



[12] Differences in this figure correspond to in-the-money options with a moneyness higher than 5%.

Figure 11: An illustration of Time-variations of Implied Index Prices[13] for the Corrected, Simplified and Modified Models Compared to the Original Corrado-Su (1996) Model - on the period 06/23/97-09/19/97 -

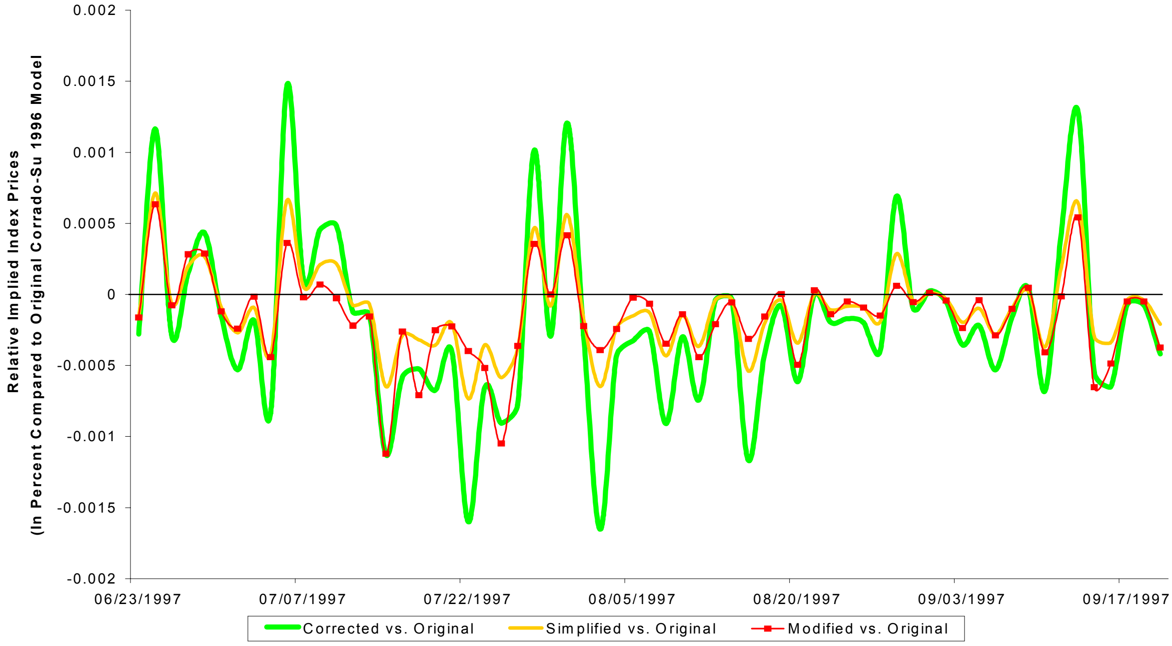


Table 1: Number of Option Contracts Needed to Delta-hedge a USD 10 Million Stock Portfolio

Number of S&P option contracts needed to delta-hedge a USD 10 million stock portfolio with a beta of one using contracts of varying strike prices. This example assumes an index level of S_0 = (USD) 700, an interest rate r = 5%, a dividend yield q = 2%, a time until option expiration $\tau = .25$. Black-Scholes (BS) model assumes a volatility of σ = 12.88% corresponding to findings of Corrado and Su (1997). The original (CS), the corrected (CS'), the modified (CS^*) and the simplified Corrado-Su (CS^{**}) models assume σ = 11.62%, and skewness and kurtosis parameters of γ_1 = -1.68 and γ_2 = 5.39 (see Corrado and Su, 1997, Exhibit 6, p.18).

In-the-money Options						Out-of-the-money Options					
Strike	(BS)	(CS)	(CS')	(CS*)	(CS**)	Strike	(BS)	(CS)	(CS')	(CS*)	(CS**)
660	167	163	163	157	157	710	303	266	266	257	257
670	179	171	171	160	159	720	370	332	332	343	342
680	197	182	182	167	167	730	465	443	443	502	501
690	221	198	198	182	181	740	601	637	637	825	824
700	256	225	225	209	209	750	802	1008	1008	1602	1598

[13] Differences in this figure correspond to in-the-money options with a moneyness higher than 5%.