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Price discovery with a richer market microstructure noise

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Abstract: We show that the standard econometric framework typically yields inconsistent estimates of price discovery measures in the presence of richer market microstructure noise dynamics. We address this errors-in-variable issue using instrumental variables. We devise valid instruments for two alternative microstructure noise settings, and then establish the asymptotic behavior of the corresponding price discovery measures. Our empirical analysis reveals that market leadership conclusions depend heavily on whether we account or not for the market microstructure noise.

JEL classification numbers: G15, G12, G32, C32, C36, C58

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1 Introduction

Recent regulatory changes set out the vision of multiple trading venues competing for order flow and liquidity (see, among others, Menkveld, 2016).[1] High-frequency trading has been contributing to scatter quotes across different exchanges, making markets much faster, with time scales dropping to microseconds or even nanoseconds (O'Hara, 2015; Hasbrouck, 2019). As a result, the intraday price dynamics depend on the interaction between learning and feedback mechanisms and (endogenous) market microstructure noises (Andersen, Archakov, Cebiroglu and Hautsch, 2021).

Market design and characteristics clearly affect the mechanism and timing of price formation. Price discovery analyses study how quickly markets (or trading platforms) impound new information into prices. If the idea is to gauge how quickly markets react to price innovations (see discussion in Putniņš, 2013), it is paramount to focus on price behavior at a very high frequency. However, microstructure noise effects typically increase with the sampling frequency, contaminating both transaction and quotes data (see, among others, Hansen and Lunde, 2006; Ait-Sahalia, 2007; Bandi and Russell, 2008; Barndorff-Nielsen, Hansen, Lunde and Shephard, 2008). Moreover, microstructure noises have a richer dynamics than the traditional martingale-plus-noise market microstructure models are able to accommodate. For instance, Andersen et al. (2021) contemplate a second layer of (endogenous) microstructure noise in the latent efficient price, apart from the typical error correction dynamics that characterizes traditional market microstructure models.

The goal of this paper is to address the implications of richer market microstructure dynamics to price discovery analyses. To do so, we start with a continuous-time model for the prices of a given asset at different trading venues. At this point, there is only one layer of market microstructure noise, corresponding to the error correction dynamics. As we move from continuous to discrete time, we add a second source of noise contamination to accommodate the empirical evidence of serial correlation in the market microstructure noise (Diebold and Strasser, 2013; Li and Linton, 2020). This additional layer of microstructure noise disrupts traditional price discovery analyses because measurement error precludes the consistent estimation by least squares (LS) of the parameters that regulate how fast prices respond to changes in the efficient price.

[1] In particular, the Regulation Alternative Trading Systems (RegATS) in 2000 and the Regulation National Market System (RegNMS) in 2007 lay the foundation for U.S. markets, whereas the Markets in Financial Instruments Directive (MiFiDin) in 2007 for trading venues in Europe.

At first glance, it might seem that reducing the sampling frequency would suffice to alleviate these market microstructure effects. This is indeed the usual fix in the realized measure literature (see, for instance, Liu, Patton and Sheppard, 2015). Bandi and Russell (2008) provide theoretical justification to this practice by deriving the optimal choice of sampling frequency in a mean squared error sense. Zhang (2011) extends their results to deal with tradeoffs between the bias and various sources of stochastic error terms, including nonsynchronous trading, microstructure noise, and time discretization. However, even if sampling at a lower frequency helps with the consistent estimation of the covariance matrix, it does not solve the measurement error problem in the estimation of the speed-of-adjustment parameters in the error correction mechanism. Although the bias in the LS estimation reduces as the sampling frequency decreases, it does not shrink to zero.

To obtain consistent estimates of the price discovery measures in the presence of microstructure noise, we propose the use of instrumental variables (IV). Instrument choice obviously depends on how market microstructure noises interact not only over time, but also across assets and trading platforms. We come up with valid instruments in two settings. The first allows for a more flexible cross-correlation structure of the microstructure noises across markets, but limits their persistence over time. The second accommodates more persistence, but constrains the dependence between microstructure noises across markets. The additional cost of allowing more persistent noise is that instruments in this setting are much weaker. As a result, we move away from the standard IV estimator to adopt a continuous-updating generalized method of moments (CU-GMM) estimator.

We establish consistency and asymptotic normality of the IV and GMM-based price discovery measures under both time-span and mixed (double) asymptotics. The former treats the number of intraday observations as fixed, while allowing the number of days in the sample to diverge. In the latter, we let the number of intraday observations grow to infinity and the number of days to diverge at a slower rate, in order to accommodate daily variation in the price discovery measures. This is in line with Hasbrouck (2003), Chakravarty, Gulen and Mayhew (2004) and Hansen and Lunde (2006), who estimate daily VEC models. To preserve robustness to microstructure noise in the mixed setting, we employ Barndorff-Nielsen, Hansen, Lunde and Shephard's (2011) realized kernel (RK) estimator of the covariance matrix to accommodate stochastic changes in the covariance matrix over time.

Monte Carlo simulations confirm the advantages of our robust-to-noise price discovery measures based on IV and GMM approaches. The LS-based price discovery measures display sizeable median bias in the presence of market microstructure noise, whereas the IV and GMM counterparts perform exceptionally well for different noise-to-signal ratios. Altogether, we find that both time-span and mixed asymptotic inferences offer excellent finite-sample approximations.

Empirically, we investigate price discovery for Alcoa (AA) on both New York Stock Exchange (NYSE) and Nasdaq Stock Market from June 2012 to May 2013. We use mid-quotes at the one-second frequency, yielding an average just shy of 18,000 observations per day. We compute price discovery measures (and their robust standard errors) using both LS and IV/GMM methods. The traditional price discovery analysis based on LS estimates indicates that Nasdaq leads the price discovery for most of the sample period. Once we account for microstructure noise, the picture changes dramatically, revealing that the NYSE steers the price discovery for the entire period.

The remainder of this paper is as follows. Section 2 describes the continuous-time setting we use for the price discovery mechanism. Section 3 shows how to estimate price discovery measures in a consistent manner accounting for the presence of market microstructure noise. Section 4 reports an extensive Monte Carlo study showing that our IV-based price discovery measures outperform the standard measures based on the daily VEC approach. Section 5 investigates how the price informativeness of the NYSE relative to the Nasdaq changes over time for Alcoa. Section 6 offers some concluding remarks.

2 Price discovery framework in continuous time

Given the interest in price discovery at the highest resolution, say seconds or microseconds, it makes sense to entertain Dias, Fernandes and Scherrer's (2019) continuous-time setting. We assume that, on any given day, the log-prices of a given asset that trades on multiple venues follow

$$\mathrm{d}P_t = \Pi P_t \,\mathrm{d}t + C \,\mathrm{d}W_t, \tag{1}$$

where $P_t = (P_{1,t}, \ldots, P_{M,t})'$ is an $M \times 1$ vector collecting the log-prices in each of the M trading venues, $\Pi = \alpha\beta'$ is an $M \times M$ reduced-rank matrix, α and β are $M \times R$ full-rank matrices, W is an $M \times 1$ vector of Brownian motions, and C is an $M \times M$ matrix such that the covariance matrix $\Sigma = CC'$ is positive definite.

Although not every discrete-time VEC model results from the exact discretization of a continuous-time reduced-rank Ornstein-Uhlenbeck (OU) process, (1) provides a very convenient framework to study the price dynamics of assets traded at multiple venues. In particular, it implies that the efficient price is a martingale, while allowing returns to follow VMA(∞) processes (Dias et al., 2019). Prices at the different markets should not drift much apart, oscillating around the (latent) efficient price. Accordingly, there is $R = M - 1$ cointegrating relationships, with log-prices sharing the asset's efficient price as the single common stochastic trend. We assume without loss of generality that β is known, taking the form of $\beta = (I_R, -\iota_R)'$, where ι_R denotes a $R \times 1$ unit vector. In turn, α determines how quickly each market reacts to deviations from the long-run equilibria given by $\beta' P_t$. Note that α reflects not only institutional characteristics of each market, such as cost structure, technological infrastructure, and traders' composition, but also liquidity aspects, such as trading activity, volume and market share (see, among others, Eun and Sabherwal, 2003).

To formalize the properties of (1), we restrict the autoregressive matrix Π so as to ensure identification, as in Kessler and Rahbek (2004).

Assumption RROU The reduced-rank OU process in (1) is such that all eigenvalues of Π are real and no elementary divisor of Π occurs more than once. In addition, α, β and $\beta'\alpha$ have full column rank R, with every eigenvalue of $\beta'\alpha$ having negative real parts.

Assumption RROU follows directly from Kessler and Rahbek (2004) and establishes the necessary conditions for (1) the uniqueness of the mapping $\theta = (\Pi, \Sigma) \overset{\psi}{\mapsto} \psi(\theta) = (\Pi_\delta, \Sigma_\delta)$ from continuous-time to discrete-time parameters at sampling interval δ; (2) the injectivity of the mapping ψ and identifiability of θ; (3) the existence of a Granger representation theorem (GRT) in continuous time; and (4) the exact discretization of the reduced-rank OU process in (1).

To work out the exact discretization of (1), we assume prices are observed regularly and equidistantly over the unit interval $[0, 1]$ that represents a trading day (calendar-time sampling, as discussed in Hansen and Lunde, 2006). Denote each interval contained in $[0, 1]$ as $[t_i, t_{i-1}]$, where $i = 1, 2, \ldots, n$ and n is the total number of intervals such that $0 = t_0 < t_1 < \ldots < t_n = 1$. The length of each interval is $\delta = t_i - t_{i-1} = 1/n$ in $[0, 1]$. For instance, the usual trading day in U.S. markets lasts for 6.5 hours, so that sampling one observation per minute yields $n = 390$

and $\delta = 1/390$. Denoting by $\exp(A)$ the matrix exponential of an $M \times M$ matrix A such that $\exp(A) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} A^{\ell}$, the exact discretization of (1) at interval length δ reads

$$\Delta P_{t_i} = \Pi_{\delta} P_{t_{i-1}} + \varepsilon_{t_i}, \tag{2}$$

where $\Pi_{\delta} = \alpha_{\delta}\beta'$ and $\alpha_{\delta} = \alpha(\beta'\alpha)^{-1}\left[\exp(\delta\beta'\alpha) - I_R\right]$, with I_R denoting a R-dimensional identity matrix, P_{t_i} is an $M \times 1$ vector of log-prices observed at discrete time, and the innovation ε_{t_i} is iid Gaussian with zero mean and covariance matrix given by $\Sigma_{\delta} = \int_0^{\delta} \exp(u\Pi)\Sigma\exp(u\Pi')\,\mathrm{d}u$. Note that $\alpha_{\delta} \equiv \alpha_{\delta}(\alpha)$ and $\Sigma_{\delta} \equiv \Sigma_{\delta}(\alpha, \Sigma)$ are known functions of the continuous-time parameters in (1). We next summarize Kessler and Rahbek's (2004) characterization of the discrete-time VEC model in (2) implied by the exact discretization of (1).

Lemma 1 Let Assumption RROU holds. The discrete-time VEC process in (2) is such that (*a*) prices are not explosive, so that the roots of the characteristic polynomial $|I_M - (\Pi_{\delta} + I_M)z| = 0$ are either outside the unit circle or equal to one; (*b*) the column space of α_{δ} is the same as the column space of α in that the number of unit roots equals $M - R$, and hence $\operatorname{rank}(\alpha\beta') = \operatorname{rank}(\alpha_{\delta}\beta')$; and (*c*) the estimation of the continuous-time parameters readily follows form the discrete-time log-likelihood in that $\ell_n^c(\alpha, \Sigma) = \ell_n^d(\alpha_{\delta}, \Sigma_{\delta})$, where ℓ_n^c and ℓ_n^d denote the continuous- and discrete-time log-likelihoods, respectively.

Computing price discovery measures requires to decompose the price vector into a permanent I(1) component and a transitory I(0) component. To do so, we consider the GRT decomposition because it ensures that the stochastic trend that reflects the efficient price P_t^* is a martingale. As in any VEC decomposition, it depends exclusively on the orthogonal complements of the speed-of-adjustment parameters and of the cointegrating vector given that they relate to the nonstationary directions of the processes.[2] In particular, the orthogonal complements of the speed-of-adjustment parameters indicate how the stochastic trend relates to the market innovations, playing a major role in any price discovery analysis.

Under Assumption RROU, Theorem 1 in Kessler and Rahbek (2001) shows that the GRT holds

[2] We denote by $\alpha_{\perp}$ and $\alpha_{\delta,\perp}$ the orthogonal complements of the speed-of-adjustment parameters in continuous and discrete times, whereas the orthogonal complement of the cointegrating vector is $\beta_{\perp} = \iota_M$ in the price discovery context.

in continuous time:

$$P_t = \Xi\left(CW_t + P_0\right) + \alpha(\beta'\alpha)^{-1}H_t, \tag{3}$$

where $\Xi = \beta_\perp\left(\alpha'_\perp\beta_\perp\right)^{-1}\alpha'_\perp$, P_0 contains initial values, and $H_t = \beta' P_t$ denotes a stationary Ornstein-Uhlenbeck process given by $\mathrm{d}H_t = \beta'\alpha H_t\,\mathrm{d}t + \beta' C\,\mathrm{d}W_t$. Similarly, as the exact discretization in (2) yields the same data generation process, the GRT also holds in discrete time

$$P_{t_i} = \Xi_\delta \sum_{h=1}^{i} \varepsilon_{t_h} + \sum_{h=0}^{\infty} \Upsilon_{\delta,h}\varepsilon_{t_{i-h}} + \Xi_\delta P_0, \tag{4}$$

where $\Xi_\delta = \beta_\perp(\alpha'_{\delta\perp}\beta_\perp)^{-1}\alpha'_{\delta\perp}$ and P_0 is a vector of initial values with elements such that $P_{m,0} = P_{m',0}$ with $m, m' = 1, \ldots, M$. Note that one can compute $\Upsilon_{\delta,h}$ from the parameters in (2) such that $\Upsilon_{\delta,h} = (I_M - \Xi_\delta)(I_M + \alpha_\delta\beta')^h$ for $h = 0, 1, 2, \ldots$ (see Corollary 2 in Hansen, 2005). The stochastic common trend given by the first term on the right-hand side of (4) is a martingale corresponding to the efficient price (Hansen and Lunde, 2006), given that $\beta_\perp = \iota_M$ implies common rows in Ξ_δ.

2.1 Price discovery metrics

The component share (CS) measure captures how the efficient price relates to market innovations. It depends exclusively on the orthogonal complement of the speed-of-adjustment parameters (see, e.g., Harris, McInish and Wood, 2002; Hansen and Lunde, 2006). Specifically, because Ξ_δ has common rows, the contribution of each market innovation to the efficient price is given by the columns of the $1 \times M$ vector $(\alpha'_{\delta,\perp}\beta_\perp)^{-1}\alpha'_{\delta,\perp}$. The orthogonal complement of α_δ is not unique and hence, without loss of generality, we impose that $\sum_{m=1}^{M} \alpha_{\delta,\perp,m} = 1$. Under this normalization, the common row of Ξ_δ reduces to $\alpha'_{\delta,\perp}$.

Dias et al. (2019) show that discretization does not affect $\alpha_{\delta,\perp}$ in view that $\alpha_{0,\perp} = \alpha_{\delta,\perp}$ for any $0 < \delta < 1$. This implies that one can estimate the continuous-time orthogonal complements of the speed-of-adjustment parameters directly from discrete-sampled prices. It then follows that the continuous-time CS measure of the mth market reads

$$\mathrm{CS}_m = \alpha_{\delta,\perp,m}, \quad m = 1, \ldots, M. \tag{5}$$

Hasbrouck's (1995) information share (IS) is the other prominent price discovery measure, gauging the contribution of each market/venue to the total variation in the efficient price innovation.

Apart from $\Xi_{\delta,\perp}$, the IS measure also takes into consideration the contemporaneous covariance between market innovations. Normalizing $\alpha_{\delta,\perp}$ such that $\sum_{m=1}^{M} \alpha_{\delta,\perp,m} = 1$ and using the fact that $\alpha_{0,\perp} = \alpha_{\delta,\perp}$ for any $0 \leq \delta < 1$, the discrete-time IS measure reads

$$IS_{\delta,m} = \frac{[\alpha'_{\perp} C_{\delta}]^2_m}{\alpha'_{\perp} \Sigma_{\delta} \alpha'_{\perp}}, \quad m = 1, \ldots, M \tag{6}$$

where $\Sigma_{\delta} = C_{\delta} C'_{\delta} = \int_0^{\delta} \exp(u\Pi) \Sigma \exp(u\Pi')\, \mathrm{d}u$ and $[\alpha'_{\perp} C_{\delta}]_m$ denotes the mth element of the $1 \times M$ vector $\alpha'_{\perp} C_{\delta}$.

It turns out that the IS becomes uninformative, converging to $1/M$, as δ increases. This happens because the contemporaneous correlation among markets increases with δ even if Σ is diagonal. This is exactly why Hasbrouck (2019) advocates for price discovery analyses in high resolution. Dias et al. (2019) formally address this issue by defining the continuous-time analogue of (6):

$$IS_m = \frac{[\alpha'_{\perp} C]^2_m}{\alpha'_{\perp} \Sigma \alpha'_{\perp}}, \quad m = 1, \ldots, M \tag{7}$$

which depends exclusively on the continuous-time parameters in (1). Altogether, computing the continuous-time CS and the continuous- and discrete-time IS measures is down to estimating α, α_{δ}, Σ, and Σ_{δ} from discrete-sampled prices.

Alternatively, one may also employ impulse response functions (IRFs) to build dynamic measures of price discovery in discrete time (Yan and Zivot, 2010; Nguenang, 2016). They depict how each market responds over time to a change in the efficient price. Because the exact discretization in (2) yields homoskedastic Gaussian innovations, it is possible to back out the IRF from the efficient price $P^*_{t_i} = \beta_{\perp} (\alpha'_{\delta\perp} \beta_{\perp})^{-1} \alpha'_{\delta\perp} \sum_{h=1}^{i} \varepsilon_{t_h}$ implied by (4). In particular,

$$\frac{\mathrm{d}P_{t_{i+\ell}}}{\mathrm{d}P^*_{t_i}} = \iota_M + \frac{\alpha'_{\perp} \beta_{\perp}}{\alpha'_{\perp} \Sigma_{\delta}\, \alpha_{\perp}}\, \Upsilon_{\delta,\ell} \Sigma_{\delta}, \alpha_{\perp}, \tag{8}$$

which converges to one in every market because $\Upsilon_{\delta,\ell} \to 0$ as $\ell \to \infty$ (see, e.g., Hansen, 2005; Hansen and Lunde, 2006).[3]

Most analyses compute price discovery measures not only at different frequencies (e.g., using tick data, 30-second, 1-minute, and 5-minute returns) but also over different time spans (e.g., day, month, quarter, and year). These combinations between frequency and time span imply different

[3] Yet another alternative is to compute Putniņš's (2013) informational leadership share, which aims to control for different levels of noise across markets. As it essentially combines CS and IS into a single measure of price discovery, it is straightforward to compute a robust-to-noise analog.

sampling schemes. Price discovery studies with long time spans (Grammig, Melvin and Schlag, 2005; Menkveld, Koopman and Lucas, 2007; Fernandes and Scherrer, 2018) require fixing the sampling interval or, equivalently, the number of intraday observations, while letting the number of days grow. To estimate daily measures of price discovery (Hansen and Lunde, 2006; Hasbrouck, 2019), it is more appropriate to consider a large number of intraday observations, while fixing the number of days to one. In the next section, we provide asymptotic results under both settings.

3 Price discovery analyses with richer microstructure dynamics

There is an extensive literature on market microstructure and volatility estimation that acknowledges the fact that observed market prices are contaminated with market microstructure noise (see, among others, Hansen and Lunde, 2006; Bandi and Russell, 2008; Diebold and Strasser, 2013). Still, whilst prices coming from the continuous-time price discovery model in (1) are consistent with partial adjustments models (see, among others, Amihud and Mendelson, 1987; Hasbrouck and Ho, 1987; Yan and Zivot, 2010), they are unable to reflect the wide spectrum of market microstructure effects.

The market microstructure noise should reflect any imperfection in the trading and data recording processes as well as asymmetric information effects. However, the iid Gaussian error in (2) is not enough to capture the persistence that microstructure effects exhibit both empirically and theoretically. For instance, fad effects as in Lehmann (1990) would entail deviations from the efficient price that take time to dissipate. Diebold and Strasser (2013) derive exactly what sort of serial correlation one should expect in the context of traditional market microstructure models. More recently, Bandi, Ren and Pirino (2017) and Li and Linton (2020) respectively argue that flat trading and herding may also induce serial correlation in the microstructure noise (see also Park and Sabourian, 2011).

We next incorporate one additional layer of noise in order to handle microstructure effects with richer discrete-time dynamics. In particular, we add a microstructure noise to the exact discretization of the continuous-time prices in (2). This is in the same spirit of Andersen et al.'s (2021) discrete-time local-regime model, which nests several important market microstructure models. The main difference lies on the focus. While they take price discovery models seriously when robustifying realized measures to market microstructure noise, we aim to accommodate richer microstructure effects in price discovery analyses. This is also in line with the consensus that it is always advantageous to entertain market microstructure noise in the estimation of continuous-time processes using high-frequency data (see, among others, Ait-Sahalia, Mykland and Zhang, 2005; Zhang, Mykland and Ait-Sahalia, 2005; Ait-Sahalia, 2007; Li and Linton, 2020). More specifically, we let p_{m,t_i} denote the observed discrete-time log-price in market m at time t_i with $m = 1, \ldots, M$. It differs from the latent log-price P_{m,t_i} due to the market microstructure noise $u_{m,t_i} = p_{m,t_i} - P_{m,t_i}$ $(m = 1 \ldots, M)$.

In the absence of such a noise, the LS estimator is consistent for α_δ, with the sample covariance matrix converging to Σ_δ under standard regularity conditions. Additionally, one could consistently estimate α by nonlinear least squares and the continuous-time covariance matrix directly from the intraday returns at their highest frequency using a realized approach. In contrast, it is no longer trivial to obtain consistent estimates of α_δ, Σ_δ, and Σ in the presence of (this second layer of) market microstructure noise. Contamination biases the LS estimation of α_δ and, as such, of Σ_δ. In fact, it affects both terms on the right-hand side of (3), leading to the misleading conclusion that the market microstructure noise has long-run effects on the efficient price and its volatility. Finally, the realized variance estimator is also biased and inconsistent. In fact, the variance of the market microstructure noise may even dominate the probability limit of the realized variance estimator.

Plugging $P_{t_i} = p_{t_i} - u_{t_i}$ with $u_{t_i} = (u_{1,t_i}, \ldots, u_{M,t_i})'$ into (2) yields a discrete-time VEC model for the observed prices, with the same autoregressive polynomial as before:

$$\Delta p_{t_i} = \alpha_\delta \beta' p_{t_{i-1}} + \mathrm{v}_{t_i}, \tag{9}$$

where Δp_{t_i} is an $M \times 1$ vector of observed price changes, $\mathrm{v}_{t_i} = \varepsilon_{t_i} + [I_M - (\alpha_\delta \beta' + I_M)L]\, u_{t_i}$ is an M-dimensional vector, and L the usual lag operator. See Hansen and Lunde (2014) for a similar result in the context of ARMA models. Andersen et al.'s (2021) discrete-time local-regime model is a particular case of (9). To see why, first rotate β such that $\beta = (\iota_R, -I_R)'$ and then impose that the price in the first market coincides with the efficient price by setting the first row of α_δ to zero and $\mathbb{E}\left[u^2_{1,t_i}\right] = 0$. Next, restrict the $R \times R$ lower block of α_δ to be a diagonal matrix. Let $\alpha_{\delta,m,m-1}$ be the $(m, m-1)$th element of α_δ, it then follows after some manipulation that

$$\Delta p_{m,t_i} = \alpha_{\delta,m,m-1}(p_{1,t_{i-1}} - p_{m,t_{i-1}}) + \varepsilon_{m,t_i} + u_{m,t_i} + (\alpha_{\delta,m,m-1} - 1)u_{m,t_{i-1}}, \qquad \text{with } m = 2, \ldots, M,$$

which coincides, up to a higher richer dynamics in the market microstructure noise, with the error correction dynamics case in Andersen et al. (2021).

It is also apparent from (9) that $\beta' p_{t-1}$ correlates with v_{t_i} because both terms contains the past microstructure noise. To appreciate why, consider the case with $M = 2$ markets for ease of exposition. After factoring the market microstructure noise, the first equation of (2) becomes

$$\Delta p_{1,t_i} = \alpha_{\delta,1} \left(p_{1,t_{i-1}} - p_{2,t_{i-1}} \right) + \mathrm{v}_{1,t_i}, \tag{10}$$

where $\mathrm{v}_{1,t_i} = \Delta u_{1,t_i} + \varepsilon_{1,t_i} - \alpha_{\delta,1}(u_{1,t_{i-1}} - u_{2,t_{i-1}})$. The LS orthogonality condition fails even in the very unrealistic case that market microstructure noises are mutually orthogonal white noises independent of the efficient price given that $\mathbb{E}\left[(p_{1,t_{i-1}} - p_{2,t_{i-1}})\, \mathrm{v}_{1,t_i} \right] = -(1 + \alpha_{\delta,1})\omega_1^2 - \alpha_{\delta,1}\omega_2^2$, with $\omega_m^2 \equiv \mathbb{E}[u_{m,t_i}^2] < \infty$ for $m = 1, 2$.[4] Interestingly, Appendix A.2 shows that the usual fix of including lags of the price changes in the regression only aggravates the problem. For instance, if $\alpha_{\delta,1} = 0$, the negative bias overestimates the reaction of the leading market, underestimating by how much the first market leads the price discovery.[5]

3.1 Noise-robust estimation

To address the error-in-variable problem that originates from the market microstructure noise, we employ instrumental variables (IV). In the presence of valid and relevant instruments, this approach yields consistent estimates for α and α_δ (and hence for $\alpha_\perp$) under both the mixed and standard asymptotic settings, regardless of the noise contamination. As a simple illustration, Figure 1 compares the box plots of 1,000 replications from estimates of the CS measures obtained from prices simulated from the exact discretization of (1) with $\alpha_{\delta=1/23400} = (0, 0.05)'$. The market microstructure noise is either totally absent or iid with variance $\omega^2 = (0.001, 0.0005)'$. As it turns out, noise plays a major role in the performance of the LS-based price discovery measures. The LS estimates of $\alpha_\perp$ falsely assign leadership to the second market in the presence of microstructure noise. In contrast, the IV-based estimators correctly identify the first market as the leader in both scenarios. Finally, estimating CS from prices sample at lower frequencies help, but do not solve the consistency problem of the LS estimator.

[4] It is straightforward to entertain daily variation in the second moment of the market microstructure noises at the cost of very heavy notation. See Appendix A for a full characterization of the LS estimator bias.

[5] Appendix A.3 provides more details on the LS bias under different market microstructure noise assumptions.

Next, we show how to construct valid instruments to consistently estimate the standard price discovery measures (CS and IS) in the presence of market microstructure noise. The key is to constrain the correlation structure of the market microstructure noises over time and/or in the cross-section. Out first setting restricts the amount of persistence in the microstructure noises, but we later show how to exploit the cross-sectional dimension as well.

Assumption MMN(TS) The market microstructure noise $u_{t_i} = \sum_{h=0}^{\bar{q}} \varpi_h \vartheta_{t_{i-h}}$ is a zero-mean invertible VMA($\bar{q}$) process, where $u_{t_i} = (u_{1,t_i}, \ldots, u_{M,t_i})'$, $\varpi_h = \text{diag}\,(\varpi_{1,h}, \ldots, \varpi_{M,h})$, and ϑ_{t_i} is a Gaussian white noise process. It then follows that u_{m,t_i} is such that (a) $\rho_{m,m'}(q) \equiv \mathbb{E}\left(\varepsilon_{m,t_{i-q}} u_{m',t_i}\right) < \infty$ for $m, m' = 1, \ldots, M$ and $q \leq \bar{q}$, zero otherwise; and (b) $\gamma_{m,m'}(q) \equiv \mathbb{E}[u_{m,t_i} u_{m',t_{i-q}}] < \infty$ for $m, m' = 1, \ldots, M$ and $q \leq \bar{q}$, zero otherwise.

Condition (a) allows for endogenous microstructure noises in that u_{m,t_i} may correlate with P_{m,t_i} through ε_{m,t_i}, in line with the empirical stylized facts in Hansen and Lunde (2006). Note also the market microstructure noise remains endogenous even if $\bar{q} = 0$. Condition (b) requires noises to exhibit neither much persistence over time, nor much correlation across markets. Assumption MMN(TS) complies with Hansen and Lunde's (2006) Assumption 4, but it is slightly more restrictive than Assumption U in Barndorff-Nielsen et al. (2011). As such, we are able to employ the realized kernel machinery to compute continuous-time IS measures in the mixed asymptotic setting.

To obtain valid instruments under Assumption MMN(TS), we note that observed price differences across markets are orthogonal to their lagged values if sufficiently in the past. Lemma 2 characterizes the properties of the $(\bar{k} - k + 1)R \times 1$ vector of instruments given by $Z_{t_i,k,\bar{k}} = \text{vec}\,(\beta' p_{t_{i-\bar{q}-k}}, \ldots, \beta' p_{t_{i-\bar{q}-\bar{k}}})$, with $2 \leq k \leq \bar{k} < \infty$.

Lemma 2 Let $Z_{t_{i-\bar{q}-\kappa}} \equiv \beta' p_{t_{i-\bar{q}-\kappa}}$ for any κ such that $k \leq \kappa \leq \bar{k}$ denote a Rth block of $Z_{t_i,k,\bar{k}}$. It follows from Assumptions RROU and MMN(TS) that (i) $\mathbb{E}[Z_{t_{i-\bar{q}-\kappa}} \mathrm{v}'_{t_i}] = 0$ and (ii) $\mathbb{E}[Z_{t_{i-\bar{q}-\kappa}} (\beta' p_{t_{i-1}})'] = \sum_{h=0}^{\infty} \beta' \Upsilon_{\delta,h} \Sigma_\delta \Upsilon'_{\delta,(\bar{q}+\kappa-1+h)} \beta + \sum_{h=\bar{q}+\kappa-1}^{2\bar{q}+\kappa-1} \beta' \rho(h - \bar{q} - \kappa + 1) \Upsilon'_{\delta,h} \beta < \infty$, with $\beta' \Upsilon_{\delta,h} = \beta' (I_M + \alpha_\delta \beta')^h$.

Lemma 2 depends exclusively on the I(0) component of the GRT in (4) because $\beta' \Xi_\delta = 0$.

Result (i) states that $Z_{t_i,k,\bar{k}}$ is a vector of valid instruments given that $\rho_{m,m'}(q) = \gamma_{m,m'}(q) = 0$ for any $q > \bar{q}$. Result (ii) establishes that $Z_{t_i,k,\bar{k}}$ is a vector of relevant instruments, with strength decaying as $\bar{q}$ and k increase given that $\lim_{h\to\infty} \Upsilon_{\delta,h} = \lim_{h\to\infty}(I_M + \alpha_\delta \beta')^h = 0$.

Although it allows for persistence in the market microstructure noises, Assumption MMN(TS) rules out MA(∞) processes as in Barndorff-Nielsen et al. (2011), Bandi et al. (2017), and Li and Linton (2020). To deal with the latter, we search for instruments in the cross-section dimension and then constrain the amount of dependence between microstructure noises across markets and assets. To come up with valid and relevant instruments, we recognize that microstructure noises are asset- and/or exchange-specific specific, as argued by Ait-Sahalia (2007). This allows us to use a cross-section of different assets traded at alternative trading platforms as instruments. Let $S_t = (S_{1,t}, \ldots, S_{J,t})'$ denote a J-dimensional vector of continuous-time log-prices of another asset that trades at J alternative trading platforms.

Assumption OTP The log-prices of the auxiliary asset follow a reduced-rank OU process: $\mathrm{d}S_t = \widetilde{\Pi} S_t \,\mathrm{d}t + \widetilde{C}\,\mathrm{d}B_t$, where $\widetilde{\Pi} = \widetilde{\alpha}_\delta \widetilde{\beta}'$ is a $J \times J$ reduced-rank matrix with rank $J-1$, B is a $J \times 1$ vector of Brownian motions, and $\widetilde{C}$ is a $J \times J$ matrix such that the covariance matrix $\widetilde{\Sigma} = \widetilde{C}\widetilde{C}'$ is positive definite. In addition, the eigenvalues of $\widetilde{\Pi}$ are real and no elementary divisor of $\widetilde{\Pi}$ occurs more than once; $\widetilde{\alpha}$ and $\widetilde{\beta}$ have full column rank $J-1$ such that $\widetilde{\beta}'\widetilde{\alpha}$ has also full rank $J-1$; and all eigenvalues of $\widetilde{\beta}'\widetilde{\alpha}$ have negative real parts.

The exact discretization of the reduced-rank OU process in Assumption OTP is

$$\Delta S_{t_i} = \widetilde{\alpha}_\delta \widetilde{\beta}' S_{t_{i-1}} + \epsilon_{t_i}, \tag{11}$$

where $\widetilde{\Pi}_\delta = \widetilde{\alpha}_\delta \widetilde{\beta}'$ and $\widetilde{\alpha}_\delta = \widetilde{\alpha}(\widetilde{\beta}'\widetilde{\alpha})^{-1}[\exp(\delta\widetilde{\beta}'\widetilde{\alpha}) - I_{J-1}]$, S_{t_i} is a $J \times 1$ vector of log-prices observed at time t_i, and ϵ_{t_i} is a Gaussian white noise with zero mean and covariance matrix $\widetilde{\Sigma}_\delta = \int_0^\delta \exp(u\widetilde{\Pi}t)\widetilde{\Sigma}\exp(u\widetilde{\Pi}')\,\mathrm{d}u$. Assumption OTP ensures that the results in Lemma 1 hold. It is straightforward to augment the vector ΔS_{t_i} to contain $V > 1$ different assets that are traded at the J alternative trading platforms at a cost of heavy notation. In such cases, (11) becomes a VJ-dimensional reduced-rank OU process with $V(J-1)$ cointegrating vectors and V stochastic trends. This is actually the specification we adopt in both our Monte Carlo study and empirical application.

As before, we observe only $s_{t_i} = S_{t_i} + \upsilon_{t_i}$, where $\upsilon_{t_i} = (\upsilon_{1,t_i}, \ldots, \upsilon_{J,t_i})'$ collects the market microstructure noises of each market $j = 1 \ldots, J$. We assume that the market microstructure noises u_m and $u_{m'}$ that plague the asset of interest in markets m and m', with $1 \leq m \neq m' \leq M$, are orthogonal to the market microstructure noises υ_j and $\upsilon_{j'}$ that affect the prices of the auxiliary assets in markets j and j', with $1 \leq j \neq j' \leq J$.

Assumption MMN(CS) The market microstructure noises u_{t_i} and υ_{t_i} are zero-mean MA(∞) processes such that $u_{t_i} = \sum_{h=0}^{\infty} \varpi_h \vartheta_{t_{i-h}}$ and $\upsilon_{t_i} = \sum_{h=0}^{\infty} \varrho_h \varphi_{t_{i-h}}$, with $\varpi_h = \text{diag}\,(\varpi_{1,h}, \ldots, \varpi_{M,h})$, $\varrho_h = \text{diag}\,(\varrho_{1,h}, \ldots, \varrho_{J,h})$ $\sum_{h=0}^{\infty} |\varpi_h| < \infty$, $\sum_{h=0}^{\infty} |\varrho_h| < \infty$, and ϑ_{m,t_i} and φ_{m,t_i} denoting Gaussian white noise processes. In addition, u_{t_i} and υ_{t_i} are such that (a) $\mathbb{E}[u_{m,t_i} \upsilon_{j,t_{i-q}}] = 0$ for $q \geq 0$ with $m = 1, \ldots, M$ and $j = 1, \ldots, J$; (b) $\mathbb{E}[\varepsilon_{m,t_{i-q}} u_{m',t_i}] < \infty$ for $q \geq 0$ with $m, m' = 1, \ldots, M$; and (c) $\mathbb{E}[\varepsilon_{m,t_{i-q}} \upsilon_{j,t_i}] = 0$ and $\mathbb{E}[\epsilon_{j,t_{i-q}} u_{m,t_i}] = 0$ for $q \geq 0$ with $m = 1, \ldots, M$ and $j = 1, \ldots, J$.

Condition (a) rules out correlation between market microstructure noises of different assets at distinct trading platforms. Condition (b) allows the market microstructure noise of the asset of interest to correlate with the efficient log-price through the idiosyncratic component in (2), whereas (c) dictates that the log-price of the asset of interest in market m, with $1 \leq m \leq M$, is orthogonal to the market microstructure of the other asset in markets $j = 1, \ldots, J$. We are now ready to define a set of valid and relevant instruments (as before, Appendix B provides for more details).

Lemma 3 Let Assumptions RROU, OTP and MMN(CS) hold, and $\mathbf{\Sigma}_\delta(0) = \mathbb{E}[\epsilon_{t_i} \varepsilon'_{t_i}]$ and $\widetilde{\Upsilon}_{\delta,h}$ denote the parameter matrices of the I(0) component of the GRT in (11). It then follows from $\widetilde{Z}_{t_{i-1}} \equiv \widetilde{\beta}' s_{t_{i-1}}$ that (i) $\mathbb{E}[\widetilde{Z}_{t_{i-1}} \mathrm{v}'_{t_i}] = 0$ and (ii) $\mathbb{E}[\widetilde{Z}_{t_{i-1}} (\beta' p_{t_{i-1}})'] = \sum_{h=0}^{\infty} \widetilde{\beta}' \widetilde{\Upsilon}_{\delta,h} \mathbf{\Sigma}_\delta(0) \Upsilon'_{\delta,h} \beta < \infty$ if $\mathbb{E}[\epsilon_{j,t_i} \varepsilon_{m,t_i}] \neq \mathbb{E}[\epsilon_{j',t_i} \varepsilon_{m',t_i}]$ for some $1 \leq j \neq j' \leq J$ and $1 \leq m \neq m' \leq M$.

Result (i) in Lemma 3 establishes that $\widetilde{\beta}' s_{t_{i-1}}$ is orthogonal to v'_{t_i}, and hence it is a valid instrument under Assumption MMN(CS). As in Lemma 2, relevance rests on the I(0) component of the GRT and some degree of heterogeneity on the contemporaneous correlation among assets innovations. Specifically, we expect price innovations to comove in response to market wide news that are impounded differently across assets and trading platforms.

Key to the asymptotic theory we discuss in the following sections is the type of dependence that the moment conditions in Lemmas 2 and 3 permit. More specifically, $Z_{t_{i-\bar{q}-\kappa}}\mathrm{v}'_{t_i}$ and $\widetilde{Z}_{t_{i-1}}\mathrm{v}'_{t_i}$ are not martingale differences sequences. This rules out central limit theorems (CLTs) for martingale differences sequences or iid MA(∞) processes. Alternatively, we exploit the fact that these IV moment conditions form L^1-mixingale sequences.[6]

Lemma 4 Assumptions MMN(TS) and MMN(CS) ensure that $\{Z_{t_{i-\bar{q}-\kappa}}, \widetilde{Z}_{t_{i-1}}, \mathrm{v}_{m,t_i}\}$ forms stationary ergodic sequences such that $\{Z_{m',t_{i-\bar{q}-\kappa}}\mathrm{v}_{m,t_i}, \mathcal{F}_{t_i}\}$ and $\{\widetilde{Z}_{j,t_{i-1}}\mathrm{v}_{m,t_i}, \mathcal{F}_{t_i}\}$ are stationary ergodic, uniformly integrable adapted L^1-mixingale sequences for $m, m' = 1, \ldots, M$ and $j = 1, \ldots, J$ with $\mathcal{F}_{t_i}$ denoting the σ-algebra generated by the entire current and past history of p_{t_i} and s_{t_i} and $Z_{m',t_{i-\bar{q}-\kappa}}$ and $Z_{j,t_{i-1}}$ denoting the m' and jth elements of $Z_{t_{i-\bar{q}-\kappa}}$ and $\widetilde{Z}_{t_{i-1}}$, respectively.

Lemma 4 essentially documents that the IV moment conditions are stationary ergodic, uniformly integrable L^1-mixingale sequences. The L^1-mixingale property then used to derive the limiting distribution of the IV-based price discovery measures in the next section.

3.2 Asymptotic theory with fixed sampling intervals

We first focus on the case in which we estimate price discovery measures over long time spans, fixing the sampling frequency. This means that we treat the number of intraday observations n as fixed, while allowing the number of days D in the sample to grow to infinity. The overall number of observations is $T = nD$, which goes to infinity as $D \to \infty$.

It follows from the discrete-time process in (2) that the observed log-prices at a fixed sampling interval δ follows

$$\Delta p_\tau = \alpha_\delta \beta' p_{\tau-1} + \mathrm{v}_\tau, \qquad \text{with} \;\; \tau = 1, \ldots, T = nD \tag{12}$$

or, in a more compact notation, $\Delta\boldsymbol{p} = \bar{\boldsymbol{p}}\,\boldsymbol{\alpha}_\delta + \mathbf{v}$, where $\Delta\boldsymbol{p} = \mathrm{vec}\,(\Delta p_1, \ldots, \Delta p_M)$ is the $M(T-1)\times 1$ vector with $\Delta p_m = (\Delta p_{m,2}, \ldots, \Delta p_{m,T})'$ for $m = 1, 2, \ldots, M$; $\bar{\boldsymbol{p}}$ is the $M(T-1)\times MR$ block diagonal matrix with M diagonal blocks given by the $(T-1)\times R$ matrix $\boldsymbol{p}'_{-1}\beta$ with $\boldsymbol{p}_{-1} = (p_1, \ldots, p_{T-1})$; $\boldsymbol{\alpha}_\delta$ is the $MR\times 1$ vector that stacks the M rows of α_δ such that $\boldsymbol{\alpha}_\delta = \mathrm{vec}\,(\alpha'_\delta)$; and $\mathbf{v} = \mathrm{vec}\,(\mathrm{v}_1, \ldots, \mathrm{v}_M)$ denotes the $M(T-1)\times 1$ matrix collecting the disturbance terms with $\mathrm{v}_m = (\mathrm{v}_{m,2}, \ldots, \mathrm{v}_{m,T})'$ for

[6] A sequence $\{\mathcal{Z}_t, \mathcal{F}_{t_i}\}_{-\infty}^{\infty}$ is said to be a L^1-mixingale with respect to $\mathcal{F}_{t_i}$ if $\mathbb{E}\left\|\mathbb{E}\left(\mathcal{Z}_t|\mathcal{F}_{t_{i-c}}\right)\right\| \leq f_{t_i}\xi_c$, where $\{f_{t_i}, \xi_c\}$ are deterministic sequences and $\lim_{c\to\infty}\xi_c = 0$ (see, for instance, Definition 16.1 in Davidson, 1994)

$m = 1, \ldots, M$. Similarly, it follows from Assumption OTP that the discrete-time process for the observed log-prices of the J auxiliary assets reads

$$\Delta s_\tau = \widetilde{\alpha}_\delta \widetilde{\beta}' s_{\tau-1} + \bar{\epsilon}_\tau, \qquad \text{with } \ \tau = 1, \ldots, T = nD. \tag{13}$$

Let $\boldsymbol{Z}$ denote a $(T-\kappa) \times \bar{r}$ matrix of valid and relevant instruments with $\bar{r} \geq R$ and $\kappa \in [1, \infty)$ denoting some appropriate lag order. The IV estimator coincides with the two-stage least-squares (2SLS) estimator that employs projections on the subspace spanned by the instruments: namely, $\widehat{\alpha}_{\delta,\text{IV}} = \left(I_M \otimes \widehat{\boldsymbol{\alpha}}'_{\delta,\text{IV}}\right)(\text{vec}\,(I_M) \otimes I_R)$, with

$$\widehat{\boldsymbol{\alpha}}_{\delta,\text{IV}} = \left[\bar{\boldsymbol{p}}' \left(I_M \otimes \boldsymbol{Z}\right) \boldsymbol{W} \left(I_M \otimes \boldsymbol{Z}\right)' \bar{\boldsymbol{p}}\right]^{-1} \bar{\boldsymbol{p}}' \left(I_M \otimes \boldsymbol{Z}\right) \boldsymbol{W} \left(I_M \otimes \boldsymbol{Z}\right)' \Delta \boldsymbol{p}. \tag{14}$$

and $\boldsymbol{W} = [(I_M \otimes \boldsymbol{Z})'(I_M \otimes \boldsymbol{Z})]^{-1}$. It is easy to appreciate that (14) also corresponds to the GMM estimator with $\boldsymbol{W}$ as weighting matrix. Although (14) holds for either Assumption MMN(TS) or MMN(CS), the dimensions of $\Delta\boldsymbol{p}$ and $\boldsymbol{Z}$ vary depending on the circumstances. For instance, $\Delta\boldsymbol{p}$, $\bar{\boldsymbol{p}}$, and $\boldsymbol{Z} = (Z_{1+(\bar{q}+\bar{k}),k,\bar{k}}, \ldots, Z_{T,k,\bar{k}})'$ respectively have dimensions $M(T-\bar{q}-\bar{k}) \times 1$, $M(T-\bar{q}-\bar{k}) \times MR$, $(T-\bar{q}-\bar{k}) \times (\bar{k}-k+1)R$, and $\bar{r} = \left(\bar{k}-k+1\right) R$ under Assumption MMN(TS).

Next, we derive the asymptotic behavior of the IV-based price discovery measures under either Assumption MMN(TS) or MMN(CS). Consistent estimation of CS requires $\widehat{\alpha}_{\delta,\text{IV}} \xrightarrow{p} \alpha_{\delta,\text{IV}}$ for any δ given that $\text{CS} = \alpha_{\delta,\perp} = \alpha_\perp$. In addition, asymptotic normality of the CS estimator readily follows from the limiting distribution of $\widehat{\boldsymbol{\alpha}}_{\delta,\text{IV}}$. As for the consistent estimation of the IS measure in discrete time, it calls for $\widehat{\Sigma}_{\delta,\text{IV}} \xrightarrow{p} \Sigma_\delta$ in addition to $\widehat{\alpha}_{\perp,\text{IV}} \xrightarrow{p} \alpha_\perp$. Because of the serial correlation in v_{t_i} due to the market microstructure noise, we estimate Σ_δ using a heteroskedasticity and autocorrelation consistent (HAC) estimator. The resulting IV-based estimator of the discrete-time IS measure then is $\widehat{IS}_{m,\delta,\text{IV}} = (\widehat{\alpha}'_{\perp,\text{IV}} \widehat{\Sigma}_\delta \ \widehat{\alpha}'_{\perp,\text{IV}})^{-1} [\widehat{\alpha}'_{\perp,\text{IV}} \widehat{C}_\delta]^2_m$, $m = 1, \ldots, M$. As long as the IV-based residuals $\widehat{\mathbf{v}} = \Delta\boldsymbol{p} - \bar{\boldsymbol{p}}\,\widehat{\boldsymbol{\alpha}}_{\delta,\text{IV}}$ are consistent, the HAC estimator will converge to the true long-run covariance matrix, which conveniently accounts for the spurious autocorrelation patterns that may arise from microstructure effects.

As for the estimation of the continuous-time IS measure, we can make use of the exact discretization of (1) to back out $\widehat{\Sigma}$ from $\widehat{\Sigma}_\delta$. In particular, we find $\widehat{\Sigma}$ such that $\widehat{\Sigma}_\delta = \int_0^\delta \exp(u\widehat{\Pi})\widehat{\Sigma}\exp(u\widehat{\Pi}')\,\mathrm{d}u$, with $\widehat{\Pi} = \delta^{-1}\log(\widehat{\alpha}_{\delta,\text{IV}}\,\beta' + I_M)$. Because $\alpha_{\delta,\perp} = \alpha_\perp$ for any $0 < \delta < 1$, we may estimate the continuous-time IS measure by

$$\widehat{IS}_{m,\mathrm{IV}} = \frac{[\widehat{\alpha}'_{\perp,\mathrm{IV}}\widehat{C}]^2_m}{\widehat{\alpha}'_{\perp,\mathrm{IV}}\widehat{\Sigma}\;\widehat{\alpha}'_{\perp,\mathrm{IV}}}, \qquad m = 1,\ldots,M, \tag{15}$$

where $\widehat{\Sigma}$ is the $\widehat{\Sigma}_\delta$-implied HAC estimator of Σ, and $\widehat{C}$ is such that $\widehat{C}\widehat{C}' = \widehat{\Sigma}$. To formalize these results, we must first establish some notation. Let

$$\Gamma_{\bar{\boldsymbol{p}}\boldsymbol{Z}} = \begin{cases} \sum_{\ell=-\bar{q}-1}^{\bar{q}+1} \mathbb{E}[\mathrm{v}_\tau \mathrm{v}'_{\tau-\ell} \otimes Z_{\tau,k,\bar{k}} Z'_{\tau-\ell,k,\bar{k}}] & \text{if under Assumption MMN(TS)} \\ \sum_{\ell=-\infty}^{\infty} \mathbb{E}[\mathrm{v}_\tau \mathrm{v}'_{\tau-\ell} \otimes \widetilde{Z}_{\tau-1} \widetilde{Z}'_{\tau-1-\ell}] & \text{if under Assumption MMN(CS),} \end{cases}$$

where $\mathrm{v}_\tau = (\mathrm{v}_{1,\tau},\ldots,\mathrm{v}_{M,\tau})'$, and $\boldsymbol{C} = \boldsymbol{C_1}\boldsymbol{C_2}\boldsymbol{C_3}$, with $\boldsymbol{C_1} = \mathcal{S}(\psi'_1 \otimes \Xi)K_{M,R}$, $\boldsymbol{C_2} = [\boldsymbol{Q}_{\bar{\boldsymbol{p}},\boldsymbol{Z}}\boldsymbol{Q}_{\boldsymbol{W}}\boldsymbol{Q}'_{\bar{\boldsymbol{p}},\boldsymbol{Z}}]^{-1}$, and $\boldsymbol{C_3} = \boldsymbol{Q}_{\bar{\boldsymbol{p}},\boldsymbol{Z}}\boldsymbol{Q}_{\boldsymbol{W}}$, where $\mathcal{S}$ is an $M \times M^2$ deterministic matrix that selects the first row of Ξ from $\mathrm{vec}\,(\Xi)$, $\psi_1 = -\alpha_\delta(\alpha'_\delta\alpha_\delta)^{-1}(\Xi - I_M)$, $K_{M,R}$ is a commutation matrix such that $K_{M,R}\,\mathrm{vec}\,(\alpha_\delta) = \boldsymbol{\alpha}_\delta$, $\boldsymbol{Q}_{\bar{\boldsymbol{p}},\boldsymbol{Z}} = \operatorname*{plim}_{T\to\infty} \frac{1}{T}\,\bar{\boldsymbol{p}}'(I_M \otimes \boldsymbol{Z})$, and $\boldsymbol{Q}_{\boldsymbol{W}} = \operatorname*{plim}_{T\to\infty} \frac{1}{T}\,\boldsymbol{W}$.

Theorem 1 Let the conditions in Lemma 2 or in Lemma 3 hold, and consider their respective vector of instruments. It then follows that, as $T = nD \to \infty$ with fixed n, (i) $\widehat{\alpha}_{\perp,\mathrm{IV}} \xrightarrow{p} \alpha_\perp$; (ii) $\sqrt{T}(\widehat{\alpha}_{\perp,\mathrm{IV}} - \alpha_\perp) \xrightarrow{d} N\,(0, \boldsymbol{C}\,\Gamma_{\bar{\boldsymbol{p}}\boldsymbol{Z}}\boldsymbol{C}')$; (iii) $\widehat{IS}_{m,\delta,\mathrm{IV}} \xrightarrow{p} IS_{\delta,m}$; and $(iv)\widehat{IS}_{m,\mathrm{IV}} \xrightarrow{p} IS_m$.

Under Assumption MMN(TS), the vector of instruments is strong for small $\bar{q}$ and k. Simulations show that the empirical distributions of the t-statistics based on Theorem 1(ii) are close to normal even when we sample one observation per second ($n = 23,400$) within a single trading day ($D = 1$). This is true regardless of the the variance of the market microstructure noise (see upper panel in Figure 2). In turn, under Assumptions OTP and MMN(CS), the vector of instruments is likely not very strong in practice, so that $\widehat{\alpha}_{\delta,\mathrm{IV}}$ might become biased for moderate-to-high values of ω^2 (see simulations and discussion in Section 4.2). This is not surprising given that the standard asymptotic theory fails for weak instruments (Bound, Jaeger and Baker, 1995; Staiger and Stock, 1997). One possible solution is to add more instruments, that is to say, to consider a larger cross-section of auxiliary assets traded at alternative exchanges, hoping to improve the instruments' relevance. Although the 2SLS estimator remains inconsistent under many weak instruments, with asymptotic bias actually increasing with the number of instruments (Stock and Yogo, 2002), one may alternatively employ the GMM-based continuous updating (CU-GMM) estimator of Hansen, Heaton and Yaron (1996).

In the context of weak instruments, the asymptotic distribution of the CU-GMM estimator depends on nuisance parameters. Stock and Wright (2000) nonetheless show how to carry out hypothesis tests and construct asymptotically-valid confidence sets directly from the objective function. At the true parameter values, the objective function has a standard χ^2 distribution under mild conditions, so that one may obtain asymptotically valid confidence sets by inverting the test of $\alpha_{\delta,m} = 0$ for $m = 1, \ldots, M$. Null hypotheses of this kind are very informative in the continuous-time price discovery context because they are equivalent to testing whether the price at market m coincides with the efficient price. Both hypothesis tests and confidence sets have asymptotic size and coverage equal to their nominal levels uniformly over the entire parameter space.

In the case of many weak instruments, the CU-GMM estimator is consistent under heteroskedastic and serially correlated moment conditions for properly chosen weighting matrices (Han and Phillips, 2006; Newey and Windmeijer, 2009; Hausman, Lewis, Menzel and Newey, 2011). In particular, consistency requires $V^2(J-1)^2/T \to 0$, whereas asymptotic normality requires the number of instruments $V(J-1)$ to satisfy $V^3(J-1)^3/T \to 0$. The many weak instruments setting fits well our framework because it is straightforward not only to employ a large cross-section of auxiliary assets that are actively traded on alternative trading platforms, but also to entertain more lagged values. The latter is not completely irrelevant in view that assets presumably impound information at different paces. The price to pay is that we cannot guarantee the CU-GMM estimator has finite moments of any order, which could lead to very wide standard errors in finite samples (Hausman et al., 2011).

The CU-GMM estimator $\widehat{\boldsymbol{\alpha}}_{\delta,\text{CU-GMM}}$ maximizes $\mathcal{C}_{\text{CU-GMM},T}(\widehat{\boldsymbol{\alpha}}_\delta) = -\boldsymbol{g}_T(\widehat{\boldsymbol{\alpha}}_\delta)'\widehat{\Psi}_\delta^{-1}(\widehat{\boldsymbol{\alpha}}_\delta)\boldsymbol{g}_T(\widehat{\boldsymbol{\alpha}}_\delta)$, where $\boldsymbol{g}_T(\widehat{\boldsymbol{\alpha}}_\delta) = \frac{1}{T-k-1}\,(I_M \otimes \boldsymbol{Z})'\widehat{\mathbf{v}}$ is the $MV(J-1) \times 1$ vector containing the average moment conditions over time with $\boldsymbol{Z} = (\widetilde{Z}_1, \ldots, \widetilde{Z}_{T-1})'$, and $\widehat{\Psi}_\delta(\widehat{\boldsymbol{\alpha}}_\delta)$ is the long-run variance estimator of the asymptotic variance $\Psi_\delta(\boldsymbol{\alpha}_\delta) = \sum_{\ell=-\infty}^{\infty} \Gamma_{\boldsymbol{g}_T}(\ell)$ of $\sqrt{T}\boldsymbol{g}_T(\boldsymbol{\alpha}_\delta)$. In particular, we employ Andrews's (1991) HAC estimator with a Parzen kernel to estimate $\Psi_\delta(\boldsymbol{\alpha}_\delta)$. As before, the consistency of the CS and IS estimators follows directly from $\widehat{\boldsymbol{\alpha}}_{\delta,\text{CU-GMM}} \xrightarrow{p} \boldsymbol{\alpha}_\delta$. Section 4 provides evidence that the CU-GMM estimator indeed performs very well under Assumption MMN(CS), yielding median biases that are not only close to zero, but also decreasing in V for different amounts of market microstructure noise.

3.3 Asymptotic theory: mixed case

We now address the case in which $n \to \infty$, while $D \to \infty$ at a slower rate. As before, we back out CS and IS measures from the estimates of the speed-of-adjustment parameters of the daily VEC models. Differently from Section 3.2, we now let $\delta \to 0$ to ensure that $n \to \infty$. Although $\alpha_\perp = \alpha_{\delta,\perp}$ for any $0 < \delta < 1$, consistency of $\alpha_\perp$ now requires consistent estimation of α given that α_δ is no longer a fixed quantity. Similarly, consistency of the continuous-time IS calls for the consistent estimation of the quadratic covariation.

As we let $\delta \to 0$, it is necessary to account for nonsynchronous trading. To this end, we synchronize tick data using the refresh-time scale. Suppose we observe the price in market m at times $t_{m,1}, t_{m,2}, \ldots$ and let $N_m(t)$ denote the counting process with the number of observations up to time t in market m, with $m = 1, \ldots, M$. The first refresh time is $\tau_1 = \max_{1 \le m \le M} t_{m,1}$, whereas the subsequent refresh times are $\tau_{i+1} = \max_{1 \le m \le M} t_{m, N_j(\tau_i)+1}$ (see more details in Barndorff-Nielsen et al., 2011). To comply with the necessary conditions for the consistent estimation of the quadratic covariation, we also carry out some jittering to the first and last prices of the day as in Jacod, Li, Mykland, Podolskij and Vetter (2009) and Barndorff-Nielsen et al. (2011).

Assumption RT Denote the durations between observation times by $\Delta_i \equiv \tau_i - \tau_{i-1} = \mathcal{A}_i/n$. We assume that $\max_{i'+1 \le i \le i'+K} D_i = o_p(\sqrt{K})$ for any i' and that $\tau_0 \le 0$ and $\tau_{m+1} \ge 1$. In addition, $\mathbb{E}\left(\mathcal{A}^r_{\lfloor nt \rfloor} \,\middle|\, \mathcal{F}_{\tau_{\lfloor nt \rfloor - 1}}\right) \xrightarrow{p} \chi_r(t)$ for $0 < r \le 2$ as $n \to \infty$, where $\chi_r(t)$ is a strictly positive, càdlàg process adapted to $\mathcal{F}_t$.

Assumption RT allows us to treat irregularly space prices as equally spaced in refresh time. The rate on $\mathcal{A}_i$ is the same as in Phillips and Yu (2008), but more restrictive than the $O_p(1)$ rate in Mykland and Zhang (2006) and Barndorff-Nielsen et al. (2008) due to random times. Empirically, the refresh-time scheme is heavily dependent on the least liquid asset/market given that one forms a new price tuple only after every price refreshes. This means that effective sample size decreases with the number of prices and with nonsynchronicity. This is especially a problem under Assumption MMN(CS) since we must synchronize VJ prices, where V is large and the J alternative exchanges are usually less liquid than the M markets of interest.

To illustrate this issue, let time stamps come from independent Bernoulli trials, with $V = 25$

and $J = 3$, so that each price process contains 9,000 observations per day, on average. Applying refresh time to synchronize these prices yields, on average, about 2,200 observations per day, with an average price duration of 10.6 seconds. The low levels of data retention in this setting affects negatively the performance of the CU-GMM estimator as we necessitate very large sample sizes to extract enough information from weak instruments and the noise-to-signal ratio increases with the average duration. Accordingly, we restrict attention to the mixed asymptotic theory under Assumption MMN(TS).

Next, we focus on the estimation of the continuous-time speed-of-adjustment parameter α. To reflect Assumption RT, we refer to the refresh-time clock using the subscript τ, with n now denoting the resulting refresh-time sample size. We then rewrite (12) as

$$\Delta p_{\tau_i} = \alpha_\delta(\boldsymbol{\alpha})\beta' p_{\tau_{i-1}} + \mathrm{v}_{\tau_i}, \qquad \text{with} \;\; i = 1, \ldots, n \tag{16}$$

where $\alpha_\delta(\boldsymbol{\alpha})$ is a function of the continuous-time speed of adjustment parameter $\boldsymbol{\alpha} = \mathrm{vec}\,(\alpha')$, such that $\alpha_\delta(\boldsymbol{\alpha}) = \alpha(\boldsymbol{\alpha})(\beta'\alpha(\boldsymbol{\alpha}))^{-1}[\exp(\delta\beta'\alpha(\boldsymbol{\alpha})) - I_R]$ with $\alpha(\boldsymbol{\alpha}) = (I_M \otimes \boldsymbol{\alpha}')[\mathrm{vec}\,(I_M) \otimes I_R]$, and $\lim_{\delta\to 0} \delta^{-1}\alpha_\delta(\boldsymbol{\alpha}) = \alpha$; $\Delta p_{\tau_i} = (\Delta p_{1,\tau_i}, \ldots, \Delta p_{M,\tau_i})'$ is an $M \times 1$ vector; and $\mathrm{v}_{\tau_i} = (\mathrm{v}_{1,\tau_i}, \ldots, \mathrm{v}_{M,\tau_i})'$ is an $M \times 1$ vector.

Recall from Lemma 1 that, in the absence of market microstructure noise, the moment condition implied by $\ell_n^c(\alpha, \Sigma)$ is $\mathbb{E}\big[\big(\Delta P_{\tau_i} - \alpha_\delta(\boldsymbol{\alpha})\beta' P_{\tau_{i-1}}\big) \otimes (\beta' P_{\tau_{i-1}})\big] = 0$, where ΔP_{τ_i} and P_{τ_i} respectively denote the noiseless counterparts of Δp_{τ_i} and $p_{\tau_{i-1}}$ in (16). It then follows that $\mathbb{E}\big[\big(\Delta P_{\tau_i} - \alpha_\delta(\boldsymbol{\alpha})\beta' P_{\tau_{i-1}}\big) \otimes \big(\beta' P_{\tau_{i-1-\kappa}}\big)\big] = 0$ for all $\kappa > 0$, so that we may estimate the continuous-time speed-of-adjustment parameters using a set of valid moment conditions with $\kappa > 0$ that meets the requirements in Assumption MMN(TS).

To formally define the GMM criterion function, let $Z_{\tau_i,k,\bar{k}} \equiv \mathrm{vec}\,(\beta' p_{\tau_{i-\bar{q}-k}}, \ldots, \beta' p_{\tau_{i-\bar{q}-\bar{k}}})$ with integers k and $\bar{k}$ such that $2 \leq k \leq \bar{k} < \infty$ and $\boldsymbol{Z} = \Big(Z_{\tau_{1+\bar{q}+k,k,\bar{k}}}, \ldots, Z_{\tau_{n,k,\bar{k}}}\Big)'$ be the $(n - \bar{q} - \bar{k}) \times (\bar{k} - k + 1)R$ matrix of instruments. The $M(\bar{k} - k + 1)R$ sample moment conditions then are

$$\boldsymbol{g}_{n_\tau}(\widehat{\boldsymbol{\alpha}}) = \frac{1}{n - \bar{q} - \bar{k}}(I_M \otimes \boldsymbol{Z})'\widehat{\mathbf{v}} = \frac{1}{n - \bar{q} - \bar{k}} \sum_{i=\bar{q}+\bar{k}+1}^{n} \widehat{\mathrm{v}}_{\tau_i} \otimes Z_{\tau_i,k,\bar{k}}, \tag{17}$$

where $\widehat{\mathbf{v}} = \boldsymbol{\Delta p} - \bar{\boldsymbol{p}}\,\boldsymbol{\alpha}_\delta(\widehat{\boldsymbol{\alpha}})$, with $\boldsymbol{\alpha}_\delta(\widehat{\boldsymbol{\alpha}}) = \mathrm{vec}\,\big(\alpha_\delta'(\widehat{\boldsymbol{\alpha}})\big)$, and $\widehat{\mathrm{v}}_{\tau_i} = \Delta p_{\tau_i} - \alpha_\delta(\widehat{\boldsymbol{\alpha}})\,\beta' p_{\tau_{i-1}}$ collect the residuals and have dimensions $M(n-\bar{q}-\bar{k}) \times 1$ and $M \times 1$, respectively; $\boldsymbol{\Delta p} = \mathrm{vec}\,(\Delta p_1, \ldots, \Delta p_n)$ is an $M(n-\bar{q}-\bar{k}) \times 1$ vector with $\Delta p_m = \big(\Delta p_{m,1+\bar{q}+\bar{k}}, \ldots, \Delta p_{m,n}\big)'$ denoting an $(n-\bar{q}-\bar{k}) \times 1$ vector for

$m = 1, \dots, M$; and $\bar{\boldsymbol{p}}$ is an $M(n-\bar{q}-\bar{k}) \times MR$ block diagonal matrix with M diagonal blocks given by the $(n-\bar{q}-\bar{k}) \times R$ matrix $\boldsymbol{p}'_{-1}\beta$, where $\boldsymbol{p}_{-1} = (p_{\bar{q}+\bar{k}}, \dots, p_{n-1})$ is an $M \times (n-\bar{q}-\bar{k})$ matrix. As for the weighting matrix, we employ the long-run variance estimator $\widehat{\Psi}$ of the asymptotic variance of $\sqrt{n}\boldsymbol{g}_{n_\tau}(\boldsymbol{\alpha})$: $\Psi_\tau = \sum_{\ell=-\bar{q}-1}^{\bar{q}+1} \Psi_\tau(\ell)$, where $\Psi_\tau(\ell) = \mathbb{E}\big[\mathrm{v}_{\tau_i}\mathrm{v}'_{\tau_{i-\ell}} \otimes Z_{\tau_i,k,\bar{k}} Z'_{\tau_{i-\ell},k,\bar{k}}\big]$. The GMM estimator then solves a nonlinear optimization problem:

$$\widehat{\boldsymbol{\alpha}}_{\text{GMM}} = \underset{\widehat{\boldsymbol{\alpha}}}{\text{argmax}} \ \mathcal{C}_{\text{GMM},n_\tau}(\widehat{\boldsymbol{\alpha}}) \equiv \underset{\widehat{\boldsymbol{\alpha}}}{\text{argmax}} \ -\boldsymbol{g}_{n_\tau}(\widehat{\boldsymbol{\alpha}})'\widehat{\Psi}^{-1}\boldsymbol{g}_{n_\tau}(\widehat{\boldsymbol{\alpha}}), \tag{18}$$

with $\widehat{\alpha}_{\text{GMM}} = \big[\big(\text{vec}\,(I_M)' \otimes I_R\big)(I_M \otimes \widehat{\boldsymbol{\alpha}}_{\text{GMM}})\big]'$. To establish the consistency and asymptotic normality of $\widehat{\boldsymbol{\alpha}}_{\text{GMM}}$, we impose the following primitive conditions.

Assumption GMM Let $\widehat{\boldsymbol{\alpha}}$ denote an arbitrary vector of parameters for which Assumption RROU holds. We assume that

(a) $\widehat{\boldsymbol{\alpha}}, \boldsymbol{\alpha} \in \mathbf{A}$, where $\mathbf{A} \subset \mathbb{R}^{MR}$ is a compact set;

(b) $\mathcal{C}_{\text{GMM},n_\tau}(\widehat{\boldsymbol{\alpha}})$ is a measurable function of the data for all $\widehat{\boldsymbol{\alpha}} \in \mathbf{A}$, and continuous in $\widehat{\boldsymbol{\alpha}} \in \mathbf{A}$ for all possible samples;

(c) $\sup_{\widehat{\boldsymbol{\alpha}} \in \mathbf{A}} |\mathcal{C}_{\text{GMM},n_\tau}(\widehat{\boldsymbol{\alpha}}) - \mathcal{C}(\widehat{\boldsymbol{\alpha}})| \xrightarrow{p} 0$ and $\sup_{\widehat{\boldsymbol{\alpha}} \in \mathbf{A}} |\mathcal{C}_{\text{GMM},n_t}(\widehat{\boldsymbol{\alpha}}) - \mathcal{C}(\widehat{\boldsymbol{\alpha}})| \xrightarrow{p} 0$, where $\mathcal{C}_{\text{GMM},n_t}(\widehat{\boldsymbol{\alpha}})$ denotes the GMM criterion function based on the latent equally-spaced clock with sampling interval $\delta = 1/n$ and $\mathcal{C}(\widehat{\boldsymbol{\alpha}})$ is a nonstochastic function;

(d) $\mathcal{C}_{\text{GMM},n_\tau}(\widehat{\boldsymbol{\alpha}})$ and $\mathcal{C}_{\text{GMM},n_t}(\widehat{\boldsymbol{\alpha}})$ are twice differentiable in a continuous and open neighbourhood of $\boldsymbol{\alpha}$;

We require conditions (a)–(c) to obtain uniform convergence of the sample moments to their population counterparts and hence of $\widehat{\boldsymbol{\alpha}}_{\text{IV}}$ to $\boldsymbol{\alpha}$. Alternatively, we could adapt Theorem 6.9 in White (2000) to establish $\widehat{\Psi} \xrightarrow{p} \Psi$ and Theorem 2.2 in Phillips and Yu (2008) to attain pointwise convergence of $\mathcal{C}_{\text{GMM},n_\tau}$ to $\mathcal{C}_{\text{GMM},n_t}$. Assuming that $\mathcal{C}_{\text{GMM},n_\tau}$ and $\mathcal{C}_{\text{GMM},n_t}$ satisfy Lipschitz-type conditions suffices to deliver stochastic equicontinuity (see, for instance, Theorem 21.10 in Davidson, 1994) and hence uniform convergence as in (c). Condition (d) is standard in the nonlinear GMM framework (see, e.g., Newey and McFadden, 1994). Note that, by Lemma 1, $\boldsymbol{\alpha}$ is the unique maximizer of $\ell_n^c(\alpha, \Sigma)$, so that it is also the unique solution to $\mathcal{C}(\widehat{\boldsymbol{\alpha}})$. Finally, Lemma 4 provides sufficient conditions for applying the CLT for L^1-mixingales (White, 2000).

As for the consistent estimation of the continuous-time information share, we estimate the daily quadratic variation directly from the intraday returns at their highest frequency using a realized kernel (RK) approach, namely,

$$\widehat{IS}_{RK,m} = \frac{[\widehat{\alpha}'_{\perp,\text{IV}}\widehat{C}_{RK}]^2_m}{\widehat{\alpha}'_{\perp,\text{IV}}\widehat{\Sigma}_{RK}\ \widehat{\alpha}'_{\perp,\text{IV}}}, \qquad m = 1, \ldots, M, \tag{19}$$

where $\widehat{\alpha}_{\perp,\text{IV}}$ is the orthogonal complement of $\widehat{\alpha}_{\text{IV}}$ in (18) and $\widehat{\Sigma}_{RK}$ is the RK estimator of Σ, with $\widehat{C}_{RK}$ such that $\widehat{C}_{RK}\widehat{C}'_{RK} = \widehat{\Sigma}_{RK}$. In particular, the RK estimator is $RK \equiv \sum_{h=-\tau}^{\tau} \mathcal{K}(h/H)\Gamma_h$, where $H \propto \tau^{\zeta}$ with $1/2 < \zeta < 1$, $\Gamma_h \equiv \sum_{i=h+1}^{m} \Delta p_{\tau_i} \Delta p'_{\tau_{i-h}}$ for $h \geq 0$, with $\Gamma_h = \Gamma'_{-h}$ for $h < 0$, and $\mathcal{K}$ is a nonstochastic kernel function. As in Barndorff-Nielsen et al. (2011), we impose the following conditions on the latter.

Assumption K The nonstochastic kernel function $\mathcal{K}$ is such that

(a) $\mathcal{K}(0) = 1$ and $\mathcal{K}'(0) = 0$;

(b) $\mathcal{K}$ is twice differentiable with continuous derivatives;

(c) $\int_0^\infty [\mathcal{K}(x)]^2 \, \mathrm{d}x < \infty$, $\int_0^\infty [\mathcal{K}'(x)]^2 \, \mathrm{d}x < \infty$, and $\int_0^\infty [\mathcal{K}''(x)]^2 \, \mathrm{d}x < \infty$;

(d) $\int_{-\infty}^{\infty} \mathcal{K}(x) \exp(ixs) \, \mathrm{d}x \geq 0$ for all $s \in \mathbb{R}$.

Condition (a) implies not only that Γ_0 gets unit weight but also that the kernel gives close-to-unit weight to Γ_h for values of h that are near the origin. Conditions (b) and (c) are purely technical, whereas (d) ensures that the resulting estimator is positive semi-definite by Bochner's theorem. The multivariate realized kernel is very similar to the standard HAC covariance matrix estimator, but with the additional assumption that $\mathcal{K}'(0) = 0$. Before documenting the mixed asymptotic behavior of the CS and IS estimators, we must first define some notation. Let $\psi_2 = \bar{\alpha}(\Xi - I_M)$, with $\bar{\alpha} = \alpha(\alpha'\alpha)^{-1}$, $\boldsymbol{G} = \mathbb{E}[\nabla_{\boldsymbol{\alpha}}\, \boldsymbol{g}_{n_{t_i}}(\boldsymbol{\alpha})]$ with $\boldsymbol{g}_{n_{t_i}}(\boldsymbol{\alpha}) = (\Delta p_{t_i} - \alpha_\delta(\boldsymbol{\alpha})\beta' p_{t_{i-1}}) \otimes Z_{t_i,k,\bar{k}}$, and $\Psi = \sum_{j=-\bar{q}-1}^{\bar{q}+1} \Psi(\ell)$ with $\Psi(\ell) = \mathbb{E}[\mathrm{v}_{t_i}\mathrm{v}'_{t_{i-\ell}} \otimes Z_{t_i,k,\bar{k}} Z'_{t_{i-\ell},k,\bar{k}}]$.

Theorem 2 Let Assumptions RROU, MMN(TS), RT, GMM and K hold. As $n \to \infty$, $D \to \infty$, and $\delta^2 D \to 0$, (i) $\widehat{\alpha}_{\perp,\text{GMM}} \xrightarrow{p} \alpha_\perp$; (ii) $\sqrt{D}\,(\widehat{\boldsymbol{\alpha}}_{\perp,\text{GMM}} - \boldsymbol{\alpha}_\perp) \xrightarrow{d} N\left(0, \mathcal{S}(\psi'_2 \otimes \Xi)(\boldsymbol{G}'\Psi^{-1}\boldsymbol{G})^{-1}(\psi'_2 \otimes \Xi)'\mathcal{S}'\right)$; and (iii) $\widehat{IS}_{RK,m} \xrightarrow{p} IS$.

The empirical distributions of the t-statistics based on Theorem 2(ii) are close to Gaussian for samples that mimic one trading day, i.e., prices recorded at random times and aggregated using the refresh-time scheme with approximately $n = 17,980$ observations. As before, this holds regardless of the variance of the market microstructure noise (see lower panel in Figure 2). In the next section, we show that the asymptotic results in Theorems 1 and 2 offer in general very good finite-sample approximations, comparing favorably to the standard price discovery analysis that does not account for the market microstructure noise.

4 Monte Carlo simulations

We next assess the relative performance of the IV-based approach under different scenarios. We show in Appendix A that bias arises in LS estimation for multiple reasons. The first is due to any dependence between efficient price and market microstructure noises. The second stems from the nonzero magnitude of the market microstructure noises, whereas the third comes from any correlation between microstructure noises across markets. Finally, any persistence in the microstructure noises contributes to the bias in the LS estimation. In what follows, we examine how the IV and GMM estimators accommodate each bias component.

We focus on the situation in which one asset trades at two different platforms from 09:30 to 16:00. We assume without loss of generality that the first market leads the price discovery process. We simulate prices at the 1-second sampling interval from the exact discretization of (1) with $\alpha_\delta = (\alpha_{\delta,1}, \alpha_{\delta,2})'$ and $\beta = (1, -1)'$. The elements of α_δ drive the price discovery for they determine how fast each market adjusts. We report results for two alternative specifications: $\alpha_\delta^{(1)} = (0.000, 0.050)'$ and $\alpha_\delta^{(2)} = (-0.025, 0.050)'$, corresponding to $\alpha_\perp^{(1)} = (1, 0)'$ and $\alpha_\perp^{(2)} = (2/3, 1/3)'$, respectively. As for the covariance matrix Σ, we assume unit variances and a contemporaneous correlation of 1/2 across markets.

The market microstructure noises follow possibly correlated MA($\bar{q}$) with lag orders $\bar{q} \in \{0, 1, \infty\}$:

$$u_{t_i} = e_{t_i} + \sum_{\ell=1}^{\bar{q}} \Lambda_{e,\ell}\, e_{t_{i-\ell}},$$

where $\Lambda_{e,\ell} = \mathrm{diag}\,(\lambda_{e,1,\ell}, \lambda_{e,2,\ell})$, e_{t_i} is normally distributed with zero mean and covariance matrix Ω_e, and $\mathrm{corr}(u_{t_i}, \varepsilon_{t_i})$ is set to either zero (exogenous noise) or -0.2 (endogenous noise). A negative correlation between market innovations and the market microstructure noise is consistent with Hansen and Lunde's (2006) empirical findings.

The type of persistence we allow in the microstructure noise depends on the assumptions in

Section 3 and hence on the type of instruments we employ. Under Assumption MMN(TS), we contemplate either iid (exogenous) noises or serially dependent market microstructure noise coming from endogenous MA(1) processes with $\rho_{m,m'}(0) = -0.2$ and $\gamma_{m,m'}(q) \neq 0$ for $q = 0, 1$. Under Assumption MMN(CS), we consider a similar noise to the latter, but with $\bar{q} = \infty$. In practice, we model the market microstructure noises as AR(1) processes (as in, e.g., Hasbrouck and Ho, 1987; Huang and Stoll, 1997) with $\lambda_{e,1,\ell} = \lambda_{e,2,\ell} = -0.5^{\ell}$ for $\ell = 0, 1, \ldots$.

The variance of the microstructure noise plays a key role in our Monte Carlo setting. Hansen and Lunde's (2006) empirical findings suggest that the magnitude of the noise is small relatively to the integrated variance: $\omega_m^2 < \Sigma_m/1000$, where Σ_m denotes the mth diagonal element of Σ. As it has also been decreasing over the years (Ait-Sahalia and Xiu, 2019), we entertain $100\,\omega^2 \in \{(0.01, 0.01)', (0.05, 0.05)', (0.10, 0.10)', (0.10, 0.05)'\}$. We account for nonsynchronous quoting/trading activity by randomly selecting about 18,000 observations for each stock using independent Bernoulli trials, and then synchronizing log-prices using the refresh-time procedure. Finally, we aggregate the resulting data at the 1- and 5-minute sampling intervals.

4.1 Finite-sample properties under Assumption MMN(TS)

This section assesses the finite-sample performance of the LS- and IV/GMM-based estimates of the CS and IS measures using tick data. We report results for exogenous iid and endogenous MA(1) market microstructure noises using instruments consistent with Assumption MMN(TS). We estimate LS-based price discovery measures using both

VEC(0) $\quad \Delta p_{t_i} = \alpha_\delta (p_{1,t_{i-1}} - p_{2,t_{i-1}}) + \mathrm{v}_{t_i}$

VEC(1) $\quad \Delta p_{t_i} = \alpha_\delta (p_{1,t_{i-1}} - p_{2,t_{i-1}}) + \Gamma_\delta \Delta p_{t_{i-1}} + \mathrm{v}_{t_i}$

whereas the IV/GMM-based estimates rest exclusively on the VEC(0) specification, bearing in mind that $\mathrm{v}_{t_i} = \Delta u_{t_i} + \varepsilon_{t_i} - \alpha_\delta (u_{1,t_{i-1}} - u_{2,t_{i-1}})$. Apart from tick-data results, we also report LS-based estimates of the CS measures at the 1- and 5-minute sampling intervals. As for the IS measures, we first compute LS estimates of the VEC parameters and the realized covariance matrix using tick-by-tick, 1-minute, and 5-minute data. In contrast, the IV/GMM-based information shares rest on the corresponding estimates of $\alpha_\perp$, considering two different estimators of the covariance matrix. We estimate either $\Sigma = CC'$ using a realized kernel approach or its exact discretization $\Sigma_\delta = C_\delta C_\delta'$

using Andrews's (1991) HAC estimator with a Parzen kernel.

Table 1 documents the performance of each estimator for the CS measure of the first market $\alpha_{\perp,1}$ in the case of iid microstructure noises. It suffices to restrict attention to a single market given that $\alpha_{\perp,1} + \alpha_{\perp,2} = 1$ by construction. The first two panels respectively display the median bias analyses for $\alpha \in \{(0, 0.05)', (-0.025, 0.050)'\}$, whereas the third and fourth panels reveal their relative root median squared error (RRMSE) with respect to the LS-based estimates from the VEC(0) model at the tick-by-tick frequency. We report different estimates of the CS measures under iid microstructure noises with variances ω_m^2 ranging from 0.0001 to 0.001 ($m = 1, 2$). The columns VEC($\jmath$)$_0$, VEC($\jmath$)$_1$, VEC($\jmath$)$_5$ refer to the LS estimates of $\alpha_\perp$ from the VEC($\jmath$) specification ($\jmath = 0, 1$) at the tick-by-tick, 1-minute and 5-minute sampling intervals, respectively.

To estimate $\alpha_\perp$ by IV and GMM, we employ the past price differentials $Z_{t_i,2,6}$, with $\bar{q}$ set either to zero or one. This is to reflect the uncertainty around the length of the MA structure in Assumption MMN(TS). In particular, the columns IV$_0$ and IV$_1$ concern the CS estimates of the VEC(0) model using tick data, with $(p_{1,t_{i-2}} - p_{2,t_{i-2}}, \ldots, p_{1,t_{i-6}} - p_{2,t_{i-6}})'$ and $(p_{1,t_{i-3}} - p_{2,t_{i-3}}, \ldots, p_{1,t_{i-7}} - p_{2,t_{i-7}})'$ as instruments, respectively. Similarly, the columns GMM$_{\bar{q}}$ with $\bar{q} = 0, 1$ correspond to the mixed-based estimates of $\alpha_\perp$ from VEC(0) models at the tick-by-tick data with instruments given by $(p_{1,t_{i-\bar{q}-2}} - p_{2,t_{i-\bar{q}-2}}, \ldots, p_{1,t_{i-\bar{q}-6}} - p_{2,t_{i-\bar{q}-6}})'$ for $\bar{q} = 0$ and $\bar{q} = 1$, respectively. Finally, we also report the F-statistic from a joint significance test on the parameter estimates from the first-stage equation of the 2SLS estimator as a measure of the relevance of the instruments.

The IV and GMM estimators provide the best performance in terms of both median bias and RRMSE, regardless of the microstructure noise level and speed-of-adjustment parameters. Notably, as our theory suggests, using VEC(1) at the tick-by-tick frequency does not help reduce the median bias and root median square error (RMSE), whereas sampling at the 1-minute interval alleviates the median bias and RMSE at least for low levels of microstructure noises. Finally, pinning down the correct value of $\bar{q}$ pays off in that IV$_0$ (GMM$_0$) performs better than IV$_1$ (GMM$_1$) due to the stronger instruments.

We next turn our attention to Table 2, which reports the results for the CS measure in the case of endogenous MA(1) microstructure noises. This is a much more interesting specification, with market microstructure noises displaying both serial dependence and endogeneity. These features affect the estimation of both α_δ and Σ. As before, the IV and GMM approaches we propose easily outclasses the extant price discovery measures. The gains in both median bias and RRMSE are considerable, highlighting the importance of accounting for the microstructure noise.

Tables 3 and 4 display the results for the continuous-time IS measures in the cases of iid and endogenous MA(1) microstructure noises, respectively. As before, the columns VEC$(\jmath)_0$, VEC$(\jmath)_1$, VEC$(\jmath)_5$ refer to the LS-based estimates of the continuous-time IS measures from the VEC$(\jmath)$ specification ($\jmath = 0, 1$) at the tick-by-tick, 1-minute and 5-minute sampling intervals, respectively. We estimate the integrated covariance matrix using the realized covariance estimator for the LS-based measures, whereas we employ either RK or HAC estimators in the case of IV/GMM-based measures. Note however that these estimators are very different in nature. While the RK approach estimates directly the continuous-time covariance matrix, the HAC estimator employs the exact discretization to back out the continuous-time (unconditional) covariance matrix. Similarly to Tables 1 and 2, we entertain instruments with $\bar{q} = 0, 1, 2$.

As expected, the median bias of the LS-based estimates of the continuous-time IS is large for all levels of microstructure noises. The $\text{IV}_{\text{RK},\bar{q}}$ and $\text{GMM}_{\text{RK},\bar{q}}$ estimators with $\bar{q} = 0, 1, 2$ have in general smaller biases and RMSE, but still quite off target regardless of the magnitude of the microstructure noise. The latter is due to the upward bias in the off-diagonal elements of the RK estimates of the integrated covariance matrix. While the median biases of the RK estimates of the diagonal elements of Σ_0 are close to zero, they are usually large and positive for the off-diagonal elements. This happens mainly because markets adjust very quickly in our setup, implying that the information on the correlation is already too weak at the tick-by-tick frequency of about one observation every 1.6 seconds. By exploiting the drift information, the $\text{IV}_{\text{HAC},\bar{q}}$ estimators with $\bar{q} = 0, 1, 2$ are able to largely outperform the other estimators.

Table 5 documents the results for the discrete-time IS measures. Because the discrete-time IS measures converge to $1/M$ as the sampling frequency decreases, we restrict our analyses to the tick-by-tick frequency. Once more, the IV-based measures outperform the competitors irrespective of the speed of adjustment in each market and market microstructure noise.

Finally, LS inconsistency affects not only the usual price discovery measures, but also the impulse response functions. Figures 3 and 4 plot the 5%, 50% and 95% empirical quantiles of the corresponding impulse response functions based on 1,000 replications for iid and endogenous MA(1) market microstructure noises, respectively. The rows contain the different estimators, while the columns relate to the microstructure noise magnitude. Dotted and solid lines correspond respectively to the true and median IRFs, with shades covering the region between the 5% and 95% quantiles. Once more, the IV-based impulse response functions behave better than their LS counterparts. Regardless of the VEC specification, there is a large positive bias in the LS-based impulse responses in the presence of market microstructure noise, overestimating the speed at which markets adjust to a change in the efficient price. A different picture emerges if we examine IRFs based on $\mathrm{IV}_{\mathrm{HAC},0}$, $\mathrm{IV}_{\mathrm{HAC},1}$, and $\mathrm{IV}_{\mathrm{HAC},2}$ estimates. They indeed exhibit little bias, correctly tracking the true IRF.

To sum up, our Monte Carlo simulations show that the IV/GMM-based price discovery measures are more reliable than their LS-based counterparts in every scenario we investigate, regardless of the speed-of-adjustment in each market, leadership pattern, and microstructure noise type (large vs small in magnitude, iid vs persistent, exogenous vs endogenous). In particular, their 90% confidence intervals increase with the magnitude of the market microstructure noise, so that fragmented markets might appear more efficient than they actually are if one does not account for the market microstructure noise. We further document that the CS estimates have better finite-sample properties than both the continuous- and discrete-time IS measures under Assumption MMN(TS).

4.2 Finite-sample performance under Assumption MMN(CS)

We next examine the finite-sample performance of the LS- and IV-based price discovery measures in the presence of endogenous MA(∞) microstructure noises. Under these circumstances, we have to select instruments in line with Assumption MMN(CS): namely, current and lagged values of price differentials for V auxiliary assets in J alternative trading platforms. Because we have the entire cross section of assets to chose from, we allow for potentially many instruments by computing IV-based estimators with $V = 1, 5, 10, 20$. The main challenge of Assumption MMN(CS) is that instruments are rather weak, and so we also report price discovery measures based on the CU-GMM estimator, with either 10 or 20 instruments.

We simulate the data as a system of 42 log-prices at the 1-second sampling interval from the exact discretization of (1). They correspond to 21 different assets, each of them trading at two exchanges. The resulting VEC(0) model has 21 cointegrating vectors. We define the 42×21 matrix α in such way that there is no Granger causality among the different assets. The remaining steps are the same as before: (1) we model the market microstructure noise as an endogenous MA(∞) process as in Assumption MMN(CS); (2) we account for nonsynchronous quoting/trading activity by randomly selecting about 18,000 observations for each stock using independent Bernoulli trials; (3) we aggregate tick data using the refresh-time technology; and then (4) aggregate prices to the 1- and 5-minute sampling intervals.

To save space, we omit the results based on the LS estimation of the VEC(1) specification, as they are essentially the same as in Section 4.1. As such, we restrict attention to the LS-based measures using the VEC(0) model at the tick-by-tick, 1-minute, and 5-minute sampling intervals, whereas we report the IV and CU-GMM estimates only for the VEC(0) specification at the tick-by-tick frequency. We organize the results as in Section 4.1, displaying median bias and RRMSE of the different estimators using least squares at the tick-by-tick frequency as a benchmark.

Table 6 shows the performance of each estimator of the first market's component share. As expected, the LS-based estimators perform very poorly regardless of the sampling frequency, microstructure noise variance and speed-of-adjustment parameters. For the lowest level of noise, $\omega^2 = (0.0001, 0.0001)'$, the IV estimator with a single instrument performs very well, with virtually no median bias and with a low RRMSE. For larger magnitudes of microstructure noise, the first-stage F-statistics and R^2 drop sharply, so that the performance of the IV estimator deteriorates accordingly. Adding more instruments is not of great help, as median bias remains moderate to high. These results are in line with the asymptotic theory in view that the IV estimator is inconsistent in the case of many weak instruments. The picture dramatically changes once we focus on CU-GMM estimators. The corresponding CS measures display not only virtually no median bias, but also the lowest RRMSE in every scenario we entertain. In particular, the CU-GMM estimator with 20 instruments exhibits the best overall performance, in line with our rationale that using many weak instruments pays off in the estimation of component shares.

Table 7 documents the results for the continuous- and discrete-time IS measures. We find that every estimator exhibits some bias provided that at least one microstructure noise is large enough in magnitude. Combining the HAC covariance matrix estimator with VEC parameter estimates coming either from the IV with 5 instruments or from the CU-GMM with 20 instruments entails the best improvements, even if it still exhibits some finite-sample bias.

As for the impulse response functions, Figure 5 depicts their true values as well as their 5%, 50% and 95% empirical quantiles according to estimator (rows) and magnitude of the market microstructure noise (columns), based on 1,000 replications. We report only the IV-based estimates with 1 to 20 instruments and the CU-GMM estimates with 10 and 20 instruments, given that the LS estimates do not differ quantitatively from those in Figure 4. As before, the IV-based estimators deliver unbiased IRF estimates only in the case of $\omega^2 = (0.0001, 0.0001)'$. For larger magnitudes of microstructure noise, we find that the CU-GMM estimator with 20 instruments performs best, especially at what concerns the coverage of the 90% empirical confidence interval.

In summary, the IV- and GMM-based estimators of α and $\alpha_\perp$ perform very well in both settings. In contrast, LS-based price discovery measures are sometimes very misleading, e.g., IRFs based on LS residuals are generally downward biased, making the convergence to the long-run effect falsely too fast.

5 Price informativeness: NYSE versus Nasdaq

In this section, we compute price discovery measures for Alcoa (AA). Although Alcoa trades on multiple venues in the US, we focus on the NYSE (N) and Nasdaq (T) in view that they are the most active listing exchanges. We extract quotes data from TAQ for the period ranging from June 2012 to May 2013. We implement the same cleaning filters as in Barndorff-Nielsen, Hansen, Lunde and Shephard (2009), discarding any observation with a zero quote, negative bid-ask spread, or outside the main trading hours (9:30 to 16:00). We also discard any data point with a bid-ask spread higher than 50 times the median spread on that day or with a midquote deviating by more than 10 mean absolute deviations from a rolling centered median of 50 observations. Finally, we take the median bid and ask quotes in the event of multiple ticks taking place within the same second.

We synchronize midquotes from both trading venues by using the refresh-time approach, and then aggregate them to regularly spaced intervals of 1 and 5 minutes. Apart from the entire sample, we also consider nonoverlapping subsamples of 12 months. We also take as potential instruments a panel of 25 random S&P 500 stocks that trade at the alternative trading venues ARCA (P), BATS (Z) and NASDAQ OMX BX Stock Exchange (B, formerly Boston Stock Exchange): AAPL, BAC, BRKB, CSCO, DAL, GM, GOOG, HPQ, IBM, JCP, JNJ, JPM, KO, MO, MRK, MRVL, MSFT, NOK, ORCL, PFE, PG, VZ, WFC, XOM, and YHOO. We handle them applying exactly the same filters as before. Table 9 reports the number of quotes we observe for each stock at each trading platform, before and after cleaning filters. It also displays the market shares of each market, as well as the corresponding average duration between quote revisions. Apart from boasting the highest trading activity, NYSE responds to the largest market share for every ticker, even if it is mostly between 20% and 25%.

We test for cointegration on a monthly basis using the same protocol as in Hansen and Lunde (2006). We first choose the lag length of the VEC models by obtaining the most parsimonious specification in which the LM test cannot reject the absence of residual autocorrelation at the 5% significance level, and then apply Johansen's maximum eigenvalue and trace cointegration tests.[7] We find strong evidence, at the 1% level of significance, that there is only one cointegrating vector in every month of the sample, regardless of the sampling interval.[8] The standard practice in price discovery analysis is to equip the VEC specification with a rich lag structure, ignoring that the presence of market microstructure noise may hatch spurious autocorrelation in the LS residuals. In contrast, our IV-based price discovery analysis always rests on the simple VEC(0) model that results from the exact discretization of continuous-time OU process we assume.

To entertain the case that Assumption MMN(TS) holds, we use past price differentials in the NYSE and Nasdaq as instruments. Table 10 reports monthly and yearly estimates of the NYSE's component share $\alpha_{\perp,\mathrm{N}}$ using data at different sampling intervals. It is striking how inference about market leadership differs markedly depending on whether we employ LS or IV estimates. The LS estimates at the tick-by-tick frequency assign Nasdaq as the market leader, whereas both IV estimates attribute market leadership to the NYSE in every month of the sample. The LS results are not robust however to the sampling frequency. Using 1- and 5-minute intervals, LS-based

[7] Qualitatively, it makes no difference in the test results if we employ the lag structure that minimizes the Bayesian information criterion. Results are available upon request.

[8] The presence of stationary market microstructure noises should not affect the consistency of the LS estimation of the cointegrating vector. Accordingly, it comes with no surprise that we find that the cointegrating vector estimates are always extremely close to $(1, -1)'$.

analyses indicate that the NYSE contributes more than Nasdaq to the price discovery. This is consistent with our results in that the LS bias decreases with the sampling interval. However, a noteworthy drawback: the order of magnitude of the standard errors. The standard errors of the LS estimates at 1- and 5-minute intervals are on average 4 and 18 times larger, respectively, than those of the IV estimates with $\bar{q} = 0$.

Instruments seem strong enough, with first-stage R^2 over 20% for virtually every month. Hausman tests also indicate that LS estimates of the speed-of-adjustment parameters are inconsistent given the significant differences with respect to the IV estimates. These differences are larger for the NYSE, perhaps hinting that the microstructure noise is larger in magnitude at the NYSE.[9] This is exactly the situation in which LS-based market leadership analyses fail, as our simulation results for $\omega^2 = (0.0010, 0.0005)'$ illustrate. To confirm whether this is the case, we compare the realized variance (RV) in each market, in view that $RV/(2n) \xrightarrow{p} \omega^2$ under the assumption of iid microstructure noises (Ait-Sahalia and Xiu, 2019). We indeed find that $\widehat{\omega}^2_{\mathrm{N}} > \widehat{\omega}^2_{\mathrm{T}}$, corroborating the indirect evidence in the difference between the LS and IV estimates. The main take way is that, in the presence of microstructure noise, one should always consider IV-based price discovery measures to make inference about market leadership.

Table 11 reveals that the LS and IV estimates of NYSE's continuous-time IS measures are markedly different. The best-performing estimators in the Monte Carlo study, $IV_{\mathrm{HAC},0}$ and $IV_{\mathrm{HAC},1}$, indicate that NYSE's continuous-time information share is stable over time, about 60%. This highlights that the speeds at which both NYSE and Nasdaq currently operate call for ultra high resolutions; otherwise, the contemporaneous correlation between markets gets very close to unit, implying uninformative information shares (Hasbrouck, 2019). Finally, a similar picture arises for the the discrete-time information share measures. In line with the previous results, the $IV_{\mathrm{HAC},0}$ and $IV_{\mathrm{HAC},1}$ estimates assign NYSE as the market leader, whereas the LS estimates attribute market leadership to Nasdaq.

We next treat the microstructure noises as if they were MA(∞) processes as in Assumption MMN(CS) by using the price differentials of auxiliary assets traded at different exchanges as instruments. Using the refresh-time scheme, we synchronize the AA midquotes with the entire set of

[9] The full set of results, including the speed-of-adjustment estimates for the NYSE and Nasdaq markets, are available upon request.

instruments at the tick-by-tick frequency. Because refresh time depends heavily on the least liquid asset, we entertain two sets of instruments with different sampling frequencies. The first selects auxiliary assets with at least 7,500 quotes on every trading day (about one quote every 3 seconds) within any given month. This means that the tickers in this set of instruments vary over time. The second set of instruments lowers the bar to at least 5,000 quotes per day (about one quote every 5 seconds). The reduction on the liquidity requirement reveals a trade-off between boosting the number of instruments and increasing the midquotes duration. Whilst increasing the number of auxiliary assets may strengthen the instruments, longer durations increases the noise-to-signal ratio, which contributes to weaken the instruments.

Table 12 reveals that the instruments are rather weak for auxiliary assets with at least 7,500 quotes, with first-stage R^2 and F-statistics well below 0 and 10, respectively. Reducing the liquidity requirement to at least $5,000$ quotes increases the number of instruments from about 14 to 28 assets. As a result, the average duration between quotes increases approximately from 8 to 17 seconds, contributing to even weaker instruments. We nonetheless find very similar CU-GMM estimates of $\alpha_{\perp,\mathrm{N}}$ for both sets of instruments. It is reassuring that the CU-GMM component share estimates also indicate that the NYSE leads the price discovery mechanism in every month of the sample period. In fact, we cannot individually reject the null that $\alpha_{\perp,\mathrm{T}} = 0$ for every month, strengthening the evidence that the NYSE indeed plays a major role in the price discovery process.

Finally, we also report results based on the mixed asymptotic theory in Section 3.3. We use tick data to compute daily component shares in order to track how the price discovery evolves over time. We treat each trading day independently by running individual VEC models. The top panel of Figure 6 displays the LS estimates of the CS measures for both VEC(0) and VEC(1) specifications, whereas the lower panel exhibits the corresponding GMM estimates for the VEC(0) model.

The results are well in line with our previous findings in that the LS- and GMM-based component shares offer utterly different pictures about market leadership. In line with our theoretical implications, the LS estimation seems to bias CS values towards 1/2. In contrast, the daily GMM estimates identify once again the NYSE as the leading market over the entire sample period. Additionally, in view of the growing interest on investigating the price discovery intraday patterns, we divide each trading days in three nonoverlapping time intervals: from 09:30:00am to 11:40:00am, 11:40:01am to 01:50:00pm, and 01:50:01pm to 04:00:00pm. We consider each trading day and time interval independently and compute LS- and GMM-based component share measures. The results are very similar to those obtained with the daily CS measures, namely the GMM-based component shares assign NYSE as the market leader, whereas the LS estimates appear to bias the CS values towards 1/2 and assigns Nasdaq as the leading market.[10]

Figure 7 displays the 5%, 50% and 95% percentiles of the daily IRF of Nasdaq to shocks in the efficient price. Given our simulation results, we expect LS-based IRFs to converge to one much faster than their IV counterparts in the presence of market microstructure noise. This is indeed the case. The LS-based IRFs of the VEC(0) models converge almost instantaneously to one, whereas the robust-to-noise IV-based IRFs take, on average, five to ten periods to incorporate all the information from the efficient price into market prices.

Altogether, this section illustrates well the importance of using price discovery measures that are robust to market microstructure noise.

6 Conclusion

It makes sense to carry out price discovery analysis at the highest frequency given our interest in how fast markets impound information. As such, it seems natural to consider a continuous-time setting as in Dias et al. (2019). Sampling at nano or even microseconds is arguably close enough to continuous-time recording. However, market microstructure effects display, both in theory and in practice, a much richer dynamics than the iid Gaussian noise that results from the reduced-rank Ornstein-Uhlenbeck process that Dias et al. (2019) entertain. We then follow the realized measure literature by adding a second layer of microstructure noise in discrete time so as to cope with serial correlation and endogeneity.

The presence of such a microstructure noise biases the least-squares estimation upon which traditional price discovery analyses rest. Accordingly, we propose IV/GMM-based approaches to deal with measurement error and derive valid sets of instruments under two different market microstructure assumptions. We establish the large sample behaviors of the IV/GMM-based estimators under two settings: by fixing the number of intraday observations and letting time span grow; and by

[10] Results are available upon request.

fixing time span and letting the number of intraday observations grow. Monte Carlo simulations confirm that the LS estimates are inconsistent, whereas our IV/GMM approaches perform very well, with virtually no median bias in finite samples. Interestingly, in our empirical application, we find that market leadership between the NYSE and Nasdaq flips once we account for microstructure noise, regardless of the type of market microstructure noise and of the asymptotic setting.

Appendix A Proofs

Proof of Theorem 1 It readily follows from $\beta = (I_r, -\iota_r)'$ that $\beta_\perp = \iota_M$ and Ξ has common rows. As (9) has no autoregressive terms and $\sum_{m=1}^{M} \alpha_{m,\perp} = 1$, $\alpha_\perp$ is equal to any row of Ξ. Let $\widehat{\Xi} = \beta_\perp[\widehat{\alpha}'_\perp \beta_\perp]^{-1}\widehat{\alpha}'_\perp$ for any estimator $\widehat{\alpha}_\perp$ and rewrite

$$\widehat{\Xi} = \beta_\perp[\alpha'_\perp \beta_\perp + (\widehat{\alpha}_\perp - \alpha_\perp)'\beta_\perp]^{-1}[\alpha'_\perp + (\widehat{\alpha}'_\perp - \alpha'_\perp)],$$

$$= \Xi - \beta_\perp(\alpha'_\perp\beta_\perp)^{-1}(\widehat{\alpha}_\perp - \alpha_\perp)'\beta_\perp(\alpha'_\perp\beta_\perp)^{-1}\alpha'_\perp + \beta_\perp(\alpha'_\perp\beta_\perp)^{-1}(\widehat{\alpha}_\perp - \alpha_\perp)'. \tag{20}$$

Without loss of generality, for any fixed $0 \leq \delta \leq 1$, consider $\widehat{\alpha}_{\perp,\text{IV}} = \alpha_\perp - \widehat{\bar{\alpha}}_\delta \widehat{\alpha}'_{\delta,\text{IV}}\alpha_\perp$, where $\widehat{\bar{\alpha}}_\delta = -\alpha_\delta(\widehat{\alpha}'_{\delta,\text{IV}}\alpha_\delta)^{-1}$ is such that $\widehat{\alpha}'_{\delta,\text{IV}}\widehat{\alpha}_{\perp,\text{IV}} = 0$. Substituting $\widehat{\alpha}_{\perp,\text{IV}} - \alpha_\perp$ into (20) and rearranging terms yields

$$\widehat{\Xi}_{\text{IV}} - \Xi = \beta_\perp(\alpha'_\perp\beta_\perp)^{-1}[\alpha'_\perp(\widehat{\alpha}_{\delta,\text{IV}} - \alpha_\delta)\widehat{\bar{\alpha}}'_\delta]\beta_\perp(\alpha'_\perp\beta_\perp)^{-1}\alpha'_\perp - \beta_\perp(\alpha'_\perp\beta_\perp)^{-1}\alpha'_\perp(\widehat{\alpha}_{\delta,\text{IV}} - \alpha_\delta)\widehat{\bar{\alpha}}'_\delta,$$

$$= \Xi(\widehat{\alpha}_{\delta,\text{IV}} - \alpha_\delta)\widehat{\bar{\alpha}}'_\delta\Xi - \Xi(\widehat{\alpha}_{\delta,\text{IV}} - \alpha_\delta)\widehat{\bar{\alpha}}'_\delta,$$

$$= \Xi(\widehat{\alpha}_{\delta,\text{IV}} - \alpha_\delta)\widehat{\bar{\alpha}}'_\delta(\Xi - I_M). \tag{21}$$

From the properties of the vec operator, we have that

$$\text{vec}\,(\widehat{\Xi}_{\text{IV}}) - \text{vec}\,(\Xi) = (\widehat{\psi}'_1 \otimes \Xi)[\text{vec}\,(\widehat{\alpha}_{\delta,\text{IV}}) - \text{vec}\,(\alpha_\delta)] = (\widehat{\psi}'_1 \otimes \Xi)K_{M,R}(\widehat{\boldsymbol{\alpha}}_{\delta,\text{IV}} - \boldsymbol{\alpha}_\delta), \tag{22}$$

where $\widehat{\psi}_1 = \widehat{\bar{\alpha}}_\delta(\Xi - I_M)$ and $K_{M,R}$ is the commutation matrix with dimensions $MR \times MR$ such that $K_{M,R}\,\text{vec}\,(\alpha_\delta) = \text{vec}\,(\alpha'_\delta) = \boldsymbol{\alpha}_\delta$. In view that Ξ has common rows, define a deterministic matrix $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_M)$ such that $\widehat{\alpha}_{\perp,\text{IV}} = \mathcal{S}\,\text{vec}\,(\widehat{\Xi}_{\text{IV}})$ and $\alpha_\perp = \mathcal{S}\,\text{vec}\,(\Xi)$, where $\mathcal{S}_j = (\iota(j), 0_{M\times R})$ with $j = 1, \ldots, M$, $\iota(j)$ denoting an $M \times 1$ vector that has its jth row equal to one and all the remaining entries equal to zero, and $0_{M\times R}$ is an $M \times R$ matrix of zeros. Pre-multiplying (22) by $\mathcal{S}$ yields

$$\widehat{\alpha}_{\perp,\text{IV}} - \alpha_\perp = \mathcal{S}\,(\widehat{\psi}'_1 \otimes \Xi)K_{M,R}(\widehat{\boldsymbol{\alpha}}_{\delta,\text{IV}} - \boldsymbol{\alpha}_\delta), \tag{23}$$

so that

$$\widehat{\boldsymbol{\alpha}}_{\delta,\text{IV}} - \boldsymbol{\alpha}_\delta = \left[\frac{\bar{\boldsymbol{p}}'(I_M \otimes \boldsymbol{Z})}{T}(T\boldsymbol{W})\frac{(I_M \otimes \boldsymbol{Z})'\bar{\boldsymbol{p}}}{T}\right]^{-1}\frac{\bar{\boldsymbol{p}}'(I_M \otimes \boldsymbol{Z})}{T}(T\boldsymbol{W})\frac{(I_M \otimes \boldsymbol{Z})'\mathbf{v}}{T}. \tag{24}$$

Assumption RROU and (ii) in Lemmata 2 and 3 ensure the first term on the right-hand-side of (24) is $O_p(1)$, meaning that $\underset{T\to\infty}{\text{plim}}\,\frac{1}{T}\bar{\boldsymbol{p}}'\,(I_M \otimes \boldsymbol{Z}) = \boldsymbol{Q}_{\bar{\boldsymbol{p}},\boldsymbol{Z}} < \infty$ and $\underset{T\to\infty}{\text{plim}}\,T\boldsymbol{W} = \boldsymbol{Q}_{\boldsymbol{W}} < \infty$ are both of order $O_p(1)$. Standard weak law of large number for stationary processes (e.g., Proposition 7.5

in Hamilton, 1994) and the validity of instruments render $\operatorname*{plim}_{T\to\infty} \frac{1}{T}(I_M \otimes \boldsymbol{Z})'\mathbf{v} = 0$, implying that $\widehat{\boldsymbol{\alpha}}_{\delta,\mathrm{IV}} \xrightarrow{p} \boldsymbol{\alpha}_\delta$, and hence $\widehat{\bar{\alpha}}_\delta \xrightarrow{p} -\alpha_\delta(\alpha_\delta'\alpha_\delta)^{-1}$ and $\widehat{\psi}_1 \xrightarrow{p} \psi_1$. As such, $\widehat{\alpha}_{\perp,\mathrm{IV}} \xrightarrow{p} \alpha_\perp$, completing the proof of (i).

To prove (ii), we use (i) and rewrite (23) such that, as $T \to \infty$,

$$\sqrt{T}(\widehat{\alpha}_{\perp,\mathrm{IV}} - \alpha_\perp) = \mathcal{S}\,(\psi_1' \otimes \Xi)\,K_{M,R}\big(\boldsymbol{Q_{\bar{p},Z}Q_W Q'_{\bar{p},Z}}\big)^{-1}\boldsymbol{Q_{\bar{p},Z}Q_W}\,\frac{(I_M \otimes \boldsymbol{Z})'\mathbf{v}}{\sqrt{T}}. \tag{25}$$

Note that $\frac{1}{\sqrt{T}}\,(I_M \otimes \boldsymbol{Z})'\,\mathbf{v} = \operatorname{vec}\left(\frac{1}{\sqrt{T}}\sum_{\tau=1}^T Z_{\tau-\kappa,\kappa}\mathrm{v}_{1,\tau}, \ldots, \frac{1}{\sqrt{T}}\sum_{\tau=1}^T Z_{\tau-\kappa,\kappa}\mathrm{v}_{M,\tau}\right)$ is a vector of serially correlated covariance stationary processes, rather than of martingale difference sequences. Lemma 4 shows that Assumptions MMN(TS) and MMN(CS) ensure that the conditions of the CLT for L^1-mixingale sequences (see, for instance, Davidson, 1994; White, 2000). Applying the Cramér-Wold device results in $\frac{1}{\sqrt{T}}\,(I_M \otimes \boldsymbol{Z})'\,\mathbf{v} \xrightarrow{d} N(0, \Gamma_{\bar{\boldsymbol{p}}\boldsymbol{Z}})$, so that the limiting distribution of (25) reads $\sqrt{T}(\widehat{\alpha}_{\perp,\mathrm{IV}} - \alpha_\perp) \xrightarrow{d} N(0, \boldsymbol{C_1}\,\boldsymbol{C_2}\,\boldsymbol{C_3}\,\Gamma_{\bar{\boldsymbol{p}}\boldsymbol{Z}}\,\boldsymbol{C_3'}\,\boldsymbol{C_2'}\,\boldsymbol{C_1'})$, with $\boldsymbol{C_1} = \mathcal{S}\,(\psi_1' \otimes \Xi)\,K_{M,R}$, $\boldsymbol{C_2} = (\boldsymbol{Q_{\bar{p},Z}Q_W Q'_{\bar{p},Z}})^{-1}$ and $\boldsymbol{C_3} = \boldsymbol{Q_{\bar{p},Z}Q_W}$, yielding the result in (ii).

In view that $\widehat{\mathrm{v}}_{t_i}$ is consistent, $\widehat{\Sigma}_{\delta,\mathrm{IV}} \xrightarrow{p} \Sigma_\delta$ ensues from standard HAC asymptotic theory (see, for instance, Corollary 6.11 in White, 2000), so that $\widehat{IS}_{\delta,\mathrm{IV}} \xrightarrow{p} IS_\delta$ in (iii) and $\widehat{IS}_{\mathrm{IV}} \xrightarrow{p} IS$ in (iv) follows from straightforward applications of the Slutsky's theorem. ■

Proof of Theorem 2 Along the same lines as in Theorem 1(i), we may write

$$\widehat{\alpha}_{\perp,\mathrm{GMM}} - \alpha_\perp = \mathcal{S}\,(\widehat{\psi}_2' \otimes \Xi)\,K_{M,R}(\widehat{\boldsymbol{\alpha}}_{\mathrm{GMM}} - \boldsymbol{\alpha}), \tag{26}$$

where $\widehat{\psi}_2 = \widehat{\bar{\alpha}}(\Xi - I_M)$ and $\widehat{\boldsymbol{\alpha}}_{\mathrm{GMM}} = \operatorname{argmax}_{\widehat{\boldsymbol{\alpha}}}\ \mathcal{C}_{\mathrm{GMM},n_\tau}(\widehat{\boldsymbol{\alpha}})$. Assumption GMM$(c)$ ensures that

$$\widehat{\boldsymbol{\alpha}}_{\mathrm{GMM}} = \operatorname*{argmax}_{\widehat{\boldsymbol{\alpha}}}\ \mathcal{C}_{\mathrm{GMM},n_t}(\widehat{\boldsymbol{\alpha}}) + o_p(1). \tag{27}$$

We now use the same arguments as in Newey and McFadden (1994). For any $\varsigma > 0$,

$$\mathcal{C}(\widehat{\boldsymbol{\alpha}}_{\mathrm{GMM}}) > \mathcal{C}_{\mathrm{GMM},n_t}(\widehat{\boldsymbol{\alpha}}_{\mathrm{GMM}}) - \varsigma/3 > \mathcal{C}_{\mathrm{GMM},n_t}(\boldsymbol{\alpha}) - 2\varsigma/3 > \mathcal{C}(\widehat{\boldsymbol{\alpha}}) - \varsigma \tag{28}$$

with probability approaching one. Next, let $\mathbf{B}$ denote any open subset of $\mathbf{A}$ that contains $\boldsymbol{\alpha}$, and $\mathbf{B}^c$ the complement of $\mathbf{B}$ in $\mathbb{R}^{MR}$. It then follows that $\mathbf{A} \cap \mathbf{B}^c$ is compact and $\operatorname{argmax}_{\widehat{\boldsymbol{\alpha}} \in \mathbf{A}\cap\mathbf{B}^c} \mathcal{C}(\widehat{\boldsymbol{\alpha}})$ exists. We now fix $\varsigma = \mathcal{C}(\boldsymbol{\alpha}) - \sup_{\widehat{\boldsymbol{\alpha}} \in \mathbf{A}\cap\mathbf{B}^c} \mathcal{C}(\widehat{\boldsymbol{\alpha}})$ in (28) given that $\mathcal{C}(\boldsymbol{\alpha}) - \sup_{\widehat{\boldsymbol{\alpha}} \in \mathbf{A}\cap\mathbf{B}^c} \mathcal{C}(\widehat{\boldsymbol{\alpha}}) > 0$ by

Assumption GMM(a,b). It then holds that $\mathcal{C}(\widehat{\boldsymbol{\alpha}}_{\text{GMM}}) > \sup_{\widehat{\boldsymbol{\alpha}}\in\mathbf{A}\cap\mathbf{B}^c}\mathcal{C}(\widehat{\boldsymbol{\alpha}})$, so that $\widehat{\boldsymbol{\alpha}}_{\text{GMM}} \in \mathbf{B}$, with probability approaching one. This means that $\widehat{\boldsymbol{\alpha}}_{\text{GMM}} \xrightarrow{p} \boldsymbol{\alpha}$, so that $\widehat{\bar{\alpha}} \xrightarrow{p} \bar{\alpha} = -\alpha(\alpha'\alpha)^{-1}$ and $\widehat{\psi}_2 \xrightarrow{p} \psi_2$. As such, $\widehat{\alpha}_{\perp,\text{GMM}} \xrightarrow{p} \alpha_\perp$ as stated in item (i).

As for the proof of (ii), it follows from (26) that $\sqrt{D}(\widehat{\alpha}_{\perp,\text{GMM}} - \alpha_\perp) = \mathcal{S}(\widehat{\psi}_2' \otimes \Xi)\, K_{M,R}\big[\sqrt{D}(\widehat{\boldsymbol{\alpha}}_{\text{GMM}} - \boldsymbol{\alpha})\big]$. As we know that $\widehat{\psi}_2 \xrightarrow{p} \psi_2$, it suffices to show the limiting distribution of $\sqrt{D}(\widehat{\boldsymbol{\alpha}}_{\text{GMM}} - \boldsymbol{\alpha})$. Next, the first-order condition of (27) implies $-2\boldsymbol{G}_{n_t}'(\widehat{\boldsymbol{\alpha}}_{\text{GMM}})\widehat{\Psi}^{-1}\boldsymbol{g}_{n_t}(\widehat{\boldsymbol{\alpha}}_{\text{GMM}}) = 0$, where $\boldsymbol{g}_{n_t}(\widehat{\boldsymbol{\alpha}})$ denotes the sample moment condition computed with prices sampled on the regularly-spaced clock with sampling interval given by $\delta = 1/n$ and $\boldsymbol{G}_{n_t}(\widehat{\boldsymbol{\alpha}}) = \nabla_{\widehat{\boldsymbol{\alpha}}}\, \boldsymbol{g}_{n_t}(\widehat{\boldsymbol{\alpha}})$ stands for the gradient of $\boldsymbol{g}_{n_t}(\widehat{\boldsymbol{\alpha}})$. Expanding $\boldsymbol{g}_{n_t}(\widehat{\boldsymbol{\alpha}}_{\text{GMM}})$ around $\boldsymbol{\alpha}$ then yields

$$\sqrt{D}(\widehat{\boldsymbol{\alpha}}_{\text{GMM}} - \boldsymbol{\alpha}) = \Big[\boldsymbol{G}_{n_t}'(\widehat{\boldsymbol{\alpha}}_{\text{GMM}})\,\widehat{\Psi}^{-1}\,\boldsymbol{G}_{n_t}'(\widehat{\boldsymbol{\alpha}}^*)\Big]^{-1}\boldsymbol{G}_{n_t}'(\widehat{\boldsymbol{\alpha}}_{\text{GMM}})\,\widehat{\Psi}^{-1}\sqrt{D}\,\boldsymbol{g}_{n_t}(\boldsymbol{\alpha}), \tag{29}$$

where $\widehat{\boldsymbol{\alpha}}^*$ is a convex combination between $\widehat{\boldsymbol{\alpha}}_{\text{GMM}}$ and $\boldsymbol{\alpha}$. Note first that $\widehat{\Psi} \xrightarrow{p} \Psi$ by Theorem 6.9 in White (2000). Next, we must show that $\boldsymbol{G}_{n_t}(\widehat{\boldsymbol{\alpha}}_{\text{GMM}}) \xrightarrow{p} \boldsymbol{G} = \mathbb{E}[\nabla_{\boldsymbol{\alpha}}\, \boldsymbol{g}_{n_{t_i}}(\boldsymbol{\alpha})]$ with $\boldsymbol{g}_{n_{t_i}}(\boldsymbol{\alpha}) = (\Delta p_{t_i} - \alpha_\delta(\boldsymbol{\alpha})\beta' p_{t_{i-1}}) \otimes Z_{t_i,k,\bar{k}}$. Taking a mean value expansion of sample and population moment conditions around $\boldsymbol{\alpha}$ and applying the triangle inequality give way to

$$\sup_{\widehat{\boldsymbol{\alpha}}^*\in\mathbf{A}^*}|\boldsymbol{G}_{n_t}(\widehat{\boldsymbol{\alpha}}^*) - \boldsymbol{G}(\widehat{\boldsymbol{\alpha}}^*)| \quad \sup_{\widehat{\boldsymbol{\alpha}}\in\mathbf{A}}|\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}| < \sup_{\widehat{\boldsymbol{\alpha}}\in\mathbf{A}}\big|\boldsymbol{g}_{n_t}(\widehat{\boldsymbol{\alpha}}) - \boldsymbol{g}(\widehat{\boldsymbol{\alpha}})\big| + \big|\boldsymbol{g}_{n_t}(\boldsymbol{\alpha}) - \boldsymbol{g}(\boldsymbol{\alpha})\big|, \tag{30}$$

where $\widehat{\boldsymbol{\alpha}}^*$ is a convex combination between an arbitrary point $\widehat{\boldsymbol{\alpha}} \in \mathbf{A}$ and $\boldsymbol{\alpha}$, and $\mathbf{A}^*$ denoting an open convex set containing $\mathbf{A}$. Assumptions GMM(c) and RROU ensure not only that both terms on the right-hand side of (30) are $o_p(1)$, but also that $\sup_{\widehat{\boldsymbol{\alpha}}\in\mathbf{A}}|\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}| = O(1)$. It then follows that $\sup_{\widehat{\boldsymbol{\alpha}}^*\in\mathbf{A}^*}|\boldsymbol{G}_{n_t}(\widehat{\boldsymbol{\alpha}}^*) - \boldsymbol{G}(\widehat{\boldsymbol{\alpha}}^*)| = o_p(1)$, so that $\boldsymbol{G}_{n_t}(\widehat{\boldsymbol{\alpha}}_{\text{GMM}})$ converges uniformly in probability to $\boldsymbol{G}$.

As before, to obtain the limiting distribution of $\sqrt{D}\boldsymbol{g}_{n_t}(\boldsymbol{\alpha})$, we use the CLT for L^1-mixingale sequences. This follows because the necessary condition in Yoshida (1992) are satisfied. Moreover, our estimator can be seen as a special case of the two-stage approach in Phillips and Yu (2009). In particular, applying the Cramér-Wold device yields $\sqrt{D}\boldsymbol{g}_{n_t}(\boldsymbol{\alpha}) \xrightarrow{d} N(0,\Psi)$, implying that $\sqrt{D}(\widehat{\boldsymbol{\alpha}}_{\text{GMM}} - \boldsymbol{\alpha}) \xrightarrow{d} N\big(0, (\boldsymbol{G}'\Psi^{-1}\boldsymbol{G})^{-1}\big)$ if we use Ψ as the weighting matrix. As such, $\sqrt{D}(\widehat{\boldsymbol{\alpha}}_{\perp,\text{GMM}} - \boldsymbol{\alpha}_\perp) \xrightarrow{d} N\big(0, \mathcal{S}(\psi_2' \otimes \Xi)(\boldsymbol{G}'\Psi^{-1}\boldsymbol{G})^{-1}(\psi_2' \otimes \Xi)'\mathcal{S}'\big)$, which proves item (ii).

To complete the proof, it suffices to establish that $RK \xrightarrow{p} \Sigma$. To this end, we must show that our assumptions satisfy the conditions in Barndorff-Nielsen et al.'s (2011) Lemma 1. This is indeed the case. The continuous-time process in (1) meets the conditions in their Assumption SH. Assumptions RT and K coincide with their Assumptions D and K, whereas their Assumption U holds under Assumption MMN(TS). It follows by Slutsky's theorem that $IS_{\text{GMM}} - IS = o_p(1)$. ■

A.1 Lemmata

Proof of Lemma 1 The equivalence between the continuous- and discrete-time log-likelihood functions and the fact that $\operatorname{rank}(\alpha\beta') = \operatorname{rank}(\alpha_\delta\beta')$ follow immediately from Theorem 1 in Kessler and Rahbek (2004). See, for instance, Johansen (1995) for the remaining results. ■

Proof of Lemma 2 Given that $p_{t_i} = P_{t_i} + u_{t_i}$ and $\mathrm{v}_{t_i} = \varepsilon_{t_i} + [I_M - (\alpha_\delta\beta' + I_M)L]\,u_{t_i}$,

$$\beta' p_{t_{i-\bar{q}-\kappa}} = \sum_{h=0}^{\infty} \beta'\Upsilon_{\delta,h}\varepsilon_{t_{i-\bar{q}-\kappa-h}} + \beta' u_{t_{i-\bar{q}-\kappa}}.$$

To prove claim (i), write $\mathbb{E}\left[\beta' p_{t_{i-\bar{q}-\kappa}} \mathrm{v}'_{t_i}\right]$ as

$$\begin{aligned}
\mathbb{E}\left[\beta' p_{t_{i-\bar{q}-\kappa}} \mathrm{v}'_{t_i}\right] &= \mathbb{E}\left[\left(\sum_{h=0}^{\infty} \beta'\Upsilon_{\delta,h}\varepsilon_{t_{i-\bar{q}-\kappa-h}} + \beta' u_{t_{i-\bar{q}-\kappa}}\right)\left(\varepsilon_{t_i} + \left[I_M - \left(\alpha_\delta\beta' + I_M\right)L\right]u_{t_i}\right)'\right], \\
&= \sum_{h=0}^{\infty} \beta'\Upsilon_{\delta,h}\mathbb{E}\left[\varepsilon_{t_{i-\bar{q}-\kappa-h}}\varepsilon'_{t_i}\right] + \sum_{h=0}^{\infty} \beta'\Upsilon_{\delta,h}\mathbb{E}\left[\varepsilon_{t_{i-\bar{q}-\kappa-h}}u'_{t_i}\right] \\
&\quad - \sum_{h=0}^{\infty} \beta'\Upsilon_{\delta,h}\mathbb{E}\left[\varepsilon_{t_{i-\bar{q}-\kappa-h}}u'_{t_{i-1}}\right]\left(\alpha_\delta\beta' + I_M\right)' + \beta'\mathbb{E}\left[u_{t_{i-\bar{q}-\kappa}}\varepsilon'_{t_i}\right] \\
&\quad + \beta'\mathbb{E}\left[u_{t_{i-\bar{q}-\kappa}}u'_{t_i}\right] - \beta'\mathbb{E}\left[u_{t_{i-\bar{q}-\kappa}}u'_{t_{i-1}}\right]\left(\alpha_\delta\beta' + I_M\right)'.
\end{aligned}$$

Under Assumptions RROU and MMN(TS), all of the above terms equal zero for any $k \le \kappa \le \bar{k}$, completing the proof of (i). We next show that the relevance of the instruments depend on the persistence in the price differentials. To this end, write

$$\begin{aligned}
\mathbb{E}\left[\beta' p_{t_{i-\bar{q}-\kappa}}(\beta' p_{t_{i-1}})'\right] &= \mathbb{E}\left[\left(\sum_{h=0}^{\infty} \beta'\Upsilon_{\delta,h}\varepsilon_{t_{i-\bar{q}-\kappa-h}} + \beta' u_{t_{i-\bar{q}-\kappa}}\right)\left(\sum_{h'=0}^{\infty} \beta'\Upsilon_{\delta,h'}\varepsilon_{t_{i-1-h'}} + \beta' u_{t_{i-1}}\right)'\right], \\
&= \sum_{h=0}^{\infty}\sum_{h'=0}^{\infty} \beta'\Upsilon_{\delta,h}\mathbb{E}\left[\varepsilon_{t_{i-\bar{q}-\kappa-h}}\varepsilon'_{t_{i-1-h'}}\right]\Upsilon'_{\delta,h'}\beta + \sum_{h=0}^{\infty} \beta'\Upsilon_{\delta,h}\mathbb{E}\left[\varepsilon_{t_{i-\bar{q}-\kappa-h}}u'_{t_{i-1}}\right]\beta \\
&\quad + \sum_{h'=0}^{\infty} \beta'\mathbb{E}\left[u_{t_{i-\bar{q}-\kappa}}\varepsilon'_{t_{i-1-h'}}\right]\Upsilon'_{\delta,h'}\beta + \beta'\mathbb{E}\left[u_{t_{i-\bar{q}-\kappa}}u_{t_{i-1}}\right]\beta.
\end{aligned}$$

It is easy to appreciate that the second and fourth terms are equal to zero under Assumption MMN(TS).

Next, it follows from $\sum_{h=0}^{\infty} |\Upsilon_{\delta,h}| < \infty$, Assumption RROU, and Assumption MMN(TS) that

$$\begin{aligned}
\mathbb{E}\left[\beta' p_{t_{i-\bar{q}-\kappa}} (\beta' p_{t_{i-1}})'\right] &= \sum_{h=0}^{\infty} \beta' \Upsilon_{\delta,h} \mathbb{E}\left[\varepsilon_{t_{i-\bar{q}-\kappa-h}} \varepsilon'_{t_{i-\bar{q}-\kappa-h}}\right] \Upsilon'_{\delta,(\bar{q}+\kappa-1+h)} \beta \\
&\quad + \sum_{h=\bar{q}+\kappa-1}^{2\bar{q}+\kappa-1} \beta' \mathbb{E}\left[u_{t_{i-\bar{q}-\kappa}} \varepsilon'_{t_{i-1-h'}}\right] \Upsilon'_{\delta,h'} \beta, \\
&= \sum_{h=0}^{\infty} \beta' \Upsilon_{\delta,h} \Sigma_\delta \Upsilon'_{\delta,(\bar{q}+\kappa-1+h)} \beta + \sum_{h=\bar{q}+\kappa-1}^{2\bar{q}+\kappa-1} \beta' \rho \left(h - \bar{q} - \kappa + 1\right) \Upsilon'_{\delta,h} \beta < \infty,
\end{aligned}$$

and hence (ii) holds for any $2 \leq k \leq \bar{k}$, completing the proof. ■

Proof of Lemma 3 First, recall that $s_{t_i} = S_{t_i} + \upsilon_{t_i}$ and that, by Hansen's (2005) representation,

$$\widetilde{\beta}' s_{t_{i-1}} = \sum_{h=0}^{\infty} \widetilde{\beta}' \widetilde{\Upsilon}_{\delta,h} \epsilon_{t_{i-1-h}} + \widetilde{\beta}' \upsilon_{t_{i-1}}.$$

As before, we first establish validity. Using $\mathrm{v}_{t_i} = \varepsilon_{t_i} + \left[I_M - \left(\alpha_\delta \beta' + I_M\right) L\right] u_{t_i}$. It then follows that $\widetilde{Z}_{t_{i-1}} = \widetilde{\beta}' s_{t_{i-1}}$ reads

$$\begin{aligned}
\mathbb{E}\left[\widetilde{Z}_{t_{i-1}} \mathrm{v}'_{t_i}\right] &= \mathbb{E}\left[\left(\sum_{h=0}^{\infty} \widetilde{\beta}' \widetilde{\Upsilon}_{\delta,h} \epsilon_{t_{i-1-h}} + \widetilde{\beta}' \upsilon_{t_{i-1}}\right) \left(\varepsilon_{t_i} + u_{t_i} - \left(\alpha_\delta \beta' + I_M\right) u_{t_{i-1}}\right)'\right] \\
&= \sum_{h=0}^{\infty} \widetilde{\beta}' \widetilde{\Upsilon}_{\delta,h} \mathbb{E}\left[\epsilon_{t_{i-1-h}} \varepsilon'_{t_i}\right] + \sum_{h=0}^{\infty} \widetilde{\beta}' \widetilde{\Upsilon}_{\delta,h} \mathbb{E}\left[\epsilon_{t_{i-1-h}} u'_{t_i}\right] \\
&\quad - \sum_{h=0}^{\infty} \widetilde{\beta}' \widetilde{\Upsilon}_{\delta,h} \mathbb{E}\left[\epsilon_{t_{i-1-h}} u'_{t_{i-1}}\right] \left(\alpha_\delta \beta' + I_M\right)' + \widetilde{\beta}' \mathbb{E}\left[\upsilon_{t_{i-1}} \varepsilon'_{t_i}\right] + \widetilde{\beta}' \mathbb{E}\left[\upsilon_{t_{i-1}} u'_{t_i}\right] \\
&\quad - \widetilde{\beta}' \mathbb{E}\left[\upsilon_{t_{i-1}} u'_{t_{i-1}}\right] \left(\alpha_\delta \beta' + I_M\right)' = 0.
\end{aligned}$$

Assumption MMN(OTP) and Assumption MMN(CS) ensure that all of the above terms are equal to zero. As for relevance, the correlation between price differentials is

$$\begin{aligned}
\mathbb{E}\left[\widetilde{Z}_{t_{i-1}} (\beta' p_{t_{i-1}})'\right] &= \mathbb{E}\left[\left(\sum_{h=0}^{\infty} \widetilde{\beta}' \widetilde{\Upsilon}_{\delta,h} \epsilon_{t_{i-1-h}} + \widetilde{\beta}' \upsilon_{t_{i-1}}\right) \left(\sum_{h'=0}^{\infty} \beta' \Upsilon_{\delta,h'} \varepsilon_{t_{i-1-h'}} + \beta' u_{t_{i-1}}\right)'\right] \\
&= \sum_{h=0}^{\infty} \sum_{h'=0}^{\infty} \widetilde{\beta}' \widetilde{\Upsilon}_{\delta,h} \mathbb{E}\left[\epsilon_{t_{i-1-h}} \varepsilon'_{t_{i-1-h'}}\right] \Upsilon'_{\delta,h'} \beta + \sum_{h=0}^{\infty} \widetilde{\beta}' \widetilde{\Upsilon}_{\delta,h} \mathbb{E}\left[\epsilon_{t_{i-1-h}} u'_{t_{i-1}}\right] \beta \\
&\quad + \sum_{h'=0}^{\infty} \widetilde{\beta}' \mathbb{E}\left[\upsilon_{t_{i-1}} \varepsilon'_{t_{i-1-h'}}\right] \Upsilon'_{\delta,h'} \beta + \widetilde{\beta}' \mathbb{E}\left[\upsilon_{t_{i-1}} u'_{t_{i-1}}\right] \beta
\end{aligned}$$

The last three terms are equal to zero under Assumption MMN(CS), whereas the first term is nonzero as long as $\mathbb{E}[\epsilon_{j,t_i} \varepsilon_{m,t_i}] \neq \mathbb{E}[\epsilon_{j',t_i} \varepsilon_{m',t_i}]$ for some $1 \leq j \neq j' \leq J$ and $1 \leq m \neq m' \leq M$ and

$\boldsymbol{\Sigma}_\delta(0) = \mathbb{E}\big[\epsilon_{t_i}\varepsilon_{t_i}'\big] < \infty$. It then follows that as $\sum_{h=0}^{\infty}\left|\widetilde{\Upsilon}_{\delta,h}\right| < \infty$ and $\sum_{h=0}^{\infty}|\Upsilon_{\delta,h}| < \infty$ hold, the first term is finite, yielding $\mathbb{E}\left[\widetilde{Z}_{t_{i-1}}(\beta' p_{t_{i-1}})'\right] = \sum_{h=0}^{\infty}\widetilde{\beta}'\widetilde{\Upsilon}_{\delta,h}\boldsymbol{\Sigma}_\delta(0)\Upsilon_{\delta,h}'\beta < \infty$, as stated. ■

Proof of Lemma 4 We first establish that $\{\beta' p_{t_i}\}$ is a stationary ergodic sequence. Stationarity comes from its Gaussian VMA(∞) representation:

$$\beta' p_{t_i} = \sum_{h=0}^{\infty}\bar{\Upsilon}_h\varepsilon_{t_{i-h}} + \sum_{h=0}^{\bar{q}}\beta'\varpi_h\vartheta_{t_{i-h}} = \sum_{h=0}^{\bar{q}}\Theta_h d_{t_{i-h}} + \sum_{h=\bar{q}+1}^{\bar{c}-1}\bar{\Theta}_h d_{t_{i-h}} + \sum_{h=\bar{c}}^{\infty}\bar{\Theta}_h d_{t_{i-h}},$$

where $\bar{\Upsilon}_h = \beta'\Upsilon_{\delta,h}$, $\Theta_h = (\bar{\Upsilon}_h, \beta'\varpi_h)$ and $\bar{\Theta}_h = (\bar{\Upsilon}_h, \mathbf{0}_{R\times M})$ are $R\times 2M$ matrices, and $d_{t_i} = \mathrm{vec}\,(\varepsilon_{t_i}, \vartheta_{t_i})$ is a multivariate Gaussian white noise with dimension $2M\times 1$. Ergodicity comes from the fact that $\bar{\Theta}_h$ is absolutely summable in that $\lim_{\bar{c}\to\infty}\sum_{h=\bar{c}}^{\infty}\|\bar{\Theta}_h\|| = 0$. To appreciate why, notice that

$$\lim_{\bar{c}\to\infty}\sum_{h=\bar{c}}^{\infty}\|\bar{\Theta}_h\| \le \lim_{\bar{c}\to\infty}\sum_{h=\bar{c}}^{\infty}\sum_{h_1=1}^{R}\sum_{h_2=1}^{2M}|\bar{\Theta}_h^{(h_1,h_2)}| \le \lim_{\bar{c}\to\infty}\sum_{h=\bar{c}}^{\infty}\sum_{h_1=1}^{R}\sum_{h_2=1}^{M}|\bar{\Upsilon}_h^{(h_1,h_2)}|, \tag{31}$$

where $|\bar{\Theta}_h^{(h_1,h_2)}|$ and $|\bar{\Upsilon}_h^{(h_1,h_2)}|$ denote the absolute values of the (h_1,h_2)th elements of $\bar{\Theta}_h$ and $\bar{\Upsilon}_h$, respectively. Assumption RROU ensures that $\bar{\Upsilon}_h$ is absolutely summable for $h \ge \bar{c}+1$, and so $\lim_{\bar{c}\to\infty}\sum_{h=\bar{c}+1}^{\infty}\sum_{h_1}^{R}\sum_{h_2}^{M}|\bar{\Upsilon}_h^{(h_1,h_2)}| = 0$. As for $\mathrm{v}_{t_i} = \varepsilon_{t_i} + [I_M - (\alpha_\delta\beta' + I_M)L]u_{t_i}$, it is a Gaussian VMA($\bar{q}+1$) process under Assumption MMN(TS), and hence stationary and ergodic.

We now turn our attention to $\widetilde{\beta}' s_{t_i} = \widetilde{\beta}' S_{t_i} + \widetilde{\beta}'\upsilon_{t_i}$ under Assumption MMN(CS). By applying the GRT, we find that $\widetilde{\beta}' s_{t_i} = \sum_{h=0}^{\bar{c}-1}\widetilde{\Theta}_h\widetilde{d}_{t_{i-h}} + \sum_{h=\bar{c}}^{\infty}\widetilde{\Theta}_h\widetilde{d}_{t_{i-h}}$ is a stationary Gaussian VMA(∞) process, where $\widetilde{\Theta}_h = (\widetilde{\beta}'\widetilde{\Upsilon}_{\delta,h}, \widetilde{\beta}'\varrho_h)$ is a $(J-1)\times 2J$ matrix and $\widetilde{d}_{t_i} = \mathrm{vec}\,(\epsilon_{t_i}, \upsilon_{t_i})$ is a $2J\times 1$ vector. In addition, $\widetilde{\beta}' s_{t_{i-1}}$ is also ergodic given that $\lim_{\bar{c}\to\infty}\sum_{h=\bar{c}}^{\infty}\|\widetilde{\Theta}_h\| = 0$. As before, we show the latter by bounding its argument from above:

$$\lim_{\bar{c}\to\infty}\sum_{h=\bar{c}}^{\infty}\|\widetilde{\Theta}_h\| \le \lim_{\bar{c}\to\infty}\sum_{h=\bar{c}}^{\infty}\sum_{h_1=1}^{J-1}\sum_{h_2=1}^{J}\left|\left(\widetilde{\beta}'\widetilde{\Upsilon}_{\delta,h}\right)^{(h_1,h_2)}\right| + \lim_{\bar{c}\to\infty}\sum_{h=\bar{c}}^{\infty}\sum_{h_1=1}^{J-1}\sum_{h_2=1}^{J}\left|\left(\widetilde{\beta}'\varrho_h\right)^{(h_1,h_2)}\right|. \tag{32}$$

Assumptions OTP and MMN(CS) respectively impose that $\widetilde{\beta}'\widetilde{\Upsilon}_{\delta,h}$ and $\widetilde{\beta}'\varrho_h$ are both absolutely summable, so that the limit in (32) equals zero. As for v_{t_i}, under Assumption MMN(CS), it is a Gaussian VMA(∞) process with absolutely summable coefficients: $\mathrm{v}_{t_i} = (I_M, I_M)d_{t_i} + \sum_{h=1}^{\infty}\bar{\Theta}_h d_{t_{i-h}}$, where $\bar{\Theta}_h = \big(0_{M\times M}, \varpi_h - (I_M + \alpha_\delta\beta')\varpi_{h-1}\big)$ is an $M\times 2M$ matrix. As before, $\bar{\Theta}_h$ inherits its absolutely summability from ϖ_h:

$$\lim_{\bar{c}\to\infty}\sum_{h'=\bar{c}}^{\infty}\|\bar{\Theta}_h\| \le \lim_{\bar{c}\to\infty}\sum_{h=\bar{c}}^{\infty}\sum_{h_1=1}^{M}\sum_{h_2=1}^{M}|\varpi_h^{(h_1,h_2)}| + \lim_{\bar{c}\to\infty}\sum_{h=\bar{c}}^{\infty}\sum_{h_1=1}^{M}\sum_{h_2=1}^{M}\sum_{h_3=1}^{M}|D_\delta^{(h_1,h_3)}|\,|\varpi_h^{(h_3,h_2)}| = 0,$$

where $\varpi_h^{(h_3,h_2)}$ and $D_\delta^{(h_1,h_3)}$ are the (h_3,h_2)th and (h_1,h_3)th elements of ϖ_h and $D_\delta = (I_M + \alpha_\delta\beta')$, respectively.

The next step is to show that $\{Z_{m',t_{i-\bar{q}-\kappa}}\mathrm{v}_{m,t_i}, \mathcal{F}_{t_i}\}$ is a stationary ergodic L^1-mixingale sequence: namely, $\mathbb{E}\Big\|\mathbb{E}\left[Z_{m',t_{i-\bar{q}-\kappa}}\mathrm{v}_{m,t_i}|\mathcal{F}_{t_{i-c}}\right]\Big\| \le f_{t_i}\xi_c$ with $\{f_{t_i}\xi_c\}$ being deterministic sequences and $\lim_{c\to\infty}\xi_c = 0$ for every $m' = 1,\dots,R$ and $m = 1,\dots,M$. We first apply the GRT to obtain the m'th row of $Z_{t_{i-\bar{q}-\kappa}}$: $\beta'_{m'}p_{t_{i-\bar{q}-\kappa}} = \sum_{h=0}^{\infty}\beta'_{m'}\Upsilon_{\delta,h}\varepsilon_{t_{i-\bar{q}-\kappa-h}} + \beta'_{m'}u_{t_{i-\bar{q}-\kappa-h}}$ for $1 \le m' \le M-1$ and $k \le \kappa \le \bar{k}$. Note that the sequence $\beta'_{m'}\Upsilon_{\delta,h}$ is absolutely summable. Under Assumption MMN(TS), it follows that $\mathbb{E}\big[\beta'_{m'}p_{t_{i-\bar{q}-\kappa}}\mathrm{v}_{m,t_i}|\mathcal{F}_{t_{i-c}}\big] = 0$ for $c > \bar{q}+1$, so that $\xi_c = 0$. Now, for $c = \bar{q}+1$,

$$\mathbb{E}\big[\beta'_{m'}p_{t_{i-\bar{q}-\kappa}}\mathrm{v}_{m,t_i}|\mathcal{F}_{t_{i-c}}\big] = \sum_{h=0}^{\infty}\beta'_{m'}\Upsilon_{\delta,h}\varepsilon_{t_{i-\bar{q}-\kappa-h}}(-\varpi_{m,\bar{q}}\vartheta_{t_{i-1-\bar{q}}} - \alpha_{\delta,m}\beta'\varpi_{\bar{q}}\vartheta_{t_{i-1-\bar{q}}})$$
$$+ \sum_{h=0}^{\bar{q}}\beta'_{m'}\varpi_h\vartheta_{t_{i-1-\bar{q}-\kappa}}(-\varpi_{m,\bar{q}}\vartheta_{t_{i-1-\bar{q}}} - \alpha_{\delta,m}\beta'\varpi_{\bar{q}}\vartheta_{t_{i-1-\bar{q}}}), \qquad (33)$$

where $\alpha_{\delta,m}$ denotes the mth row of α_δ. Given that $\sum_{h=0}^{\infty}\|\beta'_{m'}\Upsilon_{\delta,h}\| < \infty$ due to the absolute summability of $\beta'_{m'}\Upsilon_{\delta,h}$, we can interchange the order of the expectation operator to get

$$\mathbb{E}\Big\|\mathbb{E}\big[\beta'_{m'}p_{t_{i-\bar{q}-\kappa}}\mathrm{v}_{m,t_i}|\mathcal{F}_{t_{i-c}}\big]\Big\| \le \sum_{h=0}^{\infty}\|\,\beta'_{m'}\Upsilon_{\delta,h}\|\,\|\varpi'_{m,\bar{q}}\|\,\mathbb{E}\big\|\vartheta'_{t_{i-1-\bar{q}}}\otimes\varepsilon_{t_{i-\bar{q}-\kappa-h}}\big\|$$
$$+ \sum_{h=0}^{\infty}\|\beta'_{m'}\Upsilon_{\delta,h}\|\,\|\varpi'_{\bar{q}}\beta\alpha'_{\delta,m}\|\,\mathbb{E}\big\|\vartheta'_{t_{i-1-\bar{q}}}\otimes\varepsilon_{t_{i-\bar{q}-\kappa-h}}\big\|$$
$$+ \sum_{h=0}^{\bar{q}}\|\beta'_{m'}\varpi_h\|\,\|\varpi'_{m,\bar{q}}\|\,\mathbb{E}\big\|\vartheta'_{m,t_{i-1-\bar{q}}}\otimes\vartheta_{t_{i-\bar{q}-\kappa-h}}\big\|$$
$$+ \sum_{h=0}^{\bar{q}}\|\beta'_{m'}\varpi_h\|\,\|\varpi'_{\bar{q}}\beta\alpha'_{\delta,m}\|\|\,\mathbb{E}\big\|\vartheta'_{t_{i-1-\bar{q}}}\otimes\vartheta_{t_{i-\bar{q}-\kappa-h}}\big\|.$$

All terms on the right-hand side involve expectations of products of Gaussian variables, so that

$$\mathbb{E}\Big\|\mathbb{E}\big[\beta'_{m'}p_{t_{i-\bar{q}-\kappa}}\mathrm{v}_{m,t_i}|\mathcal{F}_{t_{i-c}}\big]\Big\| \le O\left(\sum_{h=0}^{\infty}\|\beta'_{m'}\Upsilon_{\delta,h}\|\right) + O\left(\sum_{h=0}^{\bar{q}}\|\beta'_{m'}\varpi_h\|\right) \le O\left(\sum_{h=0}^{\infty}\|\beta'_{m'}\Upsilon_{\delta,h}\|\right),$$

It remains to demonstrate that $\mathbb{E}\Big\|\mathbb{E}\big[\beta'_{m'}p_{t_{i-\bar{q}-\kappa}}\mathrm{v}_{m,t_i}|\mathcal{F}_{t_{i-c}}\big]\Big\| < \infty$ for $1 \le c \le \bar{q}$. Finally, for

$1 \leq c \leq \bar{q}$, it holds that

$$\mathbb{E}\Big\|\mathbb{E}\big[\beta'_{m'}p_{t_{i-\bar{q}-\kappa}}\mathrm{v}_{m,t_i}|\mathcal{F}_{t_{i-c}}\big]\Big\| = \left(\sum_{h=0}^{\infty}\beta'_{m'}\Upsilon_{\delta,h}\varepsilon_{t_{i-\bar{q}-\kappa-h}} + \sum_{h=0}^{\bar{q}}\beta'_{m'}\varpi_h\vartheta_{t_{i-1-\bar{q}-\kappa-h}}\right)$$
$$\times\left(\sum_{h_1=c}^{\bar{q}}\varpi_{m,h_1}\vartheta_{t_{i-h_1}} - \sum_{h_1=c-1}^{\bar{q}}\varpi_{m,h_1}\vartheta_{t_{i-1-h_1}}\right.$$
$$\left. - \sum_{h_1=c-1}^{\bar{q}}\varpi_{h_1}\alpha_{\delta,m}\beta'\varpi_{h_1}\vartheta_{t_{i-1-h_1}}\right),$$

so that

$$\mathbb{E}\Big\|\mathbb{E}\big[\beta'_{m'}p_{t_{i-\bar{q}-\kappa}}\mathrm{v}_{m,t_i}|\mathcal{F}_{t_{i-c}}\big]\Big\| \leq \sum_{h=0}^{\infty}\sum_{h_1=c}^{\bar{q}}\|\beta'_{m'}\Upsilon_{\delta,h}\|\,\|\varpi'_{m,h_1}\|\,\mathbb{E}\Big\|\vartheta'_{t_{i-h_1}}\otimes\varepsilon_{t_{i-\bar{q}-\kappa-h}}\Big\|$$
$$+\sum_{h=0}^{\infty}\sum_{h_1=c-1}^{\bar{q}}\|\beta'_{m'}\Upsilon_{\delta,h}\|\,\|\varpi'_{m,h_1}\|\,\mathbb{E}\Big\|\vartheta'_{t_{i-1-h_1}}\otimes\varepsilon_{t_{i-\bar{q}-\kappa-h}}\Big\|$$
$$+\sum_{h=0}^{\infty}\sum_{h_1=c-1}^{\bar{q}}\|\beta'_{m'}\Upsilon_{\delta,h}\|\,\|\varpi'_{h_1}\beta\alpha'_{\delta,m}\|\,\mathbb{E}\Big\|\vartheta'_{t_{i-1-h_1}}\otimes\varepsilon_{t_{i-\bar{q}-\kappa-h}}\Big\|$$
$$+\sum_{h=0}^{\bar{q}}\sum_{h_1=c}^{\bar{q}}\|\beta'_{m'}\varpi_h\|\,\|\varpi'_{m,h_1}\|\,\mathbb{E}\Big\|\vartheta'_{t_{i-h_1}}\otimes\vartheta_{t_{i-\bar{q}-\kappa-h}}\Big\|$$
$$+\sum_{h=0}^{\bar{q}}\sum_{h_1=c-1}^{\bar{q}}\|\beta'_{m'}\varpi_h\|\,\|\varpi'_{m,h_1}\|\,\mathbb{E}\Big\|\vartheta'_{t_{i-1-h_1}}\otimes\vartheta_{t_{i-\bar{q}-\kappa-h}}\Big\|$$
$$+\sum_{h=0}^{\bar{q}}\sum_{h_1=c-1}^{\bar{q}}\|\beta'_{m'}\varpi_h\|\,\|\varpi'_{h_1}\beta\alpha'_{\delta,m}\|\,\mathbb{E}\Big\|\vartheta'_{t_{i-1-h_1}}\otimes\vartheta_{t_{i-\bar{q}-\kappa-h}}\Big\|.$$

Using the same argument as before, this means that

$$\mathbb{E}\Big\|\mathbb{E}\big[\beta'_{m'}p_{t_{i-\bar{q}-\kappa}}\mathrm{v}_{m,t_i}|\mathcal{F}_{t_{i-c}}\big]\Big\| \leq O\left(\sum_{h=0}^{\infty}\|\beta'_{m'}\Upsilon_{\delta,h}\|\right)+O\left(\sum_{h=0}^{\bar{q}}\|\beta'_{m'}\varpi_h\|\right) \leq O\left(\sum_{h=0}^{\infty}\|\beta'_{m'}\Upsilon_{\delta,h}\|\right) < \infty,$$

given that $\sum_{h=0}^{\infty}\|\beta'_{m'}\Upsilon_{\delta,h}\| < \infty$.

We next turn our attention to show that $\{Z_{j,t_{i-1}}\mathrm{v}_{m,t_i},\mathcal{F}_{t_i}\}$ is a stationary ergodic adapted L^1-mixingale sequence: namely, $\mathbb{E}\Big\|\mathbb{E}\big[\widetilde{\beta}'_j s_{t_{i-1}}\mathrm{v}_{m,t_i}|\mathcal{F}_{t_{i-c}}\big]\Big\| \leq f_{t_i}\xi_c$ with $\lim_{c\to\infty}\xi_c = 0$, for every $j =$

$1, \ldots, J-1$ and $m = 1, \ldots, M$. We start with

$$
\begin{aligned}
\mathbb{E}\big[\widetilde{\beta}_j' s_{t_{i-1}} \mathrm{v}_{m,t_i} | \mathcal{F}_{t_{i-c}}\big] &= \mathbb{E}\big[\sum_{h=0}^{\infty} \widetilde{\beta}_j' \widetilde{\Upsilon}_{\delta,h} \epsilon_{t_{i-1-h}} \mathrm{v}_{m,t_i} | \mathcal{F}_{t_{i-c}}\big] + \mathbb{E}\big[\widetilde{\beta}_j' \upsilon_{t_{i-1}} \mathrm{v}_{m,t_i} | \mathcal{F}_{t_{i-c}}\big], \\
&= \sum_{h=c-1}^{\infty} \sum_{h_1=c-1}^{\infty} \widetilde{\beta}_j' \widetilde{\Upsilon}_{\delta,h} \epsilon_{t_{i-1-h}} \big(\varpi_{m,h_1+1} \vartheta_{t_{i+1-h_1}} - \varpi_{m,h_1} \vartheta_{t_{i-h_1}} - \alpha_{\delta,m} \beta' \varpi_{h_1} \vartheta_{t_{i-h_1}}\big) \\
&\quad + \sum_{h=c-1}^{\infty} \sum_{h_1=c-1}^{\infty} \widetilde{\beta}_j' \varrho_h \varphi_{t_{i-1-h}} \big(\varpi_{m,h_1+1} \vartheta_{t_{i+1-h_1}} - \varpi_{m,h_1} \vartheta_{t_{i-h_1}} - \alpha_{\delta,m} \beta' \varpi_{h_1} \vartheta_{t_{i-h_1}}\big).
\end{aligned}
$$

implying $\mathbb{E}\Big\|\mathbb{E}\big[\widetilde{\beta}_j' s_{t_{i-1}} \mathrm{v}_{m,t_i} | \mathcal{F}_{t_{i-c}}\big]\Big\| \le \mathbb{E}\Big\|\mathbb{E}\big[\sum_{h=0}^{\infty} \widetilde{\beta}_j' \widetilde{\Upsilon}_{\delta,h} \epsilon_{t_{i-1-h}} \mathrm{v}_{m,t_i} | \mathcal{F}_{t_{i-c}}\big]\Big\| + \mathbb{E}\Big\|\mathbb{E}\big[\widetilde{\beta}_j' \upsilon_{t_{i-1}} \mathrm{v}_{m,t_i} | \mathcal{F}_{t_{i-c}}\big]\Big\|$. The first term reads

$$
\begin{aligned}
\mathbb{E}\Big\|\mathbb{E}\Big[\sum_{h=0}^{\infty} \widetilde{\beta}_j' \widetilde{\Upsilon}_{\delta,h} \epsilon_{t_{i-1-h}} \mathrm{v}_{m,t_i} | \mathcal{F}_{t_{i-c}}\Big]\Big\| &\le \sum_{h=c-1}^{\infty} \sum_{h_1=c-1}^{\infty} \|\widetilde{\beta}_j' \widetilde{\Upsilon}_{\delta,h}\| \, \|\varpi_{m,h_1+1}'\| \, \mathbb{E}\|\vartheta_{t_{i+1-h_1}}' \otimes \epsilon_{t_{i-1-h}}\| \\
&\quad + \sum_{h=c-1}^{\infty} \sum_{h_1=c-1}^{\infty} \|\widetilde{\beta}_j' \widetilde{\Upsilon}_{\delta,h}\| \, \|\varpi_{m,h_1}'\| \, \mathbb{E}\|\vartheta_{t_{i-h_1}}' \otimes \epsilon_{t_{i-1-h}}\| \\
&\quad + \sum_{h=c-1}^{\infty} \sum_{h_1=c-1}^{\infty} \|\widetilde{\beta}_j' \widetilde{\Upsilon}_{\delta,h}\| \, \|\varpi_{h_1}' \beta \alpha_{\delta,m}'\| \, \mathbb{E}\|\vartheta_{t_{i-h_1}}' \otimes \epsilon_{t_{i-1-h}}\|.
\end{aligned}
$$

Because $\widetilde{\Upsilon}_{\delta,h}$ and ϖ_h are absolutely summable, there exists an absolutely summable sequence $\bar{D}_{1,h}$ such that $\mathbb{E}\Big\|\mathbb{E}\big[\sum_{h=0}^{\infty} \widetilde{\beta}_j' \widetilde{\Upsilon}_{\delta,h} \epsilon_{t_{i-1-h}} \mathrm{v}_{m,t_i} | \mathcal{F}_{t_{i-c}}\big]\Big\| \le O\big(\sum_{h=c-1}^{\infty} \|\bar{D}_{1,h}\|\big)$. Similarly, there exists an absolutely summable sequence $\bar{D}_{2,h}$ such that $\mathbb{E}\Big\|\mathbb{E}\big[\widetilde{\beta}_j' \upsilon_{t_{i-1}} \mathrm{v}_{m,t_i} | \mathcal{F}_{t_{i-c}}\big]\Big\| \le O\big(\sum_{h=c-1}^{\infty} \|\bar{D}_{2,h}\|\big)$. This means there exists an absolutely summable sequence $\bar{D}_h$ such that $\mathbb{E}\Big\|\mathbb{E}\big[\widetilde{\beta}_j' s_{t_{i-1}} \mathrm{v}_{m,t_i} | \mathcal{F}_{t_{i-c}}\big]\Big\|$ is at most of order $O\big(\sum_{h=c-1}^{\infty} \|\bar{D}_h\|\big)$, yielding the desired result with $\xi_c = \sum_{h=c-1}^{\infty} \|\bar{D}_h\|$, given that $\lim_{c\to\infty} \xi_c = 0$.

It now remains to establish uniform integrability. By Hamilton's (1994) Proposition 7.7, it suffices to prove that $\varsigma_{Z\mathrm{v}}^2 \equiv \mathbb{E}\big[\beta_{m'}' p_{t_{i-\bar{q}-\kappa}} \mathrm{v}_{m,t_i} \mathrm{v}_{m,t_i}' p_{t_{i-\bar{q}-\kappa}}' \beta_{m'}\big] < \infty$ under Assumption MMN(TS) and that $\varsigma_{\widetilde{Z}\mathrm{v}}^2 \equiv \mathbb{E}\big[\widetilde{\beta}_j' s_{t_{i-1}} \mathrm{v}_{m,t_i} \mathrm{v}_{m,t_i}' s_{t_{i-1}}' \widetilde{\beta}_j\big] < \infty$ under Assumption(CS). As for the former,

$$
\begin{aligned}
\varsigma_{Z\mathrm{v}}^2 = &\sum_{h=0}^{\infty} \sum_{h_1=0}^{\infty} \beta_{m'}' \Upsilon_{\delta,h} \, \mathbb{E}\Big[\varepsilon_{t_{i-\bar{q}-\kappa-h}} \mathbb{E}_{t_{i-\bar{q}-\kappa}}\big[\mathrm{v}_{m,t_i} \mathrm{v}_{m,t_i}'\big] \varepsilon_{t_{i-\bar{q}-\kappa-h_1}}'\Big] \Upsilon_{\delta,h_1}' \beta_{m'} \\
&+ \sum_{h=0}^{\infty} \beta_{m'}' \Upsilon_{\delta,h} \, \mathbb{E}\Big[\varepsilon_{t_{i-\bar{q}-\kappa-h}} \mathbb{E}_{t_{i-\bar{q}-\kappa}}\big[\mathrm{v}_{m,t_i} \mathrm{v}_{m,t_i}'\big] u_{t_{i-\bar{q}-\kappa}}'\Big] \beta_{m'} \\
&+ \sum_{h=0}^{\infty} \beta_{m'}' \mathbb{E}\Big[u_{t_{i-\bar{q}-\kappa}} \mathbb{E}_{t_{i-\bar{q}-\kappa}}\big[\mathrm{v}_{m,t_i} \mathrm{v}_{m,t_i}'\big] \varepsilon_{t_{i-\bar{q}-\kappa-h}}'\Big] \Upsilon_{\delta,h}' \beta_{m'} \\
&+ \beta_{m'}' \mathbb{E}\Big[u_{t_{i-\bar{q}-\kappa}} \mathbb{E}_{t_{i-\bar{q}-\kappa}}\big[\mathrm{v}_{m,t_i} \mathrm{v}_{m,t_i}'\big] u_{t_{i-\bar{q}-\kappa}}'\Big] \beta_{m'}.
\end{aligned}
$$

As v_{m,t_i} is a Gaussian MA$(\bar{q}+1)$ process, $\sigma^2_{\mathrm{v},\bar{q}+1} \equiv \mathbb{E}_{t_{i-\bar{q}-\kappa}}\left[\mathrm{v}_{m,t_i}\mathrm{v}'_{m,t_i}\right]$ is a finite constant, so that

$$\varsigma^2_{Z\mathrm{v}} = \sum_{h=0}^{\infty} \sigma^2_{\mathrm{v},\bar{q}+1}\beta'_{m'}\Upsilon_{\delta,h}\Sigma_\delta\Upsilon'_{\delta,h'}\beta_{m'} + \sum_{h=0}^{\bar{q}} \sigma^2_{\mathrm{v},\bar{q}+1}\beta'_{m'}\Upsilon_{\delta,h}\Gamma_{\varepsilon,u}(h)\beta_{m'} + \sum_{h=0}^{\bar{q}} \sigma^2_{\mathrm{v},\bar{q}+1}\beta'_{m'}\Gamma_{\varepsilon,u}(h)\Upsilon'_{\delta,h}\beta_{m'}$$
$$+ \sigma^2_{\mathrm{v},\bar{q}+1}\beta'_{m'}\Omega_u\beta_{m'},$$

where $\Omega_u = \mathbb{E}[u_{t_i}u'_{t_i}] < \infty$, $\Gamma_{\varepsilon,u}(h) = \mathbb{E}[\varepsilon_{t_{i-h}}u'_{t_i}] < \infty$ for $0 \leq h \leq \bar{q}$ and $\Gamma_{\varepsilon,u}(h) = \mathbf{0}_{M\times M}$ for $h > \bar{q}$. Given the last three terms involve finite sums, there exist finite constants D_1 and D such that

$$\varsigma^2_{Z\mathrm{v}} \leq \sum_{h=0}^{\infty} \|\sigma^2_{\mathrm{v},\bar{q}+1}\|\, \|\beta'_{m'}\Upsilon_{\delta,h}\| \|\Sigma_\delta\|\, \|\Upsilon'_{\delta,h_1}\beta_{m'}\| + D_1 \leq \sum_{h=0}^{\infty} \|\beta'_{m'}\Upsilon_{\delta,h}\|\, \|\Upsilon'_{\delta,h_1}\beta_{m'}\| + D < \infty. \tag{34}$$

It then follows from $\sum_{h=0}^{\infty}\|\beta'_{m'}\Upsilon_{\delta,h}\| < \infty$ that $\left\{\beta'_{m'}p_{t_{i-\bar{q}-\kappa}}\mathrm{v}_{m,t_i}, \mathcal{F}_{t_i}\right\}$ is uniformly integrable under Assumption MMN(TS). The uniform integrability of $\left\{\widetilde{\beta}'_j s_{t_{i-1}}\mathrm{v}_{m,t_i}, \mathcal{F}_{t_i}\right\}$ follows along similar lines under Assumption MMN(CS). ■

Appendix B Bias of the LS estimator

In this appendix, we derive the bias of the LS estimator in the presence of a second layer of microstructure noise. To simplify matters, we consider only $M = 2$ markets and focus on the estimation of $\alpha_{\delta,1}$ in (12), that is we treat the number of intraday observations n as fixed, meaning that the total number of observations is $T = nD$. The LS estimation returns

$$\hat{\alpha}_{\delta,1} = \frac{\sum_{\tau=1}^{T}(p_{1,\tau-1}-p_{2,\tau-1})\Delta p_{1,\tau}}{\sum_{\tau=1}^{T}(p_{1,\tau-1}-p_{2,\tau-1})^2} = \frac{\frac{1}{T}\sum_{\tau=1}^{T}(p_{1,\tau-1}-p_{2,\tau-1})\left[\alpha_{\delta,1}(p_{1,\tau-1}-p_{2,\tau-1})+\mathrm{v}_{1,\tau}\right]}{\frac{1}{T}\sum_{\tau=1}^{T}(p_{1,\tau-1}-p_{2,\tau-1})^2}$$
$$= \alpha_{\delta,1} + \frac{\frac{1}{T}\sum_{\tau=1}^{T}(p_{1,\tau-1}-p_{2,\tau-1})\mathrm{v}_{1,\tau}}{\frac{1}{T}\sum_{\tau=1}^{T}(p_{1,\tau-1}-p_{2,\tau-1})^2}.$$

It follows from $\mathrm{v}_{1,\tau} = \Delta u_{1,\tau} + \varepsilon_{1,\tau} - \alpha_{\delta,1}(u_{1,\tau-1}-u_{2,\tau-1})$ and from the fact that the mean squared difference in observed prices converges in probability to some positive finite value Q that

$$\operatorname*{plim}_{T\to\infty}\hat{\alpha}_{\delta,1} = \alpha_{\delta,1} + Q^{-1}\operatorname*{plim}_{T\to\infty}\frac{1}{T}\sum_{\tau=1}^{T}(p_{1,\tau-1}-p_{2,\tau-1})\mathrm{v}_{1,\tau}$$
$$= \alpha_{\delta,1} - \alpha_{\delta,1}\,Q^{-1}\operatorname*{plim}_{T\to\infty}\frac{1}{T}\sum_{\tau=1}^{T}(p_{1,\tau-1}-p_{2,\tau-1})(u_{1,\tau-1}-u_{2,\tau-1})$$
$$+ Q^{-1}\operatorname*{plim}_{T\to\infty}\frac{1}{T}\sum_{\tau=1}^{T}(p_{1,\tau-1}-p_{2,\tau-1})\Delta u_{1,\tau} + Q^{-1}\operatorname*{plim}_{T\to\infty}\frac{1}{T}\sum_{\tau=1}^{T}(p_{1,\tau-1}-p_{2,\tau-1})\varepsilon_{1,\tau}.$$

We next consider each term individually. Let

$$B_{p,u}(T) = \frac{1}{T}\sum_{\tau=1}^{T}(p_{1,\tau-1} - p_{2,\tau-1})(u_{1,\tau-1} - u_{2,\tau-1})$$
$$= \frac{1}{T}\sum_{\tau=1}^{T}(P_{1,\tau-1} - P_{2,\tau-1})(u_{1,\tau-1} - u_{2,\tau-1}) + \frac{1}{T}\sum_{\tau=1}^{T}(u_{1,\tau-1} - u_{2,\tau-1})^2.$$

Hansen's (2005) Corollary 2 offers the following alternative representation to (4):

$$P_\tau = \Xi_\delta \sum_{h=1}^{\tau}\varepsilon_h + \sum_{h=0}^{\infty}\Upsilon^*_{\delta,h}\varepsilon_{\tau-h} + \Xi_\delta\, P_0,$$

with $\Xi_\delta = \beta_\perp(\alpha'_{\delta\perp}\beta_\perp)^{-1}\alpha'_{\delta\perp}$ and $\Upsilon^*_{\delta,h} = (I_2 - \Xi_\delta)(I_2 + \alpha_\delta\beta')^h$. This means that

$$P_{1,\tau} - P_{2,\tau} = (\Xi_{\delta,1} - \Xi_{\delta,2})\sum_{h=1}^{\tau}\varepsilon_h + \sum_{h=0}^{\infty}(\Upsilon^*_{\delta,1,h} - \Upsilon^*_{\delta,2,h})\varepsilon_{\tau-h} + (\Xi_{\delta,1} - \Xi_{\delta,2})P_0$$
$$= \sum_{h=0}^{\infty}(\Upsilon^*_{\delta,1,h} - \Upsilon^*_{\delta,2,h})\varepsilon_{\tau-h} = \sum_{h=0}^{\infty}\bar{\Upsilon}_{\delta,h}\varepsilon_{\tau-h}$$

given that $\Xi_{\delta,1} = \Xi_{\delta,2}$. This means that

$$\operatorname*{plim}_{T\to\infty} B_{p,u}(T) = \mathbb{E}\left[\sum_{h=0}^{\infty}\bar{\Upsilon}_{\delta,h}\varepsilon_{\tau-h-1}(u_{1,\tau-1} - u_{2,\tau-1})\right] + \mathbb{E}[u_{1,\tau-1} - u_{2,\tau-1}]^2$$
$$= \sum_{h=0}^{\infty}\bar{\Upsilon}_{\delta,h}[\rho_1(h) - \rho_2(h)] + \omega_1^2 + \omega_2^2 - 2\,\omega_{12},$$

where $\rho_j(h) = \mathbb{E}[\varepsilon_{\tau-h}u_{j,\tau}]$ for $j = 1, 2$ and $\omega_{12} = \mathbb{E}[u_{1,\tau}u_{2,\tau}]$. The first component reflects the dependence between the efficient price and microstructure noises, whereas the remaining terms correspond to the extra variation due to the presence of market microstructure noises. In turn,

$$\operatorname*{plim}_{T\to\infty} B_{p,\Delta u}(T) = \mathbb{E}[(p_{1,\tau-1} - p_{2,\tau-1})\Delta u_{1,\tau}]$$
$$= \mathbb{E}\big[(P_{1,\tau-1} - P_{2,\tau-1})\Delta u_{1,\tau}\big] + \mathbb{E}\big[(u_{1,\tau-1} - u_{2,\tau-1})\Delta u_{1,\tau}\big]$$
$$= \sum_{h=0}^{\infty}\bar{\Upsilon}_{\delta,h}\mathbb{E}\big[\varepsilon_{\tau-h-1}\Delta u_{1,\tau}\big] + \mathbb{E}\big[(u_{1,\tau-1} - u_{2,\tau-1})\Delta u_{1,\tau}\big]$$
$$= \sum_{h=0}^{\infty}\bar{\Upsilon}_{\delta,h}[\rho_1(h+1) - \rho_1(h)] + \omega_{12} - \omega_1^2 + [\gamma_{11}(1) - \gamma_{12}(1)],$$

where $\gamma_{1j}(1) = \mathbb{E}[u_{1,\tau}u_{j,\tau-1}]$ for $j = 1, 2$. The last term in brackets captures an additional source of bias due to cross-autocorrelation in the microstructure noises. Lastly,

$$\operatorname*{plim}_{T\to\infty} B_{p,\varepsilon}(T) = \mathbb{E}\big[(p_{1,\tau-1} - p_{2,\tau-1})\varepsilon_{1,\tau}\big] = \mathbb{E}\big[(P_{1,\tau-1} - P_{2,\tau-1})\varepsilon_{1,\tau}\big] + \mathbb{E}\big[(u_{1,\tau-1} - u_{2,\tau-1})\varepsilon_{1,\tau}\big]$$
$$= \sum_{h=0}^{\infty}\bar{\Upsilon}_{\delta,h}\mathbb{E}\big[\varepsilon_{\tau-h-1}\varepsilon_{1,\tau}\big] + \mathbb{E}\big[u_{1,\tau-1}\varepsilon_{1,\tau}\big] - \mathbb{E}\big[u_{2,\tau-1}\varepsilon_{1,\tau}\big] = 0$$

given that price innovations are white noises that may lead, but not lag microstructure noises. The overall LS bias thus amounts to

$$\begin{aligned}\operatorname*{plim}_{T\to\infty}(\hat{\alpha}_{\delta,1}-\alpha_{\delta,1}) &= Q^{-1}\left\{\sum_{h=0}^{\infty}\bar{\Upsilon}_{\delta,h}[\rho_1(h+1)-\rho_1(h)]+\gamma_{11}(1)-\gamma_{12}(1)+\omega_{12}-\omega_1^2\right\}\\ &\quad -\alpha_{\delta,1}\,Q^{-1}\left\{\sum_{h=0}^{\infty}\bar{\Upsilon}_{\delta,h}[\rho_1(h)-\rho_2(h)]+\omega_1^2+\omega_2^2-2\,\omega_{12}\right\}. \end{aligned} \tag{35}$$

To sum up, (35) unveils four sources of bias: (1) dependence between the efficient price and market microstructure noises; (2) variation in market microstructure noises; (3) correlation between microstructure noises across markets; and (4) persistence in microstructure noises.

Finally, we obtain a similar result to the overall variation in the observed price differential:

$$Q=\mathbb{E}\big[p_{1,\tau}-p_{2,\tau}\big]^2=\mathbb{E}\big[P_{1,\tau}-P_{2,\tau}+u_{1,\tau}-u_{2,\tau}\big]^2=\mathbb{E}\left[\sum_{h=0}^{\infty}\bar{\Upsilon}_{\delta,h}\varepsilon_{\tau-h}+u_{1,\tau}-u_{2,\tau}\right]^2$$

$$=\sum_{h=0}^{\infty}\bar{\Upsilon}_{\delta,h}\Sigma_\delta\bar{\Upsilon}'_{\delta,h}+\omega_1^2+\omega_2^2-2\,\omega_{12}+2\sum_{h=0}^{\infty}\bar{\Upsilon}_{\delta,h}[\rho_1(h)-\rho_2(h)].$$

The first term corresponds to the genuine variation in the price differential, that is to say, what we would observe without any market microstructure contamination. The presence of microstructure noises gives way to the omega terms, whereas the last component arises if they display serial correlation.

B.1 Independent microstructure (white) noises

In this section, we entertain the unrealistic (and not very interesting) case in which $u_{1,\tau},\ldots,u_{M,\tau}$ are independent white noise processes with mean zero and covariance matrix $\operatorname{diag}(\omega_1^2,\ldots,\omega_M^2)$. As such, $\omega_{m_1m_2}=0$ for $1\le m_1\ne m_2\le M$ and $\gamma_{m_1m_2}(h)=0$ for $1\le m_1,m_2\le M$ and $h>0$. To simplify even further, we assume there is no dependence with the efficient price. The asymptotic bias of the LS estimator then reduces to $\operatorname{plim}_{T\to\infty}(\hat{\alpha}_{\delta,1}-\alpha_{\delta,1})=-Q^{-1}\big[(1+\alpha_{\delta,1})\omega_1^2+\alpha_{\delta,1}\omega_2^2\big]$, with $Q=\sum_{h=0}^{\infty}\bar{\Upsilon}_{\delta,h}\Sigma_\delta\bar{\Upsilon}'_{\delta,h}+\omega_1^2+\omega_2^2$. This yields

$$\operatorname*{plim}_{T\to\infty}(\hat{\alpha}_{\delta,1}-\alpha_{\delta,1})=-\frac{\omega_1^2}{\sum_{h=0}^{\infty}\bar{\Upsilon}_{\delta,h}\Sigma_\delta\bar{\Upsilon}'_{\delta,h}+\omega_1^2+\omega_2^2}-\alpha_{\delta,1}\frac{\omega_1^2+\omega_2^2}{\sum_{h=0}^{\infty}\bar{\Upsilon}_{\delta,h}\Sigma_\delta\bar{\Upsilon}'_{\delta,h}+\omega_1^2+\omega_2^2}.$$

To sum up, the mere presence of microstructure noises is enough to entail bias in the LS estimation. In particular, it depends not only on the magnitude of the market microstructure noise in that market, but also on the overall noise-to-signal ratio.

B.2 VEC models with richer lag structures

Exact discretization of the continuous-time OU process yields iid Gaussian noises, so that any residual autocorrelation must come from the discrete-time microstructure noise in our setting. In what follows, we examine what happens if one adds lags to the VEC specification in order to handle the residual autocorrelation. For simplicity, we consider only the unrealistic case of independent microstructure (white) noises and a restricted VEC(1) specification that considers only the own past price change: namely,

$$\Delta p_{1,\tau} = \alpha_{\delta,1}\,(p_{1,\tau-1} - p_{2,\tau-1}) + \gamma\,\Delta p_{1,\tau-1} + \mathrm{v}_{1,\tau},$$

so that the LS estimator of $\alpha_{\delta,1}$ converges in probability to

$$\operatorname*{plim}_{T\to\infty}\hat{\alpha}_{\delta,1} = \frac{\mathbb{E}\big[\Delta p_{1,\tau-1}^2\big]\,\mathbb{E}\big[(p_{1,\tau-1}-p_{2,\tau-1})\Delta p_{1,\tau}\big] - \mathbb{E}\big[(p_{1,\tau-1}-p_{2,\tau-1})\Delta p_{1,\tau-1}\big]\,\mathbb{E}\big[\Delta p_{1,\tau}\Delta p_{1,\tau-1}\big]}{\mathbb{E}\big[\Delta p_{1,\tau-1}^2\big]\,\mathbb{E}\big[(p_{1,\tau-1}-p_{2,\tau-1})^2\big] - \Big(\mathbb{E}\big[(p_{1,\tau-1}-p_{2,\tau-1})\Delta p_{1,\tau-1}\big]\Big)^2}.$$

As before, we treat each of the above expectations individually in order to characterize the bias:

$$\begin{aligned}
\mathbb{E}\big[(p_{1,\tau-1}-p_{2,\tau-1})\Delta p_{1,\tau}\big] &= \alpha_{\delta,1}\,\mathbb{E}\big[(p_{1,\tau-1}-p_{2,\tau-1})^2\big] + \mathbb{E}\big[(p_{1,\tau-1}-p_{2,\tau-1})\mathrm{v}_{1,\tau}\big]\\
&= \alpha_{\delta,1}\left(\sum_{h=0}^{\infty}\bar{\Upsilon}_{\delta,h}\Sigma_\delta\bar{\Upsilon}_{\delta,h}' + \omega_1^2 + \omega_2^2\right) - (1+\alpha_{\delta,1})\,\omega_1^2 - \alpha_{\delta,1}\,\omega_2^2\\
&= \alpha_{\delta,1}\sum_{h=0}^{\infty}\bar{\Upsilon}_{\delta,h}\Sigma_\delta\bar{\Upsilon}_{\delta,h}' - \omega_1^2,\\
\mathbb{E}\big[(p_{1,\tau-1}-p_{2,\tau-1})\Delta p_{1,\tau-1}\big] &= \alpha_{\delta,1}\,\mathbb{E}\big[(p_{1,\tau-1}-p_{2,\tau-1})(p_{1,\tau-2}-p_{2,\tau-2})\big] + \mathbb{E}\big[(p_{1,\tau-1}-p_{2,\tau-1})\mathrm{v}_{1,\tau-1}\big]\\
&= \alpha_{\delta,1}\,\mathbb{E}\big[(P_{1,\tau-1}-P_{2,\tau-1})(P_{1,\tau-2}-P_{2,\tau-2})\big]\\
&\quad + \mathbb{E}\big[(P_{1,\tau-1}-P_{2,\tau-1})\mathrm{v}_{1,\tau-1}\big] + \mathbb{E}\big[(u_{1,\tau-1}-u_{2,\tau-1})\mathrm{v}_{1,\tau-1}\big]\\
&= \alpha_{\delta,1}\,\mathbb{E}\left[\sum_{h=0}^{\infty}\bar{\Upsilon}_{\delta,h}\varepsilon_{\tau-h-1}\sum_{h=0}^{\infty}\bar{\Upsilon}_{\delta,h}\varepsilon_{\tau-h-2}\right] + \mathbb{E}\left[\sum_{h=0}^{\infty}\bar{\Upsilon}_{\delta,h}\varepsilon_{\tau-h-1}\Delta u_{1,\tau-1}\right]\\
&\quad + \mathbb{E}\left[(u_{1,\tau-1}-u_{2,\tau-1})\Delta u_{1,\tau-1}\right]\\
&= \alpha_{\delta,1}\sum_{h=0}^{\infty}\bar{\Upsilon}_{\delta,h}\Sigma_\delta\bar{\Upsilon}_{\delta,h+1}' + \bar{\Upsilon}_{\delta,0}\begin{pmatrix}\sigma_{\delta,1}^2\\ \sigma_{\delta,12}\end{pmatrix} + \omega_1^2,
\end{aligned}$$

with $\sigma^2_{\delta,1} = \mathbb{E}[\varepsilon^2_{1,\tau}]$ and $\sigma_{\delta,12} = \mathbb{E}\big[\varepsilon_{1,\tau}\varepsilon_{2,\tau}\big]$. In turn,

$$
\begin{aligned}
\mathbb{E}\big[\Delta p_{1,\tau}\Delta p_{1,\tau-1}\big] = {} & \alpha^2_{\delta,1}\,\mathbb{E}\big[(p_{1,\tau-1}-p_{2,\tau-1})(p_{1,\tau-2}-p_{2,\tau-2})\big] + \alpha_{\delta,1}\,\mathbb{E}\big[(p_{1,\tau-1}-p_{2,\tau-1})\mathrm{v}_{1,\tau-1}\big] \\
& + \alpha_{\delta,1}\,\mathbb{E}\big[(p_{1,\tau-2}-p_{2,\tau-2})\mathrm{v}_{1,\tau})\big] + \mathbb{E}\big[\mathrm{v}_{1,\tau}\mathrm{v}_{1,\tau-1}\big] \\
= {} & \alpha^2_{\delta,1}\,\mathbb{E}\big[(P_{1,\tau-1}-P_{2,\tau-1})(P_{1,\tau-2}-P_{2,\tau-2})\big] + \alpha_{\delta,1}\bar{\Upsilon}_{\delta,0}\begin{pmatrix}\sigma^2_{\delta,1}\\ \sigma_{\delta,12}\end{pmatrix} \\
& + \mathbb{E}\big[\Delta u_{1,\tau}\Delta u_{1,\tau-1}\big] - \alpha_{\delta,1}\,\mathbb{E}\big[(u_{1,\tau-1}-u_{2,\tau-1})\Delta u_{1,\tau-1}\big] \\
= {} & \alpha^2_{\delta,1}\sum_{h=0}^{\infty}\bar{\Upsilon}_{\delta,h}\Sigma_\delta\bar{\Upsilon}'_{\delta,h+1} + \alpha_{\delta,1}\bar{\Upsilon}_{\delta,0}\begin{pmatrix}\sigma^2_{\delta,1}\\ \sigma_{\delta,12}\end{pmatrix} - (1+\alpha_{\delta,1})\omega^2_1
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[\Delta p^2_{1,\tau}] = {} & \alpha^2_{\delta,1}\,\mathbb{E}\big[(p_{1,\tau-1}-p_{2,\tau-1})^2\big] + 2\,\alpha_{\delta,1}\,\mathbb{E}\big[(p_{1,\tau-1}-p_{2,\tau-1})\mathrm{v}_{1,\tau}\big] + \mathbb{E}\big[\mathrm{v}^2_{1,\tau}\big] \\
= {} & \alpha^2_{\delta,1}\left[\sum_{h=0}^{\infty}\bar{\Upsilon}_{\delta,h}\Sigma_\delta\bar{\Upsilon}'_{\delta,h} + \omega^2_1 + \omega^2_2\right] - 2\,\alpha_{\delta,1}\big[(1+\alpha_{\delta,1})\omega^2_1 + \alpha_{\delta,1}\omega^2_2\big] \\
& + \mathbb{E}\big[\Delta u^2_{1,\tau}\big] + \mathbb{E}\big[\varepsilon^2_{1,\tau}\big] + \alpha^2_{\delta,1}\,\mathbb{E}\big[(u_{1,\tau-1}-u_{2,\tau-1})^2\big] - 2\,\alpha_{\delta,1}\,\mathbb{E}\big[\Delta u_{1,\tau}(u_{1,\tau-1}-u_{2,\tau-1})\big] \\
= {} & \alpha^2_{\delta,1}\left[\sum_{h=0}^{\infty}\bar{\Upsilon}_{\delta,h}\Sigma_\delta\bar{\Upsilon}'_{\delta,h} + \omega^2_1 + \omega^2_2\right] - 2\,\alpha_{\delta,1}\big[(1+\alpha_{\delta,1})\omega^2_1 + \alpha_{\delta,1}\omega^2_2\big] \\
& + 2\,\omega^2_1 + \sigma^2_1 + \alpha^2_{\delta,1}\,(\omega^2_1+\omega^2_2) + 2\,\alpha_{\delta,1}\,\omega^2_1 \\
= {} & \alpha^2_{\delta,1}\sum_{h=0}^{\infty}\bar{\Upsilon}_{\delta,h}\Sigma_\delta\bar{\Upsilon}'_{\delta,h} + \sigma^2_{\delta,1} + 2\,\omega^2_1.
\end{aligned}
$$

Altogether, this yields

$$
\operatorname*{plim}_{T\to\infty}\hat{\alpha}_{\delta,1} = \alpha_{\delta,1} - \frac{Q_B}{\mathbb{E}\big[\Delta p^2_{1,\tau}\big]\mathbb{E}\big[(p_{1,\tau-1}-p_{2,\tau-1})^2\big] - \big(\mathbb{E}\big[(p_{1,\tau-1}-p_{2,\tau-1})\Delta p_{1,\tau-1}\big]\big)^2},
$$

with

$$
\begin{aligned}
Q_B = {} & \big[(1+\alpha_{\delta,1})\omega^2_1 + \omega^2_2\big]\left[\alpha^2_{\delta,1}\sum_{h=0}^{\infty}\bar{\Upsilon}_{\delta,h}\Sigma_\delta\bar{\Upsilon}'_{\delta,h} + \sigma^2_{\delta,1} + 2\,\omega^2_1\right] \\
& + (1+2\,\alpha_{\delta,1})\,\omega^2_1\left[\alpha_{\delta,1}\sum_{h=0}^{\infty}\bar{\Upsilon}_{\delta,h}\Sigma_\delta\bar{\Upsilon}'_{\delta,h+1} + \bar{\Upsilon}_{\delta,0}\begin{pmatrix}\sigma^2_{\delta,1}\\ \sigma_{\delta,12}\end{pmatrix} + \omega^2_1\right].
\end{aligned}
$$

The expression for the bias is more complex than before because it now accounts for the measurement error in the past price changes. IN fact, incorporating the other market's past price change and/or a richer lag structure would only increase the magnitude of the bias.

B.3 Bias in leading market

If the first market leads the second market at all times (i.e., $\alpha_{\delta,1} = 0$), then the LS bias becomes

$$
\operatorname*{plim}_{T\to\infty}\hat{\alpha}_{\delta,1} = Q^{-1}\left\{\sum_{h=0}^{\infty}\bar{\Upsilon}_{\delta,h}[\rho_1(h+1)-\rho_1(h)] + \gamma_{11}(1) - \gamma_{12}(1) + \omega_{12} - \omega^2_1\right\},
$$

which reduces to $-\omega_1^2/Q$ in the case of independent microstructure (white) noises. If we now add the past price changes to control for the serial correlation in the error term, the LS estimator of $\alpha_{\delta,1}$ converges in probability to

$$\operatorname*{plim}_{T\to\infty} \hat{\alpha}_{\delta,1} = -\frac{\left(\omega_1^2+\omega_2^2\right)\left(\sigma_{\delta,1}^2+2\,\omega_1^2\right)+\omega_1^2\left[\bar{\Upsilon}_{\delta,0}\left(\sigma_{\delta,1}^2,\sigma_{\delta,12}\right)'+\omega_1^2\right]}{\left(\sigma_{\delta,1}^2+2\,\omega_1^2\right)\left(\sum_{h=0}^{\infty}\bar{\Upsilon}_{\delta,h}\Sigma_\delta\bar{\Upsilon}_{\delta,h}'+\omega_1^2+\omega_2^2\right)-\left[\bar{\Upsilon}_{\delta,0}(\sigma_{\delta,1}^2,\sigma_{\delta,12})'+\omega_1^2\right]^2}.$$

It is apparent that the LS estimator overestimates the magnitude of $\alpha_{\delta,1}$, thereof underestimating by how much the first market leads the price discovery.

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Table 1: Performance of the LS, IV, and GMM estimators of $\alpha_{\perp,1}$ under Assumption MMN(TS): iid market microstructure noise

The upper panel reports the median bias of the estimates of $\alpha_{\perp,1}$ and the lower panel displays the relative median squared errors (RRMSE) with respect to the VEC$(0)_0$ estimates. The market microstructure noises are serially uncorrelated white noises with variances ω_m^2 with $m = 1, 2$ ranging from 0.0001 to 0.001. VEC$(\jmath)_0$, VEC$(\jmath)_1$, and VEC$(\jmath)_5$ refer to the LS estimates of $\alpha_\perp$ from the VEC$(\jmath)$ specification ($\jmath = 0, 1$) at the tick-by-tick, 1-minute and 5-minute sampling intervals, respectively; IV$_{\bar{q}}$ with $\bar{q} = 0, 1$ corresponds to the IV estimates from VEC(0) models fitted at tick data with instruments given by $(\beta' p_{t_{i-\bar{q}-2}}, \ldots, \beta' p_{t_{i-\bar{q}-6}})'$; GMM$_{\bar{q}}$ with $\bar{q} = 0, 1$ refers the GMM estimates of $\alpha_{\perp,1}$ that are based on the mixed asymptotic setting and are computed from VEC(0) models fitted at tick data with instruments given by $(\beta' p_{t_{i-\bar{q}-2}}, \ldots, \beta' p_{t_{i-\bar{q}-6}})'$; and R$^2_{\bar{q}}$ (F$_{\bar{q}}$) with $\bar{q} = 0, 1$ are the coefficient of determination (F statistics) of the first-stage equation of the IV$_{\bar{q}}$ estimator.

$\alpha_{\perp,1} = 1.00$

Median bias	VEC$(0)_0$	VEC$(1)_0$	VEC$(0)_1$	VEC$(1)_1$	VEC$(0)_5$	VEC$(1)_5$	IV$_0$	IV$_1$	GMM$_0$	GMM$_1$	F_0	F_1
$\omega^2 = (0.0001, 0.0001)'$	-0.43	-0.35	-0.18	-0.16	-0.15	-0.19	0.00	0.00	0.00	0.00	2655	1118
$\omega^2 = (0.0005, 0.0005)'$	-0.48	-0.46	-0.36	-0.35	-0.37	-0.33	-0.01	-0.02	0.00	-0.01	494	242
$\omega^2 = (0.0010, 0.0010)'$	-0.49	-0.48	-0.42	-0.40	-0.43	-0.45	-0.02	-0.04	-0.02	-0.01	195	102
$\omega^2 = (0.0010, 0.0005)'$	-0.65	-0.63	-0.53	-0.51	-0.52	-0.54	-0.03	-0.04	-0.02	-0.01	290	148

$\alpha_{\perp,1} = 0.67$

Median bias	VEC$(0)_0$	VEC$(1)_0$	VEC$(0)_1$	VEC$(1)_1$	VEC$(0)_5$	VEC$(1)_5$	IV$_0$	IV$_1$	GMM$_0$	GMM$_1$	F_0	F_1
$\omega^2 = (0.0001, 0.0001)'$	-0.14	-0.12	-0.07	-0.07	-0.06	-0.06	0.00	0.00	0.00	0.00	1499	596
$\omega^2 = (0.0005, 0.0005)'$	-0.16	-0.15	-0.13	-0.13	-0.14	-0.10	-0.01	-0.01	0.00	0.00	240	109
$\omega^2 = (0.0010, 0.0010)'$	-0.16	-0.16	-0.15	-0.14	-0.16	-0.18	-0.01	-0.02	0.00	0.00	87	42
$\omega^2 = (0.0010, 0.0005)'$	-0.33	-0.32	-0.28	-0.27	-0.27	-0.29	-0.02	-0.04	-0.01	-0.01	133	63

$\alpha_{\perp,1} = 1.00$

RRMSE	VEC$(1)_0$	VEC$(0)_1$	VEC$(1)_1$	VEC$(0)_5$	VEC$(1)_5$	IV$_0$	IV$_1$	GMM$_0$	GMM$_1$	R_0^2	R_1^2
$\omega^2 = (0.0001, 0.0001)'$	0.80	0.41	0.41	1.05	1.50	0.09	0.10	0.09	0.10	0.47	0.40
$\omega^2 = (0.0005, 0.0005)'$	0.94	0.74	0.71	0.83	1.01	0.14	0.14	0.13	0.16	0.14	0.13
$\omega^2 = (0.0010, 0.0010)'$	0.97	0.85	0.81	0.87	0.99	0.21	0.19	0.21	0.27	0.06	0.06
$\omega^2 = (0.0010, 0.0005)'$	0.97	0.81	0.78	0.81	0.90	0.15	0.14	0.14	0.17	0.09	0.08

$\alpha_{\perp,1} = 0.67$

RRMSE	VEC$(1)_0$	VEC$(0)_1$	VEC$(1)_1$	VEC$(0)_5$	VEC$(1)_5$	IV$_0$	IV$_1$	GMM$_0$	GMM$_1$	R_0^2	R_1^2
$\omega^2 = (0.0001, 0.0001)'$	0.80	0.71	0.92	3.19	4.44	0.23	0.23	0.21	0.24	0.33	0.26
$\omega^2 = (0.0005, 0.0005)'$	0.95	0.82	0.82	1.68	2.34	0.38	0.35	0.32	0.39	0.07	0.06
$\omega^2 = (0.0010, 0.0010)'$	0.97	0.92	0.88	1.31	1.94	0.54	0.53	0.54	0.67	0.03	0.02
$\omega^2 = (0.0010, 0.0005)'$	0.97	0.86	0.83	0.98	1.19	0.25	0.24	0.23	0.29	0.04	0.04

Table 2: Performance of the LS, IV, and GMM estimators of $\alpha_{\perp,1}$ under Assumption MMN(TS): endogenous MA(1) market microstructure noise

The upper panel reports the median bias of the estimates of $\alpha_{\perp,1}$ and the lower panel displays the relative median squared errors (RRMSE) with respect to the VEC(0)$_0$ estimates. The market microstructure noises are endogenous MA(1) processes with variances ω_m^2 with $m = 1, 2$ ranging from 0.0001 to 0.001. VEC($\jmath$)$_0$, VEC($\jmath$)$_1$, and VEC($\jmath$)$_5$ refer to the LS estimates of $\alpha_\perp$ from the VEC($\jmath$) specification ($\jmath = 0, 1$) at the tick-by-tick, 1-minute and 5-minute sampling intervals, respectively; IV$_{\bar{q}}$ with $\bar{q} = 1, 2$ corresponds to the IV estimates from VEC(0) models fitted at tick data with instruments given by $(\beta' p_{t_{i-\bar{q}-2}}, \ldots, \beta' p_{t_{i-\bar{q}-6}})'$; GMM$_{\bar{q}}$ with $\bar{q} = 1, 2$ refers to the GMM estimates of $\alpha_{\perp,1}$ that are based on the mixed asymptotic setting and are computed from VEC(0) models fitted at tick data with instruments given by $(\beta' p_{t_{i-\bar{q}-2}}, \ldots, \beta' p_{t_{i-\bar{q}-6}})'$; and R$^2_{\bar{q}}$ (F$_{\bar{q}}$) with $\bar{q} = 1, 2$ are the coefficient of determination (F statistics) of the first-stage equation of the IV$_{\bar{q}}$ estimator.

$\alpha_{\perp,1} = 1.00$

Median bias	VEC(0)$_0$	VEC(1)$_0$	VEC(0)$_1$	VEC(1)$_1$	VEC(0)$_5$	VEC(1)$_5$	IV$_1$	IV$_2$	GMM$_1$	GMM$_2$	F_1	F_2
$\omega^2 = (0.0001, 0.0001)'$	-0.36	-0.28	-0.11	-0.10	-0.10	-0.18	0.00	0.00	0.00	0.00	3770	1410
$\omega^2 = (0.0005, 0.0005)'$	-0.46	-0.43	-0.28	-0.24	-0.29	-0.29	0.00	0.00	-0.01	0.00	945	412
$\omega^2 = (0.0010, 0.0010)'$	-0.48	-0.46	-0.36	-0.32	-0.36	-0.34	-0.01	-0.01	-0.02	-0.01	402	191
$\omega^2 = (0.0010, 0.0005)'$	-0.76	-0.74	-0.54	-0.51	-0.52	-0.54	0.00	-0.02	-0.01	-0.01	548	251

$\alpha_{\perp,1} = 0.67$

Median bias	VEC(0)$_0$	VEC(1)$_0$	VEC(0)$_1$	VEC(1)$_1$	VEC(0)$_5$	VEC(1)$_5$	IV$_1$	IV$_2$	GMM$_1$	GMM$_2$	F_1	F_2
$\omega^2 = (0.0001, 0.0001)'$	-0.12	-0.09	-0.05	-0.04	-0.04	-0.09	0.00	0.00	0.00	0.00	2089	740
$\omega^2 = (0.0005, 0.0005)'$	-0.16	-0.15	-0.11	-0.10	-0.12	-0.06	0.00	0.00	-0.01	0.00	452	184
$\omega^2 = (0.0010, 0.0010)'$	-0.16	-0.16	-0.13	-0.12	-0.14	-0.12	0.00	0.00	-0.02	-0.01	176	77
$\omega^2 = (0.0010, 0.0005)'$	-0.44	-0.44	-0.35	-0.35	-0.35	-0.36	0.00	-0.02	-0.01	0.00	247	105

$\alpha_{\perp,1} = 1.00$

RRMSE	VEC(1)$_0$	VEC(0)$_1$	VEC(1)$_1$	VEC(0)$_5$	VEC(1)$_5$	IV$_1$	IV$_2$	GMM$_1$	GMM$_2$	R^2_1	R^2_2
$\omega^2 = (0.0001, 0.0001)'$	0.77	0.35	0.42	1.27	1.91	0.10	0.12	0.11	0.12	0.50	0.43
$\omega^2 = (0.0005, 0.0005)'$	0.93	0.61	0.53	0.85	1.12	0.14	0.14	0.15	0.18	0.20	0.18
$\omega^2 = (0.0010, 0.0010)'$	0.96	0.75	0.66	0.81	0.97	0.22	0.20	0.23	0.30	0.10	0.09
$\omega^2 = (0.0010, 0.0005)'$	0.97	0.71	0.67	0.71	0.81	0.13	0.11	0.14	0.19	0.13	0.12

$\alpha_{\perp,1} = 0.67$

RRMSE	VEC(1)$_0$	VEC(0)$_1$	VEC(1)$_1$	VEC(0)$_5$	VEC(1)$_5$	IV$_1$	IV$_2$	GMM$_1$	GMM$_2$	R^2_1	R^2_2
$\omega^2 = (0.0001, 0.0001)'$	0.78	0.80	1.10	3.76	5.66	0.26	0.31	0.26	0.30	0.36	0.28
$\omega^2 = (0.0005, 0.0005)'$	0.93	0.76	0.79	2.08	3.06	0.42	0.43	0.41	0.54	0.11	0.09
$\omega^2 = (0.0010, 0.0010)'$	0.96	0.83	0.80	1.65	2.43	0.65	0.62	0.69	0.92	0.04	0.04
$\omega^2 = (0.0010, 0.0005)'$	0.99	0.79	0.80	0.88	1.09	0.20	0.18	0.24	0.32	0.06	0.05

Table 3: Performance of the LS, IV, and GMM estimators of the continuous-time IS measure under Assumption MMN(TS): iid market microstructure noise

The upper panel reports the median bias of the estimates of IS_1 and the lower panel displays the relative median squared errors (RRMSE) with respect to the VEC(0)$_0$ estimates. The market microstructure noises are serially uncorrelated white noises with variances ω_m^2 with $m = 1, 2$ ranging from 0.0001 to 0.001. VEC($\jmath$)$_0$, VEC($\jmath$)$_1$, and VEC($\jmath$)$_5$ refer to estimates of IS_1 that are constructed with LS estimates of $\alpha_\perp$ from the VEC($\jmath$) specification ($\jmath = 0, 1$) at the tick-by-tick, 1-minute and 5-minute sampling intervals, respectively, and integrated covariance matrix estimated with the realized variance (RV) at these sampling frequencies; $\mathrm{IV}_{\mathrm{RK},\bar{q}}$ ($\mathrm{GMM}_{\mathrm{RK},\bar{q}}$) with $\bar{q} = 0, 1$ corresponds to the IV (continuous-time GMM-based) estimates from VEC(0) models fitted at tick data with instruments given by $(\beta' p_{t_{i-\bar{q}-2}}, \ldots, \beta' p_{t_{i-\bar{q}-6}})'$ and integrated covariance matrix estimated with the realized kernel (RK) estimator at the tick-by-tick sampling frequency; and $\mathrm{IV}_{\mathrm{HAC},\bar{q}}$ with $\bar{q} = 0, 1$ refers to estimates of IS_1 that are constructed with IV estimates of $\alpha_\perp$ from VEC(0) models fitted at tick data with instruments given by $(\beta' p_{t_{i-\bar{q}-2}}, \ldots, \beta' p_{t_{i-\bar{q}-6}})'$ and integrated covariance matrix based on the exact discretization of the HAC estimates of the discrete-time unconditional covariance matrix.

$IS_1 = 0.88$

Median bias	VEC(0)$_0$	VEC(1)$_0$	VEC(0)$_1$	VEC(1)$_1$	VEC(0)$_5$	VEC(1)$_5$	$\mathrm{IV}_{\mathrm{RK},0}$	$\mathrm{IV}_{\mathrm{RK},1}$	$\mathrm{IV}_{\mathrm{HAC},0}$	$\mathrm{IV}_{\mathrm{HAC},1}$	$\mathrm{GMM}_{\mathrm{RK},0}$	$\mathrm{GMM}_{\mathrm{RK},1}$
$\omega^2 = (0.0001, 0.0001)'$	-0.26	-0.12	-0.35	-0.25	-0.37	-0.35	-0.17	-0.17	0.01	0.01	-0.18	-0.18
$\omega^2 = (0.0005, 0.0005)'$	-0.35	-0.29	-0.36	-0.27	-0.37	-0.34	-0.22	-0.23	0.03	0.03	-0.23	-0.23
$\omega^2 = (0.0010, 0.0010)'$	-0.36	-0.33	-0.37	-0.28	-0.37	-0.36	-0.24	-0.24	0.05	0.05	-0.26	-0.26
$\omega^2 = (0.0010, 0.0005)'$	-0.52	-0.47	-0.48	-0.36	-0.41	-0.38	-0.23	-0.23	0.04	0.04	-0.25	-0.25

$IS_1 = 0.66$

Median bias	VEC(0)$_0$	VEC(1)$_0$	VEC(0)$_1$	VEC(1)$_1$	VEC(0)$_5$	VEC(1)$_5$	$\mathrm{IV}_{\mathrm{RK},0}$	$\mathrm{IV}_{\mathrm{RK},1}$	$\mathrm{IV}_{\mathrm{HAC},0}$	$\mathrm{IV}_{\mathrm{HAC},1}$	$\mathrm{GMM}_{\mathrm{RK},0}$	$\mathrm{GMM}_{\mathrm{RK},1}$
$\omega^2 = (0.0001, 0.0001)'$	-0.12	-0.07	-0.15	-0.13	-0.16	-0.15	-0.10	-0.10	0.01	0.00	-0.10	-0.10
$\omega^2 = (0.0005, 0.0005)'$	-0.15	-0.13	-0.16	-0.13	-0.16	-0.15	-0.12	-0.12	0.03	0.02	-0.12	-0.12
$\omega^2 = (0.0010, 0.0010)'$	-0.16	-0.15	-0.16	-0.14	-0.16	-0.17	-0.12	-0.12	0.04	0.04	-0.13	-0.13
$\omega^2 = (0.0010, 0.0005)'$	-0.32	-0.30	-0.27	-0.22	-0.20	-0.19	-0.12	-0.13	0.04	0.03	-0.13	-0.13

$IS_1 = 0.88$

RRMSE	VEC(1)$_0$	VEC(0)$_1$	VEC(1)$_1$	VEC(0)$_5$	VEC(1)$_5$	$\mathrm{IV}_{\mathrm{RK},0}$	$\mathrm{IV}_{\mathrm{RK},1}$	$\mathrm{IV}_{\mathrm{HAC},0}$	$\mathrm{IV}_{\mathrm{HAC},1}$	$\mathrm{GMM}_{\mathrm{RK},0}$	$\mathrm{GMM}_{\mathrm{RK},1}$
$\omega^2 = (0.0001, 0.0001)'$	0.47	1.35	0.98	1.43	1.36	0.67	0.67	0.06	0.06	0.69	0.68
$\omega^2 = (0.0005, 0.0005)'$	0.85	1.05	0.77	1.08	1.00	0.64	0.65	0.10	0.10	0.68	0.68
$\omega^2 = (0.0010, 0.0010)'$	0.92	1.02	0.78	1.04	1.01	0.66	0.67	0.16	0.15	0.71	0.71
$\omega^2 = (0.0010, 0.0005)'$	0.91	0.92	0.69	0.79	0.74	0.45	0.45	0.09	0.08	0.48	0.48

$IS_1 = 0.66$

RRMSE	VEC(1)$_0$	VEC(0)$_1$	VEC(1)$_1$	VEC(0)$_5$	VEC(1)$_5$	$\mathrm{IV}_{\mathrm{RK},0}$	$\mathrm{IV}_{\mathrm{RK},1}$	$\mathrm{IV}_{\mathrm{HAC},0}$	$\mathrm{IV}_{\mathrm{HAC},1}$	$\mathrm{GMM}_{\mathrm{RK},0}$	$\mathrm{GMM}_{\mathrm{RK},1}$
$\omega^2 = (0.0001, 0.0001)'$	0.60	1.25	1.03	1.30	1.25	0.83	0.83	0.22	0.25	0.85	0.85
$\omega^2 = (0.0005, 0.0005)'$	0.89	1.04	0.88	1.06	0.98	0.79	0.80	0.36	0.37	0.82	0.82
$\omega^2 = (0.0010, 0.0010)'$	0.95	1.01	0.88	1.03	1.07	0.78	0.80	0.64	0.61	0.84	0.83
$\omega^2 = (0.0010, 0.0005)'$	0.94	0.86	0.71	0.63	0.60	0.38	0.40	0.24	0.23	0.40	0.41

Table 4: Performance of the LS, IV, and GMM estimators of the continuous-time IS measure under Assumption MMN(TS): endogenous MA(1) market microstructure noise

The upper panel reports the median bias of the estimates of IS_1 and the lower panel displays the relative median squared errors (RRMSE) with respect to the VEC(0)$_0$ estimates. The market microstructure noises are endogenous MA(1) processes with variances ω_m^2 with $m = 1, 2$ ranging from 0.0001 to 0.001. VEC($\jmath$)$_0$, VEC($\jmath$)$_1$, and VEC($\jmath$)$_5$ refer to estimates of IS_1 that are constructed with LS estimates of $\alpha_\perp$ from the VEC($\jmath$) specification ($\jmath = 0, 1$) at the tick-by-tick, 1-minute and 5-minute sampling intervals, respectively, and integrated covariance matrix estimated with the realized variance (RV) at these sampling frequencies; $\text{IV}_{\text{RK},\bar{q}}$ ($\text{GMM}_{\text{RK},\bar{q}}$) with $\bar{q} = 1, 2$ corresponds to the IV (continuous-time GMM-based) estimates from VEC(0) models fitted at tick data with instruments given by $(\beta' p_{t_{i-\bar{q}-2}}, \ldots, \beta' p_{t_{i-\bar{q}-6}})'$ and integrated covariance matrix estimated with the realized kernel (RK) estimator at the tick-by-tick sampling frequency; and $\text{IV}_{\text{HAC},\bar{q}}$ with $\bar{q} = 1, 2$ refers to estimates of IS_1 that are constructed with IV estimates of $\alpha_\perp$ from VEC(0) models fitted at tick data with instruments given by $(\beta' p_{t_{i-\bar{q}-2}}, \ldots, \beta' p_{t_{i-\bar{q}-6}})'$ and integrated covariance matrix based on the exact discretization of the HAC estimates of the discrete-time unconditional covariance matrix.

	$IS_1 = 0.88$											
Median bias	VEC(0)$_0$	VEC(1)$_0$	VEC(0)$_1$	VEC(1)$_1$	VEC(0)$_5$	VEC(1)$_5$	$\text{IV}_{\text{RK},1}$	$\text{IV}_{\text{RK},2}$	$\text{IV}_{\text{HAC},1}$	$\text{IV}_{\text{HAC},2}$	$\text{GMM}_{\text{RK},1}$	$\text{GMM}_{\text{RK},2}$
$\omega^2 = (0.0001, 0.0001)'$	-0.21	-0.13	-0.33	-0.25	-0.37	-0.35	-0.16	-0.16	0.00	0.00	-0.16	-0.16
$\omega^2 = (0.0005, 0.0005)'$	-0.33	-0.30	-0.36	-0.26	-0.37	-0.35	-0.20	-0.20	0.01	0.01	-0.21	-0.21
$\omega^2 = (0.0010, 0.0010)'$	-0.35	-0.33	-0.36	-0.27	-0.37	-0.35	-0.22	-0.22	0.02	0.02	-0.23	-0.23
$\omega^2 = (0.0010, 0.0005)'$	-0.58	-0.56	-0.50	-0.37	-0.41	-0.38	-0.21	-0.21	0.01	0.01	-0.22	-0.22
	$IS_1 = 0.66$											
Median bias	VEC(0)$_0$	VEC(1)$_0$	VEC(0)$_1$	VEC(1)$_1$	VEC(0)$_5$	VEC(1)$_5$	$\text{IV}_{\text{RK},1}$	$\text{IV}_{\text{RK},2}$	$\text{IV}_{\text{HAC},1}$	$\text{IV}_{\text{HAC},2}$	$\text{GMM}_{\text{RK},1}$	$\text{GMM}_{\text{RK},2}$
$\omega^2 = (0.0001, 0.0001)'$	-0.11	-0.08	-0.15	-0.12	-0.16	-0.16	-0.10	-0.10	0.00	0.00	-0.09	-0.09
$\omega^2 = (0.0005, 0.0005)'$	-0.15	-0.14	-0.16	-0.13	-0.16	-0.15	-0.11	-0.11	0.01	0.01	-0.12	-0.12
$\omega^2 = (0.0010, 0.0010)'$	-0.15	-0.15	-0.16	-0.13	-0.16	-0.15	-0.11	-0.11	0.02	0.02	-0.13	-0.13
$\omega^2 = (0.0010, 0.0005)'$	-0.39	-0.38	-0.30	-0.25	-0.20	-0.19	-0.11	-0.12	0.02	0.01	-0.12	-0.12

	$IS_1 = 0.88$										
RRMSE	VEC(1)$_0$	VEC(0)$_1$	VEC(1)$_1$	VEC(0)$_5$	VEC(1)$_5$	$\text{IV}_{\text{RK},1}$	$\text{IV}_{\text{RK},2}$	$\text{IV}_{\text{HAC},1}$	$\text{IV}_{\text{HAC},2}$	$\text{GMM}_{\text{RK},1}$	$\text{GMM}_{\text{RK},2}$
$\omega^2 = (0.0001, 0.0001)'$	0.62	1.54	1.18	1.71	1.65	0.74	0.74	0.07	0.07	0.73	0.73
$\omega^2 = (0.0005, 0.0005)'$	0.89	1.07	0.78	1.11	1.04	0.61	0.61	0.07	0.07	0.63	0.63
$\omega^2 = (0.0010, 0.0010)'$	0.94	1.03	0.76	1.05	0.98	0.62	0.62	0.10	0.10	0.66	0.65
$\omega^2 = (0.0010, 0.0005)'$	0.96	0.86	0.63	0.71	0.65	0.36	0.37	0.05	0.04	0.40	0.40
	$IS_1 = 0.66$										
RRMSE	VEC(1)$_0$	VEC(0)$_1$	VEC(1)$_1$	VEC(0)$_5$	VEC(1)$_5$	$\text{IV}_{\text{RK},1}$	$\text{IV}_{\text{RK},2}$	$\text{IV}_{\text{HAC},1}$	$\text{IV}_{\text{HAC},2}$	$\text{GMM}_{\text{RK},1}$	$\text{GMM}_{\text{RK},2}$
$\omega^2 = (0.0001, 0.0001)'$	0.71	1.39	1.17	1.48	1.46	0.89	0.89	0.25	0.30	0.90	0.90
$\omega^2 = (0.0005, 0.0005)'$	0.92	1.05	0.87	1.08	1.01	0.76	0.76	0.39	0.39	0.80	0.78
$\omega^2 = (0.0010, 0.0010)'$	0.96	1.02	0.87	1.04	0.99	0.74	0.75	0.61	0.56	0.82	0.81
$\omega^2 = (0.0010, 0.0005)'$	0.99	0.76	0.64	0.52	0.49	0.29	0.30	0.20	0.17	0.33	0.33

Table 5: Performance of the LS and IV estimators of the discrete-time IS measure under Assumption MMN(TS): iid and endogenous MA(1) market microstructure noise

We report the median bias of the estimates of $IS_{\delta,1}$ and the relative root median squared error (RRMSE) with respect to the VEC(0)$_0$ estimates. The upper and lower panels report results computed from prices contaminated with iid and endogenous MA(1) market microstructure noise, respectively. The market microstructure noise variances ω_m^2 with $m = 1, 2$ range from 0.0001 to 0.001. VEC($\jmath$)$_0$ refers to estimates of $IS_{\delta,1}$ that are constructed with LS estimates of $\alpha_\perp$ from the VEC($\jmath$) specification ($\jmath = 0, 1$) at the tick-by-tick sampling interval and estimates of the unconditional covariance that are not robust to heteroskedaticity and serial correlation in the residuals; and IV$_{\text{HAC},\bar{q}}$ with $\bar{q} = 0, 1, 2$ corresponds to estimates of the discrete-time IS measures that are constructed with estimates of $\alpha_\perp$ computed by IV from VEC(0) models fitted at the tick-by-tick frequency with instruments given by $(\beta' p_{t_{i-\bar{q}-2}}, \ldots, \beta' p_{t_{i-\bar{q}-6}})'$ and estimates of the unconditional covariance computed with the HAC estimator.

$\text{IS}_{\delta,1} = 0.87$

	Median bias				RRMSE		
iid noise	VEC(0)$_0$	VEC(1)$_0$	IV$_{\text{HAC},0}$	IV$_{\text{HAC},1}$	VEC(1)$_0$	IV$_{\text{HAC},0}$	IV$_{\text{HAC},1}$
$\omega^2 = (0.0001, 0.0001)'$	-0.25	0.24	0.00	0.00	0.50	0.06	0.06
$\omega^2 = (0.0005, 0.0005)'$	-0.33	0.08	0.03	0.03	0.85	0.10	0.10
$\omega^2 = (0.0010, 0.0010)'$	-0.35	0.04	0.05	0.05	0.92	0.17	0.16
$\omega^2 = (0.0010, 0.0005)'$	-0.50	-0.10	0.04	0.04	0.93	0.09	0.09

$\text{IS}_{\delta,1} = 0.65$

	Median bias				RRMSE		
iid noise	VEC(0)$_0$	VEC(1)$_0$	IV$_{\text{HAC},0}$	IV$_{\text{HAC},1}$	VEC(1)$_0$	IV$_{\text{HAC},0}$	IV$_{\text{HAC},1}$
$\omega^2 = (0.0001, 0.0001)'$	-0.11	0.08	0.01	0.00	0.61	0.23	0.25
$\omega^2 = (0.0005, 0.0005)'$	-0.14	0.02	0.03	0.02	0.89	0.36	0.37
$\omega^2 = (0.0010, 0.0010)'$	-0.15	0.01	0.04	0.04	0.94	0.65	0.62
$\omega^2 = (0.0010, 0.0005)'$	-0.29	-0.14	0.04	0.03	0.97	0.25	0.24

$\text{IS}_{\delta,1} = 0.87$

	Median bias				RRMSE		
endogenous MA(1) noise	VEC(0)$_0$	VEC(1)$_0$	IV$_{\text{HAC},1}$	IV$_{\text{HAC},2}$	VEC(1)$_0$	IV$_{\text{HAC},1}$	IV$_{\text{HAC},2}$
$\omega^2 = (0.0001, 0.0001)'$	-0.21	0.23	0.00	0.00	0.63	0.07	0.07
$\omega^2 = (0.0005, 0.0005)'$	-0.33	0.07	0.01	0.01	0.89	0.08	0.08
$\omega^2 = (0.0010, 0.0010)'$	-0.34	0.04	0.01	0.02	0.95	0.10	0.10
$\omega^2 = (0.0010, 0.0005)'$	-0.54	-0.17	0.01	0.01	0.98	0.05	0.05

$\text{IS}_{\delta,1} = 0.65$

	Median bias				RRMSE		
endogenous MA(1) noise	VEC(0)$_0$	VEC(1)$_0$	IV$_{\text{HAC},1}$	IV$_{\text{HAC},2}$	VEC(1)$_0$	IV$_{\text{HAC},1}$	IV$_{\text{HAC},2}$
$\omega^2 = (0.0001, 0.0001)'$	-0.10	0.08	0.00	0.00	0.72	0.25	0.30
$\omega^2 = (0.0005, 0.0005)'$	-0.14	0.02	0.01	0.01	0.92	0.38	0.39
$\omega^2 = (0.0010, 0.0010)'$	-0.15	0.01	0.02	0.02	0.96	0.62	0.56
$\omega^2 = (0.0010, 0.0005)'$	-0.34	-0.20	0.02	0.01	1.01	0.21	0.18

Table 6: Performance of the LS, IV, and CU-GMM estimators of $\alpha_{\perp,1}$ under Assumption MMN(CS): endogenous MA(∞) market microstructure noise

The upper panel reports the median bias of the estimates of $\alpha_{\perp,1}$ and the lower panel displays the relative median squared errors (RRMSE) with respect to the VEC$(0)_0$ estimates. The market microstructure noises are endogenous MA(∞) processes with variances ω_m^2 with $m = 1, 2$ ranging from 0.0001 to 0.001. VEC$(0)_0$, VEC$(0)_1$, and VEC$(0)_5$ refer to the LS estimates of $\alpha_\perp$ from the VEC(0) specification at the tick-by-tick sampling intervals; IV$_V$ with $V = 1, 5, 10, 20$ corresponds to the IV estimates from VEC(0) models fitted at tick data with instruments given by $\widetilde{\beta}' s_{t_{i-1}}$; CU-GMM$_V$ with $V = 10, 20$ refers to the CU-GMM estimates from VEC(0) models fitted at tick data with instruments given by $\widetilde{\beta}' s_{t_{i-1}}$; and R$^2_V$ (F$_V$) with $V = 1, 5, 10, 20$ is the coefficient of determination (F statistics) of the first-stage equation of the IV$_V$ estimator.

	$\alpha_{\perp,1} = 1.00$												
Median bias	VEC$(0)_0$	VEC$(0)_1$	VEC$(0)_5$	IV$_1$	IV$_5$	IV$_{10}$	IV$_{20}$	CU-GMM$_{10}$	CU-GMM$_{20}$	F$_1$	F$_5$	F$_{10}$	F$_{20}$
$\omega^2 = (0.0001, 0.0001)'$	-0.40	-0.16	-0.12	0.03	0.01	0.00	-0.01	0.01	0.00	56.34	153.65	98.55	81.39
$\omega^2 = (0.0005, 0.0005)'$	-0.48	-0.38	-0.35	-0.13	-0.07	-0.11	-0.16	0.01	0.01	12.45	32.50	20.04	14.51
$\omega^2 = (0.0010, 0.0010)'$	-0.49	-0.44	-0.43	-0.31	-0.17	-0.24	-0.29	0.03	0.02	4.79	13.46	8.39	6.06
$\omega^2 = (0.0010, 0.0005)'$	-0.66	-0.56	-0.54	-0.27	-0.14	-0.22	-0.28	0.02	0.02	9.14	23.57	14.62	10.58
	$\alpha_{\perp,1} = 0.67$												
Median bias	VEC$(0)_0$	VEC$(0)_1$	VEC$(0)_5$	IV$_1$	IV$_5$	IV$_{10}$	IV$_{20}$	CU-GMM$_{10}$	CU-GMM$_{20}$	F$_1$	F$_5$	F$_{10}$	F$_{20}$
$\omega^2 = (0.0001, 0.0001)'$	-0.13	-0.08	-0.05	0.01	0.01	0.00	0.00	0.00	0.00	47.23	114.83	71.84	58.19
$\omega^2 = (0.0005, 0.0005)'$	-0.16	-0.14	-0.12	-0.05	-0.02	-0.04	-0.06	0.01	0.00	8.67	19.70	12.28	9.04
$\omega^2 = (0.0010, 0.0010)'$	-0.16	-0.15	-0.15	-0.10	-0.06	-0.09	-0.11	0.01	0.00	3.32	7.79	5.11	3.86
$\omega^2 = (0.0010, 0.0005)'$	-0.33	-0.30	-0.29	-0.14	-0.09	-0.13	-0.16	0.01	0.01	6.25	13.99	8.84	6.59

	$\alpha_{\perp,1} = 1.00$											
RRMSE	VEC$(0)_1$	VEC$(0)_5$	IV$_1$	IV$_5$	IV$_{10}$	IV$_{20}$	CU-GMM$_{10}$	CU-GMM$_{20}$	R$^2_1$	R$^2_5$	R$^2_{10}$	R$^2_{20}$
$\omega^2 = (0.0001, 0.0001)'$	0.41	0.97	0.53	0.16	0.14	0.11	0.15	0.11	0.02	0.21	0.25	0.35
$\omega^2 = (0.0005, 0.0005)'$	0.78	0.79	0.72	0.23	0.25	0.33	0.21	0.19	0.00	0.05	0.06	0.09
$\omega^2 = (0.0010, 0.0010)'$	0.89	0.87	0.89	0.37	0.49	0.60	0.30	0.28	0.00	0.02	0.03	0.04
$\omega^2 = (0.0010, 0.0005)'$	0.85	0.83	0.65	0.25	0.34	0.43	0.18	0.16	0.00	0.04	0.05	0.07
	$\alpha_{\perp,1} = 0.67$											
RRMSE	VEC$(0)_1$	VEC$(0)_5$	IV$_1$	IV$_5$	IV$_{10}$	IV$_{20}$	CU-GMM$_{10}$	CU-GMM$_{20}$	R$^2_1$	R$^2_5$	R$^2_{10}$	R$^2_{20}$
$\omega^2 = (0.0001, 0.0001)'$	0.74	3.08	1.17	0.35	0.32	0.24	0.33	0.25	0.02	0.16	0.19	0.28
$\omega^2 = (0.0005, 0.0005)'$	0.86	1.59	1.49	0.49	0.43	0.41	0.47	0.42	0.00	0.03	0.04	0.06
$\omega^2 = (0.0010, 0.0010)'$	0.94	1.27	1.75	0.63	0.62	0.66	0.68	0.68	0.00	0.01	0.02	0.02
$\omega^2 = (0.0010, 0.0005)'$	0.91	0.93	0.86	0.32	0.40	0.49	0.25	0.23	0.00	0.02	0.03	0.04

Table 7: Performance of the LS, IV, and CU-GMM estimators of the continuous-time Information Share measure under Assumption MMN(CS): endogenous MA(∞) market microstructure noise

The upper panel reports the median bias of the estimates of IS_1 and the lower panel displays the relative median squared errors (RRMSE) with respect to the VEC(0)$_0$ estimates. The market microstructure noises are endogenous MA(∞) processes with variances ω_m^2 with $m = 1, 2$ ranging from 0.0001 to 0.001. VEC(0)$_0$, VEC(0)$_1$, and VEC(0)$_5$ refer to estimates of IS_1 that are constructed with LS estimates of $\alpha_\perp$ from the VEC(0) specification at the tick-by-tick, 1-minute and 5-minute sampling intervals, respectively, and integrated covariance matrix estimated with the realized variance (RV) at these sampling frequencies; IV$_{V,\text{RK}}$ and IV$_{V,\text{HAC}}$ with $V = 1, 5, 10, 20$ correspond to estimates of IS_1 that are constructed with IV estimates of $\alpha_\perp$ from VEC(0) models fitted at tick data with $V = 1, 5, 10, 20$ assets as instruments given by $\widetilde{\beta}' st_{i-1}$ and integrated covariance matrix estimated with the realized kernel (RK) estimator and from the exact discretization of the HAC estimates of the discrete-time unconditional covariance matrix, respectively; and CU-GMM$_{V,\text{RK}}$ and CU-GMM$_{V,\text{HAC}}$ with $V = 10, 20$ refer to estimates of IS_1 that are constructed with CU-GMM estimates of $\alpha_\perp$ from VEC(0) models fitted at tick data with $V = 10, 20$ assets as instruments given by $\widetilde{\beta}' st_{i-1}$ and integrated covariance matrix estimated with the realized kernel (RK) estimator and from the exact discretization of the HAC estimates of the discrete-time unconditional covariance matrix, respectively.

$IS_1 = 0.88$

Median bias	VEC(0)$_0$	VEC(0)$_1$	VEC(0)$_5$	IV$_{1,\text{RK}}$	IV$_{5,\text{RK}}$	IV$_{10,\text{RK}}$	IV$_{20,\text{RK}}$	CU-GMM$_{10,\text{RK}}$	CU-GMM$_{20,\text{RK}}$	IV$_{1,\text{HAC}}$	IV$_{5,\text{HAC}}$	IV$_{10,\text{HAC}}$	IV$_{20,\text{HAC}}$	CU-GMM$_{10,\text{HAC}}$	CU-GMM$_{20,\text{HAC}}$
$\omega^2 = (0.0001, 0.0001)'$	-0.21	-0.25	-0.35	-0.13	-0.14	-0.14	-0.15	-0.14	-0.14	0.00	0.00	0.00	0.00	0.00	0.00
$\omega^2 = (0.0005, 0.0005)'$	-0.34	-0.29	-0.35	-0.21	-0.19	-0.20	-0.22	-0.16	-0.16	-0.03	0.03	0.02	0.01	0.04	0.04
$\omega^2 = (0.0010, 0.0010)'$	-0.36	-0.31	-0.35	-0.28	-0.22	-0.25	-0.27	-0.15	-0.15	-0.14	0.02	-0.01	-0.04	0.06	0.06
$\omega^2 = (0.0010, 0.0005)'$	-0.52	-0.40	-0.38	-0.25	-0.19	-0.23	-0.25	-0.13	-0.14	-0.07	0.02	-0.01	-0.04	0.05	0.05

$IS_1 = 0.66$

Median bias	VEC(0)$_0$	VEC(0)$_1$	VEC(0)$_5$	IV$_{1,\text{RK}}$	IV$_{5,\text{RK}}$	IV$_{10,\text{RK}}$	IV$_{20,\text{RK}}$	CU-GMM$_{10,\text{RK}}$	CU-GMM$_{20,\text{RK}}$	IV$_{1,\text{HAC}}$	IV$_{5,\text{HAC}}$	IV$_{10,\text{HAC}}$	IV$_{20,\text{HAC}}$	CU-GMM$_{10,\text{HAC}}$	CU-GMM$_{20,\text{HAC}}$
$\omega^2 = (0.0001, 0.0001)'$	-0.11	-0.13	-0.15	-0.08	-0.08	-0.08	-0.08	-0.08	-0.08	0.00	0.00	0.00	0.00	0.00	0.00
$\omega^2 = (0.0005, 0.0005)'$	-0.15	-0.14	-0.15	-0.11	-0.10	-0.11	-0.11	-0.09	-0.09	-0.01	0.02	0.01	0.00	0.04	0.03
$\omega^2 = (0.0010, 0.0010)'$	-0.16	-0.15	-0.15	-0.13	-0.11	-0.12	-0.13	-0.08	-0.08	-0.08	0.00	-0.03	-0.05	0.07	0.06
$\omega^2 = (0.0010, 0.0005)'$	-0.32	-0.25	-0.19	-0.14	-0.11	-0.13	-0.15	-0.06	-0.07	-0.07	0.00	-0.05	-0.09	0.08	0.07

$IS_1 = 0.88$

RRMSE	VEC(0)$_0$	VEC(0)$_1$	VEC(0)$_5$	IV$_{1,\text{RK}}$	IV$_{5,\text{RK}}$	IV$_{10,\text{RK}}$	IV$_{20,\text{RK}}$	CU-GMM$_{10,\text{RK}}$	CU-GMM$_{20,\text{RK}}$	IV$_{1,\text{HAC}}$	IV$_{5,\text{HAC}}$	IV$_{10,\text{HAC}}$	IV$_{20,\text{HAC}}$	CU-GMM$_{10,\text{HAC}}$	CU-GMM$_{20,\text{HAC}}$
$\omega^2 = (0.0001, 0.0001)'$		1.19	1.62	0.63	0.67	0.67	0.69	0.66	0.67	0.19	0.09	0.08	0.07	0.08	0.07
$\omega^2 = (0.0005, 0.0005)'$		0.85	1.03	0.63	0.55	0.60	0.65	0.47	0.47	0.24	0.13	0.11	0.09	0.13	0.13
$\omega^2 = (0.0010, 0.0010)'$		0.87	0.99	0.78	0.61	0.69	0.76	0.41	0.42	0.38	0.18	0.16	0.16	0.19	0.19
$\omega^2 = (0.0010, 0.0005)'$		0.76	0.74	0.48	0.37	0.43	0.49	0.26	0.26	0.20	0.09	0.08	0.10	0.11	0.11

$IS_1 = 0.66$

RRMSE	VEC(0)$_0$	VEC(0)$_1$	VEC(0)$_5$	IV$_{1,\text{RK}}$	IV$_{5,\text{RK}}$	IV$_{10,\text{RK}}$	IV$_{20,\text{RK}}$	CU-GMM$_{10,\text{RK}}$	CU-GMM$_{20,\text{RK}}$	IV$_{1,\text{HAC}}$	IV$_{5,\text{HAC}}$	IV$_{10,\text{HAC}}$	IV$_{20,\text{HAC}}$	CU-GMM$_{10,\text{HAC}}$	CU-GMM$_{20,\text{HAC}}$
$\omega^2 = (0.0001, 0.0001)'$		1.20	1.40	0.85	0.76	0.77	0.78	0.76	0.77	1.06	0.33	0.31	0.25	0.31	0.26
$\omega^2 = (0.0005, 0.0005)'$		0.93	1.01	0.83	0.66	0.71	0.76	0.58	0.59	1.36	0.53	0.44	0.36	0.49	0.44
$\omega^2 = (0.0010, 0.0010)'$		0.94	0.98	0.95	0.71	0.79	0.85	0.52	0.53	1.42	0.72	0.57	0.47	0.83	0.79
$\omega^2 = (0.0010, 0.0005)'$		0.77	0.59	0.48	0.35	0.41	0.46	0.21	0.22	0.72	0.29	0.27	0.29	0.30	0.28

Table 8: Performance of the LS, IV, and CU-GMM estimators of the discrete-time IS measure under Assumption MMN(CS): endogenous MA(∞) market microstructure noise

The upper panel reports the median bias of the estimates of $IS_{\delta,1}$ and the lower panel displays the relative root median squared errors (RRMSE) with respect to the VEC$(0)_0$ estimates. The market microstructure noises are endogenous MA(∞) processes with variances ω_m^2 with $m = 1, 2$ ranging from 0.0001 to 0.001. VEC$(0)_0$ corresponds to estimates of $IS_{\delta,1}$ that are constructed with estimates of $\alpha_\perp$ computed by LS from VEC(0) models at the tick-by-tick frequency and unconditional covariance matrix estimates that are not robust to heteroskedaticity and serial correlation in the residuals; IV$_{V,\text{HAC}}$ with $V = 1, 5, 10, 20$ refers to estimates of $IS_{\delta,1}$ that are constructed with IV-based estimates of $\alpha_\perp$ from VEC(0) models fitted at tick data with $V = 1, 5, 10, 20$ assets as instruments given by $\widetilde{\beta}' st_{i-1}$ and unconditional covariance matrix estimated with the HAC estimator at the tick-by-tick sampling frequency; and CU-GMM$_{V,\text{HAC}}$ with $V = 10, 20$ corresponds to estimates of $IS_{\delta,1}$ that are constructed with CU-GMM estimates of $\alpha_\perp$ from VEC(0) models fitted at tick data with $V = 10, 20$ assets as instruments given by $\widetilde{\beta}' st_{i-1}$ and discrete-time unconditional covariance matrix estimated with the HAC estimator at the tick-by-tick sampling frequency.

	$\text{IS}_{\delta,1} = 0.87$						
Median bias	VEC$(0)_0$	IV$_{1,\text{HAC}}$	IV$_{5,\text{HAC}}$	IV$_{10,\text{HAC}}$	IV$_{20,\text{HAC}}$	CU-GMM$_{10,\text{HAC}}$	CU-GMM$_{20,\text{HAC}}$
$\omega^2 = (0.0001, 0.0001)'$	-0.20	0.00	0.00	0.00	0.00	0.00	0.00
$\omega^2 = (0.0005, 0.0005)'$	-0.33	-0.02	0.03	0.02	0.01	0.04	0.04
$\omega^2 = (0.0010, 0.0010)'$	-0.35	-0.12	0.02	-0.01	-0.04	0.07	0.07
$\omega^2 = (0.0010, 0.0005)'$	-0.50	-0.07	0.02	-0.01	-0.05	0.06	0.06
	$\text{IS}_{\delta,1} = 0.65$						
Median bias	VEC$(0)_0$	IV$_{1,\text{HAC}}$	IV$_{5,\text{HAC}}$	IV$_{10,\text{HAC}}$	IV$_{20,\text{HAC}}$	CU-GMM$_{10,\text{HAC}}$	CU-GMM$_{20,\text{HAC}}$
$\omega^2 = (0.0001, 0.0001)'$	-0.10	0.00	0.00	0.00	0.00	0.00	0.00
$\omega^2 = (0.0005, 0.0005)'$	-0.14	0.00	0.02	0.01	0.00	0.04	0.03
$\omega^2 = (0.0010, 0.0010)'$	-0.15	-0.07	0.00	-0.03	-0.04	0.07	0.06
$\omega^2 = (0.0010, 0.0005)'$	-0.31	-0.06	0.00	-0.05	-0.09	0.08	0.07

		$\text{IS}_{\delta,1} = 0.87$					
RRMSE	VEC$(0)_0$	IV$_{1,\text{HAC}}$	IV$_{5,\text{HAC}}$	IV$_{10,\text{HAC}}$	IV$_{20,\text{HAC}}$	CU-GMM$_{10,\text{HAC}}$	CU-GMM$_{20,\text{HAC}}$
$\omega^2 = (0.0001, 0.0001)'$		0.21	0.09	0.09	0.08	0.09	0.08
$\omega^2 = (0.0005, 0.0005)'$		0.26	0.13	0.11	0.10	0.14	0.14
$\omega^2 = (0.0010, 0.0010)'$		0.37	0.19	0.16	0.16	0.22	0.21
$\omega^2 = (0.0010, 0.0005)'$		0.21	0.10	0.09	0.10	0.12	0.11
		$\text{IS}_{\delta,1} = 0.65$					
RRMSE	VEC$(0)_0$	IV$_{1,\text{HAC}}$	IV$_{5,\text{HAC}}$	IV$_{10,\text{HAC}}$	IV$_{20,\text{HAC}}$	CU-GMM$_{10,\text{HAC}}$	CU-GMM$_{20,\text{HAC}}$
$\omega^2 = (0.0001, 0.0001)'$	-0.10	1.09	0.34	0.32	0.26	0.32	0.26
$\omega^2 = (0.0005, 0.0005)'$	-0.14	1.44	0.54	0.46	0.36	0.51	0.46
$\omega^2 = (0.0010, 0.0010)'$	-0.15	1.59	0.74	0.59	0.47	0.88	0.82
$\omega^2 = (0.0010, 0.0005)'$	-0.31	0.75	0.29	0.27	0.29	0.31	0.29

Figure 1: LS and IV estimates of $\alpha_{\perp,1}$ computed from prices with (without) market microstructure noise

We report box plots of LS and IV estimates of $\alpha_{\perp,1}$. We simulate from the exact discretization of (1), with $\alpha_{\delta=1/23400} = (0.000, 0.050)'$, $\beta = (1, -1)'$, market-specific integrated variances equal to one, and $\rho = 0.5$. The first panel reports $\alpha_{\perp,1}$ estimates computed using the LS and IV estimators from VEC(0) models fitted at tick data without market microstructure noise contamination. The IV estimator uses valid and relevant instruments which are selected as $(\beta' p_{t_{i-2}}, \ldots, \beta' p_{t_{i-6}})'$. The second panel displays the box plot of the LS and IV estimators of VEC(0) models using tick data with iid market microstructure noise with variances given by $\omega^2 = (0.0010, 0.0005)'$. In the third panel, LS_1 and LS_5 refer to the LS estimator fitted at contaminated prices that are sampled at 1- and 5-minute sampling intervals. Black stars are the true values, whereas the edges of boxes in the box plots refer to the 25% and 75% percentiles, and the red line represents the median.

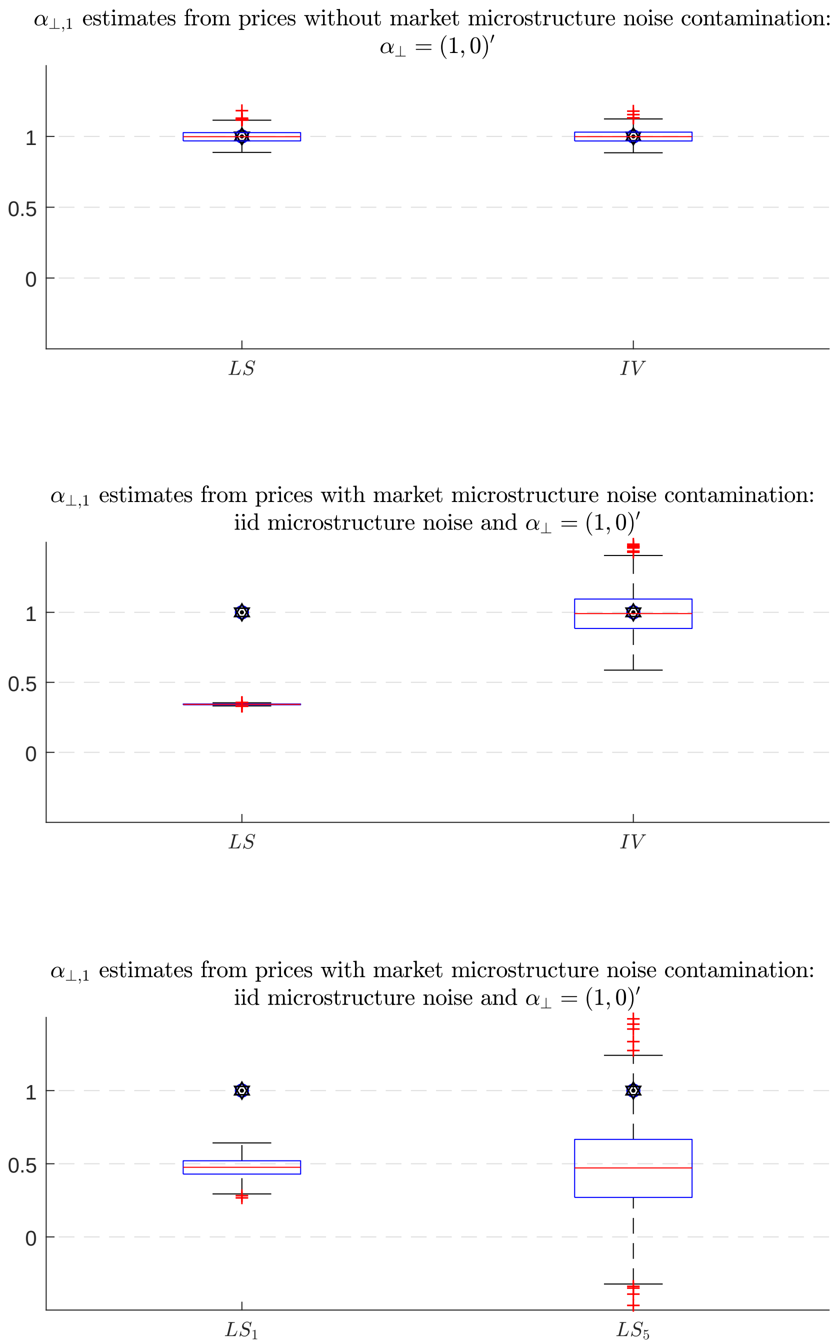

Figure 2: Empirical distributions of the t-statistics based on Theorem 1(ii) and Theorem 2(ii)

The upper and lower panels display the empirical densities of the t-statistics based on Theorem 1(ii) and Theorem 2(ii) using 1000 replications, respectively. Columns refer to alternative market microstructure levels: $\omega^2 = (0.0001, 0.0001)'$, $\omega^2 = (0.0005, 0.0005)'$, $\omega^2 = (0.001, 0.0005)'$, and $\omega^2 = (0.001, 0.001)'$, respectively. Solid and dashed lines are the empirical densities and the the standard Gaussian densities, respectively.

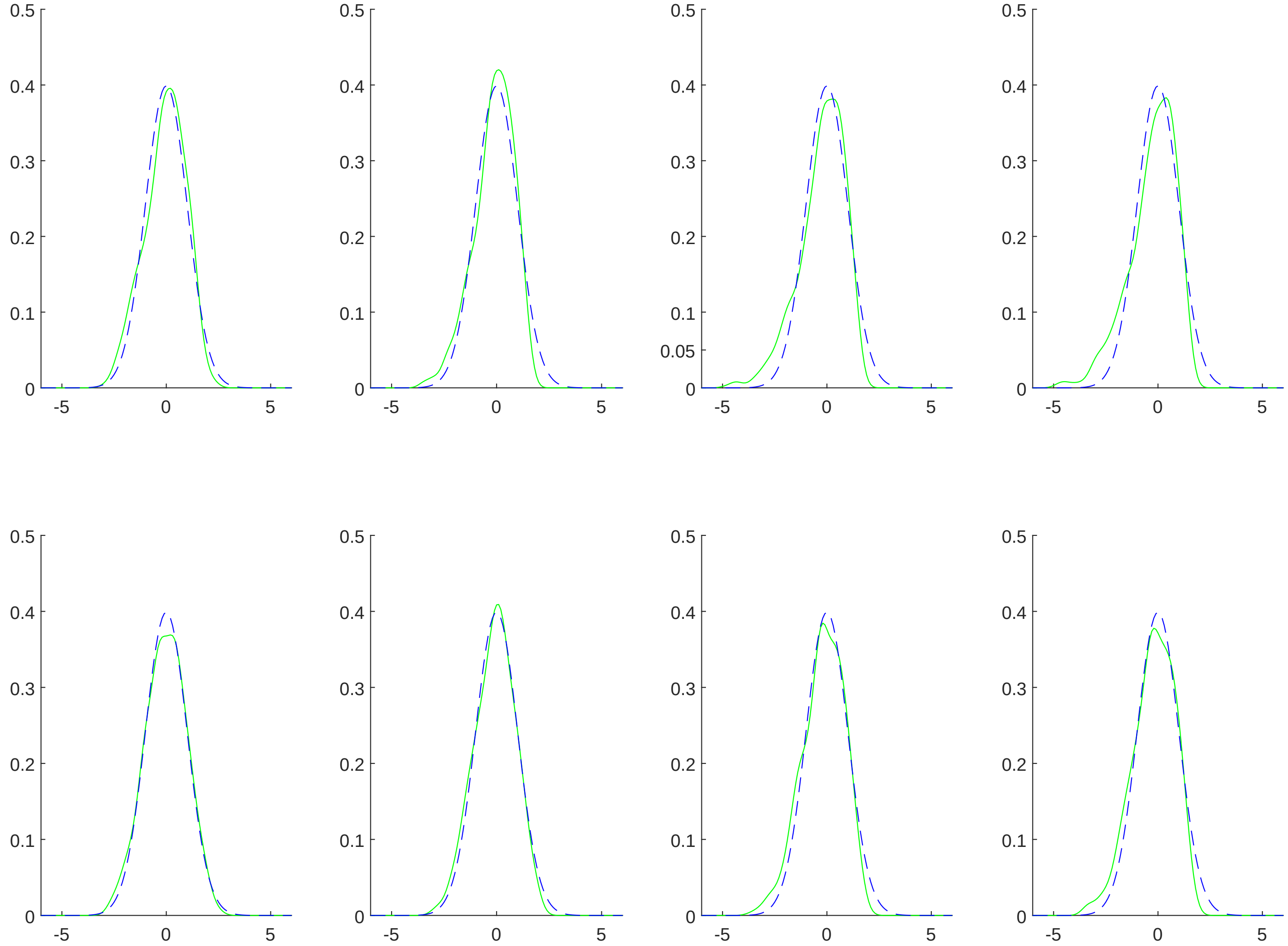

Figure 3: Impulse response function (IRF) under Assumption MMN(TS): iid market microstructure noise

We report IRFs for $p_{2,\tau}$ sampled at the tick-by-tick frequency. The market microstructure noises are serially uncorrelated white noises. Columns refer to alternative market microstructure levels: $\omega^2 = (0.0001, 0.0001)'$, $\omega^2 = (0.0005, 0.0005)'$, $\omega^2 = (0.001, 0.0005)'$, and $\omega^2 = (0.001, 0.001)'$, respectively. VEC$(\jmath)_0$ $(\jmath = 0, 1)$ refers to IRFs constructed with estimates of α_δ and $\alpha_\perp$ computed by LS from VEC$(\jmath)$ models fitted at the tick-by-tick frequency and estimates of the unconditional covariance that are not robust to heteroskedaticity and serial correlation in the residuals; $\text{IV}_{\text{HAC},\bar{q}}$ with $\bar{q} = 0, 1$ corresponds to estimates of the IRFs that are constructed with estimates of α_δ and $\alpha_\perp$ computed by IV from VEC(0) models fitted at the tick-by-tick frequency with instruments given by $(\beta' p_{t_{i-\bar{q}-2}}, \ldots, \beta' p_{t_{i-\bar{q}-6}})'$ and estimates of the unconditional covariance computed with the HAC estimator. The solid lines correspond to the median IRFs, the shaded area indicates the 90% confidence intervals, and the black dotted lines are the true IRFs.

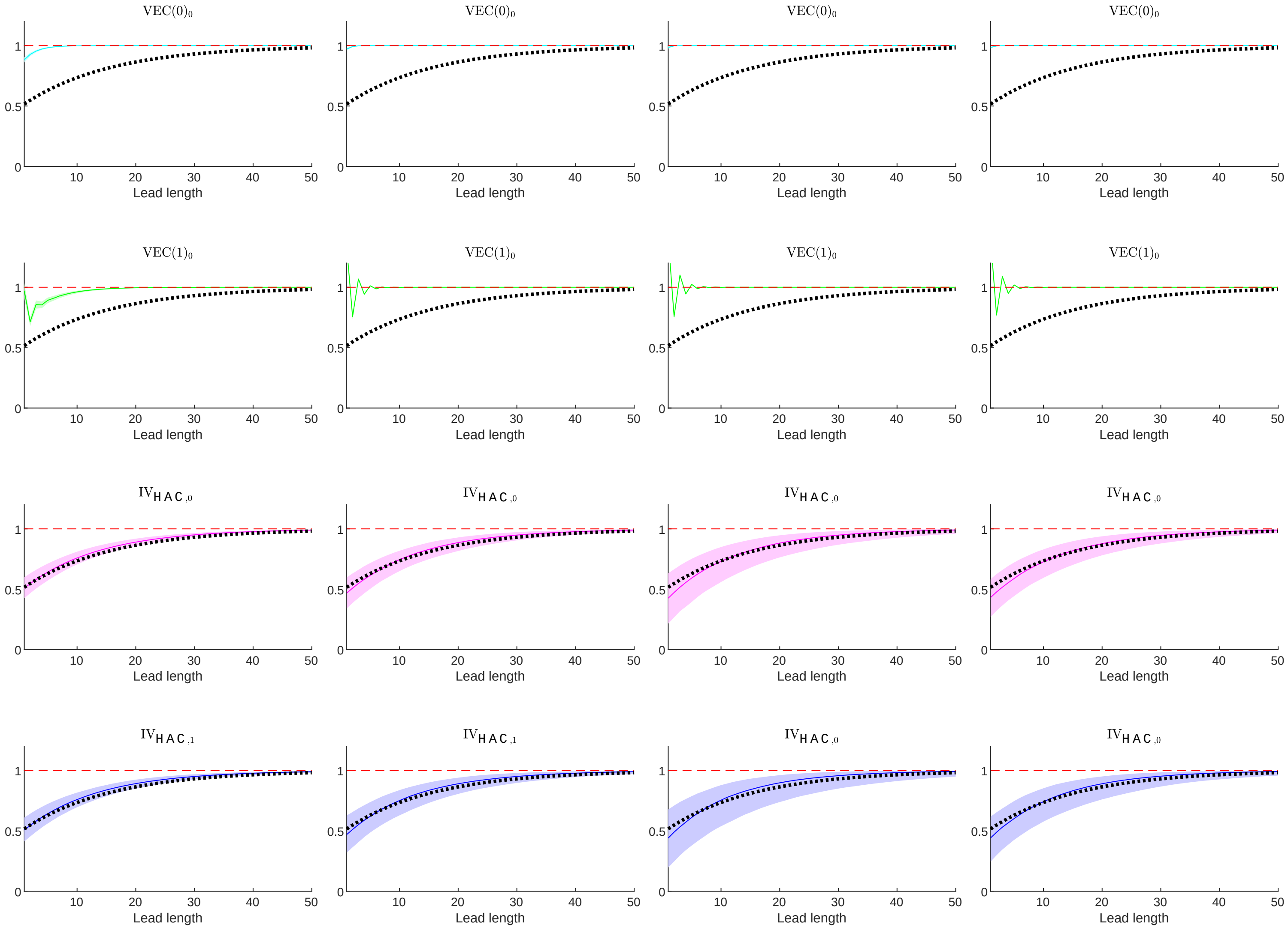

Figure 4: Impulse response function (IRF) under Assumption MMN(TS): endogenous MA(1) market microstructure noise

We report the estimated IRFs for $p_{2,\tau}$ sampled at the tick-by-tick frequency. The market microstructure noises are endogenous MA(1) processes. Columns refer to alternative market microstructure levels: $\omega^2 = (0.0001, 0.0001)'$, $\omega^2 = (0.0005, 0.0005)'$, $\omega^2 = (0.001, 0.0005)'$, and $\omega^2 = (0.001, 0.001)'$, respectively. VEC$(\jmath)_0$ $(\jmath = 0, 1)$ refers to IRFs constructed with estimates of α_δ and $\alpha_\perp$ computed by LS from VEC$(\jmath)$ models fitted at the tick-by-tick frequency and estimates of the unconditional covariance that are not robust to heteroskedaticity and serial correlation in the residuals; $\text{IV}_{\text{HAC},\bar{q}}$ with $\bar{q} = 1, 2$ corresponds to estimates of the IRFs that are constructed with estimates of α_δ and $\alpha_\perp$ computed by IV from VEC(0) models fitted at the tick-by-tick frequency with instruments given by $(\beta' p_{t_{i-\bar{q}-2}}, \ldots, \beta' p_{t_{i-\bar{q}-6}})'$ and estimates of the unconditional covariance computed with the HAC estimator. The solid lines correspond to the median IRFs, the shaded area indicates the 90% confidence intervals, and the black dotted lines are the true IRFs.

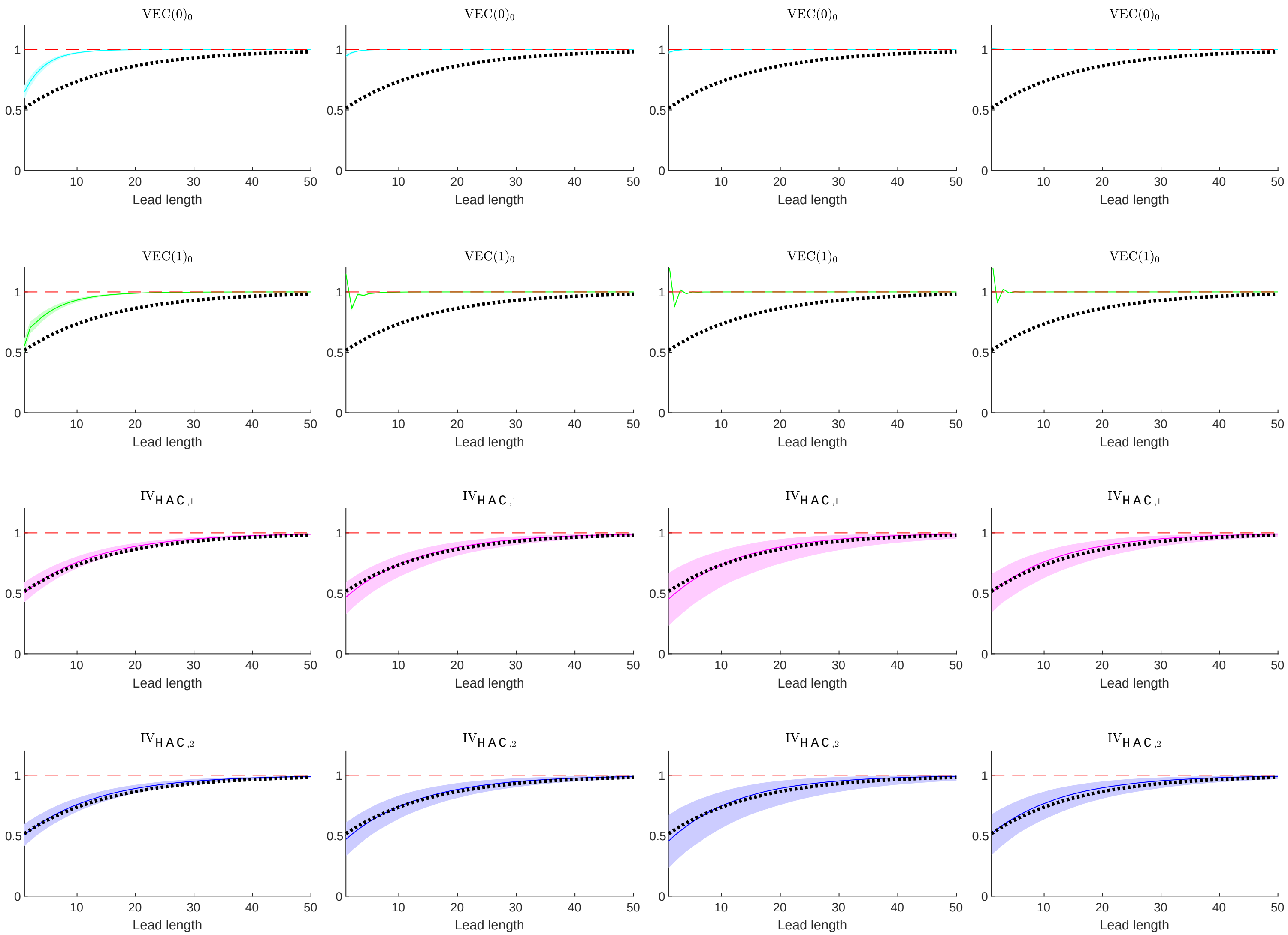

Figure 5: Impulse response function (IRF) under Assumption MMN(CS): endogenous MA(∞) market microstructure noise

We report the estimated IRFs for $p_{2,\tau}$ sampled at the tick-by-tick frequency. The market microstructure noises are endogenous MA(∞) processes. Columns refer to alternative market microstructure levels: $\omega^2 = (0.0001, 0.0001)'$, $\omega^2 = (0.0005, 0.0005)'$, $\omega^2 = (0.001, 0.0005)'$, and $\omega^2 = (0.001, 0.001)'$, respectively. VEC$(\jmath)_0$ $(\jmath = 0, 1)$ refers to IRFs constructed with estimates of α_δ and $\alpha_\perp$ computed by LS from VEC$(\jmath)$ models fitted at the tick-by-tick frequency and estimates of the unconditional covariance that are not robust to heteroskedaticity and serial correlation in the residuals; IV$_{V,\text{HAC}}$ with $V = 1, 5, 10, 20$ corresponds to estimates of the IRFs that are constructed with estimates of α_δ and $\alpha_\perp$ computed by IV from VEC(0) models fitted at the tick-by-tick frequency with $V = 1, 5, 10, 20$ assets as instruments given by $\widetilde{\beta}' st_{i-1}$ and estimates of the unconditional covariance computed with the HAC estimator; and CU-GMM$_{V,\text{HAC}}$ with $V = 10, 20$ refers to estimates of IRFs that are constructed with CU-GMM estimates of α_δ and $\alpha_\perp$ from VEC(0) models fitted at tick data with $V = 10, 20$ assets as instruments given by $\widetilde{\beta}' st_{i-1}$ and estimates of the unconditional covariance computed with the HAC estimator. The solid lines correspond to the median IRFs, the shaded area indicates the 90% confidence intervals, and the black dotted lines are the true IRFs.

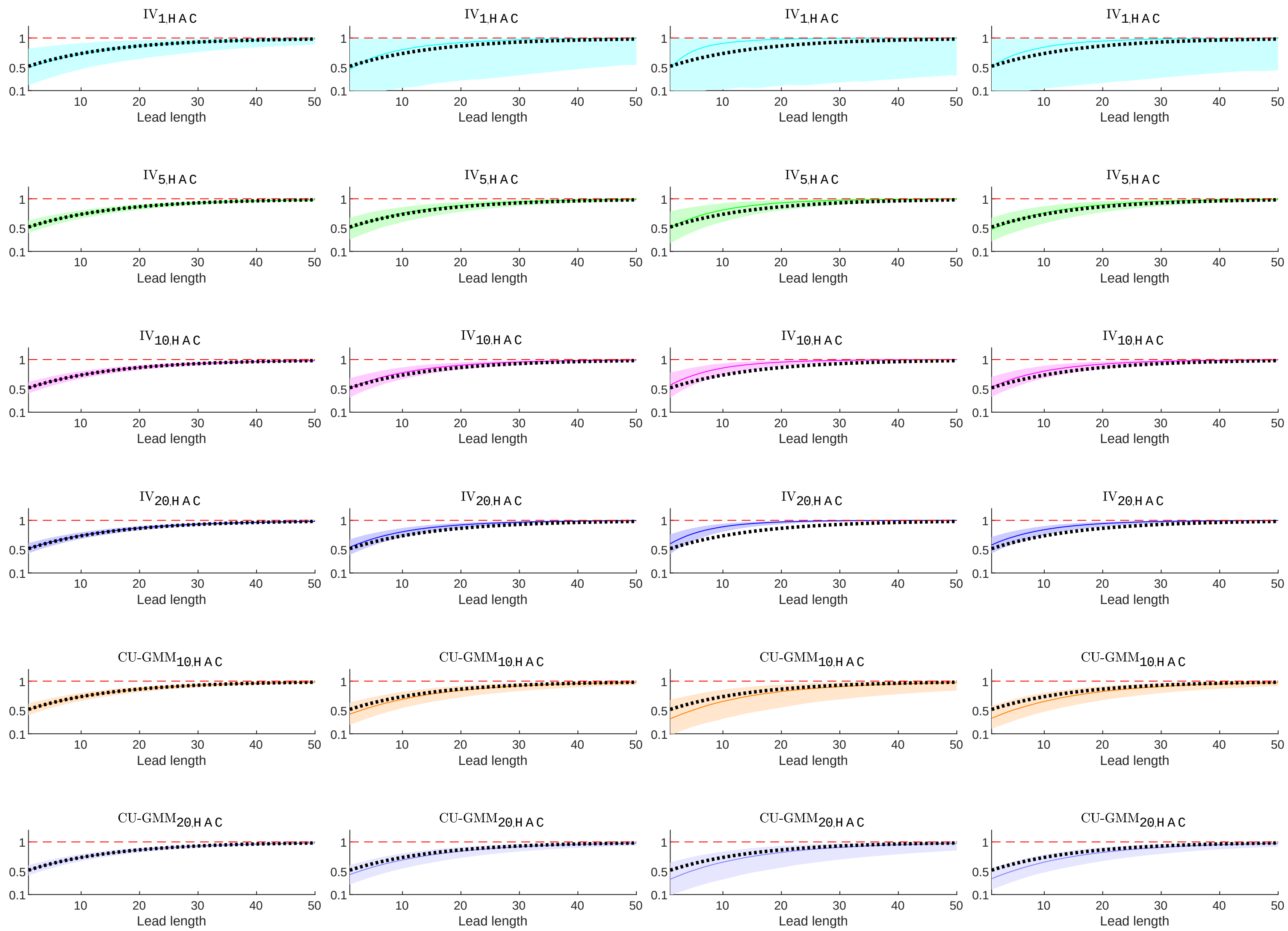

Table 9: Data description

We report summary statistics for raw and cleaned data of AA and 25 auxiliary assets traded at NYSE (N), Nasdaq (T), NASDAQ OMX BX (B), Arca (P), and BATS (Z). The first column reports the ticker symbols for all assets; the first panel (Raw quotes) presents the number of quotes (in millions) for each stock on the five trading venues before any cleaning filter; the second panel (Mkt share: raw quotes) displays the market share (in %) for each trading venue as well as the total market share of all five exchanges; the third panel (Cleaned obs) reveals the number of quotes (in millions) for each stock on the five trading venues after the implementation of the cleaning procedure; and the fourth panel (Average duration) shows the average duration (in seconds) between quotes for each trading venue. "-" denotes that the asset does not trade at this trading venue.

	Raw quotes ('000,000)					Mkt share: raw quotes (%)						Cleaned obs ('000,000)					Average duration (seconds)				
	N	T	B	P	Z	N	T	B	P	Z	Total	N	T	B	P	Z	N	T	B	P	Z
aa	35.5	9.8	9.7	11.2	9.8	27.0%	7.4%	7.4%	8.5%	7.4%	57.8%	3.8	3.0	2.1	2.8	2.6	1.6	2.0	2.7	2.1	2.3
aapl	-	17.7	32.9	25.4	17.7	-	10.4%	19.4%	15.0%	10.4%	55.3%	-	3.5	3.9	3.7	3.3	-	1.7	1.5	1.6	1.8
bac	69.5	23.8	29.6	27.2	23.8	21.6%	7.4%	9.2%	8.5%	7.4%	54.1%	4.4	3.7	3.3	3.8	3.4	1.3	1.6	1.8	1.5	1.7
brkb	30.6	16.9	3.1	6.3	16.9	33.4%	18.4%	3.4%	6.8%	18.4%	80.3%	3.7	2.3	1.0	1.8	2.4	1.6	2.6	5.6	3.3	2.4
csco	-	23.1	16.6	21.1	23.1	-	11.8%	8.5%	10.8%	11.8%	42.8%	-	1.5	1.1	1.3	1.5	-	3.9	5.2	4.5	3.8
dal	27.7	12.0	5.9	10.8	12.0	23.1%	10.0%	4.9%	9.0%	10.0%	57.0%	2.8	2.3	1.3	2.1	1.9	2.1	2.6	4.5	2.8	3.1
gm	36.5	14.2	4.9	10.7	14.2	26.4%	10.3%	3.5%	7.7%	10.3%	58.2%	3.2	2.3	1.1	2.0	2.0	1.8	2.6	5.2	2.9	3.0
goog	-	5.9	12.1	13.1	5.9	-	8.5%	17.3%	18.8%	8.5%	53.1%	-	2.0	2.0	2.1	1.7	-	2.9	2.9	2.8	3.5
hpq	50.0	21.0	14.5	20.6	21.0	21.0%	8.8%	6.1%	8.7%	8.8%	53.5%	3.7	3.2	2.1	2.9	2.6	1.6	1.9	2.7	2.0	2.2
ibm	23.2	8.8	9.8	14.9	8.8	24.2%	9.2%	10.2%	15.5%	9.2%	68.2%	3.6	2.3	2.0	2.7	2.2	1.6	2.6	2.9	2.2	2.6
jcp	26.0	16.2	5.5	14.9	16.2	19.5%	12.1%	4.2%	11.2%	12.1%	59.1%	2.9	2.5	1.3	2.3	2.1	2.0	2.3	4.6	2.5	2.7
jnj	39.4	23.0	6.2	15.6	23.0	22.0%	12.8%	3.4%	8.7%	12.8%	59.8%	3.9	3.1	1.6	2.8	3.0	1.5	1.9	3.7	2.1	2.0
jpm	120.6	58.3	21.0	42.6	58.3	25.6%	12.4%	4.5%	9.1%	12.4%	63.9%	4.8	4.2	3.0	4.1	4.2	1.2	1.4	2.0	1.4	1.4
ko	37.9	15.3	6.1	11.6	15.3	25.1%	10.2%	4.0%	7.7%	10.2%	57.1%	3.9	2.9	1.3	2.4	2.3	1.5	2.1	4.4	2.5	2.5
mo	32.1	11.7	9.5	10.5	11.7	24.4%	8.9%	7.2%	7.9%	8.9%	57.2%	3.4	2.5	1.9	2.2	2.0	1.7	2.3	3.1	2.7	2.9
mrk	42.9	19.8	9.8	14.2	19.8	23.6%	10.9%	5.4%	7.8%	10.9%	58.5%	3.9	3.0	2.0	2.6	2.6	1.5	1.9	3.0	2.3	2.2
mrvl	-	10.8	6.7	9.5	10.8	-	12.5%	7.7%	11.0%	12.5%	43.8%	-	2.5	1.3	2.1	2.1	-	2.3	4.4	2.8	2.7
msft	-	31.9	22.7	27.4	31.9	-	12.1%	8.6%	10.3%	12.1%	43.1%	-	4.1	2.9	3.7	3.6	-	1.4	2.0	1.6	1.6
nok	24.9	8.6	7.9	10.5	8.6	24.3%	8.4%	7.8%	10.2%	8.4%	59.1%	2.7	1.9	1.5	2.1	1.7	2.2	3.1	3.9	2.8	3.5
orcl	-	32.8	16.7	25.1	32.8	-	14.1%	7.2%	10.8%	14.1%	46.1%	-	3.7	2.5	3.5	3.7	-	1.6	2.3	1.7	1.6
pfe	49.9	18.5	14.5	17.7	18.5	22.9%	8.5%	6.7%	8.1%	8.5%	54.8%	4.0	3.1	2.3	3.0	2.7	1.5	1.9	2.6	2.0	2.2
pg	31.1	20.9	7.6	18.8	20.9	18.0%	12.1%	4.4%	10.9%	12.1%	57.5%	3.6	3.1	1.7	2.8	2.7	1.6	1.9	3.4	2.1	2.2
vz	41.0	17.0	9.2	15.2	17.0	23.6%	9.8%	5.3%	8.7%	9.8%	57.2%	3.9	2.9	1.9	2.7	2.5	1.5	2.0	3.1	2.2	2.3
wfc	59.6	35.7	22.2	29.6	35.7	18.9%	11.4%	7.1%	9.4%	11.4%	58.1%	4.3	3.6	2.8	3.7	3.7	1.4	1.6	2.1	1.6	1.6
xom	62.1	48.8	5.2	33.6	48.8	22.0%	17.2%	1.8%	11.9%	17.2%	70.1%	4.8	3.9	1.4	4.0	4.2	1.2	1.5	4.3	1.5	1.4
yhoo	-	15.1	10.0	12.8	15.1	-	10.6%	7.0%	9.0%	10.6%	37.2%	-	3.0	1.8	2.5	2.4	-	2.0	3.2	2.3	2.4
Total	892.6	561.4	329.1	482.6	551.6	17.4%	11.2%	6.7%	9.8%	11.2%	56.2%	67.6	72.6	49.1	71.6	69.5	1.6	2.0	2.9	2.1	2.2

Table 10: Continuous-time CS estimates for NYSE, $\alpha_{\mathrm{N},\perp}$, under assumption MMN(TS)

We estimate monthly and annual component shares for the NYSE. Specifically, $\mathrm{VEC}(0)_0$, $\mathrm{VEC}(0)_1$ and $\mathrm{VEC}(0)_5$ denote LS estimates of the CS measures from VEC(0) models at tick data, 1-minute, and 5-minute sampling intervals, respectively; IV_0 and IV_1 are the IV estimates from VEC(0) models at tick data with instruments $(p_{\mathrm{N},t_{i-k}} - p_{\mathrm{T},t_{i-k}})$ for $2 \leq \kappa \leq 6$ and $3 \leq \kappa \leq 7$, respectively; H_0 and H_1 denote the p-values of the endogeneity test using the IV_0 and IV_1 estimates of α_δ; F_0 and F_1 denote the F-statistics ($\times 10^{-3}$) from the auxiliary regressions using the two choices of instruments; R_0^2 and R_1^2 are the R^2 measures from the auxiliary regressions using the two choices of instruments; and the number in brackets are the asymptotic standard errors.

	$\mathrm{VEC}(0)_0$	$\mathrm{VEC}(0)_1$	$\mathrm{VEC}(0)_5$	IV_0	IV_1	H_0	H_1	F_0 ($\times 10^{-3}$)	R_0^2	F_1 ($\times 10^{-3}$)	R_1^2
Jun	0.47 (0.01)	1.11 (0.146)	0.76 (0.863)	0.87 (0.029)	0.95 (0.04)	0.00	0.00	23.11	0.31	12.25	0.20
Jul	0.46 (0.012)	1.09 (0.166)	1.27 (0.478)	0.79 (0.027)	0.91 (0.034)	0.00	0.00	42.82	0.48	24.51	0.35
Aug	0.40 (0.012)	0.83 (0.152)	1.07 (0.767)	0.88 (0.043)	1.02 (0.063)	0.00	0.35	20.42	0.29	11.20	0.18
Sep	0.29 (0.012)	0.66 (0.156)	0.38 (0.57)	0.90 (0.034)	1.05 (0.052)	0.00	0.10	21.11	0.32	12.63	0.22
Oct	0.29 (0.009)	0.66 (0.10)	0.41 (0.323)	0.76 (0.029)	0.83 (0.04)	0.00	0.70	28.85	0.37	17.54	0.26
Nov	0.44 (0.01)	0.68 (0.069)	0.09 (0.251)	0.85 (0.027)	0.90 (0.039)	0.00	0.00	26.89	0.39	15.71	0.28
Dec	0.36 (0.009)	0.67 (0.069)	0.79 (0.326)	0.82 (0.024)	0.98 (0.033)	0.00	0.91	30.22	0.41	17.30	0.29
Jan	0.39 (0.015)	0.39 (0.16)	−1.71 (2.588)	0.75 (0.052)	0.83 (0.064)	0.00	0.00	33.97	0.44	20.81	0.33
Feb	0.39 (0.011)	0.82 (0.075)	0.59 (0.257)	0.86 (0.028)	0.99 (0.038)	0.00	0.00	40.77	0.51	25.49	0.39
Mar	0.42 (0.013)	0.64 (0.085)	0.57 (0.38)	0.91 (0.035)	1.12 (0.053)	0.00	0.00	30.10	0.44	18.57	0.33
Apr	0.47 (0.009)	0.80 (0.106)	1.25 (0.514)	0.92 (0.028)	1.07 (0.04)	0.00	0.00	32.71	0.40	19.33	0.28
May	0.41 (0.011)	0.79 (0.102)	1.09 (0.552)	0.78 (0.031)	0.88 (0.043)	0.00	0.00	32.65	0.42	20.61	0.31
Year	0.40 (0.003)	0.75 (0.033)	0.53 (0.112)	0.84 (0.009)	0.95 (0.013)	0.00	0.00	361.08	0.40	214.84	0.28

Table 11: Continuous- and discrete-time IS estimates for NYSE under assumption MMN(TS)

We report monthly and annual averages of the daily continuous- and discrete-time IS measures estimates for the NYSE market. Specifically, the first panel presents the estimates for the continuous-time IS measure, IS_{N}, and the second panel displays the estimates for the discrete-time IS measure, $\text{IS}_{\delta,\text{N}}$. As for IS_{N}, $\text{VEC}(0)_0$ and $\text{VEC}(0)_1$ are computed by LS from VEC(0) models at tick-data and 1-minute sampling intervals, respectively; $\text{IV}_{\text{RK},\bar{q}}$ and $\text{IV}_{\text{HAC},\bar{q}}$ are constructed using the IV estimates of $\alpha_\perp$ with instruments $(p_{\text{N},t_{i-\bar{q}-2}} - p_{\text{T},t_{i-i-\bar{q}-2}}, \ldots, p_{\text{N},t_{i-\bar{q}-6}} - p_{\text{T},t_{i-i-\bar{q}-6}})'$ with $\bar{q} = 0, 1$ and estimates of the integrated covariance matrix computed with the realized kernel estimator and the exact discretization of the HAC estimate of the unconditional covariance matrix, respectively. As for $\text{IS}_{\delta,\text{N}}$, $\text{VEC}(0)_0$ is constructed by LS estimates of $\alpha_\perp$ from the VEC(0) specification at the tick-by-tick sampling interval and estimates of the unconditional covariance that are not robust to heteroskedaticity and serial correlation in the residuals; and $\text{IV}_{\text{HAC},\bar{q}}$ with $\bar{q} = 0, 1$ is constructed with estimates of $\alpha_\perp$ computed by IV from VEC(0) models fitted at the tick-by-tick frequency with instruments $(p_{\text{N},t_{i-\bar{q}-2}} - p_{\text{T},t_{i-i-\bar{q}-2}}, \ldots, p_{\text{N},t_{i-\bar{q}-6}} - p_{\text{T},t_{i-i-\bar{q}-6}})'$ and estimates of the unconditional covariance computed with the HAC estimator. Duration (sec.) is the average duration (in seconds) after aggregating prices using the refresh-time scheme.

	IS_{N}						$\text{IS}_{\delta,\text{N}}$			
	$\text{VEC}(0)_0$	$\text{VEC}(0)_1$	$\text{IV}_{\text{RK},0}$	$\text{IV}_{\text{RK},1}$	$\text{IV}_{\text{HAC},0}$	$\text{IV}_{\text{HAC},1}$	$\text{VEC}(0)_0$	$\text{IV}_{\text{HAC},0}$	$\text{IV}_{\text{HAC},1}$	Duration (sec)
Jun	0.49	0.50	0.50	0.51	0.61	0.57	0.50	0.56	0.56	1.96
	(0.034)	(0.003)	(0.003)	(0.004)	(0.063)	(0.044)	(0.028)	(0.027)	(0.037)	(0.168)
Jul	0.48	0.50	0.50	0.51	0.60	0.57	0.48	0.55	0.56	2.22
	(0.059)	(0.006)	(0.005)	(0.008)	(0.054)	(0.045)	(0.051)	(0.035)	(0.038)	(0.426)
Aug	0.43	0.50	0.50	0.50	0.59	0.56	0.44	0.55	0.55	2.17
	(0.051)	(0.004)	(0.002)	(0.004)	(0.079)	(0.064)	(0.045)	(0.05)	(0.047)	(0.247)
Sep	0.37	0.49	0.50	0.51	0.59	0.55	0.39	0.55	0.55	1.95
	(0.041)	(0.004)	(0.002)	(0.004)	(0.075)	(0.047)	(0.032)	(0.03)	(0.04)	(0.14)
Oct	0.36	0.49	0.50	0.51	0.58	0.55	0.38	0.54	0.55	2.03
	(0.041)	(0.004)	(0.003)	(0.006)	(0.066)	(0.043)	(0.036)	(0.028)	(0.038)	(0.232)
Nov	0.46	0.49	0.51	0.51	0.59	0.56	0.47	0.57	0.55	2.51
	(0.043)	(0.007)	(0.004)	(0.006)	(0.058)	(0.035)	(0.037)	(0.027)	(0.032)	(0.744)
Dec	0.41	0.49	0.51	0.51	0.64	0.60	0.42	0.57	0.59	2.29
	(0.056)	(0.009)	(0.004)	(0.008)	(0.078)	(0.049)	(0.05)	(0.042)	(0.045)	(0.833)
Jan	0.46	0.49	0.50	0.51	0.60	0.58	0.46	0.56	0.57	2.31
	(0.049)	(0.006)	(0.004)	(0.006)	(0.087)	(0.063)	(0.044)	(0.05)	(0.054)	(0.311)
Feb	0.42	0.48	0.51	0.51	0.63	0.57	0.43	0.57	0.57	2.30
	(0.108)	(0.029)	(0.007)	(0.014)	(0.098)	(0.06)	(0.101)	(0.065)	(0.056)	(0.35)
Mar	0.46	0.49	0.51	0.51	0.65	0.59	0.46	0.57	0.58	2.49
	(0.068)	(0.007)	(0.005)	(0.008)	(0.078)	(0.05)	(0.058)	(0.035)	(0.044)	(0.377)
Apr	0.50	0.50	0.51	0.51	0.64	0.58	0.50	0.58	0.57	2.15
	(0.067)	(0.006)	(0.004)	(0.006)	(0.066)	(0.047)	(0.057)	(0.038)	(0.039)	(0.266)
May	0.46	0.50	0.50	0.51	0.57	0.55	0.47	0.54	0.54	2.30
	(0.078)	(0.009)	(0.004)	(0.006)	(0.084)	(0.058)	(0.067)	(0.039)	(0.05)	(0.317)
Year	0.44	0.49	0.50	0.51	0.61	0.57	0.45	0.56	0.56	2.22
	(0.074)	(0.011)	(0.004)	(0.007)	(0.077)	(0.052)	(0.064)	(0.041)	(0.045)	(0.444)

Table 12: Continuous-time CS estimates for NYSE, $\alpha_{\mathrm{N},\perp}$, under assumption MMN(CS)

We estimate monthly and annual component shares for the NYSE using the CU-GMM estimator. Specifically, we select instruments using two rules: the left (right) panel reports the results for instruments selected from stocks which posts at least 7,500 (5,000) quotes per day within an entire month. CU-GMM$_{\mathrm{N}}$ corresponds to the CU-GMM estimates of the CS measure for NYSE; $\alpha_{\perp,\mathrm{N}} = 0$ and $\alpha_{\perp,\mathrm{T}} = 0$, denote p-values from the null hypothesis of $\alpha_{\perp,\mathrm{N}} = 0$ and $\alpha_{\perp,\mathrm{T}} = 0$, respectively; R^2 and F are the R^2 and the F-statistics from the first stage of the IV-based estimator; V is the number of instruments selected, and Duration (sec.) is the average duration (in seconds) after aggregating prices using the refresh-time scheme.

	min. 7,500 quotes per day							min. 5,000 quotes per day						
	CU-GMM$_{\mathrm{N}}$	$\alpha_{\perp,\mathrm{N}} = 0$	$\alpha_{\perp,\mathrm{T}} = 0$	R^2	F	V	Duration (sec.)	CU-GMM$_{\mathrm{N}}$	$\alpha_{\perp,\mathrm{N}} = 0$	$\alpha_{\perp,\mathrm{T}} = 0$	R^2	F	V	Duration (sec.)
Jun	1.00	0.00	1.00	0.00	9.22	19	8.27	1.00	0.52	1.00	0.00	3.22	34	19.86
Jul	1.00	0.10	1.00	0.00	10.46	3	3.89	1.00	0.04	1.00	0.00	5.98	16	10.88
Aug	1.00	0.02	1.00	0.00	10.89	7	5.59	1.00	0.50	1.00	0.01	4.39	24	17.04
Sep	1.00	0.01	1.00	0.01	3.37	13	7.20	1.00	0.09	1.00	0.01	4.82	30	19.83
Oct	0.94	0.01	0.87	0.00	2.97	15	6.72	1.00	0.04	1.00	0.00	4.33	31	18.38
Nov	1.00	0.03	1.00	0.00	12.31	3	4.37	1.00	0.03	1.00	0.00	2.36	15	8.92
Dec						0	2.11	1.00	0.02	1.00	0.00	5.59	9	9.31
Jan	1.00	0.17	1.00	0.00	2.84	15	7.90	0.75	0.25	0.72	0.00	2.23	36	24.25
Feb	1.00	0.64	1.00	0.00	2.82	15	9.30	1.00	0.04	1.00	0.01	3.58	32	19.98
Mar	1.00	0.07	1.00	0.00	12.67	17	9.64	1.00	0.62	1.00	0.01	5.11	37	25.38
Apr	1.00	0.05	1.00	0.00	9.79	23	11.13	1.00	0.00	1.00	0.01	5.34	40	20.55
May	1.00	0.00	1.00	0.01	3.27	20	11.03	1.00	0.01	1.00	0.00	4.36	36	18.42

Figure 6: Daily estimates of $\alpha_\perp$

We report the daily estimates of the continuous-time CS measures for AA computed from different estimators. The top panel displays the LS estimates of the CS measures for both VEC(0) and VEC(1) specifications, whereas the lower panel exhibits the corresponding GMM estimates for the VEC(0) model. As for the GMM estimators of the CS measures, the left graph corresponds to the continuous-time based GMM estimates with instruments given by $(p_{\mathrm{N},t_{i-2}} - p_{\mathrm{T},t_{i-2}}, \ldots, p_{\mathrm{N},t_{i-6}} - p_{\mathrm{T},t_{i-6}})'$, whereas the right plot refers to the continuous-time based GMM estimates with instruments given by $(p_{\mathrm{N},t_{i-3}} - p_{\mathrm{T},t_{i-3}}, \ldots, p_{\mathrm{N},t_{i-7}} - p_{\mathrm{T},t_{i-7}})'$. The solid lines correspond to the estimates of $\alpha_\perp$ and the shaded area is bounded by the 95% confidence interval.

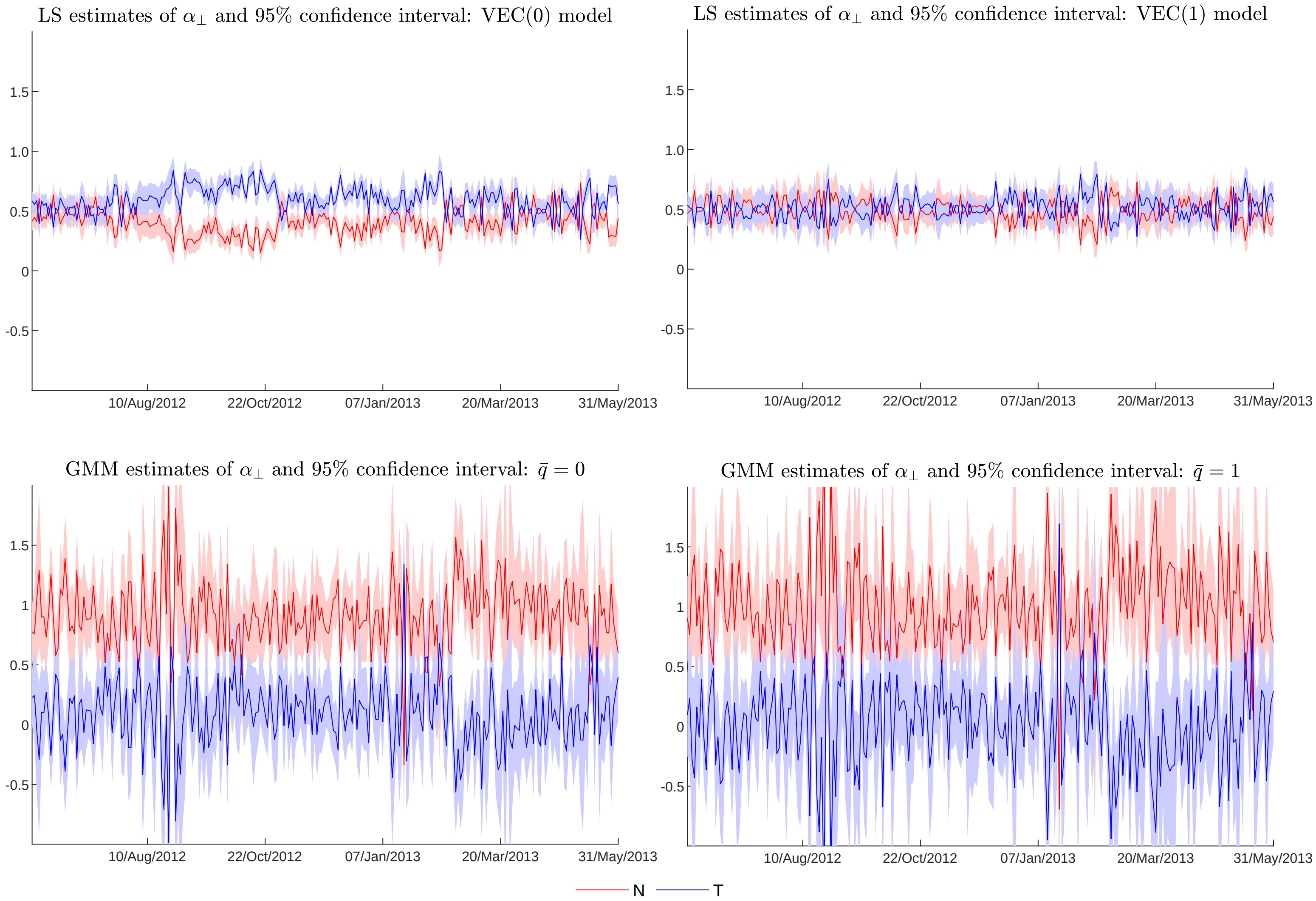

Figure 7: Impulse response function (IRF): Nasdaq

We report the IRFs estimates (median over days in the entire sample) for impulses in Nasdaq. VEC$(0)_0$ refers to the IRFs estimates that are constructed with estimates of $\alpha_\perp$ computed by LS from VEC(0) models at tick-data and unconditional covariance matrix estimates that are not robust to heteroskedastic and serially correlated residuals; VEC$(1)_0$ is the IRFs estimates that combines estimates of $\alpha_\perp$ computed by LS from VEC(1) models fitted at tick-data and unconditional covariance matrix that is not robust to heteroskedastic and serially correlated residuals; IV$_{\text{HAC},\bar{q}}$ with $\bar{q} = 0, 1$ corresponds to estimates of IRFs that are constructed with IV-based estimates of $\alpha_\perp$ from VEC(0) models at tick-data with instruments $(p_{\text{N},t_{i-\bar{q}-2}} - p_{\text{T},t_{i-\bar{q}-2}}, \ldots, p_{\text{N},t_{i-\bar{q}-6}} - p_{\text{T},t_{i-\bar{q}-6}})'$ and HAC estimates of the unconditional covariance matrix. The solid lines correspond to the median IRFs and the the shaded area indicates the 90% confidence intervals.