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Abstract

We analyze the impact of financial market fragmentation when traders differ in terms of market access and information about the asset payoff. Agents who cannot operate on all trading venues are hurt by fragmentation. Those who can trade globally can benefit. Overall liquidity peaks at an intermediate level of fragmentation; price informativeness continues to improve until a higher level. A larger investor base and less information asymmetry widen the scope for beneficial fragmentation, while greater multi-venue access narrows it. The model explains the observed non-monotonic relationship between fragmentation and liquidity, and the differential impact on large-cap versus small-cap stocks.

Journal of Economic Literature classification numbers: D47, D82, G14.

Keywords: Multiple exchanges, price impact, adverse selection, price informativeness.

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1 Introduction

Financial markets in the United States and Europe are highly fragmented. The primary stock exchanges in the US, NYSE and Nasdaq, which accounted for approximately 95% of US equity trading in 2000, had their market share fall below 30% by 2016 (Haslag and Ringgenberg 2023). This shift was driven in part by regulations designed to foster competition among trading venues, most notably Regulation National Market System (Reg NMS), which was implemented in 2007. Fragmentation has also developed in Europe since 2007, when the Markets in Financial Instruments Directive (MiFID) abolished the "concentration rule" that required domestically listed equities to be traded on the primary exchanges (Gresse 2017).

This fragmentation raises a number of important questions. How does it affect market quality? Are markets more or less liquid? Are prices more or less informative? Are traders better or worse off? In order to answer these questions, we propose a model in which a risky asset is traded on multiple exchanges.[1] Traders differ along two dimensions. A trader can be either informed or uninformed about the asset payoff, and she can be either a local trader, restricted to operating on a single exchange, or a global player able to access all exchanges.

We study the effects of market fragmentation, i.e. an increase in the number of exchanges, on the balance between informed and uninformed trade on the one hand, and between the local and global components of uninformed trade on the other. The first drives adverse selection and liquidity. The second governs the correlation between prices on different exchanges, and thereby the overall informativeness of prices. We abstract from other considerations, such as competition between exchanges, transaction fees, high frequency trading, and trading on dark pools or over-the-counter venues. All traders have a well-defined objective, so we can assess who gains and who loses from fragmentation.

Our paper makes two main contributions. First, we characterize the impact of fragmentation on market outcomes when agents are asymmetrically informed and adverse selection plays a key role, in contrast to most existing theoretical work, which assumes symmetric information. Within this framework, we identify a wedge between two natural notions of optimal fragmentation: the number of exchanges that maximizes overall liquidity is lower than the number that maximizes overall price informativeness, so that fragmentation can continue to improve price discovery even after it has begun to reduce liquidity. Second, our results provide a theoretical underpinning for previously unexplained empirical regularities,

[1]Throughout the paper, we use "exchange" to refer to a lit trading venue, i.e. a platform with pre-trade transparency. In Europe, these include both traditional exchanges and Multilateral Trading Facilities (MTFs).

in particular the inverted U-shaped relationship between overall liquidity and the degree of fragmentation (Degryse et al. 2015; Ibikunle et al. 2020), and the finding that fragmentation enhances overall liquidity for large-cap stocks but reduces it for small-cap stocks (Gresse 2017; Lausen et al. 2022; Haslag and Ringgenberg 2023).

The details of the model are as follows. The investor population consists of speculators and liquidity traders. Speculators are privately informed about the asset payoff. Each liquidity trader has a random endowment of the asset. She privately observes the realization of her endowment shock but has no information about the asset payoff. Speculators can operate on all exchanges. Some liquidity traders are also unrestricted and can trade on every exchange, while others are restricted and can trade on only one exchange. Each trader submits a market order on each exchange on which she is active, taking into account the price impact of her order. A competitive risk-neutral market-making sector sets the price on each exchange equal to the expected payoff of the asset, conditional on the aggregate order flow on that exchange. Thus our model is a multi-exchange extension of Kyle (1985), with noise traders replaced by optimizing liquidity traders.

We characterize the equilibrium in terms of local and global liquidity. Local liquidity is the inverse of "Kyle's lambda" on a single exchange, and measures the ease of trading on that exchange. In our symmetric model, it does not depend on the exchange under consideration. We define global (or overall) liquidity as local liquidity multiplied by the number of exchanges. It captures the ease of trading on all exchanges simultaneously. Market fragmentation impacts the welfare of a trader through local liquidity if this trader is local, i.e. tied to a single exchange, and through global liquidity if she is a global player and can trade across all exchanges. We find that market fragmentation always lowers local liquidity on any given exchange, making local traders worse off. If liquidity is relatively low on a single integrated exchange, market fragmentation lowers global liquidity as well, making all agents worse off. In this case, full consolidation is optimal. However, if liquidity is high on a centralized exchange, fragmentation increases global liquidity up to a threshold number of exchanges E^*, after which it declines. The welfare of global traders is maximized at E^*. The key takeaway is that the impact of fragmentation on an agent's welfare does not depend on whether she is informed or uninformed, but on whether she is a local or global player.[2]

With regard to price discovery, it is also instructive to think in terms of local and global magnitudes. Local price informativeness measures the information conveyed by the price on any given exchange, while global (or overall) price informativeness measures the infor-

[2]This holds for our baseline model with a fixed number of speculators. In Section A5 of the Online Appendix, we show that with endogenous information acquisition, fragmentation can lead to a Pareto improvement.

mativeness of prices on all exchanges taken together. In our model, market fragmentation leaves local price informativeness unchanged. But it does affect global price informativeness. Each exchange has its own set of local agents who cannot trade on other exchanges, and hence prices across exchanges are not perfectly correlated. Since multiple prices are more informative than a single price, global price informativeness is always higher when markets are fragmented compared to when they are not. However, beyond a threshold level of fragmentation, local trades taper off faster than trades by globally active agents. Consequently, prices become more correlated across exchanges and convey less information. We show that global price informativeness is increasing until the number of exchanges reaches $\hat{E}$, after which it declines.

The threshold level $\hat{E}$ is higher than E^*, the number of exchanges at which global liquidity and the welfare of global traders are maximal. Thus any fragmentation beyond $\hat{E}$ not only reduces global price informativeness, but also lowers global liquidity and makes all agents worse off.

Both E^* and $\hat{E}$ depend on the model's primitives only through two composite parameters that separate a *scale* effect from a *composition* effect. The scale parameter, which we call the *fragmentation potential* of the asset and denote by Φ, is increasing in the strength of liquidity needs and the intensity of speculator competition, and decreasing in the degree of asymmetric information about the asset payoff. A higher fragmentation potential raises both E^* and $\hat{E}$, widening the scope for fragmentation to improve market quality, whether it is measured by global liquidity, global price informativeness, or the welfare of global traders.[3] The composition parameter, denoted by φ, is the share of liquidity traders with multi-venue access, and pulls in the opposite direction: a higher share lowers both thresholds.

In an illustrative quantitative exercise, we use the available empirical evidence to infer the composite parameters (φ, Φ) for large- and small-cap stocks in the US and Europe, and compute the model-implied price-informativeness-maximizing thresholds $\hat{E}$. Observed fragmentation lies below $\hat{E}$ in essentially all segments, including those where it has reduced global liquidity.

The rest of the paper is organized as follows. We review the literature in Section 2, describe the model in Section 3, and provide conditions for equilibrium existence and uniqueness in Section 4. Sections 5, 6, and 7 are devoted in turn to liquidity, welfare, and price informativeness. We discuss the quantitative implications of the model in Section 8, and extensions in Section 9. Section 10 concludes. All proofs are in the Appendix.

[3]We use phrases such as "improvement in market quality" or "beneficial fragmentation" as shorthand for an increase in any of these three metrics. Local traders are always hurt by fragmentation in our model.

2 Literature Review

Since the work of Hamilton (1979), the theoretical and empirical literature has largely framed market fragmentation in terms of a tradeoff between the benefits of inter-exchange competition and the loss of positive liquidity externalities. We abstract from inter-exchange competition and focus instead on the interaction between liquidity externalities and adverse selection. We begin with a discussion of the theoretical literature, and then review the relevant empirical research.

2.1 Theoretical literature

Recent theoretical work on the effects of fragmentation has focused on two main themes: risk sharing and competition between exchanges, both under the assumption of symmetric information. This literature does not address the adverse selection and price discovery issues that we study. The set of papers that allow for asymmetric information about asset values, the focus of our paper, remains small. We discuss these papers first, since they are most directly related to ours, and then turn to the larger symmetric information literature.[4]

The paper that is most closely related to ours is Chowdhry and Nanda (1991), who study a multimarket version of the Kyle (1985) model of insider trading. Their model features one informed speculator; one large liquidity trader who splits an exogenously given random order across exchanges to minimize the total expected cost of trading; and small liquidity traders who are restricted to a single exchange. The aggregate order of the small liquidity traders on any exchange is given by an exogenous random variable. The authors claim that an increase in the number of exchanges improves global price informativeness and the welfare of the speculator. However, the price informativeness claim is not clearly established,[5] and there is no formal welfare analysis in the paper.

Unlike Chowdhry and Nanda (1991), we model liquidity traders as strategic utility-maximizing agents with random endowments of the asset. We find that, on any given exchange, a liquidity trader trades more if liquidity is higher on that exchange. Moreover, even for a given level of liquidity on each exchange, the aggregate trade of an unrestricted liquidity trader is increasing in the number of exchanges. Our results on liquidity and

[4] We only survey papers where the markets can be interpreted as exchanges (as opposed to dark pools or over-the-counter markets).

[5] Chowdhry and Nanda (1991) define their measure of price informativeness ψ as the ratio of the conditional variance of the asset payoff, given all prices, to its unconditional variance. This is inversely related to price informativeness as conventionally defined. The formula derived in their Proposition 2 in fact corresponds to the standard informativeness measure $1-\psi$ rather than to ψ. The comparative static in the number of markets N implicitly assumes that $\sum_{i=1}^{N}\sigma_i$ does not depend on N, where σ_i is the standard deviation of small liquidity trade on market i.

welfare are intimately tied to the effect of market fragmentation on the relative magnitude of speculative to liquidity trade. Similarly, our price discovery results bear a close connection to the effect of fragmentation on the relative size of unrestricted to restricted liquidity trade. Endogenizing liquidity trade thus yields qualitatively new results on liquidity, welfare, and price informativeness, none of which can be derived in the Chowdhry-Nanda setting with exogenous liquidity trade.

There are some other asymmetric information models of market fragmentation, but in settings that differ in significant ways from ours. Madhavan (1995) finds that fragmentation improves the welfare of large traders, but has an ambiguous effect on bid-ask spreads. Biais et al. (2015) study the interaction of fast and slow traders when exchanges differ in the liquidity that they offer. They show that more fragmentation can result in wider spreads. In Kawakami (2017), higher price informativeness negatively impacts risk sharing due to the Hirshleifer effect. Splitting a centralized market into smaller informationally separate markets can then lead to higher welfare by reducing the informativeness of prices in each market.

Most of the research that studies the impact of market fragmentation assumes that agents have symmetric information about asset payoffs. One strand of the literature focuses on risk sharing or gains from trade. Mendelson (1987) compares a consolidated market to multiple submarkets, with each trader restricted to a single submarket. He finds that fragmentation reduces expected trading volume and expected gains from trade. Pagano (1989) shows that fragmenting a single market into two, with each trader choosing one market in which to participate, is welfare-reducing. Malamud and Rostek (2017), Chen and Duffie (2021) and Rostek and Yoon (2021) show that inefficiencies in risk sharing due to imperfect competition can be ameliorated by market fragmentation under certain conditions.[6] In a similar setup, Wittwer (2021) evaluates the welfare consequences of connecting disconnected markets.

A separate strand of the symmetric information literature studies competition among exchanges, focusing on issues such as fee competition, order routing, speed, and sniping by high frequency traders (Parlour and Seppi 2003; Foucault and Menkveld 2008; Colliard and Foucault 2012; Pagnotta and Philippon 2018; Baldauf and Mollner 2021).

[6]The payoff of a liquidity trader in our model is the same as that of a strategic trader in Chen and Duffie (2021). Unlike Chen and Duffie (2021), however, we also have informed speculators, and prices convey information about the asset payoff. In their model, prices are informative about traders' endowments but not about the asset payoff.

2.2 Empirical literature

We now review the empirical evidence on the impact of fragmentation on global liquidity, local liquidity, and price informativeness. Global liquidity measures (consolidated spreads or consolidated depth) refer to liquidity across all market centers, while local liquidity measures refer to a single market center.[7] Our own notions of global and local liquidity, formally defined at the beginning of Section 5, are closer to the empirical global and local depth measures.

On global liquidity, early studies report mixed findings, with some documenting an improvement (Boehmer and Boehmer 2003 and O'Hara and Ye 2011 for the US; Foucault and Menkveld 2008 and Spankowski et al. 2012 for Europe) and others a deterioration (Bennett and Wei 2006 and Chung and Chuwonganant 2012 for the US). A more nuanced picture emerges when stocks are differentiated by size or initial liquidity: fragmentation enhances global liquidity for large-cap stocks but reduces it for small-cap or relatively illiquid stocks (Haslag and Ringgenberg 2023 for the US; Gresse 2017 and Lausen et al. 2022 for Europe).[8] The conflicting earlier studies can be partially reconciled along these lines: the sample in Chung and Chuwonganant (2012) weights small-cap stocks more heavily than that in O'Hara and Ye (2011), as noted by Haslag and Ringgenberg (2023); and Bennett and Wei (2006) themselves report stronger negative effects within the less liquid stocks in their sample. Large-cap stocks naturally have a larger investor base, of both liquidity traders and speculators, and less information asymmetry about the asset payoff. Both of these raise the composite parameter Φ in our model, which we call the fragmentation potential of the asset. The differential effect of fragmentation across stock size segments therefore aligns with our result that fragmentation improves global liquidity if and only if Φ exceeds a cutoff level (Proposition 5.2 (ii), (iii)).

Going beyond the sign of the fragmentation effect, two studies investigate its shape. Degryse et al. (2015), analyzing Dutch stocks in the 2007–2009 period, document an inverted U-shaped relationship between visible fragmentation and global depth, with the peak occurring at a higher level of fragmentation for large-cap stocks (see their Figure 4); and Ibikunle et al. (2020) identify a U-shaped relationship between fragmentation and aggregate adverse

[7]These measures relate only to venues with a visible order book. We survey the headline results, and do not focus on the details of the precise spread measure (i.e. quoted, effective, or realized spread), or the precise depth measure (at the best price or deeper in the order book).

[8]This has been acknowledged by regulators in the US: "In the equity markets, the current "one-size-fits-all" market structure is not working well for smaller companies that are currently experiencing limited liquidity for their shares. While the largest and most actively traded companies benefit from a diversity of trading venues, for the least liquid (and often smallest) companies, fragmentation of liquidity across 12 equity exchanges and 40 alternative trading systems (ATSs) may inhibit effective liquidity provision." (U.S. Department of the Treasury 2017, pp. 7–8).

selection risk for FTSE 100 stocks (see their Figure 3). Both match our theoretical results on global liquidity, which has an inverted U-shape (Proposition 5.2 (iii); Figure 1), and is maximized at a level E^* that is increasing in the fragmentation potential Φ (Proposition 5.2 (iv)).

Turning to local liquidity, Degryse et al. (2015) report a negative effect of fragmentation on the liquidity of the primary exchange; Gresse (2017) reports a similar effect for small-cap stocks; and Guo and Jain (2023) find that the entry of MEMX in the US in 2020 reduced local liquidity on the existing lit venues. All three are in line with our result that local liquidity falls with fragmentation, thereby hurting local traders (Proposition 5.1 (ii), Proposition 6.3 (i)). Other studies, however, link fragmentation to higher liquidity on the primary exchange (Boehmer and Boehmer 2003 for the US; Foucault and Menkveld 2008, Hengelbrock and Theissen 2009, Gentile and Fioravanti 2011, and Spankowski et al. 2012 for Europe), possibly reflecting increased inter-exchange competition, which we abstract from.[9]

The evidence on price informativeness is mixed. On global price informativeness, O'Hara and Ye (2011) find an increase with fragmentation, while Bennett and Wei (2006) and Chung and Chuwonganant (2012) find a decrease; Haslag and Ringgenberg (2023) report an increase for large- and mid-cap stocks and a decrease for small-cap stocks. Ibikunle et al. (2020) interpret their evidence as an inverted U-shaped relationship between fragmentation and global price efficiency, in line with our results (Proposition 7.3; Figure 2). On local price informativeness, Gentile and Fioravanti (2011) report a reduction on primary exchanges, while Boehmer and Boehmer (2003) find little effect. The latter is consistent with our result that local price informativeness is unaffected by fragmentation (Proposition 7.2 (i)).

3 The Economy

We consider a two-date economy in which all trades occur at the initial date, and payoffs are realized at the final date. There is a single risky asset with payoff v, and E exchanges on which this asset is traded. There are three groups of traders: S speculators, L^U unrestricted liquidity traders, and $\bar{L}$ restricted liquidity traders. Speculators and unrestricted liquidity traders can trade on all exchanges, while a restricted liquidity trader can only trade on a single exchange, which is exogenously specified for each such trader. The number of restricted liquidity traders on exchange i is L_i^R. With some abuse of notation, we use E, L^U, $\bar{L}$, and L_i^R to also denote the index set of exchanges, unrestricted liquidity traders, restricted liquidity traders, and restricted liquidity traders assigned to exchange i, respectively. Thus $i \in E$ is

[9] This is consistent with our model if we allow for endogenous information acquisition; see Proposition A5.3 (ii) in the Online Appendix.

exchange i, $\ell \in L^U$ is an unrestricted liquidity trader, and $\ell \in L_i^R$ is a restricted liquidity trader assigned to exchange i. The symbol ℓ serves only as an index; any given liquidity trader is either unrestricted or restricted. We assume that $E \geq 1$, $S \geq 1$, $L^U + \bar{L} \geq 1$, and $L_i^R = L^R := \bar{L}/E$. Thus there is an equal number of restricted liquidity traders on each exchange. While both L^U and $\bar{L}$ can be positive, we also consider special cases where either L^U or $\bar{L}$ is equal to zero. For simplicity of exposition, we ignore integer constraints. All the results can be restated in a form that respects these constraints.

Speculators have no endowment and observe the asset payoff v prior to trade. Liquidity trader ℓ has a stochastic endowment of the asset, which we denote by $-y_\ell$ (the negative sign is for expositional convenience and entails no loss of generality). She does not know the realization of v but privately observes y_ℓ. The random variables $(v, \{y_\ell\}_{\ell \in L^U \cup \bar{L}})$ are joint normal with mean zero, and mutually independent. We assume that $\mathrm{Var}(y_\ell) = \sigma_y^2$ for all ℓ, and denote the variance of v by σ_v^2.

Trades take the form of market orders. For exchange i, we denote the price by p_i, and the trade of a speculator, unrestricted liquidity trader $\ell \in L^U$, and restricted liquidity trader $\ell \in L_i^R$ by q_i^S, $q_{i\ell}^U$, and $q_{i\ell}^R$, respectively. Note that it is implicit in this notation that, on any given exchange, all speculators trade the same amount. Payoffs for these agents are:

$$W^S = \sum_{i \in E} (v - p_i) q_i^S, \tag{1}$$

$$W_\ell^U = \sum_{i \in E} (v - p_i) q_{i\ell}^U - y_\ell v - \frac{k}{2} \left(\sum_{i \in E} q_{i\ell}^U - y_\ell \right)^2, \tag{2}$$

$$W_{i\ell}^R = (v - p_i) q_{i\ell}^R - y_\ell v - \frac{k}{2} \left(q_{i\ell}^R - y_\ell \right)^2, \tag{3}$$

where k is a positive parameter. Thus each liquidity trader faces a holding cost, which is quadratic in her net position in the asset. The agents solve $\max_{q_i^S} \mathbb{E}(W^S | v)$, $\max_{q_{i\ell}^U} \mathbb{E}(W_\ell^U | y_\ell)$, and $\max_{q_{i\ell}^R} \mathbb{E}(W_{i\ell}^R | y_\ell)$, respectively, taking into account the price formation mechanism which we will describe shortly. We will continue to use the superscripts S, U, and R in the analysis below to identify variables for speculators, unrestricted liquidity traders, and restricted liquidity traders, respectively.

On each exchange, a competitive market-making sector sets the price equal to the expected payoff of the asset conditional on the aggregate order flow on that exchange. Let q_i denote the aggregate order flow on exchange i:

$$q_i := S q_i^S + \sum_{\ell \in L^U} q_{i\ell}^U + \sum_{\ell \in L_i^R} q_{i\ell}^R. \tag{4}$$

Equilibrium prices are determined by the condition

$$p_i = \mathbb{E}(v|q_i), \qquad i \in E. \tag{5}$$

We say that markets are fragmented if $E > 1$, and measure the degree or level of market fragmentation by E. We also use the term "market fragmentation" to refer to an increase in E. The usage is intuitive and should be clear from the context.

3.1 Discussion of the assumptions

Most of the assumptions are natural for what is essentially an extension of the Kyle (1985) model to allow for multiple exchanges, with the noise traders in Kyle's model replaced by optimizing liquidity traders. As we shall see, all agents respond to the endogenous level of liquidity, affecting it in turn. Also, explicitly modeling the objectives of all agents allows us to determine who gains and who loses from market fragmentation.

In our model some traders are local while others are global players. In European markets, we can interpret global traders as those who use smart order routing technology (SORT) to enable them to trade wherever the best price is available. In the US, the order protection rule in Reg NMS renders the fragmented market for equities as effectively consolidated for small trades. However, this is not the case for large trades, as trade-through protection only applies to the top of the book at each market center. Thus the market remains fragmented with varying access for different investors.

Traders in our model are imperfectly competitive, taking the price impact of their orders into account. Speculators and unrestricted liquidity traders can naturally be interpreted as large institutional players—hedge funds, proprietary trading desks, and asset managers with multi-venue access—for whom execution costs are a first-order consideration. Restricted liquidity traders are similarly institutional in nature, but lack the technology or relationships to access multiple venues. In the European context, this category includes asset managers and brokers without SORT, and mid-tier intermediaries whose execution is tied to a single venue determined by client relationships or operational arrangements. In the US, it includes traders of large blocks and liquidity providers using resting limit orders. Even though these traders are local, they are sufficiently large for price impact to enter into their trading decisions.

We assume that restricted liquidity traders are evenly distributed across venues. This implies that total order flow is evenly distributed as well. In actual markets, order flow tends to be concentrated on a small number of venues rather than being equally dispersed. We interpret the symmetric structure of the model as representing a market with an effective number of equal-sized venues, rather than a raw count of trading platforms; the precise mapping to observed market shares is given in Section 8.1. The mechanisms underlying our results do not require perfect symmetry; the symmetric structure is a tractable modeling choice rather than a substantive claim about the spatial distribution of order flow.

Under the pricing rule $p_i = \mathbb{E}(v|q_i)$, the price on each exchange depends only on its local order flow. This is the polar case in which information across venues is fully segmented. The opposite polar case, in which market makers on each exchange condition on order flow from all venues, is degenerate: prices on different exchanges are perfectly correlated, and fragmentation collapses to a single integrated market with no effect on liquidity, welfare, or price informativeness. The empirical evidence on fragmentation points to substantial deviation from perfect integration.[10] Our pricing rule is best read as a tractable approximation of this empirically pervasive friction. A model with partially integrated cross-venue information would dampen, but not eliminate, the channels operating in our framework.

For simplicity we assume that speculators, who observe the true value of the asset prior to trade, are risk-neutral. Liquidity traders face a quadratic cost on their net position in the asset, which can be interpreted as a cost of bearing risk or some other cost associated with holding inventory. This specification is increasingly common in the literature.[11]

The assumption that liquidity traders incur a holding cost while speculators do not simplifies the analysis. It can be justified on the grounds that speculators are often sophisticated traders with short horizons, while liquidity traders have longer holding periods. The assumption that the endowment shocks of liquidity traders are mutually independent is also for analytical tractability.

We relax some of these assumptions in extensions of the baseline model that we present in the Online Appendix. In the first extension, all agents, speculators as well as liquidity traders, pay the same holding costs. In the second extension, holding costs are not paid by any agent; instead, we assume that liquidity traders have constant absolute risk aversion. In the third extension, we allow for correlation between the endowment shocks of liquidity traders. In the fourth extension, we endogenize information acquisition by speculators.

In the first three extensions, most of the results of our baseline model on liquidity, welfare, and price informativeness continue to hold. However, endogenizing information acquisition leads to results that are qualitatively different.

[10] Evidence of imperfect integration includes better-priced quotes on individual exchanges that are not reflected in the publicly disseminated consolidated quote (Bartlett et al. 2025); economically significant price discovery on individual venues during periods when other venues are closed (Eaton et al. 2025); and the failure of cross-venue substitution during exchange outages (Degryse et al. 2025).

[11] See, for example, Vives (2011), Rostek and Weretka (2012, 2015), Du and Zhu (2017), Chen and Duffie (2021), Glebkin and Kuong (2023), and Lou and Rahi (2023).

4 Equilibrium

In this section, we describe the equilibrium notion in our model. We calculate agents' optimal strategies and provide conditions for equilibrium existence.

We look for a linear equilibrium, by which we mean an equilibrium at which $p_i = \lambda q_i$, for some $\lambda > 0$, and

$$q_i^S = \beta v, \qquad q_{i\ell}^U = \alpha^U y_\ell, \qquad q_{i\ell}^R = \alpha^R y_\ell, \tag{6}$$

for some scalars β, α^U, and α^R. At a linear equilibrium, the price on exchange i is proportional to the aggregate order flow on that exchange, q_i. The price impact parameter λ is the same for all exchanges. Agents' strategies are also symmetric—the parameters β, α^U, and α^R are the same for all speculators, unrestricted liquidity traders, and restricted liquidity traders, respectively, and do not depend on the exchange under consideration.

From (4) and (6), the aggregate order flow is

$$q_i = S\beta v + \alpha^U Q^U + \alpha^R Q_i^R, \tag{7}$$

where $Q^U := \sum_{\ell \in L^U} y_\ell$ and $Q_i^R := \sum_{\ell \in L_i^R} y_\ell$, and the price on exchange i is

$$p_i = \lambda q_i = \lambda(S\beta v + \alpha^U Q^U + \alpha^R Q_i^R). \tag{8}$$

We begin by calculating agents' optimal strategies for given λ.

Lemma 4.1 (Optimal strategies) *Suppose prices are given by $p_i = \lambda q_i$, $i \in E$, for some $\lambda > 0$. Then the following strategies are optimal for agents: $q_i^S = \beta v$, $q_{i\ell}^U = \alpha^U y_\ell$, and $q_{i\ell}^R = \alpha^R y_\ell$, where*

$$\beta = \frac{1}{(1+S)\lambda}, \tag{9}$$

$$\alpha^U = \frac{k}{kE + 2\lambda}, \tag{10}$$

$$\alpha^R = \frac{k}{k + 2\lambda}. \tag{11}$$

As one would expect, all agents trade less aggressively when facing a higher price impact parameter λ. It is instructive, however, to see how their trades vary for given λ. Each speculator trades less when there are more speculators. This is due to greater competition. In contrast, the position of a liquidity trader does not directly depend on the number of speculators or of other liquidity traders, since her value for the asset is uncorrelated with that of other traders. An increase in E induces an unrestricted liquidity trader to trade less per exchange, which reflects order-splitting to reduce total price-impact costs, though her aggregate trade across all exchanges is increasing in E, as the lower global execution cost λ/E allows her to move her total position closer to her endowment, thereby reducing holding costs. A speculator, who does not incur any holding cost, trades the same amount on each exchange regardless of the degree of fragmentation. Thus her aggregate trade across all exchanges increases linearly with E. A restricted liquidity trader operates on only one exchange, so for given λ, which reflects the liquidity of any single exchange, her trade is unaffected by the number of exchanges.

Before turning to equilibrium existence, we introduce two composite parameters that will appear throughout the analysis. While our model has six primitives (S, L^U, $\bar{L}$, k, σ_y^2, and σ_v^2), we will show in the following sections that the key equilibrium quantities depend on these primitives only through (φ, Φ), defined as follows:

$$\varphi := \frac{L^U}{L^U + \bar{L}}, \qquad \Phi := k^2(L^U + \bar{L})\sigma_y^2 \cdot \frac{(1+S)^2}{S} \cdot \frac{1}{\sigma_v^2}. \tag{12}$$

The parameter φ is the fraction of liquidity traders who are unrestricted. The parameter Φ decomposes multiplicatively into three terms, each tied to a distinct feature of the trading environment. The first term, $k^2(L^U + \bar{L})\sigma_y^2$, measures the strength of liquidity needs; stronger liquidity needs reduce adverse selection (we introduce an explicit measure of adverse selection in the next section). The second term, $(1 + S)^2/S$, is increasing in S, and captures the intensity of speculator competition. While an increase in S also raises adverse selection, it is the competition effect that dominates, as in Admati and Pfleiderer (1988). The third term, $1/\sigma_v^2$, is the inverse of the degree of asymmetric information; less information asymmetry reduces adverse selection. Framing the results in terms of (φ, Φ) allows us to separate a scale effect captured by Φ, which is increasing in the number of traders of any type, from a composition effect represented by φ, which is the share of liquidity traders with multi-venue access.

Lemma 4.1 specifies agents' optimal trades in terms of λ, which is endogenous. Thus a linear equilibrium is completely determined by λ. It exists if and only if there is a $\lambda > 0$ such that $p_i = \mathbb{E}(v|q_i) = \lambda q_i$, $i \in E$. A necessary and sufficient condition is provided by the next result.

Proposition 4.2 (Equilibrium existence and uniqueness) *For any given E, a linear equilibrium exists if and only if*

$$\frac{k^2(L^U + L^R)(1+S)^2}{4S} \cdot \frac{\sigma_y^2}{\sigma_v^2} > 1, \tag{13}$$

or equivalently,

$$\Phi\left[\varphi + \frac{1-\varphi}{E}\right] > 4. \tag{14}$$

Furthermore, if a linear equilibrium exists, it is unique (in the class of linear equilibria).

Proposition 4.2 says that a linear equilibrium exists if Φ is large enough relative to the level of fragmentation E. In terms of the three components of Φ, existence relies on adverse selection being sufficiently low (through strong liquidity needs or a low degree of asymmetric information) or on sufficiently intense speculator competition.

Note that, if $L^U = 0$ (or equivalently $\varphi = 0$), the existence condition (14) becomes $\Phi/E > 4$, which cannot hold for arbitrarily large E. We therefore need to impose an upper bound on E. Letting

$$\bar{E}_1 := \frac{k^2\bar{L}(1+S)^2}{4S} \cdot \frac{\sigma_y^2}{\sigma_v^2},$$

we define an upper bound for E as follows:

$$\bar{E} := \begin{cases} \bar{E}_1, & \text{if } L^U = 0, \\ \infty, & \text{if } L^U > 0. \end{cases} \tag{15}$$

For the rest of the paper we assume that $E \in [1, \bar{E})$ and that condition (13), or equivalently (14), holds for all E in this interval.[12]

5 Price Impact and Liquidity

In this section, we analyze price impact or its inverse, liquidity, which is the key parameter that describes an equilibrium. With multiple exchanges, there are two notions of liquidity, local and global, which apply respectively to trading on a single exchange and to trading on all exchanges simultaneously. We show that local liquidity is decreasing in the degree of market fragmentation. In contrast, if liquidity on a single centralized exchange is sufficiently high, global liquidity displays an inverted U-shaped pattern, initially rising with fragmentation, and then declining.

As noted in Section 4, a linear equilibrium is completely specified by the parameter λ. We can think of λ as the *local price impact* parameter that an agent faces when trading on a single exchange. For an agent who trades across multiple exchanges, it is convenient to consider the *global price impact* parameter $\gamma := \lambda/E$, which measures the price impact of a

[12] If $L^U > 0$, condition (14) holds for all $E \in [1, \infty)$ if and only if $\Phi\varphi \geq 4$. If $L^U = 0$, this condition is in fact equivalent to $E < \bar{E}_1$. We can make $\bar{E}_1$ as large as desired by choosing a sufficiently large $\bar{L}$.

unit trade split up equally across all exchanges. We define liquidity as the reciprocal of price impact; thus *local liquidity* is measured by λ^{-1}, and *global liquidity* by γ^{-1}.

The next result summarizes the properties of the price impact parameters λ and γ. We say that $A \propto B$ if A and B have the same sign ($A = \tau B$, for some $\tau > 0$).

Proposition 5.1 (Price impact) *We have the following results on price impact:*

i. λ *satisfies*

$$\frac{\varphi}{(kE\lambda^{-1}+2)^2} + \frac{1-\varphi}{E(k\lambda^{-1}+2)^2} = \Phi^{-1}. \tag{16}$$

Hence $\lambda = \lambda(E, k, \varphi, \Phi)$ *and* $\gamma = \gamma(E, k, \varphi, \Phi)$.

ii. $\partial\lambda/\partial E > 0$, *and* $\lim_{E \to \bar{E}} \lambda = \infty$.

iii. If $\bar{L} = 0$, γ *is independent of* E.

iv. If $\bar{L} > 0$, $\partial\gamma/\partial E \propto 2\lambda - k$.

v. $\partial\lambda/\partial\Phi < 0$, *and* $\partial\gamma/\partial\Phi < 0$.

The comparative statics results (ii)–(v) can be interpreted in terms of adverse selection. Recall that the aggregate order flow on exchange i is q_i, given by (7). We define the degree of adverse selection on exchange i as the ratio of the variance of speculative trade to the variance of liquidity trade in the order flow q_i:

$$\mathcal{A}_i := \frac{\mathrm{Var}(S\beta v)}{\mathrm{Var}(\alpha^U Q^U + \alpha^R Q_i^R)} = \frac{S^2\beta^2}{L^U(\alpha^U)^2 + L^R(\alpha^R)^2} \cdot \frac{\sigma_v^2}{\sigma_y^2}.$$

Since $\mathcal{A}_i$ does not depend on i we drop the subscript i. Using Lemma 4.1, we obtain

$$\mathcal{A} = \frac{\frac{S^2}{(1+S)^2\lambda^2}}{\frac{k^2 L^U}{(kE+2\lambda)^2} + \frac{k^2\bar{L}/E}{(k+2\lambda)^2}} \cdot \frac{\sigma_v^2}{\sigma_y^2}.$$

Dividing the numerator and denominator by $L^U + \bar{L}$, and using the definitions of φ and Φ in (12), we can rewrite this as

$$\mathcal{A}(\lambda, E, k, \varphi, \Phi) = \frac{S\Phi^{-1}}{\dfrac{\varphi}{(kE\lambda^{-1}+2)^2} + \dfrac{1-\varphi}{E(k\lambda^{-1}+2)^2}}, \tag{17}$$

where we have made explicit the dependence of $\mathcal{A}$ on the relevant variables. This formulation allows us to separate the effects of λ on the one hand and (E, k, φ, Φ) on the other. Note that

$\partial\mathcal{A}/\partial\lambda < 0$, in accordance with the intuition that market makers protect themselves from adverse selection by raising λ. In equilibrium, λ is pinned down by equation (16), which is equivalent to $\mathcal{A} = S$:

Equilibrium adverse selection. *The equilibrium price impact λ adjusts so that adverse selection on each exchange is exactly equal to the number of speculators.*

If adverse selection is high for a given λ, i.e. $\mathcal{A} > S$, λ rises until adverse selection goes down to the equilibrium level $\mathcal{A} = S$. Likewise, low adverse selection leads to a reduction in λ. The comparative statics in Proposition 5.1 can be read off (17) by tracing how $\mathcal{A}$ at fixed λ responds to a change in the underlying parameter, and then noting that equilibrium λ must adjust to restore the balance between informed and uninformed trade, as expressed by the condition $\mathcal{A} = S$.

Consider the effect of fragmentation on local price impact λ. From (17), $\partial\mathcal{A}/\partial E > 0$. Thus fragmentation leads to higher adverse selection. The reason is that, for given λ, liquidity trading is lower on any given exchange when the number of exchanges is higher, while speculative trading is unaffected (Lemma 4.1). In particular, unrestricted liquidity traders trade less, and while the amount traded by each restricted liquidity trader does not vary with the number of exchanges, the number of these traders on each exchange, $\bar{L}/E$, goes down with E. Equilibrium λ must therefore rise to bring $\mathcal{A}$ back down to S. This is part (ii) of Proposition 5.1.

Next, we examine the effect of fragmentation on global price impact $\gamma := \lambda/E$. We can rewrite (17) in terms of γ as follows:

$$\hat{\mathcal{A}}(\gamma, E, k, \varphi, \Phi) = \frac{S\Phi^{-1}}{\dfrac{\varphi}{(k\gamma^{-1}+2)^2} + \dfrac{(1-\varphi)E}{(k\gamma^{-1}+2E)^2}}. \tag{18}$$

First suppose there are no restricted liquidity traders ($\bar{L} = 0$, or equivalently $\varphi = 1$). Then $\hat{\mathcal{A}}$ does not depend on E. For fixed γ, an increase in E reduces both speculative and liquidity trade in the same proportion on any given exchange (Lemma 4.1). Fragmentation does not affect the degree of adverse selection, and hence γ is not affected either. This is part (iii) of Proposition 5.1.

Now suppose some liquidity traders are restricted ($\bar{L} > 0$, or equivalently $\varphi < 1$). From (18), $\partial\hat{\mathcal{A}}/\partial E \propto 2\lambda - k$. Since γ must adjust to a change in E to restore $\hat{\mathcal{A}} = S$, we have $\partial\gamma/\partial E \propto 2\lambda - k$, as stated in part (iv) of Proposition 5.1. This result is driven by changes in restricted liquidity trade as markets become increasingly fragmented. From (18), the relative variance of liquidity trade to speculative trade depends on E through the term

$E/(k\gamma^{-1}+2E)^2$, which is increasing in E when $\lambda < k/2$ and decreasing in E when $\lambda > k/2$. As λ increases with E, adverse selection at fixed γ falls and then rises, and equilibrium γ follows the same pattern.

Finally, consider the effect of Φ. From (17), $\partial\mathcal{A}/\partial\Phi < 0$. Since this derivative holds S fixed, and the second component of Φ depends only on S, the variation in Φ runs only through its first and third components: stronger liquidity needs raise the variance of liquidity trade, while less information asymmetry reduces the variance of speculative trade. Intuitively, both of these lower adverse selection. Equilibrium λ must therefore fall in order to push $\mathcal{A}$ back up to S. This is part (v) of Proposition 5.1.[13]

We now characterize the level of market fragmentation at which global liquidity is maximal. It is useful to make explicit the dependence of λ and γ on E by writing $\lambda(E)$ and $\gamma(E)$, $E \geq 1$, while suppressing the dependence on (k, φ, Φ). In particular $\lambda(1)$ is the value of λ when there is only one exchange.

Proposition 5.2 (Maximal global liquidity) *Suppose $\bar{L} > 0$. Then we have the following results on the number of exchanges E^* at which global liquidity is maximal:*

i. There exists $E^* \in [1, \bar{E})$ such that $\gamma(E)$ is minimized at E^*.

ii. $E^* = 1$ if and only if $\lambda(1) \geq k/2$, or equivalently $\Phi \leq 16$. Furthermore, if $E^* = 1$, then $\partial\gamma/\partial E > 0$ for $E > 1$.

iii. $E^* > 1$ if and only if $\lambda(1) < k/2$, or equivalently $\Phi > 16$. Furthermore, if $E^* > 1$, then $\partial\gamma/\partial E < 0$ for $E < E^*$, and $\partial\gamma/\partial E > 0$ for $E > E^*$.

iv. Suppose $E^* > 1$. Then

$$\frac{\varphi}{4(E^*+1)^2} + \frac{1-\varphi}{16E^*} = \Phi^{-1}. \tag{19}$$

Furthermore, $E^* = E^*(\varphi, \Phi)$, $\partial E^*/\partial\varphi < 0$, and $\partial E^*/\partial\Phi > 0$.[14]

If $\lambda(1) \geq k/2$, i.e. liquidity on a single centralized exchange is lower than the threshold level $2/k$, market fragmentation always lowers global liquidity. However, if $\lambda(1) < k/2$, i.e. liquidity in a centralized market exceeds $2/k$, fragmentation can improve global liquidity.

[13] Since λ and γ are strictly decreasing in Φ, it follows from the definition of Φ that λ and γ are strictly decreasing in S and σ_y^2, and strictly increasing in σ_v^2. Liquidity (local or global) depends on S only through Φ, which is increasing in S via the competition effect term $(1+S)^2/S$. Hence liquidity is increasing in S, even though adverse selection is increasing in S (through the term $S\Phi^{-1}$). It can be shown that λ and γ are also strictly decreasing in L^U, $\bar{L}$, and k. See the proof of Proposition 5.1 in the Appendix.

[14] Using the definition of Φ, this result implies that E^* is strictly increasing in S, $\bar{L}$, k, and σ_y^2, and strictly decreasing in σ_v^2. It can be shown that E^* is also strictly increasing in L^U. See the proof of Proposition 5.2 in the Appendix.

Global liquidity is maximized at $E^* > 1$, which is the number of exchanges such that $\lambda(E^*) = k/2$.

We call Φ the *fragmentation potential* of the asset since the range of beneficial fragmentation is increasing in Φ: fragmentation can improve global liquidity if and only if $\Phi > 16$, and the level E^* at which global liquidity is maximized is increasing in Φ (Proposition 5.2, parts (iii) and (iv)). The comparative static $\partial E^*/\partial\Phi > 0$ operates through local liquidity. By Proposition 5.1 (iv), E^* is the level at which local liquidity has fallen to the threshold $2/k$; beyond E^*, fragmentation lowers global liquidity rather than raising it. An increase in Φ improves local liquidity at every E, so the threshold $2/k$ is reached at a higher level of fragmentation.

The comparative static $\partial E^*/\partial\varphi < 0$ in Proposition 5.2 (iv) reflects the differential response of restricted and unrestricted liquidity trade to fragmentation. At fixed λ, fragmentation reduces unrestricted liquidity trade on any given exchange through order-splitting, while restricted liquidity trade on a given exchange thins only through the falling count $\bar{L}/E$. A higher φ shifts the composition of liquidity trade toward the unrestricted component, which contracts more sharply with E. Adverse selection therefore rises faster, and the threshold $2/k$ is reached at a lower level of fragmentation.

Figure 1 shows global liquidity γ^{-1} as a function of the number of exchanges E for two different values of the fragmentation potential Φ.[15]

6 Welfare

In this section, we show that the welfare of a trader depends on market fragmentation only through its effect on liquidity—local liquidity for a local trader and global liquidity for a global trader. Combining this with our liquidity results from Section 5, we characterize the welfare impact of fragmentation on each trader type. We begin by calculating equilibrium utilities for given E.

Lemma 6.1 (Equilibrium utilities) *Equilibrium utilities for given E are*

$$\mathcal{U}^S := \mathbb{E}\left(W^S\right) = \frac{1}{(1+S)^2\gamma}\sigma_v^2, \tag{20}$$

[15]It is interesting to compare our results on liquidity to those in the symmetric information models of Foucault and Menkveld (2008) and Chen and Duffie (2021). In Foucault and Menkveld (2008), entry of a competing trading venue lowers liquidity in the incumbent market and increases global liquidity. In Chen and Duffie (2021), an increase in the degree of fragmentation lowers local liquidity and raises global liquidity. While the relationship of fragmentation to local liquidity in our model is the same as in these papers, this is not the case for global liquidity, where we find a non-monotonic relationship.

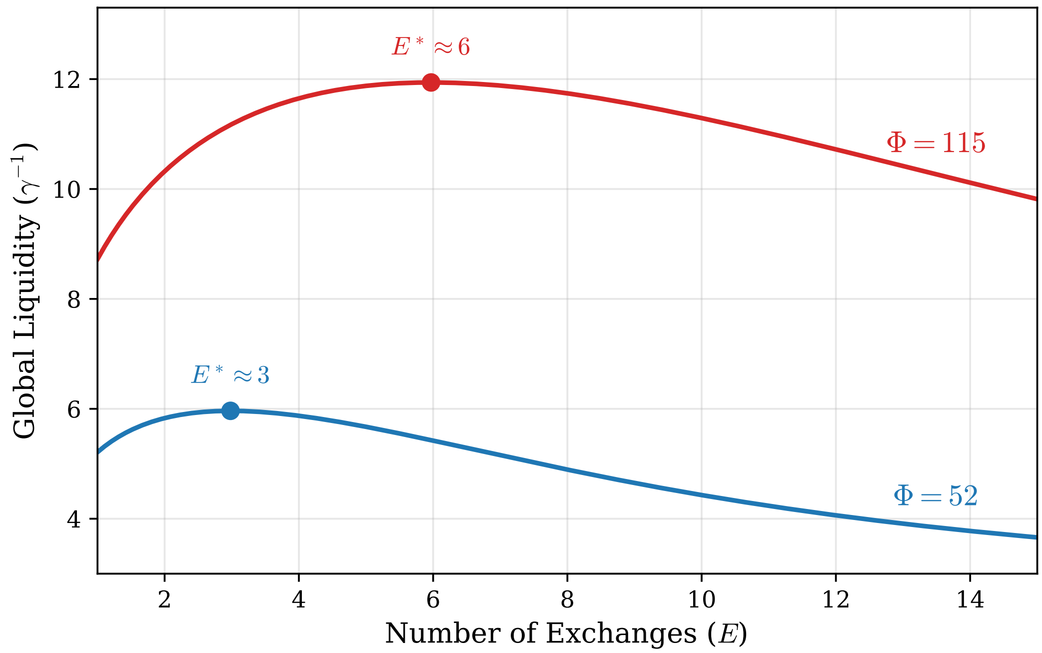


Figure 1: The figure plots global liquidity γ^{-1} as a function of the number of exchanges E, for $\varphi = 1/3$ and $\Phi \in \{52, 115\}$. The maximum value of γ^{-1} is marked by a dot, and the corresponding value of E^* is shown.

$$\mathcal{U}^U := \mathbb{E}\left(W_\ell^U\right) = -\frac{k\gamma}{k + 2\gamma}\sigma_y^2, \tag{21}$$

$$\mathcal{U}^R := \mathbb{E}\left(W_{i\ell}^R\right) = -\frac{k\lambda}{k + 2\lambda}\sigma_y^2. \tag{22}$$

Lemma 6.1 shows that the utility of a restricted liquidity trader, who trades on a single exchange, depends on E only through the local price impact parameter λ, and it is strictly decreasing in λ. An unrestricted trader, whether she is a speculator or an unrestricted liquidity trader, operates on all exchanges. Her utility depends on E only through the global price impact parameter $\gamma := \lambda/E$, and it is strictly decreasing in γ. As one would expect, if there is more than one exchange, an unrestricted liquidity trader is better off than a restricted liquidity trader.

Combining Lemma 6.1 with our results on price impact in Section 5, we can characterize the effect of market fragmentation on welfare. Consider first the case where $\bar{L} = 0$, i.e. there are no restricted liquidity traders. Then, from Proposition 5.1 (iii), γ is independent of E, and hence so is the equilibrium utility of any agent (from (20) and (21)). In other words, market fragmentation is welfare-neutral if all agents can trade on all exchanges:

Corollary 6.2 *Suppose all agents are unrestricted ($\bar{L} = 0$). Then the welfare of any agent is the same for any degree of fragmentation.*

If some liquidity traders are restricted, market fragmentation does have welfare consequences. Recall that E^* is the number of exchanges at which global liquidity is maximal (Proposition 5.2).

Proposition 6.3 (Welfare: General case) *Suppose $\bar{L} > 0$. Then market fragmentation impacts welfare as follows:*

i. The welfare of restricted liquidity traders is strictly decreasing in E.

ii. The welfare of unrestricted traders is maximized at E^*.

iii. If $\lambda(1) \geq k/2$, or equivalently $\Phi \leq 16$, then $E^* = 1$, and an increase in E makes all agents worse off.

iv. If $\lambda(1) < k/2$, or equivalently $\Phi > 16$, then $E^* > 1$. Furthermore, the welfare of unrestricted traders is strictly increasing in E for $E < E^*$, and strictly decreasing in E for $E > E^*$.

Part (i) is immediate from Proposition 5.1 (ii) and (22), and parts (ii)–(iv) follow from Proposition 5.2, (20), and (21). Market fragmentation always hurts restricted liquidity traders since it lowers local liquidity. If liquidity on a single centralized exchange is below the threshold level $2/k$, market fragmentation lowers both local and global liquidity, hurting all traders. If liquidity on a centralized exchange is higher than $2/k$, multiple exchanges lead to higher global liquidity, benefiting unrestricted traders who trade globally. The optimal number of exchanges for unrestricted traders is E^*, at which global liquidity is maximal.

7 Price Informativeness

In this section, we study the impact of market fragmentation on price informativeness. We show that fragmentation does not affect the information conveyed by the price on any given exchange. But the informativeness of the set of all prices across exchanges, which we call global price informativeness, does depend on the degree of fragmentation. If there are no unrestricted liquidity traders, global price informativeness is increasing in the number of exchanges. If there are both restricted and unrestricted liquidity traders, on the other hand, global price informativeness is increasing up to a cutoff level of fragmentation, after which it declines. Any fragmentation beyond this cutoff level not only reduces global price informativeness, but also lowers global liquidity and makes all agents worse off.

Formally, we define *local price informativeness* on exchange i by

$$\mathcal{V}_i(E) := \frac{\text{Var}(v) - \text{Var}(v|p_i)}{\text{Var}(v)}, \tag{23}$$

and *global price informativeness* by

$$\mathcal{V}(E) := \frac{\text{Var}(v) - \text{Var}(v|p_1, p_2, \ldots, p_E)}{\text{Var}(v)}. \tag{24}$$

In this notation we have made explicit the dependence of price informativeness on the number of exchanges E. However, as we shall see later, $\mathcal{V}_i$ does not depend on E. Local price informativeness measures the information about the asset payoff that is conveyed by the price on a single exchange, while global price informativeness is a metric of overall price informativeness in the economy, using all available prices.

In order to understand the effect of market fragmentation on price informativeness, we begin with the following result.

Lemma 7.1 *The equilibrium price on exchange i is given by*

$$p_i = \frac{S}{1+S}(v + \eta^U + \eta_i^R), \tag{25}$$

where $\eta^U := \alpha^U Q^U/(S\beta)$, and $\eta_i^R := \alpha^R Q_i^R/(S\beta)$. Furthermore,

$$Var(\eta^U + \eta_i^R) = Var(\eta^U) + Var(\eta_i^R) = \frac{\sigma_v^2}{S}, \tag{26}$$

and

$$Var(\eta^U) = \left(\frac{\alpha^U}{\beta}\right)^2 \frac{L^U \sigma_y^2}{S^2} = \left(\frac{\gamma}{k+2\gamma}\right)^2 \frac{k^2 L^U (1+S)^2 \sigma_y^2}{S^2}. \tag{27}$$

Recall that $Q^U := \sum_{\ell \in L^U} y_\ell$ and $Q_i^R := \sum_{\ell \in L_i^R} y_\ell$. From (7), we can interpret η^U as the relative magnitude of unrestricted liquidity trade to speculative trade on any exchange, and η_i^R as the relative magnitude of restricted liquidity trade to speculative trade on exchange i. The random variables v, η^U, and $\{\eta_i^R\}_{i \in E}$ are mutually independent. Equation (25) tells us that the price p_i is informationally equivalent to the "price signal", $v + \eta^U + \eta_i^R$, which is equal to the asset payoff v plus a noise term. The noise term in turn is the sum of a common component η^U, common across prices on different exchanges, and an idiosyncratic component η_i^R.

From (27), $\text{Var}(\eta^U)$ depends on E only through the global price impact parameter γ. An increase in γ lowers both β and α^U (from (9) and (10)), but the effect on β is more pronounced; the ratio α^U/β is increasing in γ, and hence so is $\mathrm{Var}(\eta^U)$. In other words, the variance of the common noise component in the price signal is increasing in γ because an increase in γ lowers speculative trade by more than it suppresses the trades of liquidity traders who operate across all exchanges.

A number of conclusions follow from Lemma 7.1. First, since the variance of the noise in the price signal, $\mathrm{Var}(\eta^U + \eta_i^R)$, does not depend on the exchange under consideration or on E, it follows that local price informativeness is the same across all exchanges and does not depend on the degree of market fragmentation. An increase in E leads to higher local price impact λ (Proposition 5.1 (ii)), and a lower number of restricted liquidity traders $\bar{L}/E$ on each exchange. Combining this observation with Lemma 4.1, we see that an increase in E lowers speculative as well as liquidity trade on any given exchange. The net effect on price informativeness is zero.

Second, if there are no restricted liquidity traders ($\bar{L} = 0$), then $\eta_i^R = 0$ for all i, and the price is the same on every exchange. Hence, global price informativeness is equal to local price informativeness, both of which are independent of E.

Third, if there are some restricted liquidity traders ($\bar{L} > 0$), prices on different exchanges are not perfectly correlated. Combining this with the observation that the informativeness of any one price is unaffected by market fragmentation, we see that global price informativeness is higher when markets are fragmented, compared to the case of a single centralized exchange, and this is true regardless of the degree of fragmentation.

Fourth, if there are no unrestricted liquidity traders ($L^U = 0$), then $\eta^U = 0$, and the price signals, $\{v + \eta_i^R\}_{i\in E}$, are conditionally independent signals about v. Moreover, from (26), the variance of the "error term" η_i^R is the same for all E (and for all i). Hence global price informativeness is strictly increasing in the degree of fragmentation.

We summarize these observations in the following proposition. We also derive some simple formulas for price informativeness.

Proposition 7.2 (Price informativeness)

i. $\mathcal{V}_i(E) = \mathcal{V}(1) = [1 + S^{-1}]^{-1}$*, for all* $i \in E$ *and for all* $E \in [1, \bar{E})$*.*

ii. If $\bar{L} = 0$*,* $\mathcal{V}(E) = \mathcal{V}_i(E) = \mathcal{V}(1) = [1 + S^{-1}]^{-1}$*, for all* $i \in E$ *and for all* $E \in [1, \bar{E})$*.*

iii. If $\bar{L} > 0$*,* $\mathcal{V}(E) > \mathcal{V}(1) = [1 + S^{-1}]^{-1}$*, for all* $E \in [2, \bar{E})$*.*

iv. If $L^U = 0$*,* $\mathcal{V}(E) = [1 + (ES)^{-1}]^{-1}$*, which is strictly increasing in* E*.*

Thus market fragmentation leaves local price informativeness unchanged, while the effect on global price informativeness depends on the composition of liquidity trade.

We now further investigate the impact of fragmentation on global price informativeness when both $\bar{L}$ and L^U are positive. Proposition 7.2 (iii) implies that global price informativeness is higher when markets are fragmented than when they are not, but it does not say anything about the effect on price informativeness of an increase in the degree of fragmentation. In order to determine the shape of $\mathcal{V}(E)$, it is useful to separate the direct effect of an increase in E, which simply increases the number of signals, $\{v + \eta^U + \eta_i^R\}_{i \in E}$, from the indirect effect that operates through a change in $\text{Var}(\eta^U)$. Due to (26), there is no independent effect arising from a change in $\text{Var}(\eta_i^R)$. Accordingly, we write $\mathcal{V}(E) := \nu(E, \text{Var}(\eta^U))$, so that

$$\frac{\partial \mathcal{V}}{\partial E} = \frac{\partial \nu}{\partial E} + \frac{\partial \nu}{\partial \text{Var}(\eta^U)} \frac{\partial \text{Var}(\eta^U)}{\partial E} = \underbrace{\frac{\partial \nu}{\partial E}}_{>0} + \underbrace{\frac{\partial \nu}{\partial \text{Var}(\eta^U)}}_{<0} \underbrace{\frac{\partial \text{Var}(\eta^U)}{\partial \gamma}}_{>0} \frac{\partial \gamma}{\partial E}. \tag{28}$$

Clearly, $\partial \nu / \partial E > 0$, since a higher number of price signals, that have the same variance and pairwise correlation, convey more information (see (26)). Also, $\partial \nu / \partial \text{Var}(\eta^U) < 0$, because an increase in $\text{Var}(\eta^U)$ implies that a higher proportion of the noise in price signals, as measured by $\text{Var}(\eta^U + \eta_i^R)$, is common across all price signals. Finally, $\partial \text{Var}(\eta^U)/\partial \gamma > 0$, from (27).

Recall that E^* is the number of exchanges at which $\gamma(E)$ is minimized. Moreover, if $E^* > 1$, we have $\partial \gamma / \partial E < 0$ for $E < E^*$, and $\partial \gamma / \partial E > 0$ for $E > E^*$ (Proposition 5.2 (iii)). In this case, we can conclude from (28) that $\partial \mathcal{V}/\partial E$ is positive for $E \leq E^*$, but is ambiguous in sign for $E > E^*$. The following proposition sharpens this result, and shows that the direct effect of an increase in E is eventually outweighed by the impact of a higher γ, which is accompanied by a higher $\text{Var}(\eta^U)$.

Proposition 7.3 (Maximal price informativeness) *Suppose $\bar{L} > 0$ and $L^U > 0$. Then, we have the following results on global price informativeness $\mathcal{V}(E)$:*

i. There exists $\hat{E} \in (2, \bar{E})$ such that $\mathcal{V}(E)$ is maximized at $\hat{E}$.

ii. $\partial \mathcal{V}/\partial E > 0$ for $E < \hat{E}$, and $\partial \mathcal{V}/\partial E < 0$ for $E > \hat{E}$.

iii. $\hat{E} > E^*$.

iv. $\hat{E}$ is determined jointly with λ by solving equation (16) together with

$$\varphi(1 - 4\hat{\delta}) + (1 - \varphi)(1 - 2\hat{\delta})^3 = 0, \qquad \hat{\delta} := \frac{E - 1}{E(k\lambda^{-1} + 2)}. \tag{29}$$

Furthermore, $\hat{E} = \hat{E}(\varphi, \Phi)$, $\partial \hat{E}/\partial \varphi < 0$, and $\partial \hat{E}/\partial \Phi > 0$.[16]

[16] Using the definitions of φ and Φ, this result implies that $\hat{E}$ is strictly increasing in S, $\bar{L}$, k, and σ_y^2, and strictly decreasing in σ_v^2.

Thus there is a level of market fragmentation at which global price informativeness is maximal. This level is higher than the number of exchanges at which global liquidity, and hence the welfare of speculators and unrestricted liquidity traders, is maximal. In particular, as long as market fragmentation increases the welfare of unrestricted traders, it also improves global price informativeness. The converse is not true, however. For example, suppose there is only one exchange and $\lambda(1) \geq k/2$. Then any level of fragmentation improves global price informativeness (Proposition 7.2 (iii)) but makes all agents worse off (Proposition 6.3 (iii)).

The comparative statics of $\hat{E}$ in Proposition 7.3 (iv) operate through the composition of noise in prices, just as those for E^* operate through global liquidity. From (28), $\hat{E}$ is the level at which the direct positive effect of an additional price signal is balanced by the negative indirect effect of higher $\mathrm{Var}(\eta^U)$ on cross-exchange price correlation. An increase in Φ lowers γ at every E (Proposition 5.1 (v)), and hence $\mathrm{Var}(\eta^U)$. Less common noise means less price correlation across exchanges, so the direct effect of fragmentation continues to dominate up to a higher level of fragmentation. Hence $\hat{E}$ is increasing in Φ.

The comparative static $\partial\hat{E}/\partial\varphi < 0$ in Proposition 7.3 (iv) reflects the role of unrestricted liquidity trade in determining cross-exchange price correlation. A higher φ shifts the composition of liquidity trade toward unrestricted traders, whose orders contribute to the common noise component η^U rather than to the idiosyncratic component η_i^R. The resulting increase in price correlation reduces the marginal informational gain from a new exchange, lowering the level of fragmentation at which global price informativeness is maximized.

Our next result is immediate from Propositions 5.2, 6.3 and 7.3:

Corollary 7.4 (Excessive fragmentation) *Suppose $\bar{L} > 0$, $L^U > 0$, and $E \geq \hat{E}$. Then the following metrics of market quality are strictly decreasing in E: global price informativeness, global liquidity, and the welfare of all agents.*

Any increase in the number of exchanges beyond $\hat{E}$ (the level at which global price informativeness is maximal) reduces market quality, whether it is measured in terms of global price informativeness or liquidity, or the welfare of any agent.

Finally, we summarize the comparative statics for the two composite parameters, Φ and φ, on which the entire analysis rests. Every equilibrium object considered in the paper—local and global liquidity, the welfare of each agent type, the boundary between regimes in which fragmentation can or cannot be beneficial for unrestricted traders, and the threshold levels E^* and $\hat{E}$—depends on the primitives of the model only through Φ and φ. The two parameters pull in opposite directions: Φ widens the range of beneficial fragmentation, while φ narrows it.

The *fragmentation potential* Φ determines whether fragmentation can be beneficial for any agent: $\Phi \leq 16$ implies that full consolidation is optimal for every trader, and $\Phi > 16$

is necessary for fragmentation to benefit unrestricted traders. An increase in Φ raises both local and global liquidity at every E, and pushes the thresholds E^* and $\hat{E}$ outward. We refer to Φ as a scale effect since it is increasing in the size of the investor base; it is also decreasing in the degree of asymmetric information. Cross-sectional variation in Φ provides the channel through which our results connect to the differential impact of fragmentation on large- and small-cap stocks (see Section 8).

The *composition* parameter φ plays a complementary role. In the polar case $\varphi = 1$, where all liquidity traders have multi-venue access, fragmentation has no effect on global liquidity, welfare, or global price informativeness; the multi-exchange economy is observationally equivalent to a single integrated market. For $\varphi < 1$, an increase in φ raises cross-exchange price correlation, and lowers the thresholds E^* and $\hat{E}$.

Figure 2 shows γ^{-1} and $\mathcal{V}$ as functions of E, for the same values of φ and Φ as in Figure 1 (thus the left panel of Figure 2 is the same as Figure 1). The higher value of Φ yields a higher number of exchanges E^* at which γ^{-1} is maximized, as well as a higher number $\hat{E}$ at which $\mathcal{V}$ is maximized. For both values of Φ, we have $E^* < \hat{E}$.

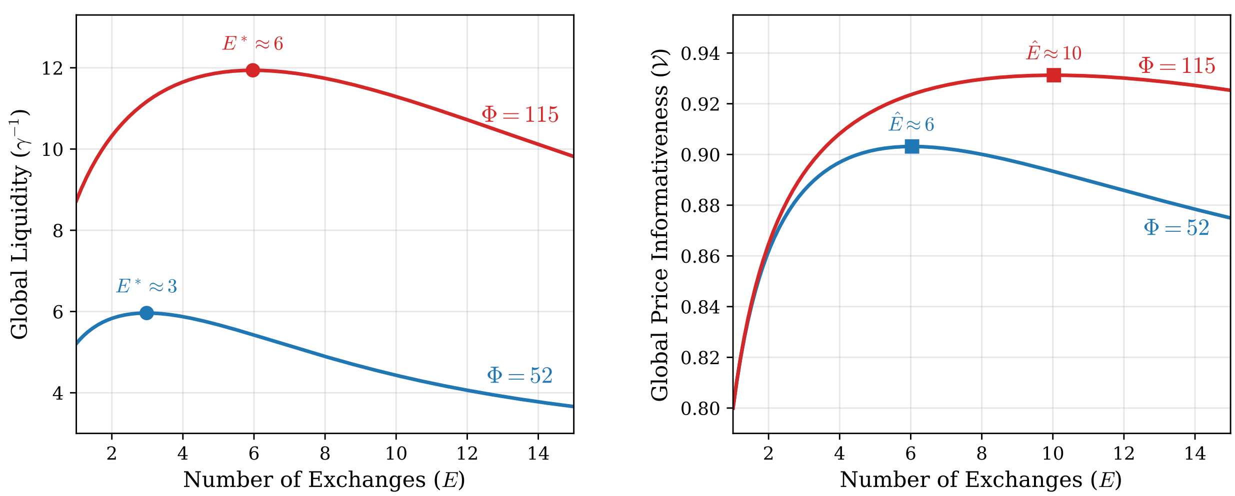


Figure 2: The figure shows global liquidity γ^{-1} and global price informativeness $\mathcal{V}$ as functions of the number of exchanges E, for $\varphi = 1/3$ and $\Phi \in \{52, 115\}$ (the same values as in Figure 1). The maximum value of γ^{-1} is marked by a dot and the maximum value of $\mathcal{V}$ by a square; the corresponding values of E^* and $\hat{E}$ are shown.

8 Quantitative Implications

In this section, we relate the model's thresholds E^* and $\hat{E}$ to empirical evidence on fragmentation in US and European equity markets. Several primitives of the model—most notably the holding cost parameter k and the variances σ_y^2 and σ_v^2—are not directly observable, so this is not a calibration in the strict sense. However, the model's predictions for E^* and $\hat{E}$ depend on the underlying primitives only through the two composite parameters φ and Φ. We pin down φ from data on multi-venue access, and Φ from empirical evidence on the location of peak global liquidity, using the characterization of E^* in Proposition 5.2. The predicted values of $\hat{E}$ are then genuine out-of-sample implications of the model, and the substantive content of the exercise lies in the comparison of $\hat{E}$ to observed fragmentation.

8.1 Mapping observed fragmentation to the model

Empirical studies typically measure fragmentation using the Herfindahl-Hirschman Index (HHI), defined as $\text{HHI} := \sum_i s_i^2$, where s_i is the market share of venue i, or the associated fragmentation index $\text{FI} := 1 - \text{HHI}$, which ranges from 0 (a single venue) to 1 (maximal fragmentation). In our model, there are E symmetric exchanges, each with a market share of $1/E$, so $\text{FI} = 1 - 1/E$. Hence the effective number of exchanges implied by an observed FI is $E_{\text{eff}} = 1/(1 - \text{FI})$. This represents the number of equal-sized venues that generate a given level of fragmentation. It is the natural empirical counterpart of the model parameter E.

In practice, market shares are concentrated in a small number of venues, so E_{eff} is lower than the actual number of active venues. We compute E_{eff} from market share data for lit exchanges as follows. For US stocks, we use the venue market shares reported in Foucault et al. (2023), Table 7.1, renormalized to lit volume only (i.e. excluding off-exchange volume), which yields $E_{\text{eff}} \approx 6$ for the aggregate market. Since large-cap stocks are more evenly distributed across venues than small-cap stocks (Haslag and Ringgenberg 2023), and the entry of MEMX in 2020 has further increased fragmentation, we posit $E_{\text{eff}} \approx 7$–8 for US large-cap stocks and $E_{\text{eff}} \approx 4$–5 for US small-cap stocks. For European stocks, we use the lit venue market shares for CAC40 and FTSE100 stocks in Foucault et al. (2023), Table 7.2, obtaining $E_{\text{eff}} \approx 2$–2.5 for large-cap stocks. European small-cap stocks are far more concentrated on the primary exchange (Gresse 2017; Lausen et al. 2022), so we set $E_{\text{eff}} \approx 1.2$–$1.5$.

8.2 Parameter values

8.2.1 Fraction of unrestricted liquidity traders φ

The parameter φ is the fraction of liquidity traders who have multi-venue access. In European markets, smart order routing technology (SORT) provides a proxy for multi-venue access. Foucault and Menkveld (2008), using a pre-MiFID Dutch sample, find that approximately 25% of trades involved SORT. Two offsetting adjustments are needed to translate this figure into a current headcount estimate of φ. First, the estimate predates MiFID and the subsequent growth of multilateral trading facilities (MTFs), which would raise current SORT use above the 25% figure. Second, the figure is volume-based and includes both speculators and unrestricted liquidity traders. Since more active traders contribute disproportionately to volume, the headcount fraction of unrestricted liquidity traders is likely smaller than the volume share of multi-venue trading. We propose the range $\varphi \in [0.05, 0.30]$ for European large-cap stocks. For European small-cap stocks, SORT adoption is lower and MTF activity is thinner, so we use a lower range of $\varphi \in [0.02, 0.10]$.

For US markets, the order protection rule in Reg NMS automatically routes marketable orders to the venue displaying the best price. This applies to all NMS stocks regardless of market capitalization. A liquidity trader whose order flow consists primarily of marketable orders is therefore effectively unrestricted. Since retail and other small liquidity traders—who dominate the investor population by headcount—submit marketable orders in most cases, the fraction of unrestricted liquidity traders in US markets is relatively high. The main exceptions are liquidity traders whose flow consists mostly of resting limit orders, or whose individual orders exceed top-of-book depth across venues (O'Hara and Ye 2011). Given the broad reach of order protection, we posit $\varphi \in [0.50, 0.75]$ for US stocks, with the same range applying to both large-cap and small-cap stocks.[17]

8.2.2 Fragmentation potential Φ

For each market segment, we use equation (19) to translate empirical evidence on E^* into a corresponding statement about Φ. For European large-cap stocks, the empirical evidence pins down E^* at a specific value; since Φ is increasing in φ for a given E^*, we report Φ at both endpoints of the φ interval. For the other three segments, the empirical evidence yields only a one-sided bound on E^*. Since the implied bound on Φ varies with φ, we use the most restrictive bound for which the constraint on E^* holds across the full calibrated φ interval.

European large-cap stocks. Degryse et al. (2015), analyzing 51 Dutch stocks over the period from November 2007 to December 2009, document an inverted U-shaped relationship between visible fragmentation and consolidated depth (see their Figure 4). The fragmentation level at which consolidated depth is maximal corresponds to E^* in our model. Using

[17] Marketable orders (market orders and marketable limit orders) correspond to the unconditional market orders in our model; order protection routes them to the best-priced venue. Resting limit orders, and orders exceeding displayed depth at the best bid or offer, remain tied to a single venue. The φ range reflects uncertainty about the relative magnitude of these two categories.

their fragmentation index, the peak occurs at FI ≈ 0.5 for large-cap stocks, corresponding to $E^* = 2$. Hence the value of Φ ranges from 32 (at $\varphi = 0.05$) to 33 (at $\varphi = 0.30$).

US large-cap stocks. For US large-cap stocks, no comparable inverted U-shape has been documented in the data. Fragmentation appears to have been beneficial for large-cap stocks throughout the sample periods studied (O'Hara and Ye 2011; Haslag and Ringgenberg 2023). This means that the US market has not yet reached E^*, even for the most fragmented stocks with $E_{\text{eff}} \approx 8$. So we take $E^* > 8$. At $\varphi = 0.75$ (the binding value of φ, which determines the most restrictive bound), the implied lower bound on Φ is 234.[18] This is about 7 times the value of Φ for European large-cap stocks, consistent with the larger investor base in the US.

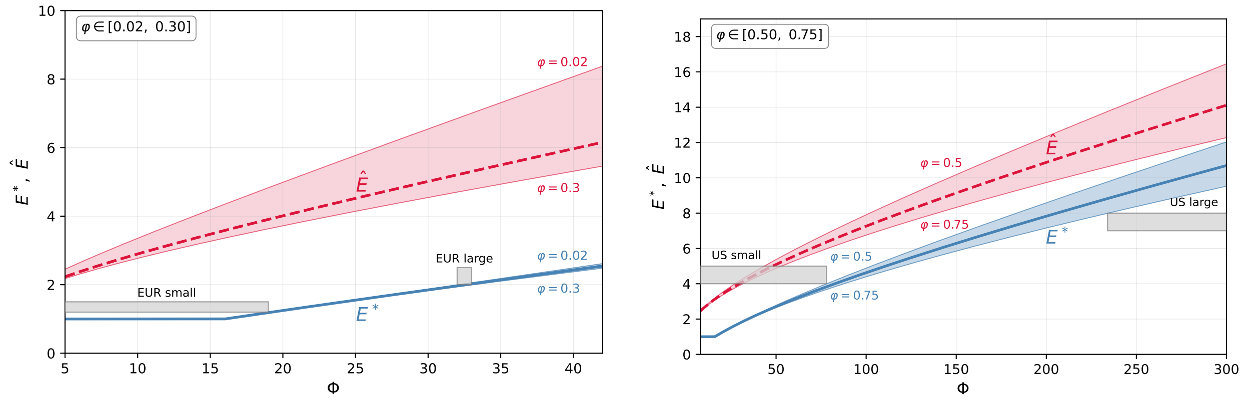
European small-cap stocks. The empirical evidence shows that fragmentation reduces global liquidity for European small-cap stocks (Gresse 2017; Lausen et al. 2022). In terms of our model, this implies that either $E^* = 1$ (so that any fragmentation reduces global liquidity, which requires $\Phi \leq 16$), or $E^* > 1$ but actual fragmentation exceeds E^*. The full-sample peak in Degryse et al. (2015) occurs at FI ≈ 0.4 ($E^* \approx 1.7$), which is below the large-cap peak. Since the full sample is a mix of large- and small-cap stocks, the small-cap peak would be lower still. Taking $E^* < 1.2$ and applying the constraint at $\varphi = 0.02$ (the binding value of φ), the implied upper bound on Φ is 19.

US small-cap stocks. For US small-cap stocks, Haslag and Ringgenberg (2023) find that fragmentation reduced global liquidity over the 2003–2016 period, when fragmentation rose from $E_{\text{eff}} \approx 1.5$ to $E_{\text{eff}} \approx 4$–5. Taking $E^* < 4$ and applying the constraint at $\varphi = 0.50$ (the binding value of φ), the implied upper bound on Φ is 78.

8.3 Results

Figure 3 displays the theoretical relationship between E^*, $\hat{E}$, and Φ for Europe and the US separately, reflecting the different ranges of φ for the two regions. In each panel, the blue and red bands show the set of values of E^* and $\hat{E}$ as φ varies over the interval indicated in the top-left corner, with the midline corresponding to the midpoint of that interval. The gray rectangles mark the observed effective number of exchanges E_{eff} for each market segment, spanning the calibrated values of Φ on the horizontal axis and E_{eff} on the vertical axis.

[18] From (19), $E^* = 8$ at $\varphi = 0.75$ and $\Phi = 234$. Since E^* is strictly decreasing in φ and strictly increasing in Φ (Proposition 5.2 (iv)), we have $E^* > 8$ for all $\varphi \in [0.50, 0.75]$ and $\Phi > 234$.



(a) Europe (b) US

Figure 3: The thresholds E^* (solid blue) and $\hat{E}$ (dashed red) as functions of the fragmentation potential Φ. Note the different axis scales in the two panels.

The figure illustrates the main features of the theoretical relationship described in Propositions 5.2 and 7.3: E^* and $\hat{E}$ are strictly increasing in Φ and strictly decreasing in φ, and $\hat{E} > E^*$ for any given (φ, Φ). Table 1 summarizes the results and compares the model-implied values of $\hat{E}$ to the estimates of E_{eff} and E^* inferred from the empirical evidence.

Table 1: Summary of Results

Market segment	φ	Φ	E^*	$\hat{E}$	E_{eff}	Assessment
Large-cap (EUR)	0.05–0.30	32–33	2	5–6	2–2.5	$E^* \leq E_{\text{eff}} < \hat{E}$
Large-cap (US)	0.50–0.75	> 234	> 8	> 11	7–8	$E_{\text{eff}} < E^* < \hat{E}$
Small-cap (EUR)	0.02–0.10	< 19	< 1.2	< 5	1.2–1.5	$E^* < E_{\text{eff}} < \hat{E}$
Small-cap (US)	0.50–0.75	< 78	< 4	< 7	4–5	$E^* < E_{\text{eff}} \lesseqgtr \hat{E}$

Note: $E^* \geq 1$ and $\hat{E} > 2$ for all (φ, Φ) (Proposition 7.3 (i)).

Three of the four segments—European large-caps, US large-caps, and European small-caps—have E_{eff} below the calibrated $\hat{E}$. Fragmentation has therefore continued to enhance global price informativeness in these segments, including those where it has reduced global liquidity. For US small-caps, the position of E_{eff} relative to $\hat{E}$ depends on where in the calibrated Φ range one looks: at the lower end, $\hat{E} \approx 3$ and E_{eff} is in the excessive-fragmentation regime; at the upper end, $\hat{E} \approx 6$–7 and E_{eff} remains below it.

A common thread runs through these results. The thresholds E^* and $\hat{E}$ differ across stock types primarily because of differences in Φ. Large-cap stocks have a larger investor base, of both liquidity traders and speculators, and less information asymmetry, each of which raises Φ and shifts the thresholds upward (Propositions 5.2 and 7.3). The calibrated Φ values inherit this ordering, with US large-caps highest and European small-caps lowest. The model thus offers a unified theoretical interpretation of the differential effects of fragmentation across the four market segments.

9 Extensions

In the Online Appendix, we extend the baseline model in several directions:

i. *Uniform holding costs:* Speculators incur the same holding cost as that borne by liquidity traders. Thus all agents have the same holding cost parameter k.

ii. *CARA utility:* Liquidity traders have constant absolute risk aversion. There is no holding cost for any agent.

iii. *Correlated endowment shocks:* The endowment shocks of liquidity traders are correlated.

iv. *Endogenous information acquisition:* The number of speculators is endogenous, with each speculator paying a fixed cost in order to learn the realization of the asset payoff.

In (ii), (iii), and (iv), we assume that there are no unrestricted liquidity traders, i.e. $L^U = 0$, in order to achieve tractability. In the first three extensions, we show that our main results on liquidity and welfare still hold. Market fragmentation reduces local liquidity λ^{-1}. Under suitable conditions, global liquidity γ^{-1} increases until the number of exchanges reaches some level E^*, and then falls, just as in Figure 1. Restricted liquidity traders are always hurt by fragmentation, while the welfare of unrestricted traders is maximized at E^*, the same level at which global liquidity is highest.

In the second and third extensions, our results on price informativeness, for the case where $L^U = 0$, also carry over. Fragmentation increases global price informativeness, while leaving local price informativeness unchanged. Since $\hat{E} = \infty$ in these extensions, the wedge $\hat{E} > E^*$ identified in Proposition 7.3 holds trivially. In the first extension, however, we find that global price informativeness may not have the inverted U-shape depicted in Figure 2; the wedge is therefore tied to our baseline assumption that only liquidity traders bear holding costs.

In our fourth extension, we show that when information acquisition is endogenous, fragmentation can induce speculator entry or exit, depending on the parameters. Entry of speculators can intensify speculator competition, raise local liquidity, and lead to a Pareto improvement. Exit of speculators can cause global price informativeness to fall as E increases. These results suggest that the long-run and short-run effects of fragmentation can differ substantially.

10 Conclusion

In this paper, we analyze the impact of financial market fragmentation on the balance between the trades of different categories of agents, who differ in terms of information and market access, and thereby on adverse selection and price discovery. Our model, despite its simplicity, generates results that match observed empirical regularities that have remained unexplained to date.

Three key insights emerge from our analysis. First, the implications of fragmentation for global liquidity and welfare depend critically on initial market conditions. If initial liquidity is sufficiently high, moderate fragmentation increases global liquidity and benefits unrestricted traders. This occurs when there is a large investor base, or when the asymmetry in information is limited. Our model provides an explanation for why the effect of fragmentation on global liquidity is positive for large-cap stocks but negative for small-cap stocks—a pattern documented empirically but not previously explained theoretically.

Second, our results provide the first theoretical rationale for the inverted U-shaped relationship between global liquidity and the degree of fragmentation documented in the empirical literature. We find that the relationship of global price informativeness to fragmentation also displays an inverted U-shape, but with a higher peak. Beyond the price informativeness peak, further fragmentation lowers both global liquidity and global price informativeness, and makes all traders worse off.

Third, fragmentation creates winners and losers. The welfare impact on an agent depends on her market access, rather than on her information, and it is tightly linked to liquidity. Local traders always suffer from reduced local liquidity, while global traders benefit from any improvement in global liquidity. This distributional impact has been neglected in the literature.

Our model provides theoretical support for regulatory concerns about excessive market fragmentation, while identifying specific thresholds beyond which intervention may be warranted. Our analysis suggests that, when evaluating proposals about market design, regulators should consider market-specific factors including existing liquidity levels, the size of the investor population, and the degree of asymmetric information. Markets with a small number of traders, or those with substantial informational asymmetry, can be negatively impacted by fragmentation. Our quantitative exercise indicates that observed fragmentation remains below the price-informativeness-maximizing threshold $\hat{E}$ in most market segments.

Our model abstracts from intermarket competition and dark trading. Both of these would be substantive extensions rather than refinements, and we leave them to future research. The wedge between liquidity-maximizing and informativeness-maximizing fragmentation, and the small set of parameters that governs it, are what we hope future work on market design can build on.

Appendix: Proofs

Proof of Lemma 4.1 The price on exchange i is given by (8). Let $Q^U_{-\ell} := \sum_{\ell' \in L^U, \ell' \neq \ell} y_{\ell'}$ and $Q^R_{i,-\ell} := \sum_{\ell' \in L^R_i, \ell' \neq \ell} y_{\ell'}$. From the perspective of an individual trader, the price function is

$$\begin{aligned}
p^S_i &:= \lambda \left[(S-1)\beta v + \alpha^U Q^U + \alpha^R Q^R_i + q^S_i \right], \\
p^U_{i\ell} &:= \lambda \left[S\beta v + \alpha^U Q^U_{-\ell} + \alpha^R Q^R_i + q^U_{i\ell} \right], \\
p^R_{i\ell} &:= \lambda \left[S\beta v + \alpha^U Q^U + \alpha^R Q^R_{i,-\ell} + q^R_{i\ell} \right],
\end{aligned}$$

for speculators, unrestricted liquidity traders, and restricted liquidity traders, respectively. From (1)–(3), we have

$$\begin{aligned}
\mathbb{E}(W^S|v) &= \sum_{i\in E} \left[[1-(S-1)\lambda\beta]\, v q^S_i - \lambda \left(q^S_i\right)^2 \right], \\
\mathbb{E}(W^U_\ell|y_\ell) &= -\lambda \sum_{i\in E} \left(q^U_{i\ell}\right)^2 - \frac{k}{2}\left(\sum_{i\in E} q^U_{i\ell} - y_\ell \right)^2, \\
\mathbb{E}(W^R_{i\ell}|y_\ell) &= -\lambda \left(q^R_{i\ell}\right)^2 - \frac{k}{2}\left(q^R_{i\ell} - y_\ell\right)^2.
\end{aligned}$$

Hence, the optimal trades are

$$q^S_i = \frac{1-(S-1)\lambda\beta}{2\lambda} v, \tag{30}$$

$$q^U_{i\ell} = \frac{k\left(y_\ell - \sum_{j\in E} q^U_{j\ell}\right)}{2\lambda}, \tag{31}$$

$$q^R_{i\ell} = \frac{k}{k+2\lambda} y_\ell. \tag{32}$$

From (30),

$$\beta = \frac{1-(S-1)\lambda\beta}{2\lambda},$$

which yields (9). From (31),

$$q^U_{i\ell} = \frac{k(y_\ell - E\alpha^U y_\ell)}{2\lambda}.$$

Hence,

$$\alpha^U = \frac{k(1 - E\alpha^U)}{2\lambda},$$

which gives us (10). Equation (11) follows from (32). $\square$

Proof of Proposition 4.2 From (5),

$$p_i = \mathbb{E}[v|q_i] = \frac{\mathrm{Cov}(v, q_i)}{\mathrm{Var}(q_i)} q_i.$$

Since $p_i = \lambda q_i$, we have

$$\lambda = \frac{\mathrm{Cov}(v, q_i)}{\mathrm{Var}(q_i)}. \tag{33}$$

From (7),

$$\begin{aligned} \mathrm{Cov}(v, q_i) &= S\beta\sigma_v^2, \qquad (34) \\ \mathrm{Var}(q_i) &= S^2\beta^2\sigma_v^2 + [L^U(\alpha^U)^2 + L^R(\alpha^R)^2]\sigma_y^2. \end{aligned}$$

Therefore,

$$\lambda = \frac{S\beta\sigma_v^2}{S^2\beta^2\sigma_v^2 + [L^U(\alpha^U)^2 + L^R(\alpha^R)^2]\sigma_y^2}. \tag{35}$$

Using (9),

$$\begin{aligned} \lambda\left[L^U(\alpha^U)^2 + L^R(\alpha^R)^2\right]\sigma_y^2 &= (1 - S\lambda\beta)S\beta\sigma_v^2 \qquad (36) \\ &= \left[1 - \frac{S}{1+S}\right]\frac{S}{(1+S)\lambda}\sigma_v^2 \\ &= \frac{S}{(1+S)^2\lambda}\sigma_v^2, \qquad (37) \end{aligned}$$

so that, using (10) and (11),

$$\begin{aligned} \frac{\sigma_v^2}{\sigma_y^2} &= \frac{\left[L^U(\alpha^U)^2 + L^R(\alpha^R)^2\right](1+S)^2}{S}\lambda^2 \\ &= \left[\frac{L^U}{(kE + 2\lambda)^2} + \frac{L^R}{(k + 2\lambda)^2}\right]\frac{k^2(1+S)^2}{S}\lambda^2 \\ &= \left[\frac{L^U}{(kE\lambda^{-1} + 2)^2} + \frac{L^R}{(k\lambda^{-1} + 2)^2}\right]\frac{k^2(1+S)^2}{S}. \end{aligned}$$

Thus λ solves $f(\lambda; E) = 0$, where

$$f(\lambda; E) := \left[\frac{L^U}{(kE\lambda^{-1}+2)^2} + \frac{L^R}{(k\lambda^{-1}+2)^2}\right]\frac{k^2(1+S)^2}{S} - \frac{\sigma_v^2}{\sigma_y^2}. \tag{38}$$

We have $\lim_{\lambda\to 0} f(\lambda; E) < 0$. Also, $\partial f(\lambda; E)/\partial\lambda > 0$. Hence, if there is a solution to $f(\lambda; E) = 0$, it must be unique. A solution exists if and only if

$$\lim_{\lambda\to\infty} f(\lambda; E) = \frac{k^2(L^U + L^R)(1+S)^2}{4S} - \frac{\sigma_v^2}{\sigma_y^2} > 0,$$

which gives us condition (13), which is easily seen to be equivalent to (14). If $L^U = 0$, (13) is equivalent to $E < \bar{E}$. $\square$

Proof of Proposition 5.1 *Proof of (i):* From (38), λ solves $f(\lambda; E) = 0$, where

$$f(\lambda; E) := \left[\frac{L^U}{(kE\lambda^{-1}+2)^2} + \frac{\bar{L}}{E(k\lambda^{-1}+2)^2}\right]\frac{k^2(1+S)^2}{S} - \frac{\sigma_v^2}{\sigma_y^2}. \tag{39}$$

Dividing through by $L^U + \bar{L}$, and using the definitions of φ and Φ in (12), gives us (16).

Proof of (ii): From (16), λ solves $\hat{f}(\lambda; E, k, \varphi, \Phi) = 0$, where

$$\hat{f}(\lambda; E, k, \varphi, \Phi) := \frac{\varphi}{(kE\lambda^{-1}+2)^2} + \frac{1-\varphi}{E(k\lambda^{-1}+2)^2} - \Phi^{-1}. \tag{40}$$

We have $\partial\hat{f}/\partial\lambda > 0$, and $\partial\hat{f}/\partial E < 0$. Hence, from

$$\frac{\partial\hat{f}}{\partial\lambda}\frac{\partial\lambda}{\partial E} + \frac{\partial\hat{f}}{\partial E} = 0,$$

we conclude that $\partial\lambda/\partial E > 0$.

Now we consider the limit of λ as E approaches its upper bound $\bar{E}$, given by (15). If $L^U > 0$, then $\bar{E} = \infty$. Since $\partial\lambda/\partial E > 0$, and $\lim_{E\to\infty}\hat{f}(\lambda; E, k, \varphi, \Phi) < 0$, we must have $\lim_{E\to\infty}\lambda(E) = \infty$. If $L^U = 0$, we have $\bar{E} = \bar{E}_1$. Furthermore, $\varphi = 0$ and $\Phi = 4\bar{E}_1$, so that

$$\hat{f}(\lambda; E, k, \varphi, \Phi) = \frac{1}{E(k\lambda^{-1}+2)^2} - \frac{1}{4\bar{E}_1}.$$

Let $\bar{\lambda} := \lim_{E\to\bar{E}_1}\lambda(E)$. Since $\hat{f}(\lambda(E); E, k, \varphi, \Phi) = 0$ for all $E \in [1, \bar{E}_1)$, we have

$$\lim_{E\to\bar{E}_1}\hat{f}(\lambda(E); E, k, \varphi, \Phi) = \hat{f}(\bar{\lambda}; \bar{E}_1, k, \varphi, \Phi) = \frac{1}{\bar{E}_1(k\bar{\lambda}^{-1}+2)^2} - \frac{1}{4\bar{E}_1} = 0,$$

implying that $\bar{\lambda} = \infty$.

Proof of (iii): From (40), γ solves $\hat{g}(\gamma; E, k, \varphi, \Phi) = 0$, where

$$\hat{g}(\gamma; E, k, \varphi, \Phi) := \frac{\varphi}{(k\gamma^{-1} + 2)^2} + \frac{(1-\varphi)E}{(k\gamma^{-1} + 2E)^2} - \Phi^{-1}. \tag{41}$$

If $\bar{L} = 0$, then $\varphi = 1$. It follows that $\hat{g}$ is independent of E, and hence so is γ.

Proof of (iv): If $\bar{L} > 0$, then from (41) we have $\partial\hat{g}/\partial\gamma > 0$, and

$$\begin{aligned} \frac{\partial \hat{g}}{\partial E} &\propto (k\gamma^{-1} + 2E)^2 - 4E(k\gamma^{-1} + 2E) \\ &\propto k\gamma^{-1} - 2E \\ &\propto k - 2\lambda. \end{aligned}$$

Hence, from

$$\frac{\partial \hat{g}}{\partial \gamma}\frac{\partial \gamma}{\partial E} + \frac{\partial \hat{g}}{\partial E} = 0,$$

we see that

$$\frac{\partial \gamma}{\partial E} \propto -\frac{\partial \hat{g}}{\partial E} \propto 2\lambda - k.$$

Proof of (v): From (40), $\partial\hat{f}/\partial\lambda > 0$, and $\partial\hat{f}/\partial\Phi > 0$. Hence, from

$$\frac{\partial \hat{f}}{\partial \lambda}\frac{\partial \lambda}{\partial \Phi} + \frac{\partial \hat{f}}{\partial \Phi} = 0,$$

we conclude that $\partial\lambda/\partial\Phi < 0$. Furthermore, $\partial\gamma/\partial\Phi = (\partial\gamma/\partial\lambda)(\partial\lambda/\partial\Phi) \propto \partial\lambda/\partial\Phi < 0$.

Comparative statics in footnote 13: From (39), $f(\lambda; E)$ is strictly increasing in λ. It is also strictly increasing in $L^U, \bar{L}$, and k. It follows that λ is strictly decreasing in these parameters. Clearly the same is true of $\gamma := \lambda/E$. $\square$

Proof of Proposition 5.2 *Proof of (i)–(iii):* From (16), $\lambda(1) < k/2$ if and only if $\Phi > 16$. The results now follow from parts (ii) and (iv) of Proposition 5.1.

Proof of (iv): Suppose $E^* > 1$. Then $\lambda(E^*) = k/2$, and (19) follows from (16). From (19), it is clear that E^* is a function of (φ, Φ). Implicitly differentiating (19), we obtain the desired comparative statics results.

Comparative statics in footnote 14: If $E^* > 1$, then E^* solves $f(k/2; E) = 0$, where f is given by (39). The function $f(k/2; E)$ is strictly decreasing in E. It is also strictly increasing in L^U. It follows that E^* is strictly increasing in L^U. $\square$

Proof of Lemma 6.1 To calculate equilibrium utilities, we use (1)–(3), (8), and Lemma 4.1. For speculators,

$$W^S = \sum_{i \in E} (v - p_i) q_i^S = \sum_{i \in E} \Big[v - \lambda(S\beta v + \alpha^U Q^U + \alpha^R Q_i^R) \Big] \beta v,$$

so that

$$\begin{aligned}
\mathbb{E}\left(W^S\right) &= (1 - S\lambda\beta) E \beta \sigma_v^2 \\
&= \left[1 - \frac{S}{1+S}\right] E\beta\sigma_v^2 \\
&= \frac{E}{(1+S)^2 \lambda} \sigma_v^2 \\
&= \frac{1}{(1+S)^2\gamma} \sigma_v^2.
\end{aligned}$$

For unrestricted liquidity traders,

$$\begin{aligned}
W_\ell^U &= \sum_{i \in E} (v - p_i) q_{i\ell}^U - y_\ell v - \frac{k}{2} \left(\sum_{i \in E} q_{i\ell}^U - y_\ell \right)^2 \\
&= \sum_{i \in E} \Big[v - \lambda(S\beta v + \alpha^U Q^U + \alpha^R Q_i^R) \Big] \alpha^U y_\ell - y_\ell v - \frac{k}{2} \left(E\alpha^U y_\ell - y_\ell \right)^2,
\end{aligned}$$

so that

$$\begin{aligned}
\mathbb{E}\left(W_\ell^U\right) &= -\lambda E (\alpha^U)^2 \sigma_y^2 - \frac{k}{2}(E\alpha^U - 1)^2 \sigma_y^2 \\
&= -\frac{\lambda k^2 E + \frac{k}{2}(4\lambda^2)}{(kE + 2\lambda)^2} \sigma_y^2 \\
&= -\frac{k\lambda}{kE + 2\lambda} \sigma_y^2 \\
&= -\frac{k\gamma}{k + 2\gamma} \sigma_y^2.
\end{aligned}$$

For restricted liquidity traders,

$$\begin{aligned}
W_{i\ell}^R &= (v - p_i) q_{i\ell}^R - y_\ell v - \frac{k}{2} \left(q_{i\ell}^R - y_\ell \right)^2 \\
&= \Big[v - \lambda(S\beta v + \alpha^U Q^U + \alpha^R Q_i^R) \Big] \alpha^R y_\ell - y_\ell v - \frac{k}{2} \left(\alpha^R y_\ell - y_\ell \right)^2,
\end{aligned}$$

so that

$$
\begin{aligned}
\mathbb{E}\left(W_{i\ell}^{R}\right) &= -\lambda(\alpha^{R})^2\sigma_y^2 - \frac{k}{2}(\alpha^{R}-1)^2\sigma_y^2 \\
&= -\frac{\lambda k^2 + \frac{k}{2}(4\lambda^2)}{(k+2\lambda)^2}\sigma_y^2 \\
&= -\frac{k\lambda}{k+2\lambda}\sigma_y^2.
\end{aligned}
$$

This completes the proof. $\square$

Proof of Lemma 7.1 From (8) and (9),

$$
p_i = \lambda S\beta\left[v + \frac{\alpha^U Q^U + \alpha^R Q_i^R}{S\beta}\right] = \frac{S}{1+S}(v + \eta^U + \eta_i^R).
$$

Using (9) and (37), we have

$$
\mathrm{Var}(\eta^U + \eta_i^R) = \frac{\left[L^U(\alpha^U)^2 + L^R(\alpha^R)^2\right]\sigma_y^2}{(S\beta)^2} = \frac{\frac{S}{(1+S)^2\lambda^2}\sigma_v^2}{\frac{S^2}{(1+S)^2\lambda^2}} = \frac{\sigma_v^2}{S}.
$$

Note that η^U and η_i^R are independent. The relations in (27) follow from the definition of η^U, and from (9) and (10). $\square$

Proof of Proposition 7.2 All the results follow from the observations in the text preceding the statement of the proposition, except for the formulas for price informativeness. We now derive these formulas.

Using (23), (33), (34), and (9), in that order, price informativeness on exchange i is given by

$$
\begin{aligned}
\mathcal{V}_i(E) &:= \frac{\mathrm{Var}(v) - \mathrm{Var}(v|q_i)}{\mathrm{Var}(v)} \\
&= \frac{\mathrm{Var}(v) - \left(\mathrm{Var}(v) - \frac{[\mathrm{Cov}(v,q_i)]^2}{\mathrm{Var}(q_i)}\right)}{\mathrm{Var}(v)} \\
&= \frac{[\mathrm{Cov}(v,q_i)]^2}{\mathrm{Var}(q_i)\sigma_v^2} \\
&= \lambda\frac{\mathrm{Cov}(v,q_i)}{\sigma_v^2} \\
&= \lambda S\beta
\end{aligned}
$$

$$= \frac{S}{1+S}.$$

Since $\mathcal{V}_i(E)$, does not depend on i or on E, we have $\mathcal{V}_i(E) = \mathcal{V}_i(1) = \mathcal{V}(1)$.

Next, we derive the formula for $\mathcal{V}(E)$ in part (iv) of the proposition. Given the form taken by p_i (equation (8)), we see that the average price across exchanges is a sufficient statistic for the set of all prices. Thus we have

$$\begin{aligned}
\operatorname{Var}(v|p_1, p_2, \ldots, p_E) &= \operatorname{Var}\left(v\middle|E^{-1}\sum_{i=1}^{E} p_i\right) \\
&= \operatorname{Var}\left(v\middle|S\beta v + \alpha^U Q^U + \frac{\alpha^R}{E}\sum_{i=1}^{E} Q_i^R\right) \\
&= \sigma_v^2 - \frac{S^2\beta^2\sigma_v^4}{S^2\beta^2\sigma_v^2 + [L^U(\alpha^U)^2 + E^{-1}L^R(\alpha^R)^2]\sigma_y^2},
\end{aligned}$$

so that, using (24) and (35),

$$\begin{aligned}
\mathcal{V} &= \frac{S^2\beta^2\sigma_v^2}{S^2\beta^2\sigma_v^2 + [L^U(\alpha^U)^2 + E^{-1}L^R(\alpha^R)^2]\sigma_y^2} \\
&= \left[\frac{S^2\beta^2\sigma_v^2 + [L^U(\alpha^U)^2 + L^R(\alpha^R)^2]\sigma_y^2}{S^2\beta^2\sigma_v^2} - \frac{(1-E^{-1})L^R(\alpha^R)^2\sigma_y^2}{S^2\beta^2\sigma_v^2}\right]^{-1} \\
&= \left[\frac{1}{S\lambda\beta} - \left(\frac{\alpha^R}{\beta}\right)^2 \frac{(E-1)\bar{L}\sigma_y^2}{E^2S^2\sigma_v^2}\right]^{-1}. \qquad (42)
\end{aligned}$$

This formula for $\mathcal{V}$ is valid for $L^U \geq 0$, and we have not yet used the optimal strategies of agents. Using these strategies (equations (9) and (11)), we obtain

$$\mathcal{V} = \left[\frac{1+S}{S} - \left(\frac{k(1+S)}{k\lambda^{-1}+2}\right)^2 \frac{(E-1)\bar{L}\sigma_y^2}{E^2S^2\sigma_v^2}\right]^{-1}. \qquad (43)$$

Setting $L^U = 0$ in (39), and noting that λ is is a zero of $f(\lambda; E)$, we have

$$\frac{1}{(k\lambda^{-1}+2)^2} = \frac{E}{\bar{L}} \cdot \frac{S}{k^2(1+S)^2} \cdot \frac{\sigma_v^2}{\sigma_y^2}.$$

Substituting this expression into (43) gives us

$$\mathcal{V} = \left[\frac{1+S}{S} - \frac{E-1}{ES}\right]^{-1} = \frac{1}{1+(ES)^{-1}},$$

as desired. $\square$

Proof of Proposition 7.3 *Proof of (i)–(iii):* From (43), we can write $\mathcal{V}$ as follows:

$$\mathcal{V} = \left[\frac{1+S}{S} - \delta^2 \left(\frac{1+S}{S} \right)^2 \frac{k^2 \bar{L} \sigma_y^2}{\sigma_v^2} \right]^{-1}, \qquad \delta := \frac{\sqrt{E-1}}{E(k\lambda^{-1}+2)}.$$

From (38), δ solves $h(\delta; E) = 0$, where

$$h(\delta; E) := \left[\frac{L^U}{\left(\frac{\sqrt{E-1}}{\delta} - 2(E-1) \right)^2} + \frac{\bar{L} E \delta^2}{E-1} \right] \frac{k^2 (1+S)^2}{S} - \frac{\sigma_v^2}{\sigma_y^2}. \tag{44}$$

We have

$$\frac{\partial h(\delta; E)}{\partial \delta} \frac{\partial \delta}{\partial E} + \frac{\partial h(\delta; E)}{\partial E} = 0.$$

Furthermore, $\partial \mathcal{V} / \partial \delta > 0$, and $\partial h(\delta; E) / \partial \delta > 0$. Therefore,

$$\begin{aligned} \frac{\partial \mathcal{V}}{\partial E} = \frac{\partial \mathcal{V}}{\partial \delta} \frac{\partial \delta}{\partial E} &\propto -\frac{\partial h(\delta; E)}{\partial E} \\ &\propto \frac{2L^U}{\left(\frac{\sqrt{E-1}}{\delta} - 2(E-1) \right)^3} \left(\frac{1}{2\delta\sqrt{E-1}} - 2 \right) + \frac{\bar{L}\delta^2}{(E-1)^2} \\ &\propto 2L^U (1 - 4\delta\sqrt{E-1}) + \frac{2\bar{L}\delta^3}{(E-1)^{3/2}} \left(\frac{\sqrt{E-1}}{\delta} - 2(E-1) \right)^3 \\ &\propto L^U (1 - 4\delta\sqrt{E-1}) + \bar{L}(1 - 2\delta\sqrt{E-1})^3. \end{aligned} \tag{45}$$

Letting

$$\hat{\delta} := \delta\sqrt{E-1} = \frac{E-1}{E(k\lambda^{-1}+2)}, \tag{46}$$

and dividing (45) by $L^U + \bar{L} > 0$, we have $\partial \mathcal{V} / \partial E \propto \xi(\hat{\delta}; \varphi)$, where

$$\xi(\hat{\delta}; \varphi) := \varphi(1 - 4\hat{\delta}) + (1-\varphi)(1 - 2\hat{\delta})^3.$$

The function $\xi(\,\cdot\,; \varphi)$ is strictly decreasing, with $\xi(1/4; \varphi) = (1-\varphi)/8 > 0$ and $\xi(1/2; \varphi) = -\varphi < 0$. Hence $\xi(\,\cdot\,; \varphi)$ has a unique zero, which lies in $(1/4, 1/2)$.

From (44), $\hat{\delta}$ solves $\hat{h}(\hat{\delta}; E) = 0$, where

$$\hat{h}(\hat{\delta}; E) := \left[\frac{L^U \hat{\delta}^2}{(E-1)^2 (1 - 2\hat{\delta})^2} + \frac{\bar{L} E \hat{\delta}^2}{(E-1)^2} \right] \frac{k^2 (1+S)^2}{S} - \frac{\sigma_v^2}{\sigma_y^2}.$$

We have

$$\frac{\partial \hat{h}(\hat{\delta}; E)}{\partial \hat{\delta}} \frac{\partial \hat{\delta}}{\partial E} + \frac{\partial \hat{h}(\hat{\delta}; E)}{\partial E} = 0.$$

Since $\partial \hat{h}/\partial \hat{\delta} > 0$, and $\partial \hat{h}/\partial E < 0$, we get $\partial \hat{\delta}/\partial E > 0$.

Given the equilibrium mapping $\lambda = \lambda(E)$, we can write $\hat{\delta} = \hat{\delta}(E)$, from (46). We have $\hat{\delta}(1) = 0$. Since $\lim_{E \to \bar{E}} \lambda = \infty$ (Proposition 5.1 (ii)) and $\bar{E} = \infty$ in the current setting with $L^U > 0$, we have $\lim_{E \to \infty} \hat{\delta} = 1/2$. Hence $\hat{\delta}(\,\cdot\,)$ is a strictly increasing bijection from $[1, \infty)$ onto $[0, 1/2)$, and there is a unique $\hat{E}$ at which $\hat{\delta}(\hat{E})$ equals the zero of $\xi(\,\cdot\,; \varphi)$. Since $\xi(\,\cdot\,; \varphi)$ is strictly decreasing in $\hat{\delta}$ and $\hat{\delta}(\,\cdot\,)$ is strictly increasing in E, $\partial \mathcal{V}/\partial E > 0$ for $E < \hat{E}$ and $\partial \mathcal{V}/\partial E < 0$ for $E > \hat{E}$.

Finally, $\hat{\delta}(\hat{E}) > 1/4$ from the bound on the zero of $\xi(\,\cdot\,; \varphi)$. From (46), $\hat{\delta}(E) < (E - 1)/(2E)$, so $\hat{\delta}(\hat{E}) > 1/4$ implies $\hat{E} > 2$. Moreover, if $E^* > 1$, then $\lambda(E^*) = k/2$ and $\hat{\delta}(E^*) = (E^* - 1)/(4E^*) < 1/4$; if $E^* = 1$, then $\hat{\delta}(E^*) = 0$. Either way, $\hat{\delta}(E^*) < 1/4 < \hat{\delta}(\hat{E})$, so $\hat{E} > E^*$.

Proof of (iv): From the proof of parts (i)–(iii), $\hat{\delta}(\hat{E})$ is the unique zero of $\xi(\,\cdot\,; \varphi)$ in $(1/4, 1/2)$, i.e. (29) holds; together with the equilibrium condition (16) on λ, this establishes the first statement of part (iv). Using (46), we can write (16) as

$$\Phi \hat{\delta}^2 \left[\frac{\varphi}{(1 - 2\hat{\delta})^2} + (1 - \varphi) E \right] = (E - 1)^2. \tag{47}$$

Since $\hat{\delta}(\hat{E})$ depends only on φ, equation (47) determines $\hat{E}$ as a function of φ and Φ.

Differentiating (47) with respect to Φ at fixed φ (and hence fixed $\hat{\delta}$), we obtain

$$\hat{\delta}^2 \left[\frac{\varphi}{(1 - 2\hat{\delta})^2} + (1 - \varphi) \hat{E} \right] + \Phi \hat{\delta}^2 (1 - \varphi) \frac{\partial \hat{E}}{\partial \Phi} = 2(\hat{E} - 1) \frac{\partial \hat{E}}{\partial \Phi},$$

i.e.

$$\left[2(\hat{E} - 1) - \Phi \hat{\delta}^2 (1 - \varphi) \right] \frac{\partial \hat{E}}{\partial \Phi} = \hat{\delta}^2 \left[\frac{\varphi}{(1 - 2\hat{\delta})^2} + (1 - \varphi) \hat{E} \right].$$

Using (47) to substitute $\Phi \hat{\delta}^2 (1 - \varphi) \hat{E} = (\hat{E} - 1)^2 - \Phi \hat{\delta}^2 \varphi / (1 - 2\hat{\delta})^2$, the coefficient on the LHS is

$$\begin{aligned} 2(\hat{E} - 1) - \Phi \hat{\delta}^2 (1 - \varphi) &= 2(\hat{E} - 1) - \frac{1}{\hat{E}} \left[(\hat{E} - 1)^2 - \frac{\Phi \hat{\delta}^2 \varphi}{(1 - 2\hat{\delta})^2} \right] \\ &= \frac{2\hat{E}(\hat{E} - 1) - (\hat{E} - 1)^2 + \Phi \hat{\delta}^2 \varphi / (1 - 2\hat{\delta})^2}{\hat{E}} \end{aligned}$$

$$= \frac{(\hat{E}-1)(\hat{E}+1) + \Phi\hat{\delta}^2\varphi/(1-2\hat{\delta})^2}{\hat{E}} > 0,$$

so $\partial\hat{E}/\partial\Phi > 0$.

We turn to $\partial\hat{E}/\partial\varphi$. Let $u := 1 - 2\hat{\delta}$, so $u \in (0, 1/2)$. In terms of u, equation (29) is $\varphi(2u-1) + (1-\varphi)u^3 = 0$, which gives us

$$\varphi = \frac{u^3}{1-2u+u^3} = \frac{u^3}{(1-u)(1-u-u^2)}. \tag{48}$$

It is straightforward to check that $d\varphi/du > 0$ on $(0, 1/2)$, so u is a strictly increasing function of φ.

Substituting (48) into (47), and using $\hat{\delta} = (1-u)/2$, $(1-2\hat{\delta})^2 = u^2$, and $1 - \varphi = (1-2u)/(1-2u+u^3)$, equation (47) becomes

$$\Phi\, b(u)\big[u + (1-2u)\hat{E}\big] = (\hat{E}-1)^2, \qquad b(u) := \frac{1-u}{4(1-u-u^2)} > 0. \tag{49}$$

Implicitly differentiating (49) with respect to u at fixed Φ yields

$$\big[2(\hat{E}-1) - \Phi b(u)(1-2u)\big]\frac{\partial\hat{E}}{\partial u} = \Phi b'(u)\big[u + (1-2u)\hat{E}\big] + \Phi b(u)(1-2\hat{E}). \tag{50}$$

We show that the coefficient on the LHS of (50) is positive. Using (49),

$$\begin{aligned}
2(\hat{E}-1) - \Phi b(u)(1-2u) &= 2(\hat{E}-1) - \frac{(\hat{E}-1)^2(1-2u)}{u + (1-2u)\hat{E}} \\
&\propto 2[u + (1-2u)\hat{E}] - (\hat{E}-1)(1-2u) \\
&= 2u + (1-2u)(\hat{E}+1) > 0.
\end{aligned}$$

Therefore, using (50), and the identity $b'(u)/b(u) = u(2-u)/(1-2u+u^3)$,

$$\begin{aligned}
\frac{\partial\hat{E}}{\partial u} &\propto \Phi b'(u)[u + (1-2u)\hat{E}] + \Phi b(u)(1-2\hat{E}) \\
&\propto \frac{u(2-u)}{1-2u+u^3}[u + (1-2u)\hat{E}] + (1-2\hat{E}) \\
&\propto u(2-u)[u + (1-2u)\hat{E}] + (1-2\hat{E})(1-2u+u^3) \\
&= (1-2u+2u^2) - (2-6u+5u^2)\hat{E} \\
&< (1-2u+2u^2) - 2(2-6u+5u^2) \\
&= -(3-10u+8u^2)
\end{aligned}$$

$$= -(4u-3)(2u-1) < 0,$$

where the first inequality in the above chain follows from the observation that $2-6u+5u^2 > 0$ everywhere (the discriminant is negative), and $\hat{E} > 2$ from part (i) of the proposition, and the second inequality is due to the fact that $u \in (0, 1/2)$. Combining $\partial\hat{E}/\partial u < 0$ with $du/d\varphi > 0$, we conclude that $\partial\hat{E}/\partial\varphi < 0$. $\square$

References

Admati, A. R. and Pfleiderer, P. (1988). A theory of intraday patterns: Volume and price variability. *Review of Financial Studies*, 1(1):3–40.

Baldauf, M. and Mollner, J. (2021). Trading in fragmented markets. *Journal of Financial and Quantitative Analysis*, 56(1):93–121.

Bartlett, R. P., McCrary, J., and O'Hara, M. (2025). The market inside the market: Odd-lot quotes. *Review of Financial Studies*, 38(3):661–711.

Bennett, P. and Wei, L. (2006). Market structure, fragmentation, and market quality. *Journal of Financial Markets*, 9(1):49–78.

Biais, B., Foucault, T., and Moinas, S. (2015). Equilibrium fast trading. *Journal of Financial Economics*, 116(2):292–313.

Boehmer, B. and Boehmer, E. (2003). Trading your neighbor's ETFs: Competition or fragmentation? *Journal of Banking and Finance*, 27(9):1667–1703.

Chen, D. and Duffie, D. (2021). Market fragmentation. *American Economic Review*, 111(7):2247–2274.

Chowdhry, B. and Nanda, V. (1991). Multimarket trading and market liquidity. *Review of Financial Studies*, 4(3):483–511.

Chung, K. H. and Chuwonganant, C. (2012). Regulation NMS and market quality. *Financial Management*, 41(2):285–317.

Colliard, J.-E. and Foucault, T. (2012). Trading fees and efficiency in limit order markets. *Review of Financial Studies*, 27(11):3389–3421.

Degryse, H., de Jong, F., and van Kervel, V. (2015). The impact of dark trading and visible fragmentation on market quality. *Review of Finance*, 19(4):1587–1622.

Degryse, H., Hagströmer, B., and Landsberg, N. (2025). Market fragmentation: A cushion against exchange outages? Working Paper.

Du, S. and Zhu, H. (2017). What is the optimal trading frequency in financial markets? *Review of Economic Studies*, 84(4):1606–1651.

Eaton, G. W., Shkilko, A., and Werner, I. M. (2025). Nocturnal trading. Working Paper.

Foucault, T. and Menkveld, A. J. (2008). Competition for order flow and smart order routing systems. *Journal of Finance*, 63(1):119–158.

Foucault, T., Pagano, M., and Röell, A. (2023). *Market Liquidity: Theory, Evidence and Policy*. Oxford University Press, 2nd edition.

Gentile, M. and Fioravanti, S. F. (2011). The impact of market fragmentation on European stock exchanges. Working paper.

Glebkin, S. and Kuong, J. C.-F. (2023). When large traders create noise. *Journal of Financial Economics*, 150(2):103709.

Gresse, C. (2017). Effects of lit and dark market fragmentation on liquidity. *Journal of Financial Markets*, 35:1–20.

Guo, L. and Jain, P. (2023). Market fragmentation and price impact. Working Paper.

Hamilton, J. L. (1979). Marketplace fragmentation, competition, and the efficiency of the stock exchange. *Journal of Finance*, 34(1):171–187.

Haslag, P. H. and Ringgenberg, M. C. (2023). The demise of the NYSE and Nasdaq: Market quality in the age of market fragmentation. *Journal of Financial and Quantitative Analysis*, 58(6):2753–2782.

Hengelbrock, J. and Theissen, E. (2009). Fourteen at one blow: The market entry of turquoise. Working Paper.

Ibikunle, G., Mare, D., and Sun, Y. (2020). The paradoxical effects of market fragmentation on adverse selection risk and market efficiency. *European Journal of Finance*, 26(14):1439–1461.

Kawakami, K. (2017). Welfare consequences of information aggregation and optimal market size. *American Economic Journal: Microeconomics*, 9(4):303–323.

Kyle, A. S. (1985). Continuous auctions and insider trading. *Econometrica*, 53(6):1315–1336.

Lausen, J., Clapham, B., Gomber, P., and Bender, M. (2022). Drivers and effects of stock market fragmentation – insights on SME stocks. Working Paper.

Lou, Y. and Rahi, R. (2023). Information, market power and welfare. *Journal of Economic Theory*, 214:105756.

Madhavan, A. (1995). Consolidation, fragmentation, and the disclosure of trading information. *Review of Financial Studies*, 8(3):579–603.

Malamud, S. and Rostek, M. (2017). Decentralized exchange. *American Economic Review*, 107(11):3320–3362.

Mendelson, H. (1987). Consolidation, fragmentation, and market performance. *Journal of Financial and Quantitative Analysis*, 22(2):189–207.

O'Hara, M. and Ye, M. (2011). Is market fragmentation harming market quality? *Journal of Financial Economics*, 100(3):459–474.

Pagano, M. (1989). Trading volume and asset liquidity. *Quarterly Journal of Economics*, 104(2):255–274.

Pagnotta, E. and Philippon, T. (2018). Competing on speed. *Econometrica*, 86(3):1067–1115.

Parlour, C. A. and Seppi, D. J. (2003). Liquidity-based competition for order flow. *Review of Financial Studies*, 16(2):301–343.

Rostek, M. and Weretka, M. (2012). Price inference in small markets. *Econometrica*, 80(2):687–711.

Rostek, M. and Weretka, M. (2015). Information and strategic behavior. *Journal of Economic Theory*, 158:536–557.

Rostek, M. and Yoon, J. H. (2021). Exchange design and efficiency. *Econometrica*, 89(6):2887–2928.

Spankowski, U. F. P., Wagener, M., and Burghof, H.-P. (2012). The role of traditional exchanges in fragmented markets. Working Paper.

U.S. Department of the Treasury (2017). A financial system that creates economic opportunities: Capital markets. Report, U.S. Department of the Treasury, Washington, DC.

Vives, X. (2011). Strategic supply function competition with private information. *Econometrica*, 79(6):1919–1966.

Wittwer, M. (2021). Connecting disconnected financial markets? *American Economic Journal: Microeconomics*, 13(1):252–282.

Online Appendix

Supplemental Material for "Market Fragmentation: Liquidity, Price Discovery, and Welfare"

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A1 Introduction

In this Online Appendix, we present extensions of the model outlined in Section 9 of the paper. These extensions do not admit a parsimonious description of the equilibrium in terms of composite parameters analogous to (φ, Φ). Nevertheless, the main results of the paper continue to hold in all except the last extension, where we endogenize information acquisition.

For ease of exposition, we state all the results even when they significantly overlap with those in the baseline model. The numbers for equations, lemmas, and propositions correspond to those in the paper, unless they are specific to the Online Appendix, in which case the numbers are prefixed by an "A". All proofs are in Section A6.

A2 Uniform Holding Costs

In this section, we study the case where speculators incur the same holding cost as liquidity traders. We show that our main results carry over to this setting. Local liquidity $\lambda^{-1}(E)$ is strictly decreasing in E. Under suitable conditions, global liquidity $\gamma^{-1}(E)$ has an inverted U-shape, and is maximized at some level E^*, as in Figure 1. The welfare of restricted and unrestricted traders is monotonically increasing in local and global liquidity, respectively. However, it is possible that global price informativeness does not have the inverted U-shape depicted in Figure 2.

The payoff of a speculator is given by

$$W^S = \sum_{i \in E} (v - p_i) q_i^S - \frac{k}{2} \left(\sum_{i \in E} q_i^S \right)^2,$$

instead of (1). The payoffs of liquidity traders are the same as in the main model, given by (2) and (3). We begin with the analog of Lemma 4.1:

Lemma A2.1 (Optimal strategies) *Suppose prices are given by $p_i = \lambda q_i$, $i \in E$, for some $\lambda > 0$. Then the following strategies are optimal for agents: $q_i^S = \beta v$, $q_{i\ell}^U = \alpha^U y_\ell$, and*

$q_{i\ell}^R = \alpha^R y_\ell$, *where*

$$\beta = \frac{1}{kE + (1+S)\lambda}, \tag{A1}$$

$$\alpha^U = \frac{k}{kE + 2\lambda}, \tag{A2}$$

$$\alpha^R = \frac{k}{k + 2\lambda}.$$

Unsurprisingly, speculators trade less when they incur holding costs. They also trade less on each exchange as the number of exchanges goes up. The optimal strategies of liquidity traders are the same as in the main model.

We have the following analog of Proposition 4.2:

Proposition A2.2 (Equilibrium existence and uniqueness) *For any given E, a linear equilibrium exists if*

$$\frac{k^2(L^U + L^R)(1+S)^2}{4S} \cdot \frac{\sigma_y^2}{\sigma_v^2} > 1. \tag{A3}$$

Furthermore, if a linear equilibrium exists, it is unique (in the class of linear equilibria) if

$$E \leq \frac{3L^U}{\bar{L}}, \tag{A4}$$

or

$$E < \frac{1+9S}{2}. \tag{A5}$$

The condition for existence is the same as in the main model ((A3) is the same as (13)). Uniqueness, on the other hand, can only be assured if the level of fragmentation is below an upper bound given by (A4) or (A5). Letting

$$\bar{E}_1 := \frac{k^2\bar{L}(1+S)^2}{4S} \cdot \frac{\sigma_y^2}{\sigma_v^2}, \tag{A6}$$

$$\bar{E}_2 := \frac{1+9S}{2}, \tag{A7}$$

we define an upper bound for E as follows:

$$\bar{E} := \begin{cases} \min\{\bar{E}_1, \bar{E}_2\}, & \text{if } L^U = 0, \\ \bar{E}_2, & \text{if } L^U > 0. \end{cases} \tag{A8}$$

For the rest of this section we assume that $E \in [1, \bar{E})$, and that condition (A3) holds. If $L^U = 0$, condition (A3) is, in fact, equivalent to $E < \bar{E}_1$. The restriction $E \in [1, \bar{E})$ also implies that $E < \bar{E}_2$, i.e. condition (A5) holds, ensuring that equilibrium is unique. We do not invoke condition (A4) in the results that follow.

The analog of Proposition 5.1 is:

Proposition A2.3 (Price impact) *We have the following results on price impact:*

i. $\partial\lambda/\partial E > 0$.

ii. If $\bar{L} = 0$, γ *is independent of* E.

iii. If $\bar{L} > 0$, $\partial\gamma/\partial E \propto 2\lambda - k$.

iv. Both λ *and* γ *are strictly decreasing in* L^U, $\bar{L}$, *and* σ_y^2, *and strictly increasing in* σ_v^2.

Comparing Proposition A2.3 to Proposition 5.1, we see that essentially all the results go through. The only difference is that the comparative statics in S and k are no longer unambiguous (cf. footnote 13), and λ does not satisfy the condition $\lim_{E\to\bar{E}} \lambda = \infty$.

For our next result, we define

$$\begin{aligned}
\Lambda_1 &:= \frac{k^2(L^U + \bar{L})(3+S)^2}{48S} \cdot \frac{\sigma_y^2}{\sigma_v^2},\\
\Lambda_2 &:= \left[\frac{4L^U}{9(1+3S)^2} + \frac{\bar{L}}{2(1+9S)}\right] \frac{k^2(1+5S)^2}{S(2+9S)} \cdot \frac{\sigma_y^2}{\sigma_v^2}.
\end{aligned}$$

Proposition A2.4 (Maximal global liquidity) *Suppose* $\bar{L} > 0$, *and* $\Lambda_2 < 1$. *Then we have the following results on the number of exchanges* E^* *at which global liquidity is maximal:*

i. There exists $E^* \in [1, \bar{E})$ *such that* $\gamma(E)$ *is minimized at* E^*.

ii. $E^* = 1$ *if and only if* $\lambda(1) \geq k/2$, *or equivalently* $\Lambda_1 \leq 1$. *Furthermore, if* $E^* = 1$, *then* $\partial\gamma/\partial E > 0$ *for* $E > 1$.

iii. $E^* > 1$ *if and only if* $\lambda(1) < k/2$, *or equivalently* $\Lambda_1 > 1$. *Furthermore, if* $E^* > 1$, *then* $\partial\gamma/\partial E < 0$ *for* $E < E^*$, *and* $\partial\gamma/\partial E > 0$ *for* $E > E^*$.

iv. If $E^* > 1$, *it is strictly increasing in* L^U, $\bar{L}$, *and* σ_y^2, *and strictly decreasing in* σ_v^2.

Proposition A2.4 is the same as Proposition 5.2 except for the following differences. First, the inequality $\lambda(1) < k/2$ is equivalent to $\Lambda_1 > 1$ instead of $\Phi > 16$ in the main paper. Second, we impose an additional condition, $\Lambda_2 < 1$. The role of this condition is to ensure that λ exceeds $k/2$ before E reaches its upper bound $\bar{E}$; without it, $\gamma(E)$ is strictly decreasing on the entire interval $[1, \bar{E})$, and there is no interior E^* at which global liquidity is maximal.

Since Λ_2 is strictly decreasing in S and converges to zero as S goes to infinity, we have $\Lambda_2 < 1$ for sufficiently large S. More generally, $\Lambda_2 < 1$ when the number of speculators, and the degree of asymmetric information as measured by σ_v^2, are large relative to the number of liquidity traders and the size of the liquidity shock σ_y^2. Third, we do not have the result that E^* is increasing in S or in k (cf. footnote 14).

Equilibrium utilities are the same as in the main model, as stated in Lemma 6.1, except for speculators:

Lemma A2.5 (Equilibrium utilities) *Equilibrium utilities for given E are*

$$\mathcal{U}^S := \mathbb{E}\left(W^S\right) = \frac{k+2\gamma}{2\big[k+(1+S)\gamma\big]^2}\sigma_v^2,$$
$$\mathcal{U}^U := \mathbb{E}\left(W_\ell^U\right) = -\frac{k\gamma}{k+2\gamma}\sigma_y^2,$$
$$\mathcal{U}^R := \mathbb{E}\left(W_{i\ell}^R\right) = -\frac{k\lambda}{k+2\lambda}\sigma_y^2.$$

While the expression for $\mathcal{U}^S$ differs from that in the main model, it has the same key properties: $\mathcal{U}^S$ depends on E only through γ and it is strictly decreasing in γ. It is easy to check that Corollary 6.2 holds with no modification, and Proposition 6.3 holds with the conditions on Φ replaced by the corresponding conditions on Λ_1.

Corollary A2.6 *Suppose all agents are unrestricted ($\bar{L} = 0$). Then the welfare of any agent is the same for any degree of fragmentation.*

Proposition A2.7 (Welfare: General case) *Suppose $\bar{L} > 0$. Then market fragmentation impacts welfare as follows:*

i. The welfare of restricted liquidity traders is strictly decreasing in E.

ii. The welfare of unrestricted traders is maximized at E^*.

iii. If $\lambda(1) \geq k/2$, or equivalently $\Lambda_1 \leq 1$, then $E^* = 1$, and an increase in E makes all agents worse off.

iv. If $\lambda(1) < k/2$, or equivalently $\Lambda_1 > 1$, then $E^* > 1$. Furthermore, the welfare of unrestricted traders is strictly increasing in E for $E < E^*$, and strictly decreasing in E for $E > E^*$.

Finally, we turn to price informativeness. We have the following analog of Lemma 7.1:

Lemma A2.8 *The equilibrium price on exchange i is given by*

$$p_i = \frac{S\gamma}{k + (1+S)\gamma}(v + \eta^U + \eta_i^R),$$

where $\eta^U := \alpha^U Q^U/(S\beta)$, and $\eta_i^R := \alpha^R Q_i^R/(S\beta)$. Furthermore,

$$Var(\eta^U + \eta_i^R) = Var(\eta^U) + Var(\eta_i^R) = \frac{k+\gamma}{S\gamma}\sigma_v^2, \tag{A9}$$

and

$$Var(\eta^U) = \left(\frac{\alpha^U}{\beta}\right)^2 \frac{L^U \sigma_y^2}{S^2} = \left[\frac{k + (1+S)\gamma}{k+2\gamma}\right]^2 \frac{k^2 L^U \sigma_y^2}{S^2}. \tag{A10}$$

Comparing Lemma A2.8 to Lemma 7.1, the main difference is that the variance of the noise in the price signal, $\mathrm{Var}(\eta^U + \eta_i^R)$, is not independent of E. It depends on E through γ, and it is strictly decreasing in γ. The variance of the common noise component of the price signal, $\mathrm{Var}(\eta^U)$, is still strictly increasing in γ, unless $S = 1$, in which case it does not depend on γ and hence is independent of E.

Since $\mathrm{Var}(\eta^U + \eta_i^R)$ is strictly decreasing in γ, local price informativeness on any exchange is strictly increasing in γ. This is in contrast to the main model, where local price informativeness does not depend on liquidity, local or global.

If $\bar{L} = 0$, then $\eta_i^R = 0$ for all i, and the price is the same on every exchange. Hence, global price informativeness is equal to local price informativeness. Moreover, since γ is independent of E when $\bar{L} = 0$ (Proposition A2.3 (ii)), local and global price informativeness are also independent of E.

If $\bar{L} > 0$, prices on different exchanges are not perfectly correlated, as in the main model. However, unlike the main model, fragmentation can lower local price informativeness, so that Proposition 7.2 (i) does not carry over.

If $L^U = 0$, then $\eta^U = 0$, and the price signals, $\{v + \eta_i^R\}_{i\in E}$, are conditionally independent and identically distributed signals about v. We will show below that $\mathcal{V}(E)$ is strictly increasing, even though $\mathrm{Var}(\eta_i^R)$ is not invariant with respect to E as in the main model (it is, in fact, non-monotonic in E; see (A9)).

We can now state the analog of Proposition 7.2. Parts (i) and (ii) follow from the above observations. Part (iii) follows from part (i) combined with our previous results on the number of exchanges E^* at which γ is minimized (Proposition A2.4). A proof of part (iv) is in Section A6.

Proposition A2.9 (Price informativeness)

i. $\mathcal{V}_i$ is strictly increasing in γ.

ii. *If $\bar{L} = 0$, $\mathcal{V}(E) = \mathcal{V}_i(E) = \mathcal{V}(1)$ for all $i \in E$, and for all $E \in [1, \bar{E})$.*

iii. *If $\bar{L} > 0$, $\mathcal{V}_i$ is minimized at E^*. If $E^* > 1$, $\mathcal{V}_i$ is strictly decreasing for $E < E^*$ and strictly increasing for $E \geq E^*$.*

iv. *If $L^U = 0$, $\mathcal{V}(E)$ is strictly increasing in E.*

As in the main model, local price informativeness $\mathcal{V}_i$ is the same for all $i \in E$ when $\bar{L} = 0$. However, unlike the main model, it varies with E when $\bar{L} > 0$. In fact, it is inversely related to global liquidity. Local price informativeness is minimized when global liquidity is maximized.

The results for global price informativeness in Proposition A2.9 are the same as in the main model, except for Proposition 7.2 (iii). If there are no restricted liquidity traders, global price informativeness is the same as local price informativeness and does not depend on the degree of market fragmentation. If all liquidity traders are restricted, on the other hand, global price informativeness is increasing in the level of fragmentation.

For the case where some liquidity traders are restricted, it is difficult to fully characterize the shape of $\mathcal{V}(E)$ as we did in Proposition 7.3 for the main model. However, we can show that if there is only one speculator, $\mathcal{V}(E)$ is strictly increasing for $E \geq E^*$, which contrasts with the result in Proposition 7.3 that $\mathcal{V}(E)$ attains a maximum at $\hat{E}$.

Proposition A2.10 (Global price informativeness) *Suppose $S = 1$ and $\bar{L} > 0$. Then $\partial \mathcal{V}/\partial E > 0$ for $E \geq E^*$.*

The result follows from two observations. First, the variance of the noise in the price signal on any exchange, $\mathrm{Var}(\eta^U + \eta_i^R)$, is strictly decreasing in γ (equation (A9)), and hence strictly decreasing in E for $E \geq E^*$. Second, with regard to the common component of the noise, either $L^U = 0$ and hence $\eta^U = 0$, or $L^U > 0$, in which case $\mathrm{Var}(\eta^U)$ does not depend on E when $S = 1$ (equation (A10)).

A3 CARA Utility

In this section, we assume that liquidity traders have constant absolute risk aversion, with the same coefficient r. There is no holding cost for any agent. For tractability we assume that there are no unrestricted liquidity traders ($L^U = 0$).

Our main results, for the case where $L^U = 0$, continue to hold in this setting. Local liquidity λ^{-1} is strictly decreasing in E. Under suitable conditions, global liquidity $\gamma^{-1}(E)$ has an inverted U-shape, and is maximized at some level E^*, as in Figure 1. The welfare of restricted liquidity traders is strictly decreasing in E, while that of unrestricted traders (speculators) is maximized at E^*. Market fragmentation increases global price informativeness, but has no effect on local price informativeness.

A restricted liquidity trader ℓ operating on exchange i solves

$$\max_{q_{i\ell}^R} \mathbb{E}[-\exp(-rW_{i\ell}^R)|y_\ell], \tag{A11}$$

where

$$W_{i\ell}^R = (v - p_{i\ell}^R)q_{i\ell}^R - y_\ell v.$$

We assume throughout this section that $L^R \geq 1$, which is a natural assumption, but one that we did not need to invoke in the main model. We begin with the analog of Lemma 4.1.

Lemma A3.1 (Optimal strategies) *Suppose prices are given by $p_i = \lambda q_i$, $i \in E$, for some $\lambda > 0$. Then the following strategies are optimal for agents: $q_i^S = \beta v$, and $q_{i\ell}^R = \alpha^R y_\ell$, where*

$$\beta = \frac{1}{(1+S)\lambda}, \tag{A12}$$

$$\alpha^R = \frac{rL^R(1+S)\sigma_v^2}{r\left[L^R + (L^R-1)S\right]\sigma_v^2 + 2L^R(1+S)^2\lambda}. \tag{A13}$$

The optimal strategy of a speculator is the same as in the main model, but that of a restricted liquidity trader is different. Unlike the main model, the coefficient α^R depends on E through λ as well as through L^R. It is decreasing in both λ and L^R. The dependence on L^R is due to the variance term in the denominator of the optimal asset position, given by (A39) in the proof of Lemma A3.1 in Section A6. In particular, for a given value of λ, an increase in E lowers L^R, thereby reducing $\mathrm{Var}[v - p_{i\ell}^R|y_\ell]$, and hence increasing α^R.

For existence of equilibrium, we need to impose an upper bound on E, given by

$$\bar{E} := \frac{\bar{L}}{4S}. \tag{A14}$$

Note that this is not the same upper bound as the one given by (15) but, as in the main model, we can make it as large as desired with an appropriate choice of $\bar{L}$. We have the following analog of Proposition 4.2:

Proposition A3.2 (Equilibrium existence and uniqueness) *For any $E \in [1, \bar{E})$, a linear equilibrium exists if and only if*

$$z < r\sigma_v\sigma_y < Z, \tag{A15}$$

where

$$z := 2\left(\frac{ES}{\bar{L}}\right)^{\frac{1}{2}}, \tag{A16}$$

$$Z := \frac{\left[\bar{L}^2 + (\bar{L}-E)^2 S\right]^{\frac{1}{2}} - (\bar{L}E)^{\frac{1}{2}}}{(\bar{L}-E)S^{\frac{1}{2}}}. \tag{A17}$$

We have $\partial z/\partial E > 0$, $\partial Z/\partial E < 0$, and $\lim_{E\to\bar{E}} z = \lim_{E\to\bar{E}} Z = 1$. Furthermore, if a linear equilibrium exists, it is unique (in the class of linear equilibria).

We can interpret $r\sigma_v\sigma_y$ as a measure of the hedging need of a restricted liquidity trader. Existence of equilibrium requires this hedging need to be sufficiently high ($r\sigma_v\sigma_y > z$). This is analogous to the existence condition in the main model which is satisfied if liquidity needs are strong enough. Unlike the main model, however, we also need an upper bound on the hedging need ($r\sigma_v\sigma_y < Z$) to ensure that the expected utility of a restricted liquidity trader is well-defined. The proof shows that $z < Z$ for all $E \in [1, \bar{E})$. The interval (z, Z) collapses to a singleton as $E \to \bar{E}$. Henceforth, we assume that the existence condition (A15) is satisfied.

Let

$$Q(E) := \frac{4(\bar{L}ES)^{\frac{1}{2}}(1+S)}{\bar{L} + (\bar{L}+E)S}. \tag{A18}$$

Then we can state the analog of Proposition 5.1 as follows:

Proposition A3.3 (Price impact) *We have the following results on price impact:*

i. $\partial\lambda/\partial E > 0$, *and* $\lim_{E\to\bar{E}} \lambda = \infty$.

ii. $\partial\gamma/\partial E \propto Q(E) - r\sigma_v\sigma_y$.

iii. $\partial Q/\partial E > 0$, $\partial Q/\partial \bar{L} < 0$, $\partial Q/\partial S > 0$, $\lim_{\bar{L}\to\infty} Q = 0$, *and* $\lim_{E\to\bar{E}} Q > r\sigma_v\sigma_y$.

iv. Both λ *and* γ *are strictly decreasing in* $\bar{L}$, r, *and* σ_y^2.

Proposition A3.3 is broadly similar to Proposition 5.1 (recall that we are assuming here that $L^U = 0$). The main difference is that the effect of fragmentation on global price impact γ is pinned down by the relationship $\partial\gamma/\partial E \propto Q(E) - r\sigma_v\sigma_y$, instead of $\partial\gamma/\partial E \propto 2\lambda - k$. Like λ, Q is a strictly increasing function of E. This implies that the function $\gamma(E)$ has the same shape as in the main model, as can be seen in the next proposition.

Proposition A3.4 (Maximal global liquidity) *We have the following results on the number of exchanges E^* at which global liquidity is maximal:*

i. There exists $E^* \in [1, \bar{E})$ such that $\gamma(E)$ is minimized at E^*.

ii. $E^* = 1$ if and only if $Q(1) \geq r\sigma_v\sigma_y$. Furthermore, if $E^* = 1$, then $\partial\gamma/\partial E > 0$ for $E > 1$.

iii. $E^* > 1$ if and only if $Q(1) < r\sigma_v\sigma_y$. Furthermore, if $E^* > 1$, then $\partial\gamma/\partial E < 0$ for $E < E^*$, and $\partial\gamma/\partial E > 0$ for $E > E^*$.

iv. $Q(1) < r\sigma_v\sigma_y$, and hence $E^* > 1$, for sufficiently large $\bar{L}$.

v. If $E^* > 1$, it is strictly decreasing in S, and strictly increasing in $\bar{L}$ and $r\sigma_v\sigma_y$.

This result is analogous to Proposition 5.2, with $Q(1) - r\sigma_v\sigma_y$ replacing $\lambda(1) - k/2$. However, it differs in two respects. First, E^* is increasing in S in the main model (see footnote 14), while here it is decreasing in S. As discussed in the paper, an increase in S leads to higher adverse selection as well as more competition among speculators. The competition effect dominates in the main model, but that is not the case here. Second, the comparative static of E^* with respect to σ_v is reversed. In the current setting, σ_v measures not only the degree of asymmetric information, but also scales the hedging need $r\sigma_v\sigma_y$. The latter effect is stronger.

Next, we calculate equilibrium utilities and analyze the welfare consequences of fragmentation.

Lemma A3.5 (Equilibrium utilities) *Equilibrium utilities for given E are*

$$\mathcal{U}^S := \mathbb{E}\left(W^S\right) = \frac{1}{(1+S)^2\gamma}\sigma_v^2, \tag{A19}$$

$$\mathcal{U}^R := \mathbb{E}[-\exp(-rW_{i\ell}^R)] = -(1-2\sigma^2)^{-\frac{1}{2}}, \tag{A20}$$

where

$$\sigma^2 := \frac{r\sigma_v\sigma_y}{2}\left[\frac{r\sigma_v\sigma_y(\bar{L}-E)S + 2(\bar{L}ES)^{\frac{1}{2}}}{\bar{L} + (\bar{L}-E)S}\right].$$

The equilibrium utility of a speculator is the same as in the main model. The calculations for $\mathcal{U}^R$ are in the proof of Proposition A3.2 (see (A47) and (A48) in Section A6).

The following result is almost exactly as in the main model (Proposition 6.3), for the case where $L^U = 0$, with $Q(1) - r\sigma_v\sigma_y$ replacing $\lambda(1) - k/2$.

Proposition A3.6 (Welfare) *Market fragmentation impacts welfare as follows:*

i. The welfare of restricted liquidity traders is strictly decreasing in E.

ii. The welfare of speculators is maximized at E^*.

iii. If $Q(1) \geq r\sigma_v\sigma_y$, then $E^* = 1$, and an increase in E makes all agents worse off.

iv. If $Q(1) < r\sigma_v\sigma_y$, then $E^* > 1$. Furthermore, the welfare of speculators is strictly increasing in E for $E < E^*$, and strictly decreasing in E for $E > E^*$.

Finally, we turn to price informativeness. We have the following analog of Proposition 7.2:

Proposition A3.7 (Price informativeness)

i. $\mathcal{V}_i(E) = \mathcal{V}(1) = [1 + S^{-1}]^{-1}$, for all $i \in E$, and for all $E \in [1, \bar{E})$.

ii. $\mathcal{V}(E) = [1 + (ES)^{-1}]^{-1}$, which is strictly increasing in E.

This result is exactly the same as Proposition 7.2 for the case where $L^U = 0$. Also, since we have set $L^U = 0$, there are no analogs of Proposition 7.3 and Corollary 7.4.

A4 Correlated Endowment Shocks

In this section, we consider the case where the endowment shocks of liquidity traders are correlated, with

$$\operatorname{corr}(y_\ell, y_{\ell'}) = \rho, \qquad \forall \ell \neq \ell', \tag{A21}$$

where $\rho \in [0, 1)$. As in the main model, $\operatorname{corr}(v, y_\ell) = 0$ for all ℓ. For tractability we assume that there are no unrestricted liquidity traders ($L^U = 0$).

The results of the main model, for the case where $L^U = 0$, carry over to this setting. Local liquidity λ^{-1} is strictly decreasing in E. Under suitable conditions, global liquidity $\gamma^{-1}(E)$ has an inverted U-shape, and is maximized at some level E^*, as in Figure 1. The welfare of restricted liquidity traders is strictly decreasing in E, while that of unrestricted traders (speculators) is maximized at E^*. Market fragmentation increases global price informativeness, while leaving local price informativeness unchanged.

As in Section A3, we assume throughout this section that $L^R \geq 1$. We begin with the analog of Lemma 4.1.

Lemma A4.1 (Optimal strategies) *Suppose prices are given by $p_i = \lambda q_i$, $i \in E$, for some $\lambda > 0$. Then the following strategies are optimal for agents: $q_i^S = \beta v$, and $q_{i\ell}^R = \alpha^R y_\ell$,* *where*

$$\beta = \frac{1}{(1+S)\lambda}, \tag{A22}$$

$$\alpha^R = \frac{k}{k + 2\lambda + \lambda\rho(L^R - 1)}. \tag{A23}$$

The optimal strategy of a speculator is the same as in the main model, but that of a restricted liquidity trader is different. As in the main model, α^R is decreasing in λ. But for given λ, it is also decreasing in the correlation coefficient ρ. The reason is that any given liquidity trader expects other liquidity traders to trade in the same direction as her, since their endowments are positively correlated with hers. For example, if the realized value of y_ℓ is positive, she buys and expects other liquidity traders to buy as well, increasing the expected price. This leads her to buy less, with the amount decreasing in ρ. An analogous argument applies if $y_\ell < 0$.

When $\rho > 0$, the coefficient α^R is also increasing in E. An increase in E lowers the number of liquidity traders L^R who compete with each other on any given exchange. This competition effect is missing when $\rho = 0$ since in that case the trading motives of liquidity traders are mutually independent.

We have the following analog of Proposition 4.2:

Proposition A4.2 (Equilibrium existence and uniqueness) *For any given E, a linear equilibrium exists if and only if*

$$H := \frac{k^2(1-\rho+\rho L^R)L^R(1+S)^2}{(2-\rho+\rho L^R)^2 S} \cdot \frac{\sigma_y^2}{\sigma_v^2} > 1. \tag{A24}$$

Furthermore, if a linear equilibrium exists, it is unique (in the class of linear equilibria).

It is easy to check that H, as defined in condition (A24), is strictly increasing in L^R. As in the main model, a linear equilibrium exists if liquidity needs (as measured by k, L^R, or σ_y^2) are relatively high, if competition among speculators (as measured by $(1+S)^2/S$) is sufficiently intense, or if the degree of asymmetric information (as measured by σ_v^2) is sufficiently low. We assume that $H > 1$ for some $E \geq 1$.

Since H is strictly increasing in L^R, and $L^R = \bar{L}/E$, it follows that H is strictly increasing in $\bar{L}$ and strictly decreasing in E. Therefore, there exists $\bar{E} > 1$ such that $H(\bar{E}) = 1$. The existence condition $H > 1$ is equivalent to $E < \bar{E}$. Accordingly, we assume for the rest of this section that $E \in [1, \bar{E})$. We can make $\bar{E}$ as high as we like by choosing a sufficiently high value of $\bar{L}$.

Let

$$\omega(E) := (2-\rho)\lambda(E) + \frac{\rho(3-\rho)}{1-\rho}\bar{L}\gamma(E). \tag{A25}$$

The analog of Proposition 5.1 is:

Proposition A4.3 (Price impact) *We have the following results on price impact:*

i. $\partial\lambda/\partial E > 0$, *and* $\lim_{E\to\bar{E}}\lambda = \infty$.

ii. $\partial\gamma/\partial E \propto \omega(E) - k$.

iii. Both λ *and* γ *are strictly decreasing in* S, $\bar{L}$, k, *and* σ_y^2, *and strictly increasing in* σ_v^2.

This result is broadly similar to Proposition 5.1, under the assumption that $L^U = 0$. The main difference is that the effect of market fragmentation on global price impact γ is pinned down by the relationship $\partial\gamma/\partial E \propto \omega - k$, instead of $\partial\gamma/\partial E \propto 2\lambda - k$. Note that $\omega = 2\lambda$ when $\rho = 0$. For any $\rho \in [0, 1)$, ω shares a key property with 2λ: once ω exceeds k for some level of fragmentation, it remains greater than k at higher levels of fragmentation. Consequently, the function $\gamma(E)$ exhibits the same qualitative shape as in the main model, as shown in the next proposition.

Proposition A4.4 (Maximal global liquidity) *We have the following results on the number of exchanges* E^* *at which global liquidity is maximal:*

i. There exists $E^* \in [1, \bar{E})$ *such that* $\gamma(E)$ *is minimized at* E^*.

ii. $E^* = 1$ *if and only if* $\omega(1) \geq k$. *Furthermore, if* $E^* = 1$, *then* $\partial\gamma/\partial E > 0$ *for* $E > 1$.

iii. $E^* > 1$ *if and only if* $\omega(1) < k$. *Furthermore, if* $E^* > 1$, *then* $\partial\gamma/\partial E < 0$ *for* $E < E^*$, *and* $\partial\gamma/\partial E > 0$ *for* $E > E^*$.

iv. $\omega(1) < k$, *and hence* $E^* > 1$, *if either* S *or* σ_y^2 *are sufficiently large, or* σ_v^2 *is sufficiently small.*

v. If $E^* > 1$, *it is strictly increasing in* S, k, *and* σ_y^2, *and strictly decreasing in* σ_v^2.

This result is analogous to Proposition 5.2, with ω replacing 2λ. The only part that is materially different is that E^* may not be increasing in $\bar{L}$ (cf. footnote 14).

Next, we calculate equilibrium utilities and analyze the welfare consequences of fragmentation.

Lemma A4.5 (Equilibrium utilities) *Equilibrium utilities for given E are*

$$\mathcal{U}^S := \mathbb{E}\left(W^S\right) = \frac{1}{(1+S)^2\gamma}\sigma_v^2, \tag{A26}$$

$$\mathcal{U}^R := \mathbb{E}\left(W^R_{i\ell}\right) = -\frac{k}{2}\left[1 - \frac{k(k+2\lambda)E^2}{[(k+2\lambda-\lambda\rho)E+\lambda\rho\bar{L}]^2}\right]\sigma_y^2. \tag{A27}$$

The equilibrium utility of a speculator is the same as in the main model. The equilibrium utility of a restricted liquidity trader is decreasing λ, as in the main model, but it is also increasing in E when $\rho > 0$. This is due to the competition effect alluded to earlier (see the discussion of α^R that follows Lemma A4.1). Thus an increase in E has two opposing effects on the utility of a restricted liquidity trader, a direct positive effect and an indirect negative effect through λ. The net effect is negative, just as in the main model, as we show in the next result.

The following result is the same as Proposition 6.3, for the case where $L^U = 0$, except that ω replaces 2λ.

Proposition A4.6 (Welfare) *Market fragmentation impacts welfare as follows:*

i. The welfare of restricted liquidity traders is strictly decreasing in E.

ii. The welfare of speculators is maximized at E^*.

iii. If $\omega(1) \geq k$, then $E^* = 1$, and an increase in E makes all agents worse off.

iv. If $\omega(1) < k$, then $E^* > 1$. Furthermore, the welfare of speculators is strictly increasing in E for $E < E^*$, and strictly decreasing in E for $E > E^*$.

Finally, we turn to price informativeness. The following result is the analog of Proposition 7.2.

Proposition A4.7 (Price informativeness)

i. $\mathcal{V}_i(E) = \mathcal{V}(1) = [1+S^{-1}]^{-1}$, for all $i \in E$ and for all $E \in [1, \bar{E})$.

ii. $\mathcal{V}(E)$ is strictly increasing in E, and strictly decreasing in ρ.

The effect of market fragmentation on price informativeness is the same as in the main model, for the case where $L^U = 0$. Local price informativeness does not depend on E, while global price informativeness is strictly increasing in E. Global price informativeness is also strictly decreasing in ρ. This is because a higher ρ makes prices more correlated across exchanges. Since we assume here that $L^U = 0$, there are no analogs of Proposition 7.3 and Corollary 7.4.

A5 Endogenous Information Acquisition

In the paper, as well as in the preceding extensions, the number of speculators is exogenous and each speculator observes the fundamental v at no cost. In this section, we endogenize information acquisition by speculators. For analytical tractability, we set $L^U = 0$, so that all liquidity traders are restricted. As we shall see, endogenizing private information leads to results on liquidity, welfare, and price informativeness that differ markedly from those that we derived in the paper. This is because market fragmentation can lead to entry or exit of speculators, depending on the parameters and on the equilibrium under consideration.

First, we consider welfare. With exogenous information, fragmentation leads to lower local liquidity (higher λ). Hence restricted liquidity traders are always worse off. When information is endogenous, on the other hand, fragmentation can result in more speculators entering the market. Greater competition among a higher number of speculators can in turn raise liquidity on each exchange, which is Pareto improving.

Next, we look at price informativeness. When private information is exogenous, and all liquidity traders are restricted, market fragmentation always increases global price informativeness. In contrast, when information is endogenous, market fragmentation can lower price informativeness, due to exit of speculators from the market. This happens when liquidity is relatively low to begin with, and fragmentation reduces it further.

A5.1 Equilibrium

We assume that each speculator must incur a cost $c > 0$ in order to learn the realization of the fundamental v. The number of speculators S is determined by the zero-profit condition $\mathcal{U}^S = c$, where $\mathcal{U}^S$ is the utility that a speculator derives from trading, given by (20). Thus we have

$$\frac{E\sigma_v^2}{(1+S)^2\lambda} = c. \tag{A28}$$

We define equilibrium as in Section 4, with the additional condition that $S \geq 1$, where S is determined by (A28).

A key feature of this setup is that there may be multiple equilibria with different comparative statics properties. Existence and multiplicity of equilibrium can be characterized in terms of the magnitude of liquidity needs, as measured by $\mathcal{L} := k^2\bar{L}\sigma_y^2$; this is the same as the first of component of the parameter Φ in the paper, for the special case where $L^U = 0$.

Let

$$\mathcal{L}_1 := \frac{4ck(ck + E\sigma_v^2)}{E\sigma_v^2},$$
$$\mathcal{L}_2 := \frac{(2ck + E\sigma_v^2)^2}{E\sigma_v^2},$$
$$\mathcal{L}_3 := \frac{(2ck + E\sigma_v^2)(6ck + E\sigma_v^2)}{E\sigma_v^2}.$$

Note that $\mathcal{L}_1 < \mathcal{L}_2 < \mathcal{L}_3$. By "generic subset of (a, b)" we mean "subset of (a, b) of full Lebesgue measure".

Proposition A5.1 (Equilibrium: Existence) *There exists $\bar{E}$ such that there is no equilibrium if $E > \bar{E}$. For given $E \in [1, \bar{E}]$, we have the following results:*

i. For $\mathcal{L} \leq \mathcal{L}_1$, there is no equilibrium.

ii. For $\mathcal{L}$ in a generic subset of $(\mathcal{L}_1, \mathcal{L}_2)$, either there is no equilibrium, or there are two equilibria.

iii. For $\mathcal{L}$ in a generic subset of $[\mathcal{L}_2, \mathcal{L}_3)$, either there are three equilibria, or there is a unique equilibrium.

iv. For $\mathcal{L} \geq \mathcal{L}_3$, there is a unique equilibrium.

Thus there is no equilibrium if the magnitude of liquidity needs is low and a unique equilibrium if it is high, while for intermediate values there are multiple equilibria. In contrast, in the exogenous information case, equilibrium is always unique when it exists (Proposition 4.2).

We identify an equilibrium by the price impact parameter λ associated with it. If there is a unique equilibrium, we denote it by λ_u. If there are two equilibria, we denote them by λ_ℓ and λ_m, and if there are three equilibria, by λ_ℓ, λ_m, and λ_h $(0 < \lambda_\ell < \lambda_m < \lambda_h)$. The subscripts ℓ, m, and h denote low, medium, and high price impact, respectively.

It turns out that a λ_ℓ equilibrium has the same comparative statics properties regardless of whether the number of equilibria is 2 or 3. The same is true of a λ_m equilibrium. When we say that a certain property applies to a λ_ℓ, λ_h, or λ_u equilibrium, it should be kept in mind that these three types of equilibrium arise for different sets of parameters. Thus, if there are two equilibria, there is no λ_h or λ_u equilibrium, and if the equilibrium is unique, there is no λ_ℓ, λ_m, or λ_h equilibrium.

Of particular interest is the case where there are multiple equilibria. We already know from Proposition A5.1 that a necessary condition for there to be multiple equilibria is that the magnitude of liquidity needs, as measured by $\mathcal{L}$, is neither too low nor too high. In the next result, we strengthen this condition to ensure that equilibrium is not unique.

Proposition A5.2 (Equilibrium: Multiplicity) *Sufficient conditions for there to be multiple equilibria are as follows:*

i. There are two equilibria if (a) c is sufficiently small, and (b) $\mathcal{L}_1 < \mathcal{L} < \mathcal{L}_2$.

ii. There are three equilibria if (a) c is sufficiently small, and (b) $\mathcal{L}_2 \leq \mathcal{L} < \mathcal{L}_2 + \epsilon$, for sufficiently small ϵ.

Relative to the necessary conditions for multiplicity provided in Proposition A5.1, the sufficient conditions in Proposition A5.2 further restrict the values of $\mathcal{L}$, and specify that the cost of acquiring information, c, is low.

A5.2 Liquidity and welfare

We now turn to the consequences of fragmentation for liquidity and welfare.

Proposition A5.3 (Liquidity and welfare) *The effect of market fragmentation on liquidity and welfare is as follows:*

i. At a λ_ℓ, λ_h, or λ_u equilibrium, an increase in E lowers local liquidity. Market fragmentation is Pareto worsening.

ii. At a λ_m equilibrium, an increase in E raises local liquidity. Market fragmentation is Pareto improving.

Figure A1 illustrates equilibria for increasing levels of $\mathcal{L}$ (see Proposition A5.2). We show in Lemma A6.1 in Section A6 that an equilibrium value of λ solves $f(\lambda; E) = 0$, where the function f is defined by

$$f(\lambda; E) = \frac{k^2 \bar{L}}{E(k\lambda^{-1} + 2)^2} \frac{(1+S)^2}{S} - \frac{\sigma_v^2}{\sigma_y^2}.$$

From (A28), for given E, the number of speculators is inversely related to λ. The maximum level of λ corresponds to $S = 1$, and is given by $\lambda_{\max} = E\sigma_v^2/(4c)$. In the proof of Proposition A5.3, we show that $\partial f(\lambda; E)/\partial E < 0$. In Figure A1, this corresponds to a downward shift in the graph of f when there is an increase in E. We can see from the figure that a higher E leads to an increase in λ at a λ_ℓ, λ_h, or λ_u equilibrium, and a decrease in λ at a λ_m equilibrium, as stated in Proposition A5.3.

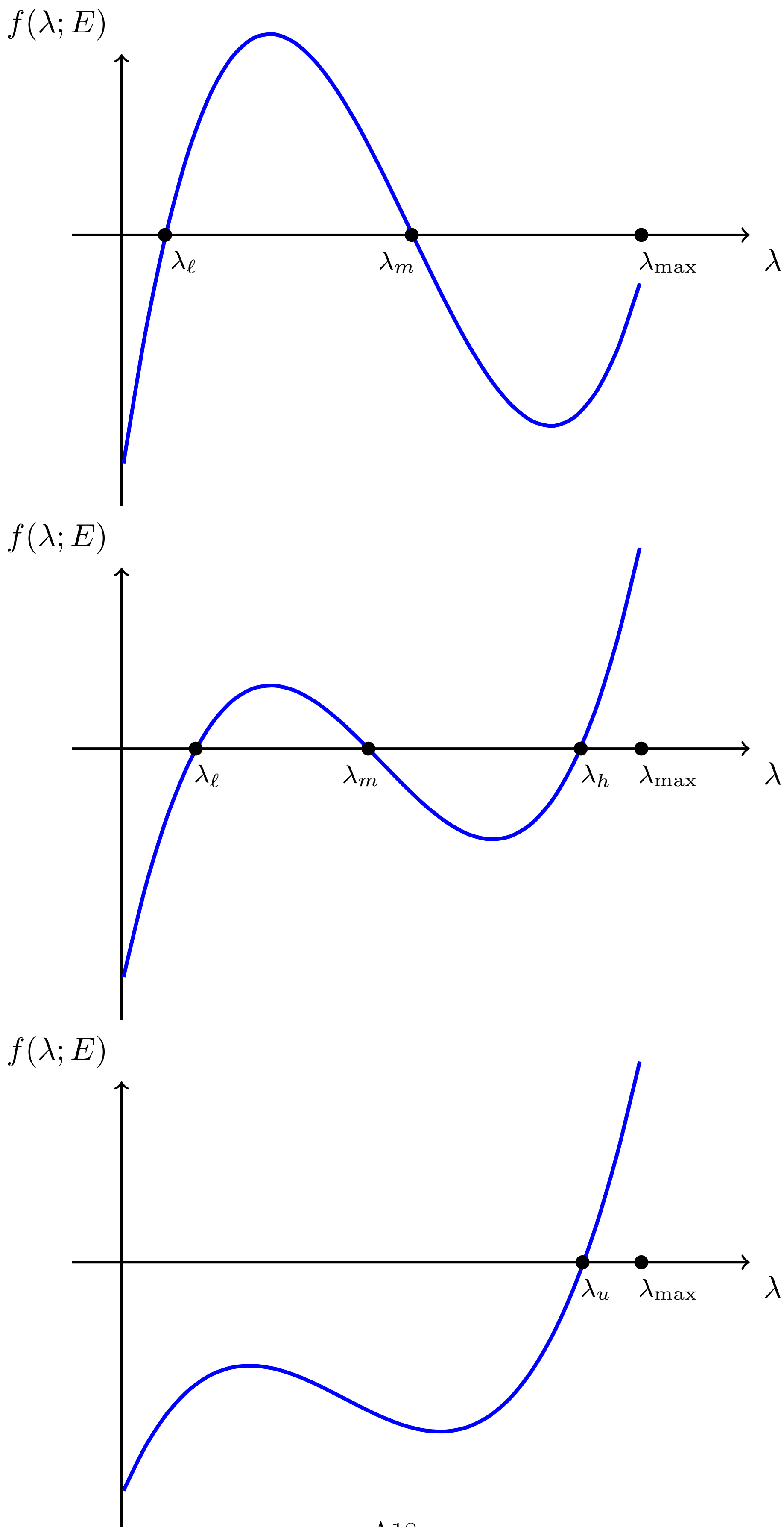

Figure A1: Equilibria associated with increasing levels of $\mathcal{L}$

Recall that, when private information is exogenous, market fragmentation always lowers local liquidity (Proposition 5.1 (ii)), making restricted liquidity traders worse off (Proposition 6.3 (i)). With endogenous information acquisition, market fragmentation can lead to entry of more speculators accompanied by higher liquidity, both local and global. This is what happens at a λ_m equilibrium. As a result, restricted liquidity traders are better off (from (22)). Speculators break even due to the zero-profit condition (A28). Therefore, market fragmentation is Pareto improving.

A5.3 Price informativeness

Finally, we turn to the effect of market fragmentation on price informativeness. From Proposition 7.2, parts (i) and (iv), local and global price informativeness are given by $\mathcal{V}_i = [1 + S^{-1}]^{-1}$ and $\mathcal{V} = [1 + (ES)^{-1}]^{-1}$, respectively. Hence an increase in the number of exchanges E leads to higher local price informativeness $\mathcal{V}_i$ if and only if the number of speculators S goes up, and higher global price informativeness $\mathcal{V}$ if and only if the product ES goes up. When information is exogenous, S is fixed, and hence an increase in E leads to a higher $\mathcal{V}$, while leaving $\mathcal{V}_i$ unchanged, as we noted in our discussion of Proposition 7.2. But when speculators acquire information at a cost, S can increase or decrease when E goes up. From Proposition A5.3 (ii), we see that, at a λ_m equilibrium, an increase in E results in lower price impact λ, attracting a higher number of speculators. Therefore, at a λ_m equilibrium, market fragmentation leads to higher local and global price informativeness. However, Proposition A5.3 (i) shows that λ is increasing in E at an equilibrium of any other type. This gives rise to the possibility of a fall in the number of speculators as E increases, to such an extent that ES, and hence $\mathcal{V}$, falls. If global price informativeness is lower, local price informativeness must be lower as well (the converse is not true, however).

Proposition A5.4 (Price informativeness) *Suppose $\mathcal{L}_2 < \mathcal{L} < \mathcal{L}_2 + \epsilon$, for sufficiently small ϵ, and $6ck < E\sigma_v^2$. Then there exists an equilibrium such that market fragmentation reduces both local and global price informativeness.*

An equilibrium at which global (and hence also local) price informativeness is decreasing in E is one where liquidity is relatively low, as is the number of speculators (this can be seen from the proof in Section A6).[1] An increase in E lowers liquidity further.

[1]The proof also shows that this equilibrium is either a λ_u equilibrium or a λ_h equilibrium.

A6 Proofs

Proof of Lemma A2.1 We calculate β as in the proof of Lemma 4.1. A speculator maximizes

$$\mathbb{E}(W^S|v) = \sum_{i\in E}\left[[1-(S-1)\lambda\beta]\, vq_i^S - \lambda\left(q_i^S\right)^2\right] - \frac{k}{2}\left(\sum_{i\in E} q_i^S\right)^2.$$

Using the first-order condition with respect to q_i^S, we get

$$\begin{aligned} q_i^S &= \frac{[1-(S-1)\lambda\beta]\, v - k\sum_{j\in E} q_j^S}{2\lambda} \\ &= \frac{[1-(S-1)\lambda\beta]\, v - kE\beta v}{2\lambda}. \end{aligned}$$

Hence

$$\beta = \frac{1-(S-1)\lambda\beta - kE\beta}{2\lambda},$$

which yields (A1). The optimal trades for liquidity traders are unaffected. □

Proof of Proposition A2.2 Substituting (A1) into (36), we have

$$\lambda\left[L^U(\alpha^U)^2 + L^R(\alpha^R)^2\right]\sigma_y^2 = \frac{(kE+\lambda)S\sigma_v^2}{[kE+(1+S)\lambda]^2},$$

so that, using (10) and (11),

$$\begin{aligned} \frac{\sigma_v^2}{\sigma_y^2} &= \frac{\left[L^U(\alpha^U)^2 + L^R(\alpha^R)^2\right][kE+(1+S)\lambda]^2}{S(kE+\lambda)}\lambda \\ &= \left[\frac{L^U}{(kE+2\lambda)^2} + \frac{L^R}{(k+2\lambda)^2}\right]\frac{k^2[kE+(1+S)\lambda]^2}{S(kE+\lambda)}\lambda. \end{aligned}$$

Thus λ is a solution to $f(\lambda;E)=0$, where

$$\begin{aligned} f(\lambda;E) &:= \left[\frac{L^U}{(kE+2\lambda)^2} + \frac{\bar{L}}{E(k+2\lambda)^2}\right]\frac{k^2[kE+(1+S)\lambda]^2}{S(kE+\lambda)}\lambda - \frac{\sigma_v^2}{\sigma_y^2} && \text{(A29)} \\ &= \frac{\lambda k^2\left[L^U E(k+2\lambda)^2 + \bar{L}(kE+2\lambda)^2\right][kE+(1+S)\lambda]^2}{ES(kE+\lambda)(kE+2\lambda)^2(k+2\lambda)^2} - \frac{\sigma_v^2}{\sigma_y^2}. && \text{(A30)} \end{aligned}$$

Since $\lim_{\lambda \to 0} f(\lambda; E) < 0$, a solution exists if

$$\lim_{\lambda \to \infty} f(\lambda; E) = \frac{k^2(L^U + L^R)(1+S)^2}{4S} - \frac{\sigma_v^2}{\sigma_y^2} > 0,$$

which is equivalent to (A3).

Provided a solution exists, it must be unique if $\partial f(\lambda; E)/\partial \lambda > 0$. We will show that this is indeed the case under either of the two conditions (A4) or (A5). We have

$$\begin{aligned}
\frac{\partial f(\lambda; E)}{\partial \lambda} &\propto (kE+\lambda)(kE+2\lambda)^2(k+2\lambda)^2 \\
&\quad \cdot \Big(\lambda\big[kE+(1+S)\lambda\big]^2\big[4L^U E(k+2\lambda) + 4\bar{L}(kE+2\lambda)\big] \\
&\qquad + \big[[kE+(1+S)\lambda]^2 + 2(1+S)\lambda\,[kE+(1+S)\lambda]\big]\big[L^U E(k+2\lambda)^2 + \bar{L}(kE+2\lambda)^2\big]\Big) \\
&\quad - \lambda\left[L^U E(k+2\lambda)^2 + \bar{L}(kE+2\lambda)^2\right]\left[kE+(1+S)\lambda\right]^2 \\
&\qquad \cdot \Big((kE+2\lambda)^2(k+2\lambda)^2 + 4(kE+\lambda)(kE+2\lambda)(k+2\lambda)^2 \\
&\qquad\quad + 4(kE+\lambda)(kE+2\lambda)^2(k+2\lambda)\Big) \\
&\propto (kE+\lambda)(kE+2\lambda)(k+2\lambda) \\
&\quad \cdot \Big(4\lambda\left[kE+(1+S)\lambda\right]\left[L^U E(k+2\lambda) + \bar{L}(kE+2\lambda)\right] \\
&\qquad + \big[kE+(1+S)\lambda + 2(1+S)\lambda\big]\big[L^U E(k+2\lambda)^2 + \bar{L}(kE+2\lambda)^2\big]\Big) \\
&\quad - \lambda\left[L^U E(k+2\lambda)^2 + \bar{L}(kE+2\lambda)^2\right]\left[kE+(1+S)\lambda\right] \\
&\qquad \cdot \Big((kE+2\lambda)(k+2\lambda) + 4(kE+\lambda)(k+2\lambda) + 4(kE+\lambda)(kE+2\lambda)\Big) \\
&= L^U E(k+2\lambda)^2\Big[4\lambda[kE+(1+S)\lambda](kE+\lambda)(kE+2\lambda) \\
&\qquad + \big[kE+(1+S)\lambda+2(1+S)\lambda\big](kE+\lambda)(kE+2\lambda)(k+2\lambda) \\
&\qquad - \lambda\big[kE+(1+S)\lambda\big] \\
&\qquad\quad \cdot \Big((kE+2\lambda)(k+2\lambda) + 4(kE+\lambda)(k+2\lambda) + 4(kE+\lambda)(kE+2\lambda)\Big)\Big] \\
&\quad + \bar{L}(kE+2\lambda)^2\Big[4\lambda[kE+(1+S)\lambda](kE+\lambda)(k+2\lambda) \\
&\qquad + \big[kE+(1+S)\lambda+2(1+S)\lambda\big](kE+\lambda)(kE+2\lambda)(k+2\lambda) \\
&\qquad - \lambda\big[kE+(1+S)\lambda\big] \\
&\qquad\quad \cdot \Big((kE+2\lambda)(k+2\lambda) + 4(kE+\lambda)(k+2\lambda) + 4(kE+\lambda)(kE+2\lambda)\Big)\Big]
\end{aligned}$$

$$
\begin{aligned}
&= L^U E(k+2\lambda)^2 \Big[\big[kE + (1+S)\lambda + 2(1+S)\lambda\big](kE+\lambda)(kE+2\lambda)(k+2\lambda) \\
&\qquad - \lambda\big[kE+(1+S)\lambda\big]\Big((kE+2\lambda)(k+2\lambda) + 4(kE+\lambda)(k+2\lambda)\Big)\Big] \\
&\quad + \bar{L}(kE+2\lambda)^2 \Big[\big[kE + (1+S)\lambda + 2(1+S)\lambda\big](kE+\lambda)(kE+2\lambda)(k+2\lambda) \\
&\qquad - \lambda\big[kE+(1+S)\lambda\big]\Big((kE+2\lambda)(k+2\lambda) + 4(kE+\lambda)(kE+2\lambda)\Big)\Big] \\
&= L^U E(k+2\lambda)^3 \Big[2(1+S)\lambda(kE+\lambda)(kE+2\lambda) \\
&\qquad + \big[kE+(1+S)\lambda\big]\Big((kE+\lambda)(kE+2\lambda) - \lambda(kE+2\lambda) - 4\lambda(kE+\lambda)\Big)\Big] \\
&\quad + \bar{L}(kE+2\lambda)^3 \Big[2(1+S)\lambda(kE+\lambda)(k+2\lambda) \\
&\qquad + \big[kE+(1+S)\lambda\big]\Big((kE+\lambda)(k+2\lambda) - \lambda(k+2\lambda) - 4\lambda(kE+\lambda)\Big)\Big] \\
&= L^U E(k+2\lambda)^3 \Big[2(1+S)\lambda(kE+\lambda)(kE+2\lambda) \\
&\qquad + \big[kE+(1+S)\lambda\big]\Big(kE(kE+2\lambda) - 4\lambda(kE+\lambda)\Big)\Big] \\
&\quad + \bar{L}(kE+2\lambda)^3 \Big[2(1+S)\lambda(kE+\lambda)(k+2\lambda) \\
&\qquad + \big[kE+(1+S)\lambda\big]\Big(kE(k+2\lambda) - 4\lambda(kE+\lambda)\Big)\Big] \\
&= L^U E(k+2\lambda)^3 \Big[2\lambda(kE+\lambda)\big[(1+S)(kE+2\lambda) - 2[kE+(1+S)\lambda]\big] \\
&\qquad + [kE+(1+S)\lambda]kE(kE+2\lambda)\Big] \\
&\quad + \bar{L}(kE+2\lambda)^3 \Big[2\lambda(kE+\lambda)\big[(1+S)(k+2\lambda) - 2[kE+(1+S)\lambda]\big] \\
&\qquad + [kE+(1+S)\lambda]kE(k+2\lambda)\Big] \\
&= kA,
\end{aligned}
$$

where

$$
\begin{aligned}
A := {} & L^U E^2(k+2\lambda)^3 \Big[2\lambda(kE+\lambda)(S-1) + [kE+(1+S)\lambda](kE+2\lambda)\Big] \\
& + \bar{L}(kE+2\lambda)^3 \Big[2\lambda(kE+\lambda)(1+S-2E) + [kE+(1+S)\lambda]\,(k+2\lambda)E\Big].
\end{aligned}
\tag{A31}
$$

Since A is increasing in S, it is greater than or equal to its value at $S = 1$. Thus we have

$$
\begin{aligned}
A &\geq L^U E^2(k+2\lambda)^3(kE+2\lambda)^2 + \bar{L}(kE+2\lambda)^3\left[4\lambda(kE+\lambda)(1-E) + (kE+2\lambda)(k+2\lambda)E\right] \\
&\propto L^U E^2(k+2\lambda)^3 + \bar{L}(kE+2\lambda)(k^2E^2 + 4\lambda^2 + 6k\lambda E - 2k\lambda E^2) \\
&> L^U E^2(k+2\lambda)^3 - 2\bar{L}k\lambda E^2(kE+2\lambda) \\
&\propto L^U(k^3 + 6k^2\lambda + 12k\lambda^2 + 8\lambda^3) - \bar{L}(2k^2\lambda E + 4k\lambda^2) \\
&> L^U(6k^2\lambda + 12k\lambda^2) - \bar{L}E(2k^2\lambda + 4k\lambda^2) \\
&= (3L^U - \bar{L}E)(2k^2\lambda + 4k\lambda^2).
\end{aligned}
$$

If condition (A4) is satisfied, i.e. $E \leq 3L^U/\bar{L}$, we have $A > 0$, and hence $\partial f(\lambda; E)/\partial\lambda > 0$.

Going back to (A31), we see that $A > 0$ for all $L^U \geq 0$ if the term multiplying $\bar{L}$ is positive, i.e. if $B > 0$, where

$$
\begin{aligned}
B &:= 2\lambda(kE+\lambda)(1+S-2E) + \left[kE + (1+S)\lambda\right](k+2\lambda)E \\
&= 2[1+S+(S-1)E]\lambda^2 + kE[3(1+S)-2E]\lambda + k^2E^2.
\end{aligned}
$$

This is a quadratic in λ, which is positive for all λ if and only if the discriminant Δ is negative. We have

$$
\begin{aligned}
\Delta &= k^2E^2[3(1+S)-2E]^2 - 8[1+S+(S-1)E]k^2E^2 \\
&\propto [3(1+S)-2E]^2 - 8[1+S+(S-1)E] \\
&= 9S^2 + 10(1-2E)S + (2E-1)^2,
\end{aligned}
$$

which is negative if and only if $(2E-1)/9 < S < 2E-1$. Moreover, if $S \geq 2E-1$, $B > 0$ for all $\lambda > 0$. Therefore, if condition (A5) is satisfied so that $S > (2E-1)/9$, then $B > 0$ for all $\lambda > 0$, and hence $\partial f(\lambda; E)/\partial\lambda > 0$. We conclude that, under either (A4) or (A5),

$$
\frac{\partial f(\lambda; E)}{\partial \lambda} > 0. \tag{A32}
$$

This completes the proof. $\square$

Proof of Proposition A2.3 *Proof of (i):* Differentiating (A30) with respect to E, we get

$$
\begin{aligned}
\frac{\partial f(\lambda; E)}{\partial E} \propto E(kE+2\lambda)^2(kE+\lambda)\Big(&\left[L^U(k+2\lambda)^2 + 2k\bar{L}(kE+2\lambda)\right]\left[kE+(1+S)\lambda\right]^2 \\
&+ 2k\left[L^U E(k+2\lambda)^2 + \bar{L}(kE+2\lambda)^2\right]\left[kE+(1+S)\lambda\right]\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \left[L^U E(k+2\lambda)^2 + \bar{L}(kE+2\lambda)^2\right]\left[kE+(1+S)\lambda\right]^2 \\
&\qquad \cdot \Big(2kE(kE+2\lambda)(kE+\lambda) + (kE+2\lambda)^2(2kE+\lambda)\Big) \\
&\propto E(kE+2\lambda)(kE+\lambda)\Big(\left[L^U(k+2\lambda)^2 + 2k\bar{L}(kE+2\lambda)\right]\left[kE+(1+S)\lambda\right] \\
&\qquad + 2k\left[L^U E(k+2\lambda)^2 + \bar{L}(kE+2\lambda)^2\right]\Big) \\
&\quad - \left[L^U E(k+2\lambda)^2 + \bar{L}(kE+2\lambda)^2\right]\left[kE+(1+S)\lambda\right] \\
&\qquad \cdot \Big(2kE(kE+\lambda) + (kE+2\lambda)(2kE+\lambda)\Big) \\
&= L^U E(k+2\lambda)^2\Big[(kE+2\lambda)(kE+\lambda)\left[kE+(1+S)\lambda+2kE\right] \\
&\qquad - \left[kE+(1+S)\lambda\right]\Big(2kE(kE+\lambda)+(kE+2\lambda)(2kE+\lambda)\Big)\Big] \\
&\quad + \bar{L}(kE+2\lambda)^2\Big[2kE(kE+\lambda)\left[kE+(1+S)\lambda+(kE+2\lambda)\right] \\
&\qquad - \left[kE+(1+S)\lambda\right]\Big(2kE(kE+\lambda)+(kE+2\lambda)(2kE+\lambda)\Big)\Big] \\
&= -L^U kE^2(k+2\lambda)^2\left[(kE+2\lambda)^2 + (S-1)\lambda(3kE+4\lambda)\right] \\
&\quad - \bar{L}(kE+2\lambda)^2\left[S\lambda(kE+2\lambda)(2kE+\lambda) + \lambda(kE+\lambda)(kE+2\lambda)\right] \\
&< 0.
\end{aligned}
$$

Furthermore, $\partial f(\lambda; E)/\partial\lambda > 0$, from (A32). Hence, from

$$
\frac{\partial f(\lambda;E)}{\partial\lambda}\frac{\partial\lambda}{\partial E} + \frac{\partial f(\lambda;E)}{\partial E} = 0,
$$

we conclude that $\partial\lambda/\partial E > 0$.

Proof of (ii): From (A30), $\gamma := \lambda/E$ solves $g(\gamma; E) = 0$, where

$$
g(\gamma;E) = \frac{k^2\gamma\left[L^U(k+2\gamma E)^2 + \bar{L}E(k+2\gamma)^2\right]\left[k+(1+S)\gamma\right]^2}{S(k+\gamma)(k+2\gamma)^2(k+2\gamma E)^2} - \frac{\sigma_v^2}{\sigma_y^2}.
$$

If $\bar{L} = 0$,

$$
g(\gamma;E) = \frac{k^2\gamma L^U\left[k+(1+S)\gamma\right]^2}{S(k+\gamma)(k+2\gamma)^2} - \frac{\sigma_v^2}{\sigma_y^2}.
$$

Since g is independent of E, so is γ.

Proof of (iii): Suppose $\bar{L} > 0$. Then, we have

$$
\frac{\partial g(\gamma;E)}{\partial E} \propto \frac{\partial \frac{E}{(k+2\gamma E)^2}}{\partial E}
$$

$$
\begin{aligned}
&\propto k - 2\gamma E \\
&= k - 2\lambda.
\end{aligned}
$$

Furthermore, using (A32),

$$\frac{\partial g(\gamma; E)}{\partial \gamma} = \frac{\partial f(\lambda; E)}{\partial \lambda}\frac{\partial \lambda}{\partial \gamma} \propto \frac{\partial f(\lambda; E)}{\partial \lambda} > 0.$$

Hence, from

$$\frac{\partial g(\gamma; E)}{\partial \gamma}\frac{\partial \gamma}{\partial E} + \frac{\partial g(\gamma; E)}{\partial E} = 0,$$

we see that

$$\frac{\partial \gamma}{\partial E} \propto -\frac{\partial g(\gamma; E)}{\partial E} \propto 2\lambda - k.$$

Proof of (iv): From (A32), $f(\lambda; E)$ is strictly increasing in λ. From (A29), it is strictly increasing in $L^U, \bar{L}$, and σ_y^2, and strictly decreasing in σ_v^2. It follows that λ is strictly decreasing in $L^U, \bar{L}$, and σ_y^2, and strictly increasing in σ_v^2. Clearly the same is true of $\gamma := \lambda/E$. $\square$

Proof of Proposition A2.4 The proof is analogous to that of Proposition 5.2, except that here we do not have the result that $\lim_{E\to\bar{E}} \lambda = \infty$, which ensured that $\lambda > k/2$ for sufficiently large E. Instead, we use condition that $\Lambda_2 < 1$ to show that $\lim_{E\to\bar{E}} \lambda > k/2$. Since (A5) is satisfied, we have $\partial f(\lambda; E)/\partial \lambda > 0$, from (A32). Therefore, $\lim_{E\to\bar{E}} \lambda > k/2$ if and only if $\lim_{E\to\bar{E}} f(k/2; E) < 0$.

First, suppose $L^U > 0$. Then, from (A8), $\bar{E} = \bar{E}_2$. Using (A29), (A7), and the condition $\Lambda_2 < 1$, we have

$$
\begin{aligned}
\lim_{E\to\bar{E}_2} f(k/2; E) &= \left[\frac{L^U}{(k\bar{E}_2 + k)^2} + \frac{\bar{L}}{\bar{E}_2(k+k)^2}\right]\frac{k^2[k\bar{E}_2 + (1+S)k/2]^2}{S(k\bar{E}_2 + k/2)}\frac{k}{2} - \frac{\sigma_v^2}{\sigma_y^2} \\
&= \left[\frac{L^U}{(\bar{E}_2 + 1)^2} + \frac{\bar{L}}{4\bar{E}_2}\right]\frac{k^2[2\bar{E}_2 + (1+S)]^2}{4S(2\bar{E}_2 + 1)} - \frac{\sigma_v^2}{\sigma_y^2} \\
&= \left[\frac{4L^U}{9(1+3S)^2} + \frac{\bar{L}}{2(1+9S)}\right]\frac{k^2(1+5S)^2}{S(2+9S)} - \frac{\sigma_v^2}{\sigma_y^2} \\
&\propto \Lambda_2 - 1 \\
&< 0.
\end{aligned}
$$

We also require condition (A3) for all $E \in [1, \bar{E}_2)$, i.e.

$$\frac{k^2 \left(L^U + \frac{\bar{L}}{\bar{E}_2}\right)(1+S)^2}{4S} > \frac{\sigma_v^2}{\sigma_y^2},$$

or

$$\left[\frac{L^U}{4} + \frac{\bar{L}}{2(1+9S)}\right] \frac{k^2(1+S)^2}{S} > \frac{\sigma_v^2}{\sigma_y^2}.$$

In order for this inequality to be consistent with $\Lambda_2 < 1$, we need to ensure that

$$\left[\frac{4L^U}{9(1+3S)^2} + \frac{\bar{L}}{2(1+9S)}\right] \frac{k^2(1+5S)^2}{S(2+9S)} < \left[\frac{L^U}{4} + \frac{\bar{L}}{2(1+9S)}\right] \frac{k^2(1+S)^2}{S},$$

which holds since $\frac{4}{9(1+3S)^2} < \frac{1}{4}$ and $(1+5S)^2 < (2+9S)(1+S)^2$ for any $S \geq 1$.

Next, suppose $L^U = 0$. If $\bar{E} = \bar{E}_2$, the above calculations for $\lim_{E \to \bar{E}_2} f(k/2; E)$ still apply, so that this limit is negative. If $\bar{E} = \bar{E}_1$, we can show directly that $\bar{\lambda} := \lim_{E \to \bar{E}_1} \lambda > k/2$. From (A30) and (A6),

$$\begin{aligned} f(\lambda; E) &= \frac{\lambda k^2 \bar{L} \left[kE + (1+S)\lambda\right]^2}{ES(kE+\lambda)(k+2\lambda)^2} - \frac{\sigma_v^2}{\sigma_y^2} \\ &= \frac{4\lambda \bar{E}_1 \left[kE + (1+S)\lambda\right]^2}{E(1+S)^2(kE+\lambda)(k+2\lambda)^2} \cdot \frac{\sigma_v^2}{\sigma_y^2} - \frac{\sigma_v^2}{\sigma_y^2} \\ &= \frac{4\bar{E}_1(k\lambda^{-1}E + 1 + S)^2}{E(1+S)^2(k\lambda^{-1}E+1)(k\lambda^{-1}+2)^2} \cdot \frac{\sigma_v^2}{\sigma_y^2} - \frac{\sigma_v^2}{\sigma_y^2}. \end{aligned}$$

Since $f(\lambda(E); E) = 0$ for all $E \in [1, \bar{E}_1)$, we have

$$\lim_{E \to \bar{E}_1} f(\lambda(E); E) = f(\bar{\lambda}; \bar{E}_1) = \frac{4(k\bar{\lambda}^{-1}\bar{E}_1 + 1 + S)^2}{(1+S)^2(k\bar{\lambda}^{-1}\bar{E}_1+1)(k\bar{\lambda}^{-1}+2)^2} \cdot \frac{\sigma_v^2}{\sigma_y^2} - \frac{\sigma_v^2}{\sigma_y^2} = 0.$$

Moreover, since $\partial f(\lambda; E)/\partial \lambda > 0$, $f(\bar{\lambda}; \bar{E}_1)$ is non-decreasing in $\bar{\lambda}$, so the above equation has a unique solution. The equation is satisfied if $\bar{\lambda}^{-1} = 0$. Hence $\bar{\lambda} = \infty$. $\square$

Proof of Lemma A2.5 We proceed as in the proof of Lemma 6.1. We have

$$\begin{aligned} W^S &= \sum_{i \in E} (v - p_i) q_i^S - \frac{k}{2} \left(\sum_{i \in E} q_i^S\right)^2 \\ &= \sum_{i \in E} \Big[v - \lambda \left[S\beta v + \alpha^U Q^U + \alpha^R Q_i^R\right]\Big] \beta v - \frac{k}{2} \left(E\beta v\right)^2, \end{aligned}$$

so that, using (A1),

$$\begin{aligned}
\mathbb{E}\left(W^S\right) &= \left[1 - S\lambda\beta - \frac{k}{2}E\beta\right] E\beta\sigma_v^2 \\
&= \frac{(kE + 2\lambda)E\sigma_v^2}{2\big[kE + (1+S)\lambda\big]^2} \\
&= \frac{k + 2\gamma}{2\big[k + (1+S)\gamma\big]^2}\sigma_v^2.
\end{aligned}$$

The calculations for the utilities of the liquidity traders are identical to those in the proof of Lemma 6.1. □

Proof of Lemma A2.8 From (8) and (A1),

$$\begin{aligned}
p_i &= \lambda S\beta\left[v + \frac{\alpha^U Q^U + \alpha^R Q_i^R}{S\beta}\right] \\
&= \frac{S\gamma}{k + (1+S)\gamma}(v + \eta^U + \eta_i^R).
\end{aligned}$$

Using (36) and (A1), we have

$$\begin{aligned}
\mathrm{Var}(\eta^U + \eta_i^R) &= \frac{\left[L^U(\alpha^U)^2 + L^R(\alpha^R)^2\right]\sigma_y^2}{(S\beta)^2} \\
&= \frac{(1 - S\lambda\beta)\sigma_v^2}{S\lambda\beta} \\
&= \frac{k+\gamma}{S\gamma}\sigma_v^2.
\end{aligned}$$

The relations in (A10) follow from the definition of η^U, and from (A1) and (A2). □

Proof of Proposition A2.9 *Proof of (i)–(iii):* See the paragraph immediately preceding the statement of the proposition.

Proof of (iv): If $L^U = 0$, the price signals, $\{v + \eta_i^R\}_{i\in E}$, are conditionally i.i.d. signals about v. Hence,

$$\mathrm{Var}(v|p_1, \ldots, p_E) = \mathrm{Var}\left(v\middle|v + \frac{1}{E}\sum_{i\in E}\eta_i^R\right).$$

Using (A9),

$$\begin{aligned}\text{Var}\left(\frac{1}{E}\sum_{i\in E}\eta_i^R\right) &= \frac{\text{Var}(\eta_i^R)}{E}\\ &= \frac{k+\gamma}{ES\gamma}\sigma_v^2\\ &= \frac{k\sigma_v^2}{S\lambda}+\frac{\sigma_v^2}{ES},\end{aligned}$$

which is strictly decreasing in E, since $\partial\lambda/\partial E > 0$ (Proposition A2.3 (i)). Therefore, from (24), $\mathcal{V}(E)$ is strictly increasing in E. $\square$

Proof of Lemma A3.1 The optimization problem of a speculator is the same as in the main model, leading to the same coefficient β ((A12) is the same as (9)).

Consider liquidity trader ℓ on exchange i. The price function from the perspective of this trader is the same as in the proof of Lemma 4.1, with the order flow coming from unrestricted liquidity traders set equal to zero:

$$p_{i\ell}^R := \lambda\left[S\beta v + \alpha^R Q_{i,-\ell}^R + q_{i\ell}^R\right], \tag{A33}$$

where $Q_{i,-\ell}^R := \sum_{\ell'\in L_i^R,\ell'\neq\ell} y_{\ell'}$. Her optimization problem (A11) is equivalent to maximizing the following mean-variance objective:

$$\begin{aligned}\mathcal{M}_{i\ell}^R &:= \mathbb{E}\left[W_{i\ell}^R|y_\ell\right] - \frac{r}{2}\text{Var}\left[W_{i\ell}^R|y_\ell\right] \qquad\qquad (\text{A34})\\ &= \mathbb{E}\left[v-p_{i\ell}^R|y_\ell\right]q_{i\ell}^R - \frac{r}{2}\Big((q_{i\ell}^R)^2\text{Var}\left[v-p_{i\ell}^R|y_\ell\right] + y_\ell^2\text{Var}(v) - 2y_\ell q_{i\ell}^R\text{Cov}\left[v-p_{i\ell}^R, v|y_\ell\right]\Big).\end{aligned}$$

We have

$$v - p_{i\ell}^R = (1-S\lambda\beta)v - \lambda\alpha^R Q_{i,-\ell}^R - \lambda q_{i\ell}^R,$$

so that

$$\mathbb{E}\left[v-p_{i\ell}^R|y_\ell\right] = -\lambda q_{i\ell}^R, \tag{A35}$$

$$\text{Var}\left[v-p_{i\ell}^R|y_\ell\right] = (1-S\lambda\beta)^2\sigma_v^2 + \lambda^2(\alpha^R)^2(L^R-1)\sigma_y^2, \tag{A36}$$

$$\text{Cov}\left[v-p_{i\ell}^R, v|y_\ell\right] = (1-S\lambda\beta)\sigma_v^2. \tag{A37}$$

Therefore, using (A35),

$$\mathcal{M}_{i\ell}^R = -\lambda(q_{i\ell}^R)^2 - \frac{r}{2}\Big((q_{i\ell}^R)^2\text{Var}\left[v-p_{i\ell}^R|y_\ell\right] + y_\ell^2\text{Var}(v) - 2y_\ell q_{i\ell}^R\text{Cov}\left[v-p_{i\ell}^R, v|y_\ell\right]\Big). \tag{A38}$$

From the first-order condition with respect to $q_{i\ell}^R$, we see that

$$q_{i\ell}^R = \frac{r\text{Cov}\left[v - p_{i\ell}^R, v|y_\ell\right]}{r\text{Var}\left[v - p_{i\ell}^R|y_\ell\right] + 2\lambda} y_\ell. \tag{A39}$$

From (A12),

$$1 - S\lambda\beta = 1 - \frac{S}{1+S} = \frac{1}{1+S}.$$

From (37),

$$\lambda^2(\alpha^R)^2\sigma_y^2 = \frac{S}{L^R(1+S)^2}\sigma_v^2. \tag{A40}$$

Hence, from (A36) and (A37),

$$\begin{aligned}\text{Var}\left[v - p_{i\ell}^R|y_\ell\right] &= (1+S)^{-2}[1 + (L^R)^{-1}(L^R-1)S]\sigma_v^2,\\ \text{Cov}\left[v - p_{i\ell}^R, v|y_\ell\right] &= (1+S)^{-1}\sigma_v^2.\end{aligned} \tag{A41}$$

Substituting into (A39), we obtain

$$\begin{aligned}q_{i\ell}^R &= \frac{r(1+S)^{-1}\sigma_v^2}{r(1+S)^{-2}\left[1 + (L^R)^{-1}(L^R-1)S\right]\sigma_v^2 + 2\lambda} y_\ell\\ &= \frac{rL^R(1+S)\sigma_v^2}{r\left[L^R + (L^R-1)S\right]\sigma_v^2 + 2L^R(1+S)^2\lambda} y_\ell.\end{aligned}$$

Therefore, α^R is given by (A13). $\square$

Proof of Proposition A3.2 *Step 1:* We first derive a necessary and sufficient condition for existence of a positive λ. Substituting the expression for α^R given by (A13) into (A40), we obtain

$$\frac{\lambda rL^R(1+S)\sigma_v^2\sigma_y}{r\left[L^R + (L^R-1)S\right]\sigma_v^2 + 2L^R(1+S)^2\lambda} = \frac{S^{\frac{1}{2}}}{(L^R)^{\frac{1}{2}}(1+S)}\sigma_v,$$

which gives us

$$\lambda = \frac{rS\big[L^R + (L^R-1)S\big]\sigma_v^2}{L^R(1+S)^2\big[r(L^RS)^{\frac{1}{2}}\sigma_v\sigma_y - 2S\big]} \tag{A42}$$

$$= \frac{r\big[\bar{L} + (\bar{L}-E)S\big]\sigma_v^2}{\bar{L}(1+S)^2\big[r\bar{L}^{\frac{1}{2}}(ES)^{-\frac{1}{2}}\sigma_v\sigma_y - 2\big]}. \tag{A43}$$

Thus there exists a positive λ (in which case it is also unique) if and only if

$$r\bar{L}^{\frac{1}{2}}(ES)^{-\frac{1}{2}}\sigma_v\sigma_y - 2 > 0,$$

which gives us the first inequality in (A15).

Step 2: Next, we consider the ex ante expected utility of a restricted liquidity trader:

$$\begin{aligned}\mathcal{U}^R := \mathbb{E}[-\exp(-rW_{i\ell}^R)] &= -\mathbb{E}\big[\mathbb{E}(\exp(-rW_{i\ell}^R)|y_\ell)\big] \\ &= -\mathbb{E}\big[\exp(-r\mathcal{M}_{i\ell}^R)\big],\end{aligned} \tag{A44}$$

where $\mathcal{M}_{i\ell}^R$ is the mean-variance objective given by (A34). Substituting (A42) into (A13), we obtain:

$$\alpha^R = \frac{L^R(1+S)[r(L^R S)^{\frac{1}{2}}\sigma_v\sigma_y - 2S]}{[L^R + (L^R-1)S]r(L^R S)^{\frac{1}{2}}\sigma_v\sigma_y}. \tag{A45}$$

From (A38), (A39), (A41), and (A45),

$$\begin{aligned}\mathcal{M}_{i\ell}^R &= -\frac{1}{2}(q_{i\ell}^R)^2\Big(r\mathrm{Var}\left[v - p_{i\ell}^R|y_\ell\right] + 2\lambda\Big) - \frac{r}{2}\mathrm{Var}(v)y_\ell^2 + rq_{i\ell}^R\mathrm{Cov}\left[v - p_{i\ell}^R, v|y_\ell\right]y_\ell \\ &= -\frac{r}{2}q_{i\ell}^R\mathrm{Cov}\left[v - p_{i\ell}^R, v|y_\ell\right]y_\ell - \frac{r}{2}\mathrm{Var}(v)y_\ell^2 + rq_{i\ell}^R\mathrm{Cov}\left[v - p_{i\ell}^R, v|y_\ell\right]y_\ell \\ &= \frac{r}{2}q_{i\ell}^R\mathrm{Cov}\left[v - p_{i\ell}^R, v|y_\ell\right]y_\ell - \frac{r}{2}\mathrm{Var}(v)y_\ell^2 \\ &= \frac{r}{2}\alpha^R(1+S)^{-1}\sigma_v^2 y_\ell^2 - \frac{r}{2}\sigma_v^2 y_\ell^2 \\ &= -\frac{r}{2}\left[1 - \frac{L^R[r(L^R S)^{\frac{1}{2}}\sigma_v\sigma_y - 2S]}{[L^R + (L^R-1)S]r(L^R S)^{\frac{1}{2}}\sigma_v\sigma_y}\right]\sigma_v^2 y_\ell^2 \\ &= -\frac{r}{2}\left[\frac{r\sigma_v\sigma_y(L^R-1)S + 2(L^R S)^{\frac{1}{2}}}{r\sigma_v\sigma_y[L^R + (L^R-1)S]}\right]\sigma_v^2 y_\ell^2 \\ &= -\frac{\sigma_v}{2\sigma_y}\left[\frac{r\sigma_v\sigma_y(L^R-1)S + 2(L^R S)^{\frac{1}{2}}}{L^R + (L^R-1)S}\right]y_\ell^2.\end{aligned} \tag{A46}$$

We now invoke the fact that if $x \sim N(0, \sigma^2)$, then $\mathbb{E}(e^{x^2}) = (1-2\sigma^2)^{-\frac{1}{2}}$, provided $\sigma^2 < 1/2$. Using (A44) and (A46),

$$\mathcal{U}^R = -(1-2\sigma^2)^{-\frac{1}{2}}, \tag{A47}$$

where

$$\begin{aligned}\sigma^2 &:= \frac{r\sigma_v\sigma_y}{2}\left[\frac{r\sigma_v\sigma_y(L^R-1)S + 2(L^R S)^{\frac{1}{2}}}{L^R + (L^R-1)S}\right] \\ &= \frac{r\sigma_v\sigma_y}{2}\left[\frac{r\sigma_v\sigma_y(\bar{L}-E)S + 2(\bar{L}ES)^{\frac{1}{2}}}{\bar{L} + (\bar{L}-E)S}\right].\end{aligned} \tag{A48}$$

For any given E, a necessary and sufficient condition for the expected utility of a restricted liquidity trader to be well-defined is $\sigma^2 < 1/2$. Letting $x := r\sigma_v\sigma_y$, we can write this condition as $\chi(x) < 0$, where

$$\chi(x) := (L^R - 1)Sx^2 + 2(L^R S)^{\frac{1}{2}}x - [L^R + (L^R - 1)S]. \tag{A49}$$

The quadratic χ has a unique positive zero, given by

$$\begin{aligned} x^* &= \frac{-2(L^R S)^{\frac{1}{2}} + \Big(4L^R S + 4(L^R - 1)S[L^R + (L^R - 1)S]\Big)^{\frac{1}{2}}}{2(L^R - 1)S} \\ &= \frac{\big[(L^R)^2 + (L^R - 1)^2 S\big]^{\frac{1}{2}} - (L^R)^{\frac{1}{2}}}{(L^R - 1)S^{\frac{1}{2}}}, \end{aligned} \tag{A50}$$

which is equal to Z, defined by (A17). Since the parabola defined by χ opens upwards, $\chi(x) < 0$ if and only if $x < x^*$, or $r\sigma_v\sigma_y < Z$, which is the second inequality in (A15).

Step 3: We need to ensure that the interval (z, Z) is non-degenerate. We have

$$\begin{aligned} Z - z &= \frac{\big[(L^R)^2 + (L^R - 1)^2 S\big]^{\frac{1}{2}} - (L^R)^{\frac{1}{2}}}{(L^R - 1)S^{\frac{1}{2}}} - 2\left(\frac{S}{L^R}\right)^{\frac{1}{2}} \\ &\propto (L^R)^{\frac{1}{2}}\big[(L^R)^2 + (L^R - 1)^2 S\big]^{\frac{1}{2}} - L^R - 2(L^R - 1)S \\ &\propto L^R\big[(L^R)^2 + (L^R - 1)^2 S\big] - [L^R + 2(L^R - 1)S]^2 \\ &= L^R\big[(L^R)^2 + (L^R - 1)^2 S\big] - (L^R)^2 - 4(L^R - 1)^2 S^2 - 4L^R(L^R - 1)S \\ &= (L^R)^3 - (L^R)^2 + L^R(L^R - 1)^2 S - 4L^R(L^R - 1)S - 4(L^R - 1)^2 S^2 \\ &= (L^R)^2(L^R - 1) + L^R(L^R - 1)^2 S - 4L^R(L^R - 1)S - 4(L^R - 1)^2 S^2 \\ &\propto (L^R)^2 + L^R(L^R - 1)S - 4L^R S - 4(L^R - 1)S^2 \\ &= (1 + S)(L^R)^2 - S(5 + 4S)L^R + 4S^2, \end{aligned}$$

which is a quadratic in L^R. The discriminant is

$$\begin{aligned} \Delta &= S^2(5 + 4S)^2 - 16(1 + S)S^2 \\ &= S^2(9 + 24S + 16S^2) \\ &= S^2(3 + 4S)^2, \end{aligned}$$

and the zeros are

$$L^R = \frac{S(5 + 4S) \pm S(3 + 4S)}{2(1 + S)},$$

i.e. $S(1+S)^{-1}$ and $4S$. Since the parabola opens upward, $Z - z > 0$ when $L^R < S(1+S)^{-1}$ or $L^R > 4S$. Given that $L^R \geq 1$, the only feasible restriction is $L^R > 4S$. From (A14),

$$L^R = \frac{\bar{L}}{E} > \frac{\bar{L}}{\bar{E}} = 4S.$$

Thus the restriction $L^R > 4S$ is satisfied for all $E \in [1, \bar{E})$.

Step 4: Finally, we check the properties of the bounds, z and Z. From (A16), it is obvious that $\partial z/\partial E > 0$, and the limits are easy to verify. Using the expression for Z given by (A50),

$$\begin{aligned}
\frac{\partial Z}{\partial L^R} &\propto \frac{L^R - 1}{2}\big[(L^R)^2 + (L^R-1)^2 S\big]^{-\frac{1}{2}}\big[2L^R + 2(L^R-1)S\big] \\
&\qquad - \frac{L^R - 1}{2}(L^R)^{-\frac{1}{2}} - \big[(L^R)^2 + (L^R-1)^2 S\big]^{\frac{1}{2}} + (L^R)^{\frac{1}{2}} \\
&\propto (L^R)^{\frac{1}{2}}(L^R - 1)\big[2L^R + 2(L^R-1)S\big] - (L^R-1)\big[(L^R)^2 + (L^R-1)^2 S\big]^{\frac{1}{2}} \\
&\qquad - 2(L^R)^{\frac{1}{2}}\big[(L^R)^2 + (L^R-1)^2 S\big] + 2L^R\big[(L^R)^2 + (L^R-1)^2 S\big]^{\frac{1}{2}} \\
&= -2(L^R)^{\frac{3}{2}} + (L^R+1)\big[(L^R)^2 + (L^R-1)^2 S\big]^{\frac{1}{2}} \\
&\propto (L^R+1)^2\big[(L^R)^2 + (L^R-1)^2 S\big] - 4(L^R)^3 \\
&> 0,
\end{aligned}$$

and, therefore, $\partial Z/\partial E < 0$. $\square$

Proof of Proposition A3.3 *Proof of (i):* Differentiating (A43) with respect to E, we have

$$\begin{aligned}
\frac{\partial \lambda}{\partial E} &\propto -S\big[r\bar{L}^{\frac{1}{2}}(ES)^{-\frac{1}{2}}\sigma_v\sigma_y - 2\big] + \frac{1}{2}r\bar{L}^{\frac{1}{2}}S^{-\frac{1}{2}}E^{-\frac{3}{2}}\sigma_v\sigma_y\big[\bar{L} + (\bar{L} - E)S\big] \\
&\propto -2ES\big[r\bar{L}^{\frac{1}{2}}\sigma_v\sigma_y - 2(ES)^{\frac{1}{2}}\big] + r\bar{L}^{\frac{1}{2}}\sigma_v\sigma_y\big[\bar{L} + (\bar{L} - E)S\big] \\
&= 4(ES)^{\frac{3}{2}} + r\bar{L}^{\frac{1}{2}}\sigma_v\sigma_y\big[\bar{L} + (\bar{L} - 3E)S\big],
\end{aligned}$$

which is positive, given our assumption that $L^R > 4S$, which is required for existence of equilibrium (see the proof of Proposition A3.2).

From Proposition A3.2, $\lim_{E\to\bar{E}} z = \lim_{E\to\bar{E}} Z = 1$. Therefore, for the existence condition (A15) to be satisfied for E arbitrarily close to $\bar{E}$, we must have $r\sigma_v\sigma_y = 1$. From (A14) and (A43), it follows that $\lim_{E\to\bar{E}} \lambda = \infty$.

Proof of (ii): From (A43),

$$\gamma := \frac{\lambda}{E} = \frac{r\big[\bar{L} + (\bar{L} - E)S\big]\sigma_v^2}{\bar{L}(1+S)^2\big[r\bar{L}^{\frac{1}{2}}S^{-\frac{1}{2}}E^{\frac{1}{2}}\sigma_v\sigma_y - 2E\big]}.$$

Differentiating, we get

$$\begin{aligned}
\frac{\partial \gamma}{\partial E} &\propto -S\big[r\bar{L}^{\frac{1}{2}}S^{-\frac{1}{2}}E^{\frac{1}{2}}\sigma_v\sigma_y - 2E\big] - \big[\bar{L} + (\bar{L} - E)S\big]\left[\frac{1}{2}r\bar{L}^{\frac{1}{2}}(ES)^{-\frac{1}{2}}\sigma_v\sigma_y - 2\right] \\
&\propto -2ES\big[r\bar{L}^{\frac{1}{2}}(ES)^{-\frac{1}{2}}\sigma_v\sigma_y - 2\big] - \big[\bar{L} + (\bar{L} - E)S\big]\left[r\bar{L}^{\frac{1}{2}}(ES)^{-\frac{1}{2}}\sigma_v\sigma_y - 4\right] \\
&= 4\bar{L}(1+S) - r\bar{L}^{\frac{1}{2}}(ES)^{-\frac{1}{2}}\sigma_v\sigma_y[\bar{L} + (\bar{L} + E)S] \\
&\propto \frac{4(\bar{L}ES)^{\frac{1}{2}}(1+S)}{\bar{L} + (\bar{L} + E)S} - r\sigma_v\sigma_y \\
&= Q - r\sigma_v\sigma_y,
\end{aligned}$$

where

$$\begin{aligned}
Q &:= \frac{4(\bar{L}ES)^{\frac{1}{2}}(1+S)}{\bar{L} + (\bar{L} + E)S} \\
&= \frac{4(L^R S)^{\frac{1}{2}}(1+S)}{L^R + (L^R + 1)S}.
\end{aligned} \tag{A51}$$

Proof of (iii): Differentiating (A51), we have

$$\begin{aligned}
\frac{\partial Q}{\partial L^R} &\propto \frac{1}{2}(L^R)^{-\frac{1}{2}}[L^R + (L^R + 1)S] - (L^R)^{\frac{1}{2}}(1+S) \\
&\propto L^R + (L^R + 1)S - 2L^R(1+S) \\
&\propto S - L^R(1+S) \\
&< 0.
\end{aligned}$$

Since $L^R = \bar{L}/E$, we conclude that $\partial Q/\partial E > 0$, $\partial Q/\partial \bar{L} < 0$. Furthermore,

$$\begin{aligned}
\frac{\partial Q}{\partial S} &\propto [L^R + (L^R + 1)S]\left[\frac{1}{2}S^{-\frac{1}{2}} + \frac{3}{2}S^{\frac{1}{2}}\right] - (L^R + 1)S^{\frac{1}{2}}(1+S) \\
&\propto [L^R + (L^R + 1)S](1 + 3S) - 2(L^R + 1)S(1+S) \\
&= L^R(1 + 3S) + (L^R + 1)S(S - 1) \\
&> 0.
\end{aligned}$$

It is easy to check that $\lim_{\bar{L}\to\infty} Q = 0$. From (A14) and (A18),

$$\lim_{E\to\bar{E}} Q = \frac{4\big(\bar{L}S\frac{\bar{L}}{4S}\big)^{\frac{1}{2}}(1+S)}{\bar{L}+\big(\bar{L}+\frac{\bar{L}}{4S}\big)S} = \frac{8(1+S)}{5+4S} > 1.$$

Since $r\sigma_v\sigma_y < Z$, and $\lim_{E\to\bar{E}} Z = 1$ (Proposition A3.2), it follows that $\lim_{E\to\bar{E}} Q > r\sigma_v\sigma_y$.

Proof of (iv): From (A43), we can easily see that both λ and γ are strictly decreasing in r and σ_y^2. Differentiating (A43) with respect to $\bar{L}$, and using (A14), we have

$$\begin{aligned}\frac{\partial\lambda}{\partial\bar{L}} &\propto (1+S)\bar{L}\big[r\bar{L}^{\frac{1}{2}}(ES)^{-\frac{1}{2}}\sigma_v\sigma_y - 2\big] - \big[\bar{L}+(\bar{L}-E)S\big]\left[\frac{3}{2}r\bar{L}^{\frac{1}{2}}(ES)^{-\frac{1}{2}}\sigma_v\sigma_y - 2\right]\\ &= \frac{1}{2}\left[3ES - \bar{L}(1+S)\right] r\bar{L}^{\frac{1}{2}}(ES)^{-\frac{1}{2}}\sigma_v\sigma_y - 2ES\\ &= \frac{1}{2}\left[3ES - 4S\bar{E}(1+S)\right] r\bar{L}^{\frac{1}{2}}(ES)^{-\frac{1}{2}}\sigma_v\sigma_y - 2ES\\ &< 0,\end{aligned}$$

as desired. $\square$

Proof of Proposition A3.4 *Proof of (i)–(iv):* These results are immediate from parts (ii) and (iii) of Proposition A3.3.

Proof of (v): If $E^* > 1$, then E^* solves $\hat{Q}(E) := Q(E) - r\sigma_v\sigma_y = 0$. Using Proposition A3.3, we have $\partial\hat{Q}/\partial E > 0$, $\partial\hat{Q}/\partial\bar{L} < 0$, and $\partial\hat{Q}/\partial S > 0$. Furthermore, $\partial\hat{Q}/\partial(r\sigma_v\sigma_y) < 0$. The result follows. $\square$

Proof of Proposition A3.6 *Proof of (i):* Differentiating (A48), and using (A15), we obtain

$$\begin{aligned}\frac{\partial\sigma^2}{\partial E} &\propto \big[\bar{L}+(\bar{L}-E)S\big]\big[-r\sigma_v\sigma_y S + (\bar{L}S)^{\frac{1}{2}}E^{-\frac{1}{2}}\big] + S\big[r\sigma_v\sigma_y(\bar{L}-E)S + 2(\bar{L}ES)^{\frac{1}{2}}\big]\\ &= -r\sigma_v\sigma_y\bar{L}S + \big[\bar{L}+(\bar{L}-E)S\big](\bar{L}S)^{\frac{1}{2}}E^{-\frac{1}{2}} + 2S(\bar{L}ES)^{\frac{1}{2}}\\ &\propto -r\sigma_v\sigma_y + \big[\bar{L}+(\bar{L}-E)S\big](\bar{L}ES)^{-\frac{1}{2}} + 2ES(\bar{L}ES)^{-\frac{1}{2}}\\ &= (\bar{L}ES)^{-\frac{1}{2}}[\bar{L}+(\bar{L}+E)S] - r\sigma_v\sigma_y\\ &> (\bar{L}ES)^{-\frac{1}{2}}[\bar{L}+(\bar{L}+E)S] - Z.\end{aligned} \tag{A52}$$

From the proof of Proposition A3.2, in particular (A49), we see that Z is a zero of the quadratic $\hat{\chi}(x)$, where

$$\hat{\chi}(x) := (\bar{L} - E)Sx^2 + 2(\bar{L}ES)^{\frac{1}{2}}x - [\bar{L} + (\bar{L} - E)S].$$

Moreover,

$$\hat{\chi}\Big((\bar{L}ES)^{-\frac{1}{2}}[\bar{L} + (\bar{L} + E)S]\Big) > 2[\bar{L} + (\bar{L} + E)S] - [\bar{L} + (\bar{L} - E)S] > 0.$$

Therefore, $(\bar{L}ES)^{-\frac{1}{2}}[\bar{L} + (\bar{L} + E)S] > Z$, so that $\partial\sigma^2/\partial E > 0$, from (A52). Using (A20), we conclude that $\partial\mathcal{U}^R/\partial E \propto -\partial\sigma^2/\partial E < 0$.

Proof of (ii): This is immediate from Proposition A3.4 (i) and (A19).

Proof of (iii) and (iv): These results are immediate from parts (ii) and (iii) of Proposition A3.4. □

Proof of Proposition A3.7 *Proof of (i):* It easy to check that the proof of this result in the proof of Proposition 7.2 is still valid in the current setting. In particular, we use the fact that β is the same as in the main model.

Proof of (ii): From (A12), (A42), and (A45),

$$\begin{aligned}\frac{\alpha^R}{\beta} &= \lambda\frac{L^R(1+S)^2[r(L^RS)^{\frac{1}{2}}\sigma_v\sigma_y - 2S]}{[L^R + (L^R-1)S]r(L^RS)^{\frac{1}{2}}\sigma_v\sigma_y} \\ &= \frac{S\sigma_v^2}{(L^RS)^{\frac{1}{2}}\sigma_v\sigma_y} \\ &= \frac{(ES)^{\frac{1}{2}}\sigma_v}{(\bar{L})^{\frac{1}{2}}\sigma_y}.\end{aligned}$$

Substituting this expression into (42), and using (A12), we obtain

$$\begin{aligned}\mathcal{V} &= \left[\frac{1+S}{S} - \frac{E-1}{ES}\right]^{-1} \\ &= \frac{1}{1+(ES)^{-1}},\end{aligned}$$

as desired. □

Proof of Lemma A4.1 The optimization problem of a speculator is the same as in the main model, leading to same coefficient β ((A22) is the same as (9)).

Consider liquidity trader ℓ on exchange i. The price function from the perspective of this trader, $p_{i\ell}^R$, is given by (A33). Using (A21),

$$\mathbb{E}(y_{\ell'}|y_\ell) = \frac{\text{Cov}(y_\ell, y_{\ell'})}{\sigma_y^2} y_\ell = \rho y_\ell.$$

Therefore, from (3),

$$\begin{aligned}
\mathbb{E}(W_{i\ell}^R|y_\ell) &= -\mathbb{E}\left[p_{i\ell}^R|y_\ell\right] q_{i\ell}^R - \frac{k}{2}\left(q_{i\ell}^R - y_\ell\right)^2 \\
&= -\lambda\left[(L^R-1)\alpha^R\rho y_\ell + q_{i\ell}^R\right] q_{i\ell}^R - \frac{k}{2}\left(q_{i\ell}^R - y_\ell\right)^2.
\end{aligned}$$

Maximizing this objective with respect to $q_{i\ell}^R$, we obtain the optimal portfolio:

$$q_{i\ell}^R = \frac{k - \lambda\rho(L^R-1)\alpha^R}{k+2\lambda} y_\ell.$$

Hence

$$\alpha^R = \frac{k - \lambda\rho(L^R-1)\alpha^R}{k+2\lambda},$$

which gives us (A23). $\square$

Proof of Proposition A4.2 The aggregate order flow on exchange i is

$$\begin{aligned}
q_i &= S\beta v + \alpha^R \sum_{\ell \in L_i^R} y_\ell \\
&= \zeta^\top \theta,
\end{aligned}$$

where $\theta^\top := \left(v, (y_\ell)_{\ell \in L_i^R}\right)$, $\zeta^\top := \left(S\beta, \alpha^R \mathbf{1}\right)$, and $\mathbf{1}$ is a row vector of ones with L^R elements. We have

$$\text{Var}(q_i) = \zeta^\top \text{Var}(\theta)\zeta = \zeta^\top \begin{bmatrix} \sigma_v^2 & 0 & \dots & 0 \\ 0 & \sigma_y^2 & \dots & \rho\sigma_y^2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \rho\sigma_y^2 & \dots & \sigma_y^2 \end{bmatrix} \zeta$$

$$= \zeta^\top \begin{bmatrix} S\beta\sigma_v^2 \\ [1+\rho(L^R-1)]\alpha^R\sigma_y^2 \\ \vdots \\ [1+\rho(L^R-1)]\alpha^R\sigma_y^2 \end{bmatrix}$$

$$= S^2\beta^2\sigma_v^2 + [1+\rho(L^R-1)]L^R(\alpha^R)^2\sigma_y^2. \tag{A53}$$

Also,

$$\mathrm{Cov}(v, q_i) = S\beta\sigma_v^2.$$

Therefore,

$$\lambda = \frac{\mathrm{Cov}(v, q_i)}{\mathrm{Var}(q_i)} = \frac{S\beta\sigma_v^2}{S^2\beta^2\sigma_v^2 + [1+\rho(L^R-1)]L^R(\alpha^R)^2\sigma_y^2}. \tag{A54}$$

Using (A22),

$$\begin{aligned} \lambda[1+\rho(L^R-1)]L^R(\alpha^R)^2\sigma_y^2 &= (1-S\lambda\beta)S\beta\sigma_v^2 \\ &= \left[1 - \frac{S}{1+S}\right]\frac{S}{(1+S)\lambda}\sigma_v^2 \\ &= \frac{S}{(1+S)^2\lambda}\sigma_v^2, \end{aligned}$$

so that, using (A23),

$$\begin{aligned} \frac{\sigma_v^2}{\sigma_y^2} &= \frac{[1+\rho(L^R-1)]L^R(\alpha^R)^2(1+S)^2}{S}\lambda^2 \\ &= \frac{k^2[1+\rho(L^R-1)]L^R(1+S)^2}{[k+2\lambda+\lambda\rho(L^R-1)]^2 S}\lambda^2 \\ &= \frac{k^2(1-\rho+\rho L^R)L^R(1+S)^2}{[k\lambda^{-1}+2-\rho+\rho L^R]^2 S}. \end{aligned} \tag{A55}$$

Thus λ is a solution to $f(\lambda; E) = 0$, where

$$f(\lambda; E) := \frac{k^2(1-\rho+\rho L^R)L^R(1+S)^2}{[k\lambda^{-1}+2-\rho+\rho L^R]^2 S} - \frac{\sigma_v^2}{\sigma_y^2}. \tag{A56}$$

We have $\lim_{\lambda\to 0} f(\lambda; E) < 0$. Also, $\partial f(\lambda; E)/\partial\lambda > 0$. Hence, if there is a solution to $f(\lambda; E) = 0$, it must be unique. A solution exists if and only if

$$\lim_{\lambda\to\infty} f(\lambda; E) = \frac{k^2(1-\rho+\rho L^R)L^R(1+S)^2}{(2-\rho+\rho L^R)^2 S} - \frac{\sigma_v^2}{\sigma_y^2} > 0,$$

which gives us condition (A24). $\square$

Proof of Proposition A4.3 *Proof of (i):* From (A56), λ solves $f(\lambda;E)=0$, where

$$f(\lambda;E) := \frac{k^2[(1-\rho)E+\rho\bar{L}]\bar{L}(1+S)^2}{[(k\lambda^{-1}+2-\rho)E+\rho\bar{L}]^2 S} - \frac{\sigma_v^2}{\sigma_y^2}. \tag{A57}$$

Clearly, $\partial f(\lambda;E)/\partial\lambda > 0$. Also,

$$\begin{aligned}
\frac{\partial f(\lambda;E)}{\partial E} &\propto [(k\lambda^{-1}+2-\rho)E+\rho\bar{L}]^2(1-\rho) \\
&\qquad - 2[(1-\rho)E+\rho\bar{L}][(k\lambda^{-1}+2-\rho)E+\rho\bar{L}](k\lambda^{-1}+2-\rho) \\
&\propto [(k\lambda^{-1}+2-\rho)E+\rho\bar{L}](1-\rho) - 2[(1-\rho)E+\rho\bar{L}](k\lambda^{-1}+2-\rho) \\
&= -(1-\rho)(k\lambda^{-1}+2-\rho)E - \rho\bar{L}(2k\lambda^{-1}+3-\rho) \\
&< 0.
\end{aligned}$$

Hence, from

$$\frac{\partial f(\lambda;E)}{\partial\lambda}\frac{\partial\lambda}{\partial E} + \frac{\partial f(\lambda;E)}{\partial E} = 0,$$

we conclude that $\partial\lambda/\partial E > 0$.

Now we consider the limit of λ as E approaches its upper bound $\bar{E}$, which satisfies $H(\bar{E}) = 1$, where H is given by (A24), i.e.

$$\frac{k^2[(1-\rho)\bar{E}+\rho\bar{L}]\bar{L}(1+S)^2}{[(2-\rho)\bar{E}+\rho\bar{L}]^2 S}\cdot\frac{\sigma_y^2}{\sigma_v^2} = 1.$$

Let $\bar{\lambda} := \lim_{E\to\bar{E}}\lambda(E)$. Since $f(\lambda(E);E)=0$ for all $E\in[1,\bar{E})$, we have, from (A57),

$$\lim_{E\to\bar{E}} f(\lambda(E);E) = f(\bar{\lambda};\bar{E}) = \frac{[(2-\rho)\bar{E}+\rho\bar{L}]^2}{[(k\bar{\lambda}^{-1}+2-\rho)\bar{E}+\rho\bar{L}]^2}\cdot\frac{\sigma_v^2}{\sigma_y^2} - \frac{\sigma_v^2}{\sigma_y^2} = 0.$$

It follows that $\bar{\lambda} = \infty$.

Proof of (ii): From (A57), $\gamma := \lambda/E$ solves $g(\gamma;E)=0$, where

$$g(\gamma;E) := \frac{k^2[(1-\rho)E+\rho\bar{L}]\bar{L}(1+S)^2}{[k\gamma^{-1}+(2-\rho)E+\rho\bar{L}]^2 S} - \frac{\sigma_v^2}{\sigma_y^2}.$$

It is easy to see that $\partial g(\gamma; E)/\partial\gamma > 0$. We also have

$$\begin{aligned}\frac{\partial g(\gamma;E)}{\partial E} &\propto [k\gamma^{-1} + (2-\rho)E + \rho\bar{L}]^2(1-\rho) \\ &\qquad - 2[(1-\rho)E + \rho\bar{L}][k\gamma^{-1} + (2-\rho)E + \rho\bar{L}](2-\rho) \\ &\propto [k\gamma^{-1} + (2-\rho)E + \rho\bar{L}](1-\rho) - 2[(1-\rho)E + \rho\bar{L}](2-\rho) \\ &= (1-\rho)[k\gamma^{-1} - (2-\rho)E] + \rho\bar{L}[1-\rho-2(2-\rho)] \\ &\propto (1-\rho)[k-(2-\rho)\lambda] - \rho\bar{L}(3-\rho)\gamma.\end{aligned}$$

Hence, from

$$\frac{\partial g(\gamma;E)}{\partial\gamma}\frac{\partial\gamma}{\partial E} + \frac{\partial g(\gamma;E)}{\partial E} = 0,$$

we see that

$$\frac{\partial\gamma}{\partial E} \propto -\frac{\partial g(\gamma;E)}{\partial E} \propto \omega(E) := (2-\rho)\lambda + \frac{\rho(3-\rho)}{1-\rho}\bar{L}\gamma - k.$$

Proof of (iii): As noted above, $f(\lambda; E)$ is strictly increasing in λ. From (A57),

$$\begin{aligned}\frac{\partial f(\lambda;E)}{\partial\bar{L}} &\propto [(k\lambda^{-1} + 2-\rho)E + \rho\bar{L}]^2[(1-\rho)E + 2\rho\bar{L}] \\ &\qquad - 2\rho\bar{L}[(1-\rho)E + \rho\bar{L}][(k\lambda^{-1} + 2-\rho)E + \rho\bar{L}] \\ &\propto [(k\lambda^{-1} + 2-\rho)E + \rho\bar{L}][(1-\rho)E + 2\rho\bar{L}] - 2\rho\bar{L}[(1-\rho)E + \rho\bar{L}] \\ &= [(k\lambda^{-1} + 2-\rho)E + \rho\bar{L}](1-\rho)E + 2\rho\bar{L}(k\lambda^{-1} + 1)E \\ &> 0.\end{aligned}$$

Moreover, it is easy to see that $f(\lambda; E)$ is strictly increasing in S, k, and σ_y^2, and strictly decreasing in σ_v^2. It follows that λ is strictly decreasing in S, $\bar{L}$, k, and σ_y^2, and strictly increasing in σ_v^2. Clearly the same is true of $\gamma := \lambda/E$. $\square$

Proof of Proposition A4.4 *Proof of (i)–(iii):* These results follow from parts (i) and (ii) of Proposition A4.3. In particular, since $\lim_{E\to\bar{E}} \lambda = \infty$, it follows from (A25) that $\omega(E) > k$ for sufficiently large E. Moreover, since $\partial\lambda/\partial E > 0$, and $\partial\gamma/\partial E \propto \omega(E) - k$, we see that if $\omega(\tilde{E}) > k$ for some $\tilde{E}$, then $\omega(E) > k$ for all $E \geq \tilde{E}$.

Proof of (iv): Let x denote either S or σ_y^2/σ_v^2. From (A57), we have $\lim_{x\to\infty} \lambda = 0$, and therefore, from (A25), $\lim_{x\to\infty} \omega = 0$. The result follows.

Proof of (v): If $E^* > 1$, then E^* solves $\Omega(E) := \omega(E) - k = 0$, where $\omega(E)$ is given by (A25). While $\Omega(E)$ is not necessarily monotone in E, it is strictly increasing in E for $E \geq E^*$, and the graph of Ω must cut the E-axis (at E^*) from below. From Proposition A4.3 (iii) and (A25), Ω is strictly decreasing in S, k, and σ_y^2, and strictly increasing in σ_v^2. Therefore, E^* is strictly increasing S, k, and σ_y^2, and strictly decreasing in σ_v^2. $\square$

Proof of Lemma A4.5 The calculations for the utility of a speculator are identical to those in the proof of Lemma 6.1. For a restricted liquidity trader, we have

$$\begin{aligned} W_{i\ell}^R &= (v - p_i) q_{i\ell}^R - y_\ell v - \frac{k}{2}\left(q_{i\ell}^R - y_\ell\right)^2 \\ &= \Big[v - \lambda(S\beta v + \alpha^R Q_i^R)\Big]\alpha^R y_\ell - y_\ell v - \frac{k}{2}\left(\alpha^R y_\ell - y_\ell\right)^2, \end{aligned}$$

so that, using (A23),

$$\begin{aligned} \mathbb{E}\left(W_{i\ell}^R\right) &= -\lambda[1 + \rho(L^R - 1)](\alpha^R)^2 \sigma_y^2 - \frac{k}{2}(\alpha^R - 1)^2 \sigma_y^2 \\ &= -\frac{\lambda k^2 [1 + \rho(L^R - 1)] + \frac{k}{2}[2\lambda + \lambda\rho(L^R - 1)]^2}{[k + 2\lambda + \lambda\rho(L^R - 1)]^2}\sigma_y^2 \\ &= -\frac{k}{2}\frac{2k\lambda[1 + \rho(L^R - 1)] + [k + 2\lambda + \lambda\rho(L^R - 1) - k]^2}{[k + 2\lambda + \lambda\rho(L^R - 1)]^2}\sigma_y^2 \\ &= -\frac{k}{2}\frac{2k\lambda[1 + \rho(L^R - 1)] + [k + 2\lambda + \lambda\rho(L^R - 1)]^2 + k^2 - 2k[k + 2\lambda + \lambda\rho(L^R - 1)]}{[k + 2\lambda + \lambda\rho(L^R - 1)]^2}\sigma_y^2 \\ &= -\frac{k}{2}\left[1 - \frac{k(k + 2\lambda)}{[k + 2\lambda + \lambda\rho(L^R - 1)]^2}\right]\sigma_y^2 \\ &= -\frac{k}{2}\left[1 - \frac{k(k + 2\lambda)E^2}{[(k + 2\lambda - \lambda\rho)E + \lambda\rho\bar{L}]^2}\right]\sigma_y^2, \end{aligned}$$

as desired. $\square$

Proof of Proposition A4.6 *Proof of (i):* We have $f(\lambda(E); E) = 0$ for all E, where f is given by (A57). Calculating the total derivative of f with respect to E, and setting it equal to zero, we obtain

$$\begin{aligned} 0 &= [(k\lambda^{-1} + 2 - \rho)E + \rho\bar{L}]^2(1 - \rho) \\ &\quad - 2[(1 - \rho)E + \rho\bar{L}][(k\lambda^{-1} + 2 - \rho)E + \rho\bar{L}][k\lambda^{-1} + kE(-\lambda^{-2}\lambda') + 2 - \rho], \end{aligned}$$

where $\lambda' := \partial\lambda/\partial E$. Therefore,

$$\begin{aligned} 0 &= [(k\lambda^{-1} + 2 - \rho)E + \rho\bar{L}](1 - \rho) - 2[(1 - \rho)E + \rho\bar{L}][k\lambda^{-1} + kE(-\lambda^{-2}\lambda') + 2 - \rho] \\ &= \rho(1 - \rho)\bar{L} - (k\lambda^{-1} + 2 - \rho)[(1 - \rho)E + 2\rho\bar{L}] + 2kE\lambda^{-2}\lambda'[(1 - \rho)E + \rho\bar{L}], \end{aligned}$$

which gives us

$$\lambda' = \lambda \frac{(k+2\lambda-\lambda\rho)(1-\rho)E + (2k+3\lambda-\lambda\rho)\rho\bar{L}}{2kE[(1-\rho)E+\rho\bar{L}]}.$$

Differentiating (A27), and using the above expression for λ',

$$\begin{aligned}
\frac{d\mathcal{U}^R}{dE} &\propto [(k+2\lambda-\lambda\rho)E+\lambda\rho\bar{L}]^2[2(k+2\lambda)E+2\lambda' E^2] \\
&\qquad - 2(k+2\lambda)E^2[(k+2\lambda-\lambda\rho)E+\lambda\rho\bar{L}][k+2\lambda-\lambda\rho+2\lambda' E-\lambda'\rho E+\lambda'\rho\bar{L}] \\
&\propto [(k+2\lambda-\lambda\rho)E+\lambda\rho\bar{L}](k+2\lambda+\lambda' E)-(k+2\lambda)E[k+2\lambda-\lambda\rho+2\lambda' E-\lambda'\rho E+\lambda'\rho\bar{L}] \\
&= (k+2\lambda)\lambda\rho\bar{L}-\lambda' E\big[-(k+2\lambda-\lambda\rho)E-\lambda\rho\bar{L}+(k+2\lambda)(2E-\rho E+\rho\bar{L})\big] \\
&= (k+2\lambda)\lambda\rho\bar{L}-\lambda' E\big[\lambda\rho E+(k+2\lambda)(1-\rho)E+(k+\lambda)\rho\bar{L}\big] \\
&= (k+2\lambda)\lambda\rho\bar{L} \\
&\qquad -\lambda\frac{(k+2\lambda-\lambda\rho)(1-\rho)E+(2k+3\lambda-\lambda\rho)\rho\bar{L}}{2k[(1-\rho)E+\rho\bar{L}]}\big[\lambda\rho E+(k+2\lambda)(1-\rho)E+(k+\lambda)\rho\bar{L}\big] \\
&\propto 2k[(1-\rho)E+\rho\bar{L}](k+2\lambda)\rho\bar{L} \\
&\qquad -\big[(k+2\lambda-\lambda\rho)(1-\rho)E+(2k+3\lambda-\lambda\rho)\rho\bar{L}\big]\big[\lambda\rho E+(k+2\lambda)(1-\rho)E+(k+\lambda)\rho\bar{L}\big] \\
&< 2k[(1-\rho)E+\rho\bar{L}](k+2\lambda)\rho\bar{L}-(2k+3\lambda-\lambda\rho)\rho\bar{L}\big[(k+2\lambda)(1-\rho)E+(k+\lambda)\rho\bar{L}\big] \\
&\propto 2k(1-\rho)E(k+2\lambda)+2k(k+2\lambda)\rho\bar{L} \\
&\qquad -(2k+3\lambda-\lambda\rho)(k+2\lambda)(1-\rho)E-(2k+3\lambda-\lambda\rho)(k+\lambda)\rho\bar{L} \\
&= -\lambda(3-\rho)(k+2\lambda)(1-\rho)E-\lambda[(1-\rho)k+\lambda(3-\rho)]\rho\bar{L} \\
&< 0.
\end{aligned}$$

Proof of (ii)–(iv): These results follow from Proposition A4.4 and (A26). □

Proof of Proposition A4.7 *Proof of (i):* It is easy to verify that the proof of this result in the proof of Proposition 7.2 is still valid in the current setting. In particular, we use the fact that β is the same as in the main model.

Proof of (ii): From (A53), we see that $\text{Var}(\sum_{\ell\in L_i^R} y_\ell) = [1+\rho(L^R-1)]L^R\sigma_y^2$. Similarly, $\text{Var}(\sum_{\ell\in\bar{L}} y_\ell) = [1+\rho(\bar{L}-1)]\bar{L}\sigma_y^2$. Given the form taken by p_i (equation (8), with $L^U=0$), we have

$$\begin{aligned}
\text{Var}(v|p_1,p_2,\ldots,p_E) &= \text{Var}\left(v\middle|E^{-1}\sum_{i=1}^{E}p_i\right) \\
&= \text{Var}\left(v\middle|S\beta v+\frac{\alpha^R}{E}\sum_{\ell\in\bar{L}}y_\ell\right)
\end{aligned}$$

$$= \sigma_v^2 - \frac{S^2\beta^2\sigma_v^4}{S^2\beta^2\sigma_v^2 + \left(\frac{\alpha^R}{E}\right)^2 [1+\rho(\bar{L}-1)]\bar{L}\sigma_y^2}$$
$$= \sigma_v^2 - \frac{S^2\beta^2\sigma_v^4}{S^2\beta^2\sigma_v^2 + [(1-\rho)E^{-1} + \rho L^R]L^R(\alpha^R)^2\sigma_y^2},$$

so that, using (24), (A54), (A26) and (A27),

$$\begin{aligned}
\mathcal{V} &= \frac{S^2\beta^2\sigma_v^2}{S^2\beta^2\sigma_v^2 + [(1-\rho)E^{-1} + \rho L^R]L^R(\alpha^R)^2\sigma_y^2} \\
&= \left[\frac{S^2\beta^2\sigma_v^2 + [1+\rho(L^R-1)]L^R(\alpha^R)^2\sigma_y^2}{S^2\beta^2\sigma_v^2} - \frac{(1-\rho)(1-E^{-1})L^R(\alpha^R)^2\sigma_y^2}{S^2\beta^2\sigma_v^2}\right]^{-1} \\
&= \left[\frac{1}{S\lambda\beta} - \left(\frac{\alpha^R}{\beta}\right)^2 \frac{(1-\rho)(E-1)\bar{L}\sigma_y^2}{E^2S^2\sigma_v^2}\right]^{-1}. \qquad \text{(A58)}
\end{aligned}$$

From (A22), (A23), and then (A55),

$$\begin{aligned}
\left(\frac{\alpha^R}{\beta}\right)^2 &= \left[\frac{k(1+S)\lambda}{k+2\lambda+\lambda\rho(L^R-1)}\right]^2 \\
&= \frac{S}{[1+\rho(L^R-1)]L^R}\frac{\sigma_v^2}{\sigma_y^2} \\
&= \frac{E^2S}{[(1-\rho)E+\rho\bar{L}]\bar{L}}\frac{\sigma_v^2}{\sigma_y^2}.
\end{aligned}$$

Substituting this expression into (A58), and using (A26), we obtain

$$\begin{aligned}
\mathcal{V} &= \left[\frac{1+S}{S} - \frac{(1-\rho)(E-1)}{S[(1-\rho)E+\rho\bar{L}]}\right]^{-1} \\
&= \left[1 + \frac{1-\rho+\rho\bar{L}}{S[(1-\rho)E+\rho\bar{L}]}\right]^{-1},
\end{aligned}$$

which is strictly increasing in E, and strictly decreasing in ρ. □

Proof of Proposition A5.1 Let $x := S-1$. From the zero-profit condition (A28),

$$\lambda = \frac{E\sigma_v^2}{c(x+2)^2}.$$

Setting $L^U = 0$ in (39), we obtain

$$\begin{aligned} f(\lambda; E) &= \frac{k^2 \bar{L}}{E(k\lambda^{-1}+2)^2} \frac{(1+S)^2}{S} - \frac{\sigma_v^2}{\sigma_y^2} \\ &= \frac{k^2 \bar{L} E \sigma_v^4}{[ck(x+2)^2 + 2E\sigma_v^2]^2} \frac{(x+2)^2}{x+1} - \frac{\sigma_v^2}{\sigma_y^2}. \end{aligned}$$

Defining $\mathcal{L} := k^2 \bar{L} \sigma_y^2$, we can write the equation $f(\lambda; E) = 0$ as $G(x; E) = 0$, where

$$\begin{aligned} G(x; E) &:= [ck(x+2)^2 + 2E\sigma_v^2]^2 (x+1) - E\sigma_v^2 \mathcal{L}(x+2)^2 \\ &= c^2k^2(x+2)^4(x+1) + 4ckE\sigma_v^2(x+2)^2(x+1) \\ &\quad + 4E^2\sigma_v^4(x+1) - E\sigma_v^2\mathcal{L}(x+2)^2 \\ &= c^2k^2(x^4 + 8x^3 + 24x^2 + 32x + 16)(x+1) + 4ckE\sigma_v^2(x^2+4x+4)(x+1) \\ &\quad + 4E^2\sigma_v^4(x+1) - E\sigma_v^2\mathcal{L}(x^2+4x+4) \\ &= c^2k^2(x^5 + 9x^4 + 32x^3 + 56x^2 + 48x + 16) + 4ckE\sigma_v^2(x^3 + 5x^2 + 8x + 4) \\ &\quad + 4E^2\sigma_v^4(x+1) - E\sigma_v^2\mathcal{L}(x^2+4x+4) \\ &= c^2k^2x^5 + 9c^2k^2x^4 + 4ck(8ck + E\sigma_v^2)x^3 + (56c^2k^2 + 20ckE\sigma_v^2 - E\sigma_v^2\mathcal{L})x^2 \\ &\quad + 4(12c^2k^2 + 8ckE\sigma_v^2 + E^2\sigma_v^4 - E\sigma_v^2\mathcal{L})x \\ &\quad + 4(4c^2k^2 + 4ckE\sigma_v^2 + E^2\sigma_v^4 - E\sigma_v^2\mathcal{L}) \\ &= c^2k^2x^5 + 9c^2k^2x^4 + 4ck(8ck + E\sigma_v^2)x^3 + (56c^2k^2 + 20ckE\sigma_v^2 - E\sigma_v^2\mathcal{L})x^2 \\ &\quad + 4\big[(2ck + E\sigma_v^2)(6ck + E\sigma_v^2) - E\sigma_v^2\mathcal{L}\big]x + 4\big[(2ck + E\sigma_v^2)^2 - E\sigma_v^2\mathcal{L}\big] \\ &= a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0, \end{aligned}$$

where the coefficients a_i are given by

$$\begin{aligned} a_0 &:= 4E\sigma_v^2(\mathcal{L}_2 - \mathcal{L}), \\ a_1 &:= 4E\sigma_v^2(\mathcal{L}_3 - \mathcal{L}), \\ a_2 &:= 56c^2k^2 + E\sigma_v^2(20ck - \mathcal{L}), \\ a_3 &:= 4ck(8ck + E\sigma_v^2), \\ a_4 &:= 9c^2k^2, \\ a_5 &:= c^2k^2. \end{aligned}$$

An equilibrium corresponds to a nonnegative zero of G, i.e. a nonnegative solution x to $G(x; E) = 0$ (note that $x \geq 0$ if and only if $S \geq 1$). The solution $x = 0$ exists if and only if $a_0 = 0$.

Since $\lim_{E\to\infty}[G(x;E)/E^2] > 0$ for all $x > 0$, there exists $\bar{E}$ such that there is no equilibrium if $E > \bar{E}$. For a given E, we can determine the number of positive zeros of G by using Descartes' rule of signs: ordering the nonzero coefficients of G by descending variable exponent, the number of positive zeros is either equal to the number of sign changes between consecutive (nonzero) coefficients, or is less than it by an even number. A zero of multiplicity m is counted as m zeros.

Note that the coefficients a_3, a_4, and a_5 are positive. The signs of a_0, a_1, and a_2 depend on the parameters as described below.

Proof of (i): If $\mathcal{L} \leq \mathcal{L}_1$, then $a_i > 0$ for all i, and hence there is no nonnegative zero of G.

Proof of (ii): If $\mathcal{L}_1 < \mathcal{L} < \mathcal{L}_2$, then $a_1 > a_0 > 0$, and $x = 0$ is not a zero of G. If $a_2 \geq 0$, there is no positive zero of G. If $a_2 < 0$, there are two changes in sign in the coefficients of G. Hence, the number of positive zeros of G is 0 or 2. If there are two positive zeros, they are distinct for $\mathcal{L}$ in a generic subset of $(\mathcal{L}_1, \mathcal{L}_2)$. It follows that, for $\mathcal{L}$ in this generic subset, the number of equilibria is 0 or 2.

Proof of (iii): Since the result is stated for $\mathcal{L}$ in a generic subset of $[\mathcal{L}_2, \mathcal{L}_3)$, we can disregard the case where $\mathcal{L} = \mathcal{L}_2$, and assume that $\mathcal{L}_2 < \mathcal{L} < \mathcal{L}_3$. Then $a_0 < 0$ and $a_1 > 0$, and $x = 0$ is not a zero of G. If $a_2 \geq 0$, there is only one sign change in the coefficients of G, and hence it has a unique positive zero. If $a_2 < 0$, there are three sign changes, and hence the number of positive zeros is either 1 or 3. In the case of 3 zeros, they are all distinct for $\mathcal{L}$ in a generic subset of $(\mathcal{L}_2, \mathcal{L}_3)$.

Proof of (iv): If $\mathcal{L} \geq \mathcal{L}_3$, then $a_0 < a_1 \leq 0$. Regardless of the sign of a_2, there is only one sign change in the coefficients of G. Hence there is a unique positive zero of G. Furthermore, $x = 0$ is not a zero, and hence there is a unique equilibrium. $\square$

Proof of Proposition A5.2 *Proof of (i):* Under the condition $\mathcal{L}_1 < \mathcal{L} < \mathcal{L}_2$, we see from the proof of Proposition A5.1 (ii) that $x = 0$ is not a zero of G, and the number of positive zeros is either 0 or 2. In order to ensure that there are two equilibria, i.e. two distinct positive zeros, it suffices to show that there exists a positive zero with multiplicity 1. Note that $G(0;E) = a_0 > 0$, and

$$\lim_{c\to 0} G(x;E) = E\sigma_v^2\big[-\mathcal{L}x^2 + 4(E\sigma_v^2 - \mathcal{L})(x+1)\big],$$

which is negative for large positive x. Therefore, there exists $\kappa_1 > 0$ such that for $c \in (0, \kappa_1)$, there exists $x_0 > 0$ with the property that $G(x_0;E) = 0$. Moreover, the graph of G intersects the x-axis at x_0, thus ruling out the possibility that this zero has multiplicity 2.

Proof of (ii): If $\mathcal{L} = \mathcal{L}_2$, then $a_1 > a_0 = 0$. Factoring out x, we can write $G(x; E) = x\hat{G}(x; E)$, where

$$\hat{G}(x; E) := a_5 x^4 + a_4 x^3 + a_3 x^2 + a_2 x + a_1.$$

One equilibrium corresponds to $x = 0$. There exists $\kappa_2 > 0$ such that for $c \in (0, \kappa_2)$, we have $a_2 < 0$, and hence the number of positive zeros of $\hat{G}$ is either 0 or 2. Note that $\hat{G}(0; E) = a_1 > 0$, and $\lim_{c \to 0} \hat{G}(x; E) = -E^2 \sigma_v^4 x$, which is negative for all $x > 0$. Therefore, there exists $\kappa_3 \in (0, \kappa_2)$ such that for $c \in (0, \kappa_3)$, $\hat{G}$ has a positive zero with multiplicity 1 (the graph of $\hat{G}$ intersects the x-axis at this zero). Since the number of positive zeros of $\hat{G}$ is 0 or 2, $\hat{G}$ must have two distinct positive zeros. Therefore, the number of equilibria is 3.

For any $c \in (0, \kappa_3)$, there exists $\epsilon > 0$ (which depends on c) such that, for $\mathcal{L}$ in an ϵ-neighborhood of $\mathcal{L}_2$, there are either two or three equilibria (two if the equilibrium corresponding to $x = 0$ when $\mathcal{L} = \mathcal{L}_2$ becomes negative when $\mathcal{L}$ is perturbed away from $\mathcal{L}_2$). In particular, for sufficiently small ϵ, the number of equilibria is two or three if $\mathcal{L} \in [\mathcal{L}_2, \mathcal{L}_2 + \epsilon)$, and hence also if $\mathcal{L}$ is in any generic subset of $[\mathcal{L}_2, \mathcal{L}_2 + \epsilon)$. By Proposition A5.1 (iii), the number of equilibria is one or three if $\mathcal{L}$ is in a generic subset of $[\mathcal{L}_2, \mathcal{L}_3)$. Therefore, for $c \in (0, \kappa_3)$ and sufficiently small ϵ, the number of equilibria is three if $\mathcal{L} \in [\mathcal{L}_2, \mathcal{L}_2 + \epsilon)$. $\square$

The following result will be useful for proving Proposition A5.3.

Lemma A6.1 *An equilibrium λ is a positive zero of*

$$f(\lambda; E) = \frac{k^2 \bar{L}}{E(k\lambda^{-1} + 2)^2} \frac{[1 + S(\lambda; E)]^2}{S(\lambda; E)} - \frac{\sigma_v^2}{\sigma_y^2}, \tag{A59}$$

where $S(\lambda; E) \geq 1$, and satisfies

$$[1 + S(\lambda; E)]^2 = \frac{E\sigma_v^2}{c\lambda}. \tag{A60}$$

We have

i. $\partial f(\lambda; E)/\partial \lambda > 0$ at $\lambda = \lambda_\ell, \lambda_h, \lambda_u$.

ii. $\partial f(\lambda; E)/\partial \lambda < 0$ at $\lambda = \lambda_m$.

Proof of Lemma A6.1 The function $f(\lambda; E)$ is obtained from (39) by setting $L^U = 0$, with S given by (A28). The equation $f(\lambda; E) = 0$ is equivalent to $G(x; E) = 0$, where $x := S - 1$. Using (A59) and (A60), we have

$$f(\lambda; E) = \frac{k^2 \bar{L} \lambda^2}{E(k + 2\lambda)^2} \frac{\frac{E\sigma_v^2}{c\lambda}}{S(\lambda; E)} - \frac{\sigma_v^2}{\sigma_y^2}$$

$$= \frac{k^2 \bar{L} \lambda \sigma_v^2}{c(k+2\lambda)^2 S(\lambda; E)} - \frac{\sigma_v^2}{\sigma_y^2}. \tag{A61}$$

From (A60), $\lim_{\lambda \to 0} S(\lambda; E) = \infty$. Therefore, $\lim_{\lambda \to 0} f(\lambda; E) < 0$.

Let $\lambda_{\max}$ be such that $S(\lambda_{\max}; E) = 1$. From (A60), we obtain $\lambda_{\max} = E\sigma_v^2/(4c)$. We have $S \geq 1$ if and only if $\lambda \leq \lambda_{\max}$. From (A59),

$$\begin{aligned} f(\lambda_{\max}; E) &= \frac{4k^2 \bar{L}}{E\left(\frac{4ck}{E\sigma_v^2} + 2\right)^2} - \frac{\sigma_v^2}{\sigma_y^2} \\ &= \frac{k^2 E \bar{L} \sigma_v^4}{(2ck + E\sigma_v^2)^2} - \frac{\sigma_v^2}{\sigma_y^2} \\ &\propto \mathcal{L} - \mathcal{L}_2. \end{aligned}$$

It is evident from the proofs of Propositions A5.1 and A5.2 that the equilibria described in these results correspond to nonnegative zeros of G at which the graph of G intersects the x-axis. The price impact parameters corresponding to these equilibria are zeros of f, lying in the interval $(0, \lambda_{\max}]$, at which the graph of f intersects the λ-axis.

If there are two equilibria, λ_ℓ and λ_m, then $\mathcal{L} < \mathcal{L}_2$ (Proposition A5.1 (ii)), and hence $f(\lambda_{max}; E) < 0$. It follows that the graph of f cuts the λ-axis from below at λ_ℓ, and from above at λ_m.

If there are three equilibria, λ_ℓ, λ_m, and λ_h, then $\mathcal{L} \in [\mathcal{L}_2, \mathcal{L}_3)$ (Proposition A5.1 (iii)), and hence $f(\lambda_{\max}; E) \geq 0$. This implies that the graph of f cuts the λ-axis from below at λ_ℓ and λ_h, and from above at λ_m.

Finally, if there is a unique equilibrium λ_u, then $\mathcal{L} \geq \mathcal{L}_2$ (Proposition A5.1 (iii) and (iv)), and hence $f(\lambda_{\max}; E) \geq 0$. We conclude that the graph of f cuts the λ-axis from below at λ_u.

To summarize, the graph of $f\colon \lambda \mapsto f(\lambda; E)$, cuts the λ-axis from below at λ_ℓ, λ_h, and λ_u, and from above at λ_m. This gives us the desired properties of $\partial f(\lambda; E)/\partial \lambda$. $\square$

Proof of Proposition A5.3 From (A60) and (A61),

$$\frac{\partial f(\lambda; E)}{\partial E} \propto -\frac{\partial S(\lambda; E)}{\partial E} < 0.$$

In equilibrium, $f(\lambda; E) = 0$ for all E, which implies that

$$\frac{\partial f(\lambda; E)}{\partial \lambda} \frac{\partial \lambda}{\partial E} + \frac{\partial f(\lambda; E)}{\partial E} = 0.$$

Therefore, if $\partial f(\lambda; E)/\partial\lambda \neq 0$, we have

$$\frac{\partial\lambda}{\partial E} = -\frac{\partial f(\lambda; E)}{\partial E}\left[\frac{\partial f(\lambda; E)}{\partial\lambda}\right]^{-1} \propto \frac{\partial f(\lambda; E)}{\partial\lambda}.$$

Now we invoke Lemma A6.1 to conclude that $\partial\lambda/\partial E > 0$ at $\lambda = \lambda_\ell, \lambda_h, \lambda_u$; and $\partial\lambda/\partial E < 0$ at $\lambda = \lambda_m$. This establishes the results on the effect of market fragmentation on liquidity. The welfare results follow from the inverse relationship between λ and the utility of restricted liquidity traders (see (22)), and the fact that the expected net payoff of speculators is zero for all E. $\square$

Proof of Proposition A5.4 From Proposition 7.2, $\mathcal{V} = [1 + (ES)^{-1}]^{-1}$. It follows that

$$\frac{\partial\mathcal{V}}{\partial E} \propto \frac{\partial\mu}{\partial E}, \tag{A62}$$

where $\mu := ES$. We will show that, under the conditions stated in the proposition, there exists an equilibrium such that $\partial\mu/\partial E < 0$. From (A28), we have

$$\begin{aligned}\lambda &= \frac{E\sigma_v^2}{c(1+S)^2}\\ &= \frac{E\sigma_v^2}{c(1+\mu E^{-1})^2}\\ &= \frac{E^3\sigma_v^2}{c(E+\mu)^2}.\end{aligned}$$

Therefore, from (A59),

$$\begin{aligned}f(\lambda; E) &= \frac{k^2\bar{L}}{E(k\lambda^{-1}+2)^2}\frac{(1+S)^2}{S} - \frac{\sigma_v^2}{\sigma_y^2}\\ &= \frac{k^2\bar{L}}{E\left[\frac{ck(E+\mu)^2}{E^3\sigma_v^2}+2\right]^2}\frac{(1+\mu E^{-1})^2}{\mu E^{-1}} - \frac{\sigma_v^2}{\sigma_y^2}\\ &= \frac{k^2\bar{L}E^4\sigma_v^4}{[ck(E+\mu)^2+2E^3\sigma_v^2]^2}\frac{(E+\mu)^2}{\mu} - \frac{\sigma_v^2}{\sigma_y^2}.\end{aligned}$$

Hence $f(\lambda; E) = 0$ is equivalent to $F(\mu; E) = 0$, where

$$F(\mu; E) := \frac{1}{\mu}\left[\frac{E^2(E+\mu)}{ck(E+\mu)^2+2E^3\sigma_v^2}\right]^2 - \frac{1}{k^2\bar{L}\sigma_v^2\sigma_y^2}.$$

We have

$$\begin{aligned}\frac{\partial F(\mu;E)}{\partial E} &\propto \left[ck(E+\mu)^2+2E^3\sigma_v^2\right](3E^2+2E\mu)-E^2(E+\mu)\left[2ck(E+\mu)+6E^2\sigma_v^2\right]\\ &\propto ck(E+\mu)^2(E+2\mu)-2E^3\mu\sigma_v^2\\ &\propto ck(S+1)^2(2S+1)-2ES\sigma_v^2.\end{aligned}$$

Also,

$$\begin{aligned}\frac{\partial F(\mu;E)}{\partial \mu} &\propto -\frac{1}{\mu^2}\left[\frac{E+\mu}{ck(E+\mu)^2+2E^3\sigma_v^2}\right]^2\\ &\quad+\frac{2}{\mu}\left[\frac{E+\mu}{ck(E+\mu)^2+2E^3\sigma_v^2}\right]\frac{ck(E+\mu)^2+2E^3\sigma_v^2-2ck(E+\mu)^2}{[ck(E+\mu)^2+2E^3\sigma_v^2]^2}\\ &\propto -(E+\mu)[ck(E+\mu)^2+2E^3\sigma_v^2]+2\mu[2E^3\sigma_v^2-ck(E+\mu)^2]\\ &=-ck(E+\mu)^2(E+3\mu)+2E^3\sigma_v^2(\mu-E)\\ &\propto -ck(S+1)^2(3S+1)+2E(S-1)\sigma_v^2.\end{aligned}$$

Since $F(\mu;E)=0$ for all E, we have

$$\frac{\partial F(\mu;E)}{\partial E}+\frac{\partial F(\mu;E)}{\partial \mu}\frac{\partial \mu}{\partial E}=0.$$

Assuming that $\partial F(\mu;E)/\partial\mu\neq 0$, we have

$$\begin{aligned}\frac{\partial \mu}{\partial E} &= -\frac{\partial F(\mu;E)}{\partial E}\left[\frac{\partial F(\mu;E)}{\partial \mu}\right]^{-1}\\ &\propto \frac{ck(S+1)^2(2S+1)-2ES\sigma_v^2}{ck(S+1)^2(3S+1)-2E(S-1)\sigma_v^2}.\end{aligned}$$

Note that $ck(S+1)^2(3S+1)-2E(S-1)\sigma_v^2 > ck(S+1)^2(2S+1)-2ES\sigma_v^2$. Therefore, $\partial\mu/\partial E<0$ if and only if $ck(S+1)^2(2S+1)-2ES\sigma_v^2<0$ and $ck(S+1)^2(3S+1)-2E(S-1)\sigma_v^2>0$. Using (A28) and (A62), we see that $\partial\mathcal{V}/\partial E<0$ if and only if

$$k\frac{2S+1}{2S}<\lambda<k\frac{3S+1}{2(S-1)}. \tag{A63}$$

Now we adapt the arguments in the proof of Proposition A5.2 (ii). If $\mathcal{L}=\mathcal{L}_2$, then $G(0;E)=0$. Moreover, for any $\phi>0$, there exists $\epsilon>0$, such that, for $\mathcal{L}\in(\mathcal{L}_2,\mathcal{L}_2+\epsilon)$, we have $G(x_0(\mathcal{L});E)=0$, where $x_0(\mathcal{L})\in(0,\phi)$. In other words, the equilibrium can be parametrized as $x_0(\mathcal{L})$, where $x_0(\mathcal{L}_2)=0$ and $x_0(\mathcal{L})$ is a small positive number when $\mathcal{L}$ is greater than but close to $\mathcal{L}_2$. As seen in the proof of Proposition A5.2 (ii), there may be two additional equilibria. But for our purposes here it does not matter if the equilibrium we have identified is unique or not.

At the equilibrium x_0, the number of speculators is greater than but close to one and, from (A28), λ is less than but close to $\lambda_{\max} = E\sigma_v^2/(4c)$. Therefore, at this equilibrium, (A63) holds if $E\sigma_v^2/(4c) > 3k/2$, or $6ck < E\sigma_v^2$. $\square$