# Dynamic portfolio and mortgage choice for homeowners[*]

Otto van Hemert        Frank de Jong        Joost Driessen

University of Amsterdam

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## Abstract

We investigate the impact of owner-occupied housing on financial portfolio and mortgage choice under stochastic inflation and real interest rates. To this end we develop a dynamic framework in which investors can invest in stocks and bonds with different maturities. We use a continuous-time model with CRRA preferences and calibrate the model parameters using data on inflation rates and equity, bond, and house prices. For the case of no short-sale constraints, we derive an implicit solution and identify the main channels through which the housing to total wealth ratio and the horizon affect financial portfolio choice. This solution is used to interpret numerical results that we provide when the investor has short-sale constraints. We also use our framework to investigate optimal mortgage size and type. A moderately risk-averse investor prefers an adjustable-rate mortgage (ARM), while a more risk-averse investor prefers a fixed-rate mortgage (FRM). A combination of an ARM and an FRM further improves welfare. Choosing a suboptimal mortgage leads to utility losses up to 6%.

# 1  Introduction

For homeowners a house is not only a place to live in, but also a risky financial investment. Often the owner-occupied house is a large, if not the largest, asset for a household. It can therefore have a major impact on financial portfolio choice. Moreover, homeowners face the decision how to finance the house. The financial portfolio and mortgage choice are both important elements of overall financial planning and are therefore closely related. Even though many investors own the house they live in, most literature on financial portfolio choice ignores housing and mortgages altogether.

This paper is the first (to the best of our knowledge) to provide an integrated analysis of a homeowner's optimal financial portfolio and mortgage choice. We take a long-term investment perspective, where the investor derives utility from the services supplied by the house and from the consumption of other goods which is equal to total terminal wealth. We interpret current investor's wealth as including human capital, i.e. we assume that future labor income is capitalized and invested in the house and the available menu of financial assets. The housing investment is taken as fixed and given, while positions in financial assets are rebalanced dynamically.

Our setup incorporates that housing differs from financial assets in at least four crucial respects. First, the total amount of housing is often dictated by consumption motives rather than investment motives. Second, the housing investment is far less liquid than financial investments because of high monetary and effort costs involved with moving. Third, the house can serve as collateral for a mortgage loan up to the market value of the house. Fourth, the expected housing return will be lower than on a hypothetical pure financial asset with comparable risk characteristics, because the market will recognize that a house also provides housing services.

Another important part of our setup is a realistic model for the term structure of interest rates, with expected inflation and real interest rate as factors. Our model thus extends the Brennan and Xia (2002) model by including housing. The model gives a rationale for holding nominal bonds with different maturities. Importantly, it also allows us to investigate the implications of different types of mortgage loans. In addition to the two term structure factors, we model unexpected inflation, house risk and stock market risk, leading to a total of five sources of uncertainty. This structure enables us to realistically examine the interaction of financial asset prices and the house price. We also investigate mortgage choice. We link the optimal mortgage size and type to the coefficient of relative risk aversion, house

size and horizon, and study the interplay of financial portfolio choice and mortgage choice.

This paper relates to several strands in the portfolio choice literature. First, Brennan and Xia (2002) and Campbell and Viceira (2001) illustrate the importance of bonds for a long-term investor. Both use a two-factor model similar to ours for the nominal interest rate. A long-term investor holds bonds not only to exploit the risk premium, but also to hedge changes in the investment opportunity set. Our paper extends this work by taking into account the impact of an owner-occupied house on portfolio choice and by optimally choosing the mortgage size and type.

Another strand of literature focuses on the housing and financial portfolio choice in a static one-period mean-variance setting (Brueckner (1997) and Flavin and Yamashita (2002)). These articles focus on the so-called 'housing constraint': the investment in housing must be at least as large as the consumption of housing. The classical mutual fund separation theorem is no longer valid in the presence of the housing constraint. In contrast to the static one-period mean-variance setting, we use CRRA preferences, allow for dynamic strategies, and focus on long horizons. More importantly, these papers ignore the long term real interest rate and inflation risks, and therefore have little role for bond investments. In addition, the one period framework gives no advice for mortgage choice.

A number of papers have extended the literature on portfolio choice over the life cycle by examining the optimal house size decision. Examples are Cocco (2004), Hu (2003), and Yao and Zhang (2003). The latter two papers also model the house tenure decision. Our paper can be seen as complementing this literature. We have a much richer asset menu, we study the choice for mortgage type, and implement more sophisticated modelling of the interaction of the return on the house with financial asset returns. We do not, however, model life cycle features like the labor income profile or labor income risk.

Finally, this paper relates to Campbell and Cocco (2003) who examine the choice between a Fixed-Rate Mortgage (FRM) and an Adjustable-Rate Mortgage (ARM). They consider a model which has persistent shocks to the expected inflation rate only. In this case an FRM has a variable real value and therefore has wealth risk. An ARM has a fixed real value, but has income risk that is important in their life-cycle setup with an explicit borrowing constraint. A positive shock to interest rates may cause a temporary reduction in consumption, which can be very costly in utility terms. Our mortgage analysis differs from Campbell and Cocco (2003) in two important ways. First, in our model setup, there is no income risk but wealth risk is more complex since we incorporate persistent shocks to

both real interest rates and inflation. This makes the choice between an ARM and FRM nontrivial. Second, we model a much richer asset menu with stocks and bonds of various maturity, and provide an integrated analysis of portfolio and mortgage choice, where the mortgage is modeled as a short position in one or several bonds. Our analysis therefore sheds light on a different, but important, determinant of mortgage choice.

The main results of the paper can be summarized as follows. In the absence of short-sale constraints, we are able to derive an implicit analytical expression for the investor's optimal financial portfolio. This portfolio is composed of (i) the nominal mean-variance tangency portfolio; (ii) a portfolio that most closely resembles an inflation-indexed bond; and (iii) a portfolio that best offsets the risk of the illiquid house. The first two portfolios were also derived by Brennan and Xia (2002). The size of the position in the three portfolios depends on the coefficient of relative risk aversion, the financial wealth ratio and the effective housing to total wealth ratio. Because the provided housing services make a house not a pure financial asset, and because the position in the house is fixed, it is not the current market value of the house that appears in the expressions for optimal portfolio choice, but a value we refer to as effective housing wealth. The third component can be seen as a position offsetting the effective housing exposure. Our analysis shows that house ownership affects the optimal financial portfolio in several ways. First, it gives rise to a portfolio hedging the house risk. Second, it determines a factor to leverage the financial portfolio weights in order to maintain an appropriate absolute financial risk exposure. Third, it determines the effective to total wealth ratio. We show that horizon effects in the portfolio choice arise due to a horizon dependent hedge against real interest changes and due to the effective to total wealth ratio which decreases with the investment horizon. In the more realistic case that the investor can only hold nonnegative positions, we use numerical techniques to calculate the optimal portfolio. The interpretation of the numerical results is greatly enhanced by the lessons learned from the no-constraints case.

We estimate the model parameters using data on equity, bond, and house prices, and study the optimal financial portfolios for different investor horizons and house sizes. As the house size increases, two effects determine the change in financial portfolio weights. First, in order to maintain an approximately constant absolute stock market exposure, the financial portfolio weights are levered up. Second, since the risk-averse investor is exposed to undiversified risk of the fixed house position, he will decrease his exposure to stock and bond market risk. In case of short-sale constraints, an additional effect is that stock and bond positions compete in terms of their hedging and return benefits. As a result, the

weighted average duration of the two bonds and cash position increases with the housing to total wealth ratio, which is consistent with a desire for a leveraged exposure to real interest rate and expected inflation rate shocks. The horizon effects can be understood from the fact that the effective to total wealth ratio decreases substantially with horizon. For moderately risk-averse investors this results in a decrease of the fraction invested in stocks and for more risk-averse investors in a decrease of the fraction invested in long-term bonds with horizon. We show that neglecting house ownership in portfolio choice can lead to utility losses up to 2%, which illustrates the importance of incorporating house ownership in calculating optimal financial portfolios.

Finally, we allow for mortgage loans, and investigate the choice between fixed-rate and adjustable-rate mortgages. We find that a moderately risk-averse investor prefers an ARM, in order to avoid paying the risk premium on long-term bonds. A more risk-averse investor is more concerned about hedging inflation and interest rate risk, and rather chooses an FRM. An even better mortgage for this investor is a hybrid mortgage, which is a combination of an FRM and an ARM. Choosing a suboptimal mortgage leads to utility losses up to 6% for long horizons. This illustrates that the mortgage choice should play a central role in a household's financial planning.

The structure of the paper is as follows. Section 2 presents the investor's portfolio allocation problem, describes the price processes of the available assets, and discusses the optimal portfolio choice. In section 3 we discuss the estimation of the model parameters. Section 4 contains our main results for unconstrained investors as well as for investors with short-sale constraints. We also investigate how sensitive our results are to alternative parameters for the house price process. In section 5 we introduce mortgage loans. Section 6 concludes.

## 2 Optimal asset allocation

In this section we present the investor's portfolio allocation problem. We describe the economy and the price processes of the available assets, and discuss the optimal portfolio choice. For the case with no constraints on the size of positions in financial assets, we are able to provide an analytical expression for the optimal investment in stocks, bonds with different maturities and cash. In the special case that housing risk is perfectly hedgeable we can solve for the optimal investment in closed form. Finally, we discuss the numerical techniques to analyze the model with short sale constraints.

## 2.1 The investor's optimization problem

We consider optimal financial portfolio choice for an investor from time 0 until time $T$, his horizon. We assume that besides financial assets, the investor owns the house he lives in,[1] which has given size $H$. The house size is interpreted as a one-dimensional representation of the quality of the house. The investor sells his house at time $T$, which could be interpreted as the moment of retirement. The possibility to sell his house or buy a second house before time $T$ is ignored. The nominal price of a unit of housing at time $t$ is denoted $Q_t$. We normalize $Q_0$ to 1. Nominal housing wealth is denoted $W_t^H \equiv Q_t H$. We define $W_t^F$ as nominal financial wealth. $W_0^F$ includes human capital, but excludes housing wealth. Labor income risk and moral hazard issues involved in capitalizing labor income are ignored. We make the simplifying assumption that maintenance costs are capitalized and paid in advance, which means they do not play an explicit role in our analysis. Taking into account that labor income is capitalized, we like to think of the housing to total wealth ratio, $h$, as being in the order of magnitude of 0.2, and in our tables it typically ranges from 0 to 0.5.[2]

Total nominal wealth is denoted $W_t \equiv W_t^F + W_t^H \equiv W_t^F + Q_t H$. At time $T$, the investor uses his total wealth for consumption of other goods. The real price of these consumption goods is chosen to be the numeraire. The nominal price level at time $t$ is denoted as $\Pi_t$ and we normalize $\Pi_0 = 1$. We use uppercase letters for nominal variables and the corresponding lowercase letter for their real counterpart, so total real wealth is $w_t = w_t^F + w_t^H$. Following Cocco (2004), Hu (2003) and Yao and Zhang (2003) we represent preferences over housing consumption to other goods by the Cobb-Douglas function

$$u\left(w_T, H\right) = \frac{\left(w_T^\psi H^{1-\psi}\right)^{1-\tilde{\gamma}}}{1-\tilde{\gamma}} = \frac{w_T^{1-\gamma}}{1-\gamma}\nu_H \tag{1}$$

with $\nu_H \equiv \psi H^{(1-\psi)(1-\tilde{\gamma})}$, $\gamma \equiv 1 - \psi\left(1-\tilde{\gamma}\right)$. We have $\gamma = -w_t u_{ww}/u_w$, which is the coefficient of relative risk aversion given a fixed position in housing.

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[1] Owning might be preferred to renting a house because of favorable tax treatment or frictions in the rental market. For example, Henderson and Ioannides (1983) focus on a friction in the rental market which they refer to as 'renter externality'. This externality is due to moral hazard issues, like property abuse by renters.

[2] Heaton and Lucas (2000, Table V) report housing to total wealth ratios in the range 0.1 to 0.3 while including capitalised labor income, social security and pension benefits in the total wealth measure. In contrast, Flavin and Yamashita (2002) ignore human capital as part of total wealth. Their housing to total wealth ratio ranges from 0.65 to 3.51.

At $t \in [0, T]$ the investor solves the portfolio allocation problem

$$\max_{\substack{x(\tau) \in A, \\ t \leq \tau \leq T}} E_t \left[ u \left( w_T^F (x) + w_T^H, H \right) \right] \tag{2}$$

subject to the constraint that $w_T^F$ is financed by the strategy for financial portfolio weights $x(\tau)$, with wealth at time $t$ equal to $w_t^F$, and where $A$ is the set of admissible financial portfolio weights. Throughout the paper, we assume that this set is independent of total wealth $w_t$ and the real interest rate. We consider three specific cases for the set $A$: no constraints, short sale constraints on all assets, and the case with a mortgage up to the initial value of the house.

Four remarks regarding the investors objective are in place here. First, the model can easily be embedded into a model where just before time $t = 0$ the investor also optimizes over house size. In this case the preference parameter $\psi$ is crucial for determining the optimal house size. Because our focus is on financial portfolio choice, we choose to condition on a given house size (or equivalently, housing wealth at $t = 0$) directly. Second, defining utility over interim consumption instead of over terminal wealth is not likely to change the results qualitatively, but basically reduces the effective horizon.[3] Third, we can ignore the multiplicative factor $\nu_H$ of the utility function when solving for optimal portfolio choice $x$. So given the house size, utility is of the power utility form over terminal wealth. This also means that we only have to know $\gamma$ and not $\psi$ and $\tilde{\gamma}$ separately. Fourth, throughout we will assume that $W_0^F \geq 0$, implying that it is always possible to have $W_t^F \geq 0$ for all $t \in [0, T]$, which ensures that the investor's problem is well defined.

## 2.2   Asset price dynamics

We consider an economy with five sources of uncertainty represented by innovations in five Brownian motions. This is similar to Brennan and Xia (2002, henceforth BX), except that we have an extra source of uncertainty to capture idiosyncratic risk in house price changes. As stated earlier, we focus on the financial portfolio choice of a single investor who takes

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[3]This is a well-known result in the absence of an owner occupied house (see e.g. Brennan and Xia (2002) for a further discussion). We argue that this result also holds in our set-up, which enhances the comparison between this paper and e.g. Cocco (2004), Hu (2003) and Yao and Zhang (2003) who use interim consumption. This relies on the observation that also for interim consumption the factor $\nu_H$ can be separated in the utility function and on the assertion that the difference in house size between the interim consumption and the terminal wealth formulation will be small. To expand on the latter, in a world without uncertainty it is easy to show that the Cobb-Douglas preference structure implies that the optimal house size is the same in the interim consumption and terminal wealth cases.

price processes as given. Furthermore, we assume that the risk premia on the sources of uncertainty are constant.

The nominal stock price follows a geometric Brownian motion

$$\frac{dS}{S} = (R_f + \sigma_S \lambda_S)\, dt + \sigma_S dz_S, \tag{3}$$

where $R_f$ is the nominal interest rate and $\lambda_S$ is the nominal risk premium. Both the instantaneous real riskless rate $r$ and the instantaneous expected rate of inflation $\pi$ follow Ornstein-Uhlenbeck processes

$$dr = \kappa\left(\bar{r} - r\right) dt + \sigma_r dz_r, \tag{4}$$

$$d\pi = \alpha\left(\bar{\pi} - \pi\right) dt + \sigma_\pi dz_\pi. \tag{5}$$

The total expected return on owner-occupied housing is the expected house price appreciation plus a convenience yield representing the benefits from the housing services. We refer to this convenience yield as imputed rent and denote it by $r^{imp}$. We assume that the imputed rent is a constant fraction of the house value.[4] The nominal price process for the house price follows a geometric Brownian motion

$$\frac{dQ}{Q} + r^{imp} dt = (R_f + \sigma_Q \lambda_Q)\, dt + \sigma_Q dz_Q. \tag{6}$$

where $\sigma_Q \lambda_Q$ is the premium for the financial risk of exposure to the $\sigma_Q dz_Q$ shocks. We can rewrite and orthogonalize this equation as

$$\frac{dQ}{Q} = \left(R_f + \theta'\lambda - r^{imp}\right) dt + \theta' dz, \tag{7}$$

where $z = (z_S, z_r, z_\pi, z_v)$ and $z_v$ is orthogonal to $z_S$, $z_r$ and $z_\pi$. In addition, $\theta = (\theta_S, \theta_r, \theta_\pi, \theta_v)'$ are the loadings on the various Brownian motions and $\lambda = (\lambda_S, \lambda_r, \lambda_\pi, \lambda_v)'$ is the vector of nominal risk premia on the sources of uncertainty. The price level follows

$$\frac{d\Pi}{\Pi} = \pi dt + \sigma_\Pi dz_\Pi = \pi dt + \xi' dz + \xi_u dz_u, \tag{8}$$

where $\xi = (\xi_S, \xi_r, \xi_\pi, \xi_v)'$ and $dz_u$ is orthogonal to $dz$.

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[4]Flavin and Yamashita (2002) also specify the imputed rent as a constant fraction of the house value, but it has a slightly different interpretation. In their mean-variance set-up it reflects the monetary value of the utility an individual derives from the housing services. In contrast, in our case it represents the return differential between the house and a (hypothetical) pure financial asset with comparable risk characteristics, as determined by the market.

Defining the covariance matrix of $dz$ by

$$\rho = \begin{pmatrix} \rho_{S,r,\pi} & 0 \\ 0 & 1 \end{pmatrix},$$

we have $\sigma_Q^2 = \theta'\rho\theta$ and $\sigma_\Pi^2 = \xi'\rho\xi + \xi_u^2$.

We will sometimes use the real risk premia given by $\lambda^* = \lambda - \rho\xi$ and $\lambda_u^* = \lambda_u - \xi_u$. For notational convenience we introduce the notation $\phi = (\phi_S, \phi_r, \phi_\pi, \phi_v)' \equiv -\rho^{-1}\lambda^*$ and $\phi_u = -\lambda_u^*$. For an interpretation of these parameters, recall from BX that the real pricing kernel, $M$, evolves as

$$\frac{dM}{M} = -rdt + \phi'dz + \phi_u dz_u. \tag{9}$$

We assume that the available assets are nominal bonds with different maturities (including an instantaneous bond which will be referred to as cash), stocks and the nontradable house. Also notice that we assume that there are no assets available whose nominal return have nonzero loading on $dz_u$, and therefore that there are no inflation-indexed assets available. We also assume that there are no tradable financial assets available whose nominal return have nonzero loading on $dz_v$, i.e. no house price dependent contracts. This means that we do not need information on $\lambda_v$ and $r^{imp}$ separately because they are only relevant for the expected house price appreciation. That is, information on $\theta_v\lambda_v - r^{imp}$ is sufficient. Only in Theorem III where we consider the special $\theta_v = 0$ case, we can illustrate the implications of the imputed rent in isolation.

BX show that the nominal price at time $t$ of a discount bond maturing at time $T$, denoted as $P_{tT}$, satisfies

$$\frac{dP}{P} = \left(R_f - B\sigma_r\lambda_r - C\sigma_\pi\lambda_\pi\right)dt - B\sigma_r dz_r - C\sigma_\pi dz_\pi, \tag{10}$$

where $B_{tT} = \kappa^{-1}\left(1 - e^{-\kappa(T-t)}\right)$, $C_{tT} = \alpha^{-1}\left(1 - e^{-\alpha(T-t)}\right)$, and $R_f = r + \pi - \xi'\lambda - \xi_u\lambda_u$ is the return on the instantaneous nominal risk free asset (cash). Using Ito's lemma one can show that the real return on a nominal bond is given by

$$\frac{dp}{p} = \left(r - B\sigma_r\lambda_r^* - C\sigma_\pi\lambda_\pi^* - \xi'\lambda^* - \xi_u\lambda_u^*\right)dt - B\sigma_r dz_r - C\sigma_\pi dz_\pi - \xi'dz - \xi_u dz_u. \tag{11}$$

A first point to notice is that the real return on a nominal bond of a given maturity has a fixed real risk premium. In particular it does not depend on the expected inflation rate, $\pi$. It is straightforward to show that the same holds for the real risk premia on stocks and the

house. Since utility is defined over real wealth this implies that the investor's indirect utility function will depend on the real riskless interest rate, $r$, but not on the current expected rate of inflation, $\pi$. A second point to notice is that the return processes of bonds with different maturities differ only in their loadings on $dz_r$ and $dz_\pi$. When there are no constraints on position size, any desired combination of loadings on $dz_r$ and $dz_\pi$ can be accomplished by positions in any two bonds with different maturities. In the unconstrained case we therefore first characterize optimal portfolio choice by optimal allocation to factor assets, whose nominal return has a nonzero loading on exactly one factor (source of uncertainty). Only thereafter we choose two particular bond maturities and describe the optimal portfolio choice in terms of the weights on these bonds.

## 2.3 Optimal portfolio choice

Since utility is defined over real wealth and all available assets have constant real risk premia over the real riskless asset, we can choose $w$, $h$, $r$, and $t$ as state variables, where $h \equiv w^H / \left( w^F + w^H \right)$ is the housing to total wealth ratio. Here we use that $\left( w^F, w^H \right)$ maps one-to-one to $(w, h)$, where as before $w \equiv w^F + w^H$ is total real wealth. Notice also that there are no financial assets available whose nominal return has nonzero loading on $dz_u$. Given these restrictions on the menu of assets, the evolution of real wealth is given by

$$
\begin{aligned}
dw/w &= \left[ r + \mu_w^e \left( x, h \right) - \xi_u \lambda_u^* \right] dt + \sigma_w' \left( x, h \right) dz - \xi_u dz_u, \qquad (12) \\
\mu_w^e \left( x, h \right) &\equiv \left( 1 - h \right) \mu_F^e \left( x \right) + h \mu_q^e, \\
\sigma_w' \left( x, h \right) &\equiv \left( 1 - h \right) \sigma_F' \left( x \right) + h \sigma_q',
\end{aligned}
$$

where $x$ are fractions of financial wealth invested in available assets. The $\mu^e$ variables denote expected excess returns, for example $\mu_q^e = \left( \theta - \xi \right)' \lambda^* - r^{imp}$, and $\sigma_q = \theta - \xi$. $\mu_F^e$ and $\sigma_F$ are simple linear functions of $x$. We can now prove a theorem that will turn out to be very useful.

**Theorem I**

If the set of admissible portfolio weights, $A$, is independent of $w_t$ and $r_t$, then the indirect utility function can be written as

$$J(w, h, r, t) \equiv \frac{w_t^{1-\gamma}}{1-\gamma} * \nu_H * \exp\left\{(1-\gamma)(r_t - \bar{r}) B_{tT}\right\} *$$
$$\exp\left\{(1-\gamma)\left(-\xi_u \lambda_u^* - \frac{\gamma}{2}\xi_u^2\right)(T-t)\right\} * I(h_t, t), \qquad (13a)$$

where $I$ satisfies

$$I(h_t, t) = \min_{x \in A} E_t\left[\left(\exp\left\{\int_t^T \left(\tilde{r}_s + \mu_w^e - \frac{1}{2}\sigma_w'\rho\sigma_w\right)ds + \int_t^T \sigma_w'dz\right\}\right)^{1-\gamma}\right] \quad (13b)$$
$$\text{s.t.}$$
$$dh = h\left[\left(\mu_q^e - \mu_w^e\right)dt + \left(\sigma_q' - \sigma_w'\right)dz\right] - h\sigma_w'\rho\left(\sigma_q - \sigma_w\right)dt \qquad (13c)$$
$$d\tilde{r} = \kappa(\bar{r} - \tilde{r})dt + \sigma_r dz_r \qquad (13d)$$
$$\tilde{r}_t = \bar{r} \qquad (13e)$$

with $I(h, T) = 1$ for all $h$ and where $B_{tT}$ was defined before.

**Proof**: see Appendix A.

The fact that indirect utility is separable in wealth is a well-known consequence of power utility. It is more surprising that it is also separable in the real interest rate. The assumption that the variance of increments in $r$ is independent of the level of $r$ is key for this to hold. Notice that we have not yet specified whether short positions in available assets are possible or not. We only assumed that restrictions do not depend on $w_t$ and $r_t$.

Theorem I has two important implications for financial portfolio choice. First, financial portfolio choice is independent of the current value of real wealth, $w_t$, and the current value of the real interest rate, $r_t$. Second, it implies that the degree of market incompleteness caused by the lack of financial assets with nominal returns with nonzero loading on $dz_u$, as measured by $\xi_u$, has no impact on the financial asset allocation. The reason is that $dz_u$ is orthogonal to $dz$ and that the financial asset allocation does not influence the future degree of market incompleteness due to the lack of inflation-indexed assets.

We now derive the expressions for the asset allocations. We assume that there are no assets with non-zero loading on $dz_v$ and $dz_u$, i.e. no inflation-indexed assets and no house price dependent contracts. With the available assets we can get any combination of loadings

on $dz_S$, $dz_r$ and $dz_\pi$. More precisely, we assume that there are four financial assets available whose instantaneous variance-covariance matrix has rank three. Stocks, two bonds with different maturities and cash would be obvious choices to achieve this. If we assume that the investor can unconstrained allocate fractions $x_S$, $x_r$ and $x_\pi$ of his financial wealth to three factor assets whose nominal returns have stochastic components $\sigma_S dz_S$, $\sigma_r dz_r$ and $\sigma_\pi dz_\pi$ respectively, and allocate a fraction $1 - x_S - x_r - x_\pi$ to cash, then we can derive an implicit expression for the optimal asset allocation.[5]

**Theorem II**

Let the set of admissible portfolio weights, $A$, be such that effectively the investor can allocate unconstrained fractions $x_S$, $x_r$ and $x_\pi$ of his financial wealth to three factor assets whose nominal returns have stochastic components $\sigma_S dz_S$, $\sigma_r dz_r$ and $\sigma_\pi dz_\pi$ respectively, and allocate a fraction $1 - x_S - x_r - x_\pi$ to cash. The optimal fractions are given by

$$x_S = -\frac{J_{w^F}}{w^F J_{w^F w^F}} \gamma \left[ \frac{1}{\gamma} \frac{\xi_S - \phi_S}{\sigma_S} + \left(1 - \frac{1}{\gamma}\right) \frac{\xi_S}{\sigma_S} \right] - \frac{J_{w^F w^H}}{J_{w^F w^F}} \frac{h}{1-h} \frac{\theta_S}{\sigma_S}, \tag{14a}$$

$$x_r = -\frac{J_{w^F}}{w^F J_{w^F w^F}} \gamma \left[ \frac{1}{\gamma} \frac{\xi_r - \phi_r}{\sigma_r} + \left(1 - \frac{1}{\gamma}\right) \left(\frac{\xi_r}{\sigma_r} - B_{tT}\right) \right] - \frac{J_{w^F w^H}}{J_{w^F w^F}} \frac{h}{1-h} \frac{\theta_r}{\sigma_r}, \tag{14b}$$

$$x_\pi = -\frac{J_{w^F}}{w^F J_{w^F w^F}} \gamma \left[ \frac{1}{\gamma} \frac{\xi_\pi - \phi_\pi}{\sigma_\pi} + \left(1 - \frac{1}{\gamma}\right) \frac{\xi_\pi}{\sigma_\pi} \right] - \frac{J_{w^F w^H}}{J_{w^F w^F}} \frac{h}{1-h} \frac{\theta_\pi}{\sigma_\pi}, \tag{14c}$$

where partial derivatives are determined using $w = w^F + w^H$, $h = w^H / \left(w^F + w^H\right)$ and applying the chain rule.

**Proof**: see Appendix B.

The intuition for these portfolio weights is fairly simple. The asset positions consist of two components. The expression in square brackets is exactly the same as the long-term investment portfolio derived by Brennan and Xia (2002). The first term of this BX portfolio can be seen as a position in the nominal mean-variance tangency portfolio. The second term is the projection of an inflation-indexed bond with maturity $T$ on $dz$, which is the best possible hedge against unexpected inflation plus a hedge against real interest changes, captured by $B_{tT}$. This BX portfolio is pre-multiplied by the ratio of the coefficient of relative risk aversion associated with total wealth changes $\gamma$, which equals $-w J_{ww}/J_w$, to the coefficient of relative risk aversion associated with financial wealth changes $-w^F J_{w^F w^F}/J_{w^F}$.

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[5] The authors thank Yihong Xia for pointing out to us that solving for an implicit expression is informative and that it boils down to solving a system of three linear equations in three unknowns.

The correction factor takes into account that increases in total and financial wealth have different consequences for $h$ and therefore different utility implications.[6] For our parameter values this correction factor will generally be smaller than one. It captures the well-known effect that in the presence of an illiquid asset an investor effectively behaves more risk averse in his liquid asset allocation.[7]

The second component is a hedge term, which arises when the financial asset return and the housing return are correlated. Taking into account the relative size of financial and housing wealth, a one-for-one hedge against non-idiosyncratic house risk would give $(h/(1-h))\theta_i/\sigma_i$. However, changes in financial and housing wealth have different consequences for $h$, and therefore different utility implications. This gives rise to the correction factor $J_{w^F w^H}/J_{w^F w^F}$.

The correction factors for the long-term portfolio and the hedge portfolio are related and can be further worked out. This is shown in Corollary I.

**Corollary I**

From equations (14a)-(14c) in Theorem II we can derive the following expression

$$(1-h)\,x_S \;=\; (1-h+\omega h)\left[\frac{1}{\gamma}\frac{\xi_S-\phi_S}{\sigma_S}+\left(1-\frac{1}{\gamma}\right)\frac{\xi_S}{\sigma_S}\right]-\omega h\frac{\theta_S}{\sigma_S}, \tag{15a}$$

$$(1-h)\,x_r \;=\; (1-h+\omega h)\left[\frac{1}{\gamma}\frac{\xi_r-\phi_r}{\sigma_r}+\left(1-\frac{1}{\gamma}\right)\left(\frac{\xi_r}{\sigma_r}-B_{tT}\right)\right]-\omega h\frac{\theta_r}{\sigma_r}, \tag{15b}$$

$$(1-h)\,x_\pi \;=\; (1-h+\omega h)\left[\frac{1}{\gamma}\frac{\xi_\pi-\phi_\pi}{\sigma_\pi}+\left(1-\frac{1}{\gamma}\right)\frac{\xi_\pi}{\sigma_\pi}\right]-\omega h\frac{\theta_\pi}{\sigma_\pi}, \tag{15c}$$

where $\omega\left(h,\tau\right)=1+\frac{\gamma I_h+h I_{hh}}{\gamma(1-\gamma)I-2\gamma h I_h-h^2 I_{hh}}$.

**Proof**: see Appendix B.

These equations show that the investor behaves *as if* the value of the house is different from the prevailing price in the market. The investor acts as if his house is worth $\omega h W$. We refer to $\omega h$ as the effective housing to total wealth ratio. This means that an investor acts as if the value of his assets is effectively only a fraction $1-h+\omega h$ of total wealth. We will refer to this ratio as the effective to total wealth ratio. Another point to notice is that

---

[6] Here a change in total wealth means a wealth change leaving the housing to total wealth ratio, $h$, the same. That is, a \$1 increase in $w$ corresponds to a \$h increase $w^H$ and a \$1 − h increase $w^F$. In contrast, a change in financial wealth does affect $h$.

[7] See e.g. Grossman and Laroque (1990).

financial wealth is a fraction $1 - h$ of total wealth, so that the financial portfolio should be leveraged up by a factor $1/(1 - h)$ to get the desired exposure for the total portfolio. For expositional easy the leverage factor is put on the left-hand side of equations (15a)-(15c). Observe that there are two distinct horizon effects. First, $B_{tT}$ captures the horizon dependent hedge against changes in the real interest rate. Second, as we will show below, the effective housing to total wealth ratio changes substantially with horizon. Both effects make the asset allocation change over time. In addition, with a fixed position in the house, the housing to total wealth ratio $h$ is stochastic and generates time-varying asset allocations.

For the parameter choice presented in section 3, we find that $\omega$ is between zero and one, and declining with the investment horizon. In Figure I we plot $\omega$ as a function of horizon for a $\gamma = 3$ investor (Panel A) and a $\gamma = 7$ investor (Panel B) for various housing to total wealth ratios. Comparing panel A and B, notice the intuitive result that the effective housing wealth, and therefore $\omega$, is lower for the more risk-averse $\gamma = 7$ investor (ceteris paribus).

To further illustrate the effective housing to total wealth ratio we provide an explicit closed-form solution for $\omega$ in the special case that the house price risk is spanned by available assets, i.e. $\theta_v = 0$.

**Theorem III**

If (i) the nominal housing return is perfectly hedgeable, i.e. $\theta_v = 0$ and (ii) the investment opportunity set, $A$, is such that effectively the investor can allocate unconstrained fractions $x_S$, $x_r$ and $x_\pi$ of his financial wealth to three factor assets whose nominal returns have stochastic components $\sigma_S dz_S$, $\sigma_r dz_r$ and $\sigma_\pi dz_\pi$ respectively, and allocate a fraction $1 - x_S - x_r - x_\pi$ to cash, then

$$
\begin{aligned}
I\left(h, t\right) \quad = \quad & [1 - h + \omega h]^{1-\gamma} \exp\{(1 - \gamma)\,[\bar{r}\tau + \frac{1}{2}\frac{1}{\gamma}\phi'\rho\phi\tau + \\
& \frac{1}{\kappa}\left(1 - \frac{1}{\gamma}\right)\sigma_r\phi'\rho e_2\left(\tau - B\right) - \frac{1}{4\kappa^3}\left(1 - \frac{1}{\gamma}\right)\sigma_r^2\left(2\kappa\tau - 3 + 4e^{-\kappa\tau} - e^{-2\kappa\tau}\right)]\}
\end{aligned}
\tag{16}
$$

and $\omega = e^{-r^{imp}\tau}$.

**Proof**: see Appendix C.

In Theorem III the only market incompleteness is the absence of inflation-indexed assets. Notice that the real time-$t$ value of a unit of housing to be delivered at time $T$ is

**Figure I:** $\omega(h, \tau)$ **for unconstrained financial asset allocation (** $\gamma = 3$ **and** $\gamma = 7$ **).**
The figure shows $\omega(h, \tau)$ as a function of horizon for various housing to total wealth ratios.
We use the parameter values presented in section 3.
Panel A: the investor has risk aversion $\gamma = 3$



Panel B: the investor has risk aversion $\gamma = 7$



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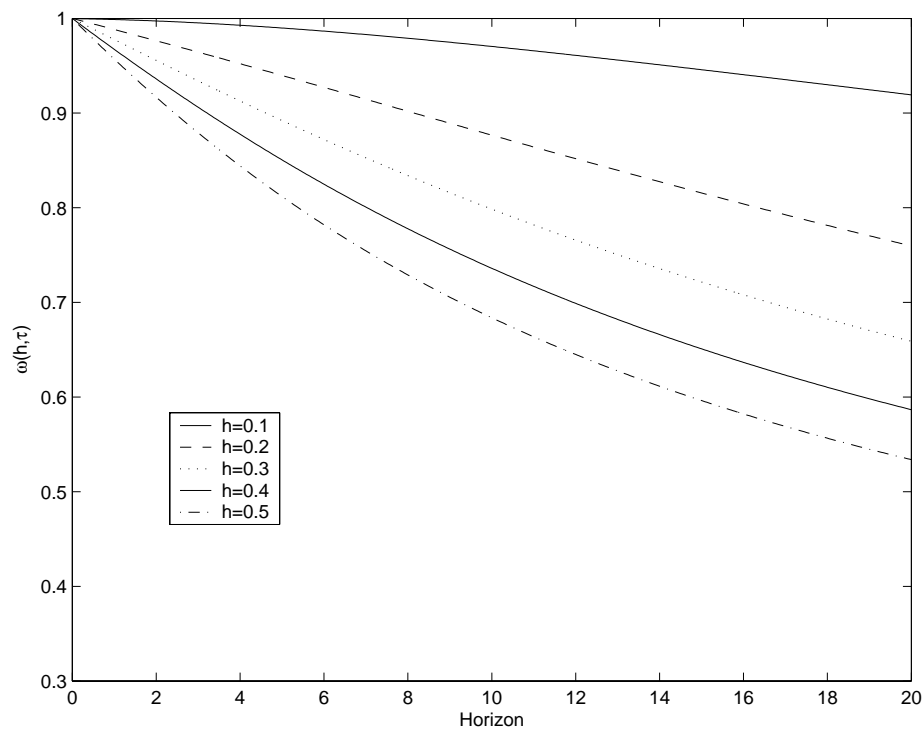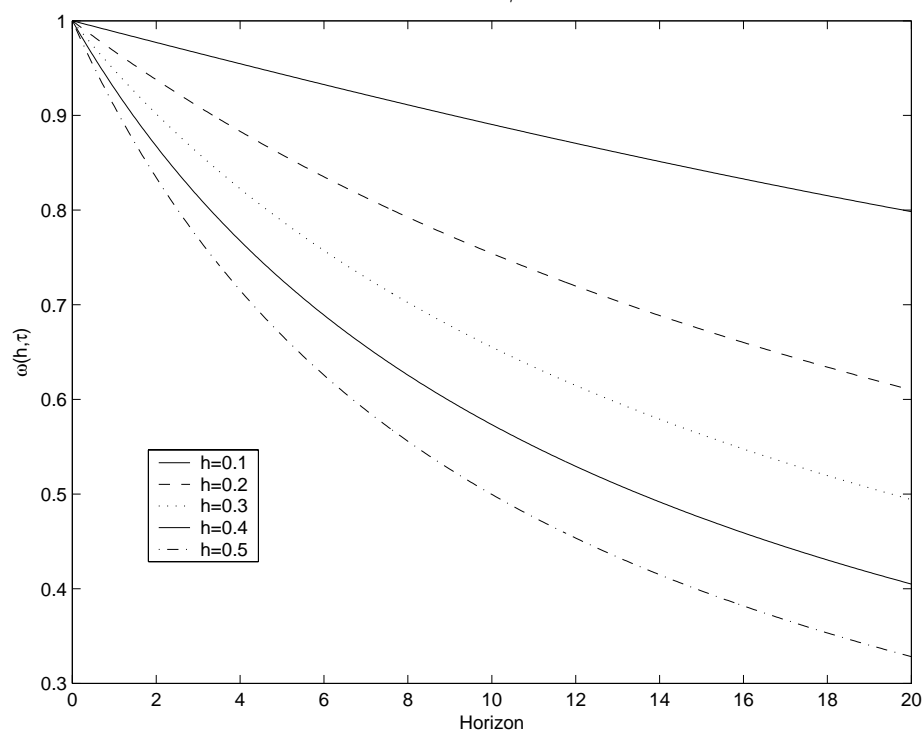
$e^{-r^{imp}(T-t)}q_t < q_t$ for a positive convenience yield, $r^{imp}$. This means that $\omega = e^{-r^{imp}(T-t)}$. Consistent with Figure I, the effective to total wealth ratio is smaller than one and declining in horizon. This makes sense since the house has a lower expected return than a portfolio of pure financial assets with the same risk characteristics. The longer the horizon, the lower the effective financial value of the house. In case $\theta_v \neq 0$, it is easy to see that the functional form $\omega = e^{-r^{imp}\tau}$ again obtains if a 'house price dependent asset' is available, that has a nonzero loading on $dz_v$, zero loading on $dz_u$, arbitrary loadings on $dz_S$, $dz_r$, $dz_\pi$, and a constant real risk premium.

In Section 4 and 5 we will use numerical techniques to evaluate optimal asset allocation for the general case where no house price dependent asset exists and $\theta_v \neq 0$. Then effective housing wealth differs from the market value of the house not only because of the imputed rent, but also because housing involves an exposure to unhedgeable, idiosyncratic house risk. Assuming unconstrained allocations to available assets are possible, we decompose the numerical solution into the three components of Theorem II. Subsequently, we also investigate the optimal asset allocation when there are short-sale constraints and when the investor can take a mortgage loan.

For the numerical results we continue to assume that the investment opportunity set, $A$, is independent of $w_t$ and $r_t$. In this case, we can use Theorem I to see that the only part of the indirect utility function that is not known in closed form is $I(h, t)$. We know that $I(h, T) = 1$ for all $h$. A grid over $h$ and $t$ is chosen and we solve for $I(h, t)$ and the optimal asset allocation backwards in time. More precisely, without loss of generality at node $(h, t)$ we normalize $w_t = 1$ and $r_t = \bar{r}$, and set $\xi_u = \lambda_u^* = 0$. The latter reduces the number of Brownian motions from five to four in the numerical procedure. Thus we determine $I(h, t)$ by solving

$$I(h_t, t) = \max_{x \in A} E\left[ \frac{w_{t+dt}^{1-\gamma}(x)}{1-\gamma} e^{(\tilde{r}_{t+dt}-\bar{r})B_{t+dt,T}} I(h_{t+dt}(x), t+dt) \,|\, w_t = 1, \tilde{r}_t = \bar{r}, h_t \right] \quad (17)$$

where $dt$ is the step size of the grid over time.[8]

---

[8] To determine $I(h, t+dt)$ for values of $h$ that are not on the grid, we use cubic spline interpolation. The expectation is evaluated using Gaussian quadrature with 5-points for the unconstrained portfolio choice and 3-points for the constrained portfolio choice. Increasing the number of points did not alter results in the presented precision. For the optimization over $x$ we use a search algorithm that does not use any derivative information and is robust to different starting values. The grid on $h$ and $t$ is chosen fine enough to ensure precision up to the presented number of decimals.

# 3   Calibration

To illustrate the impact of an owner-occupied house on the financial portfolio and mortgage choice for different horizons and housing to total wealth ratios, we calibrate the model parameters to quarterly data on stock returns, inflation, T-bill rates, long bond yields, and house price returns.

We first estimate a term structure model on quarterly data on nominal interest rates and inflation from 1973Q1 to 2003Q4. We use a Kalman filter to extract the real interest rate and expected inflation rate from the data, and estimate the model by Quasi Maximum Likelihood.[9] This procedure provides estimates of the mean reversion parameters and provides time series of innovations in the real interest rate and expected inflation, and a time series of unexpected inflation. The values for the mean reversion parameters of real interest and expected inflation rate, $\kappa = 0.6501$ and $\alpha = 0.0548$, imply half-lives of 1.1 and 12.6 years respectively.[10]

In the second step of the calibration, we fit the means, standard deviations and correlations of stock returns, real interest rates, expected and unexpected inflation and house prices, and the market prices of risk. The sample period for this second step is limited due to the availability of house price data, and runs from 1980Q2 until 2003Q4. The reason to estimate the mean reversion parameters over a longer sample period than the other parameters is that we need a long sample to obtain good estimates of the mean reversions; all the other parameters are best fitted to the more recent common sample period, taking the estimated mean reversions from the first step as given. Table I provides all the (annualized) parameter estimates.

We now give some more detail on the second step of the calibration process. To estimate the stock return process we use quarterly stock returns on an index comprising all NYSE, AMEX and NASDAQ firms.[11] Following Fama and French (2002), amongst others, we believe that the equity premium, $\sigma_S \lambda_S$ in our model, is lower than the realized premium when measured over the past few decades. While the realized excess return in our data is 6.4%, we set $\lambda_S$ such that the equity premium is 4.0%.

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[9] Details on the procedure are provided in Appendix D.

[10] Using different sample periods, Brennan and Xia (2002) and Campbell and Viceira (2001) also find a half-life of around 1 year for innovations in the real rate and a much longer half-life for expected inflation.

[11] The authors would like to thank Kenneth R. French for making this data available at his website.

## Table I. Choice of model parameters.

The table reports calibrated parameter values for the dynamics of the asset prices, the inflation rate and real interest rate (as described section 2.2). The parameter values are obtained using quarterly data.

| Parameter | Estimate |
|---|---|
| Stock return process: $dS/S = (R_f + \sigma_S \lambda_S)\,dt + \sigma_S dz_S$ | |
| $\sigma_S$ | 0.1748 |
| $\lambda_S$ | 0.2288 |
| Real riskless interest rate process: $dr = \kappa(\bar{r} - r)\,dt + \sigma_r dz_r$ | |
| $\bar{r}$ | 0.0226 |
| $\kappa$ | 0.6501 |
| $\sigma_r$ | 0.0183 |
| $\lambda_r$ | −0.3035 |
| Expected inflation process: $d\pi = \alpha(\bar{\pi} - \pi)\,dt + \sigma_\pi dz_\pi$ | |
| $\bar{\pi}$ | 0.0351 |
| $\alpha$ | 0.0548 |
| $\sigma_\pi$ | 0.0191 |
| $\lambda_\pi$ | −0.1674 |
| House price process: $dQ/Q = \left(R_f + \sigma_Q \lambda_Q - r^{imp}\right)dt + \sigma_Q dz_Q = \left(R_f + \theta'\lambda - r^{imp}\right)dt + \theta'dz$ | |
| $\theta_S$ | 0.0077 |
| $\theta_r$ | 0.0198 |
| $\theta_\pi$ | 0.0295 |
| $\theta_v$ | 0.1465 |
| $\theta_v \lambda_v - r^{imp}$ | 0.0038 |
| $\sigma_Q$ | 0.1500 |
| $\sigma_Q \lambda_Q - r^{imp}$ | −0.0054 |
| Realized inflation process: $d\Pi/\Pi = \pi dt + \sigma_\Pi dz_\Pi = \pi dt + \xi'dz + \xi_u dz_u$ | |
| $\xi_S$ | −0.0033 |
| $\xi_r$ | 0.0067 |
| $\xi_\pi$ | 0.0012 |
| $\xi_v$ | −0.0184 |
| $\xi_u$ | 0.0497 |
| $\sigma_\Pi$ | 0.0535 |
| Correlations: | |
| $\rho_{Sr}$ | −0.1643 |
| $\rho_{S\pi}$ | 0.0544 |
| $\rho_{r\pi}$ | −0.2323 |

## Table II Correlation matrix for $(dz_S, dz_r, dz_\pi, dz_Q, dz_\Pi)'$

| | $dz_S$ | $dz_r$ | $dz_\pi$ | $dz_Q$ | $dz_\Pi$ |
|---|---|---|---|---|---|
| $dz_s$ | 1 | | | | |
| $dz_r$ | −0.1643 | 1 | | | |
| $dz_\pi$ | 0.0544 | −0.2323 | 1 | | |
| $dz_Q$ | 0.0402 | 0.0781 | 0.1686 | 1 | |
| $dz_\Pi$ | −0.0809 | 0.1294 | −0.0090 | −0.3251 | 1 |

For the interest rate and inflation part of the model, the first step already provides us with the mean reversion parameters. The other parameters are estimated as follows. The mean expected inflation is estimated by the mean increase of the CPI. For the mean real interest rate we take the difference between the means of the T-bill rate and the expected inflation, minus a 0.5% correction to reflect the premium on unexpected inflation.[12] The standard deviations of the real interest rate, expected inflation and unexpected inflation are determined using the time series generated by the Kalman filter.[13] We estimate the risk premia $\lambda_r$ and $\lambda_\pi$ by matching the average yields of two bond portfolios with a constant time to maturity of 3.4 and 10.4 years. For this we use formulas derived by Brennan and Xia (2002), Appendix A.

We estimate the house price process using repeated sales data at the city level for Atlanta, Boston, Chicago and San Francisco from 1980Q2 to 2003Q4. There are quite some differences in the price processes for the four cities. We choose to focus on general house price characteristics and constructed a nation-wide return index by weighing the cities equally. Case and Shiller (1989) argue that the standard deviation of individual house price changes are close to 15%, like individual stocks. Because price changes of different houses are far from perfectly correlated, aggregation leads to a considerable reduction of the variability. In our nation-wide index we find a standard deviation of 2.67%. Since we are interested in the dynamics of an individual house, we correct this series by simply scaling house price shocks with a factor $15.00\%/2.67\% = 5.6$ around its mean.

Finally, we estimate the correlation matrix $\rho$ and the coefficient vectors $\xi$ and $\theta$ using quarterly stock returns, house price returns, the innovations in the real interest rate, expected inflation, and unexpected inflation. We have data on house prices, but not on imputed rent. Therefore we calibrate $\theta_v \lambda_v - r^{imp}$ and not $\lambda_v$ and $r^{imp}$ separately. As discussed in section 2 this is sufficient to determine the optimal asset allocation. Table II provides the implied correlation matrix of the stochastic vector $(dz_S, dz_r, dz_\pi, dz_Q, dz_\Pi)'$.

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[12] The 50 basis points unexpected inflation risk premium is based on the estimate of Campbell and Viceira (2002, p.72)). With this assumption there is no further need to estimate the market price of risk for unexpected inflation, $\lambda_u$, because it does not influence the asset allocation in our set-up with only nominal securities (it does however influence indirect utility).

[13] The discrete-time standard deviations are converted to the continuous-time counterparts, incorporating the effect of mean reversion in the processes.

# 4 Portfolio Choice without Mortgage

In this section we first present the unconstrained optimal portfolio choice for a moderately risk-averse investor ($\gamma = 3$). We do this first in terms of factor assets and split the solution into the three components discussed in section 2. Thereafter we translate this to portfolio choice in terms of available assets. Equipped with the intuition of the unconstrained case, we tackle the portfolio allocation problem for an investor who is constrained to holding nonnegative positions in available assets. Here we consider a moderately risk-averse investor ($\gamma = 3$) as well as a more risk-averse investor ($\gamma = 7$). Finally we investigate how sensitive our results are to alternative parameters for the house price process.

## 4.1 Unconstrained portfolio choice

We present the optimal portfolio for the situation where there are no financial assets whose nominal return has nonzero loading on $dz_v$ and $dz_u$. That is, there is no house price dependent contract nor an inflation-indexed security. Table III shows the optimal allocation to stocks, real interest and expected inflation factor assets, whose nominal returns have stochastic components $\sigma_S dz_S$, $\sigma_r dz_r$ and $\sigma_\pi dz_\pi$ respectively. We use the parameter values presented in Table I. Panel A shows the allocation as fraction of financial wealth, i.e. the sum corresponds to $x_S$, $x_r$ and $x_\pi$ respectively. Panel B shows the allocation as fraction of total wealth, i.e. the sum corresponds to $(1 - h) x_S$, $(1 - h) x_r$ and $(1 - h) x_\pi$ respectively. The table also shows the optimal allocation split into the three components given in Theorem II.

The first component comprises the positions in the mean-variance tangency portfolio. The fraction allocated to the real interest factor asset and the expected inflation factor asset are much larger in absolute terms than the fraction allocated to the stocks factor asset. The main reason is that the returns on the real interest and expected inflation factor asset both have a relatively low standard deviation, resulting in large investments to obtain the optimal risk exposure. As a fraction of total wealth at the 1-month horizon the allocation is the same for all housing to total wealth ratios, but as a fraction of financial wealth there is a leverage effect. All position sizes decrease in horizon because the effective housing to total wealth ratio and therefore the effective to total wealth ratio decreases in horizon. This horizon effect is more profound for a larger housing to total wealth ratio, as $\omega$ is smaller for larger housing to total wealth ratios (see Figure 1). This can be understood from the fact that the risk of the fixed house position can only be hedged partially. As the house size

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## Table III. Unconstrained factor asset allocation ($\gamma = 3$).

The table presents the optimal allocation to stocks, real interest and expected inflation factor assets, whose nominal returns have stochastic components $\sigma_S dz_S$, $\sigma_r dz_r$ and $\sigma_\pi dz_\pi$ respectively. Panel A shows the allocation as fraction of financial wealth, i.e. the sum corresponds to $x_S$, $x_r$ and $x_\pi$ respectively. Panel B shows the allocation as fraction of total wealth, i.e. the sum corresponds to $(1-h)\,x_S$, $(1-h)\,x_r$ and $(1-h)\,x_\pi$ respectively. The table also shows the optimal allocation split into the three components given in Theorem II. The investor has risk aversion $\gamma = 3$.

Panel A: as fraction of financial wealth

| Horizon | Factor Asset | h = 0.2 Components 1st | 2nd | 3rd | Sum | h = 0.4 Components 1st | 2nd | 3rd | Sum |
|---|---|---|---|---|---|---|---|---|---|
| 1 month | Stocks | 0.45 | −0.02 | −0.01 | 0.42 | 0.60 | −0.02 | −0.03 | 0.55 |
| | Real interest | −7.55 | 0.24 | −0.27 | −7.58 | −10.06 | 0.32 | −0.72 | −10.46 |
| | Exp. inflation | −5.56 | 0.05 | −0.39 | −5.89 | −7.40 | 0.07 | −1.03 | −8.36 |
| 5 years | Stocks | 0.44 | −0.02 | −0.01 | 0.42 | 0.56 | −0.02 | −0.03 | 0.52 |
| | Real interest | −7.46 | −0.92 | −0.25 | −8.63 | −9.47 | −1.16 | −0.61 | −11.25 |
| | Exp. inflation | −5.49 | 0.05 | −0.36 | −5.80 | −6.97 | 0.07 | −0.88 | −7.78 |
| 20 years | Stocks | 0.43 | −0.02 | −0.01 | 0.40 | 0.50 | −0.02 | −0.02 | 0.46 |
| | Real interest | −7.19 | −0.93 | −0.21 | −8.33 | −8.41 | −1.09 | −0.42 | −9.92 |
| | Exp. inflation | −5.29 | 0.05 | −0.29 | −5.53 | −6.18 | 0.06 | −0.60 | −6.73 |

Panel B: as fraction of total wealth

| Horizon | Factor Asset | h = 0.2 Components 1st | 2nd | 3rd | Sum | h = 0.4 Components 1st | 2nd | 3rd | Sum |
|---|---|---|---|---|---|---|---|---|---|
| 1 month | Stocks | 0.36 | −0.01 | −0.01 | 0.34 | 0.36 | −0.01 | −0.02 | 0.33 |
| | Real interest | −6.04 | 0.19 | −0.22 | −6.07 | −6.04 | 0.19 | −0.43 | −6.28 |
| | Exp. inflation | −4.44 | 0.04 | −0.31 | −4.71 | −4.44 | 0.04 | −0.62 | −5.01 |
| 5 years | Stocks | 0.35 | −0.01 | −0.01 | 0.33 | 0.34 | −0.01 | −0.02 | 0.31 |
| | Real interest | −5.97 | −0.73 | −0.20 | −6.91 | −5.68 | −0.70 | −0.37 | −6.75 |
| | Exp. inflation | −4.39 | 0.04 | −0.29 | −4.64 | −4.18 | 0.04 | −0.53 | −4.67 |
| 20 years | Stocks | 0.34 | −0.01 | −0.01 | 0.32 | 0.30 | −0.01 | −0.01 | 0.28 |
| | Real interest | −5.75 | −0.74 | −0.16 | −6.66 | −5.04 | −0.65 | −0.25 | −5.95 |
| | Exp. inflation | −4.23 | 0.04 | −0.23 | −4.43 | −3.71 | 0.04 | −0.36 | −4.04 |

## Table IV. Unconstrained financial portfolio choice ($\gamma = 3$).

The table presents optimal financial portfolio weights for stocks, bonds with maturities of 5 and 20 years, and cash. The investor has risk aversion $\gamma = 3$.

| Horizon | Asset | h = 0 | h = 0.1 | h = 0.2 | h = 0.3 | h = 0.4 | h = 0.5 |
|---|---|---|---|---|---|---|---|
| 1 month | Stocks | 0.35 | 0.38 | 0.42 | 0.48 | 0.55 | 0.65 |
| | 5 year bond | 5.72 | 6.46 | 7.39 | 8.57 | 10.16 | 12.37 |
| | 20 year bond | −1.70 | −1.91 | −2.17 | −2.51 | −2.97 | −3.60 |
| | Cash | −3.37 | −3.93 | −4.64 | −5.54 | −6.74 | −8.42 |
| 5 years | Stocks | 0.35 | 0.38 | 0.42 | 0.46 | 0.52 | 0.59 |
| | 5 year bond | 6.73 | 7.57 | 8.54 | 9.68 | 11.10 | 12.98 |
| | 20 year bond | −2.06 | −2.31 | −2.60 | −2.93 | −3.35 | −3.91 |
| | Cash | −4.02 | −4.64 | −5.36 | −6.21 | −7.27 | −8.66 |
| 20 years | Stocks | 0.35 | 0.38 | 0.40 | 0.43 | 0.46 | 0.51 |
| | 5 year bond | 6.77 | 7.57 | 8.26 | 8.97 | 9.81 | 10.91 |
| | 20 year bond | −2.08 | −2.31 | −2.52 | −2.73 | −2.98 | −3.30 |
| | Cash | −4.04 | −4.64 | −5.14 | −5.67 | −6.29 | −7.12 |

increases, the presence of this undiversifiable risk makes a risk-averse investor decrease his exposure to risky financial assets.

The second component comprises positions in the hedge portfolio against real interest changes and unexpected inflation. At the 1-month horizon there is mainly hedging against unexpected inflation and positions are limited. Only real interest rate changes are considerably (positively) correlated with unexpected inflation leading to a positive hedge demand for the real interest factor asset. At longer horizons the hedge against real interest changes becomes important, resulting in a substantially negative allocation to the real interest factor asset. If the real interest rate shocks would be more persistent than in our calibration (where the half life is just 1.1 year) the value would become even more negative (see e.g. Brennan and Xia (2002)). The sign for the real interest factor asset at longer horizons in this second component is negative, because a short term gain compensates for deteriorating investment opportunities when confronted with a downward shock to the real interest rate. The same remarks regarding the horizon and housing to total wealth ratio effect as in the analysis of the first component apply.

In our calibration the return on housing is positively correlated to changes in the real interest rate and expected inflation rate. The third component, the position in the portfolio that hedges the effective housing wealth, therefore involves negative values for the real interest and expected inflation factor assets. At the 1-month horizon magnitudes increase about linear in $h$ as a fraction of total wealth and more than linear in $h$ as fraction of financial wealth. The position size decreases in horizon because the effective housing wealth ratio decreases in horizon. Again, this is more profound for a larger housing to total wealth ratio.

Summed over all three components, Table III shows that in particular the allocation to the real interest factor asset (as fraction of financial wealth denoted by $x_r$) is large and negative. Note that Theorem II implies that for a more risk-averse investor (say $\gamma = 7$ instead of $\gamma = 3$), the positions in the first component become (a factor 7/3) smaller and the positions in the second component become (a factor 9/7) larger. In this case the size of $x_r$ would stand out even more.

In Table IV, we translate the factor asset positions to portfolio weights in financial assets. If we assume that the investor can invest unconstrained in stocks, two bonds with different maturities and cash, any combination of loadings on $dz_S$, $dz_r$ and $dz_\pi$ can be accomplished. Table IV reports the optimal total portfolio choice for various values for $h$,

when bonds of 5 and 20 years maturity are available.

To interpret the bond positions, first note that in our calibration we have $\kappa > \alpha$. That is, the mean reversion in the real interest rate is quicker than the mean reversion in the expected inflation rate. Remember that the stochastic component of the nominal return on a nominal bond is given by

$$\frac{dP_{tT}}{P_{tT}} = [...] \, dt - B_{tT}\sigma_r dz_r - C_{tT}\sigma_\pi dz_\pi.$$

Since $\kappa > \alpha$, for any horizon $T - t$ we have $0 < B_{tT} < C_{tT}$. Moreover, we have that $\frac{B_{tT}}{C_{tT}}$ is decreasing in $\tau \equiv T - t$.[14] This implies that to obtain a negative value for $x_r$ in the same order of magnitude (or even larger, in size) than $x_\pi$, one needs a long position in a short-term bond and a short position in a long-term bond. Because $B$ and $C$ are larger for longer horizons, the size of the short position will be smaller than the size of the long position. This is exactly what we see in Table IV. We also observe in Table IV that the optimal bond positions are very large in size.

## 4.2  Constrained portfolio choice

The unconstrained results in Table IV exhibit large short positions in the 20-year bond and cash. In practice, such positions can not be easily achieved for a typical investor who faces short sale constraints. Table V therefore shows the results when we constrain the fraction invested in stocks, the two bonds and cash to be positive.[15] Panel A shows the optimal portfolio for a moderately risk averse investor ($\gamma = 3$) and Panel B for a fairly risk averse investor ($\gamma = 7$).

For the moderately risk-averse investor ($\gamma = 3$), the constrained allocation to stocks approximately equals the unconstrained allocation to stocks for $h \leq 0.3$. Since almost all stockholdings in the unconstrained case originate from the first component (i.e. the

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[14]To see this notice that $\frac{d}{d\tau}\left[\frac{B_{tT}}{C_{tT}}\right] = \frac{e^{-\kappa\tau}C_{tT} - e^{-\alpha\tau}B_{tT}}{(C_{tT})^2}$. We have $e^{-\kappa\tau}C_{tT} = e^{-\kappa\tau}\int_0^\tau e^{-\alpha s} ds$ and $e^{-\alpha\tau}B_{tT} = e^{-\alpha\tau}\int_0^\tau e^{-\kappa s} ds$. Because for $s \in ]0, t[$ we have $e^{-\alpha s - \kappa\tau} < e^{-\kappa s - \alpha\tau}$ if and only if $\kappa > \alpha$, it follows that $\frac{d}{d\tau}\left[\frac{B_{tT}}{C_{tT}}\right] < 0$. For a maturity of 5 years we have: $B = 1.48$ and $C = 4.37$. For a 20 year maturity we have: $B = 1.54$ and $C = 12.15$.

[15]In the unconstrained case the available bond maturities have no impact on indirect utility as long as there are at least two different maturities available at any time. In the constrained case available maturities do matter for indirect utility. This in turn makes future available bond maturities relevant for current portfolio choice. In the remainder of this paper we assume that the maturities of the available bonds are constant. In practice this would mean that the investor can invest in two bond portfolios that are rebalanced in such a way that the duration is always 5 and 20 years.

**Table V. Constrained portfolio choice ($\gamma = 3$ and $\gamma = 7$).**

The table presents optimal financial portfolio weights for stocks, bonds with maturities of 5 and 20 years, and cash in the presence of short-sale constraints, using the base case parameter set in Table I. The investor has risk aversion $\gamma = 3$. The bond maturities are assumed to be constant over the investment period.

Panel A: the investor has risk aversion $\gamma = 3$

| Horizon | Asset | $h = 0$ | $h = 0.1$ | $h = 0.2$ | $h = 0.3$ | $h = 0.4$ | $h = 0.5$ |
|---|---|---|---|---|---|---|---|
| | Stocks | 0.35 | 0.39 | 0.43 | 0.48 | 0.51 | 0.47 |
| 1 month | 5 year bond | 0.58 | 0.47 | 0.32 | 0.14 | 0.00 | 0.00 |
| | 20 year bond | 0.07 | 0.15 | 0.25 | 0.38 | 0.49 | 0.53 |
| | Cash | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | Stocks | 0.35 | 0.38 | 0.42 | 0.46 | 0.51 | 0.52 |
| 5 years | 5 year bond | 0.61 | 0.50 | 0.38 | 0.24 | 0.07 | 0.00 |
| | 20 year bond | 0.04 | 0.11 | 0.20 | 0.30 | 0.41 | 0.48 |
| | Cash | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | Stocks | 0.35 | 0.38 | 0.40 | 0.43 | 0.45 | 0.48 |
| 20 years | 5 year bond | 0.61 | 0.52 | 0.45 | 0.37 | 0.28 | 0.19 |
| | 20 year bond | 0.04 | 0.10 | 0.15 | 0.21 | 0.27 | 0.33 |
| | Cash | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |

Panel B: the investor has risk aversion $\gamma = 7$

| Horizon | Asset | $h = 0$ | $h = 0.1$ | $h = 0.2$ | $h = 0.3$ | $h = 0.4$ | $h = 0.5$ |
|---|---|---|---|---|---|---|---|
| | Stocks | 0.17 | 0.18 | 0.20 | 0.22 | 0.21 | 0.23 |
| 1 month | 5 year bond | 0.46 | 0.55 | 0.66 | 0.78 | 0.79 | 0.65 |
| | 20 year bond | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.11 |
| | Cash | 0.37 | 0.27 | 0.14 | 0.00 | 0.00 | 0.00 |
| | Stocks | 0.19 | 0.20 | 0.22 | 0.24 | 0.23 | 0.23 |
| 5 years | 5 year bond | 0.49 | 0.57 | 0.67 | 0.76 | 0.77 | 0.77 |
| | 20 year bond | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | Cash | 0.33 | 0.22 | 0.11 | 0.00 | 0.00 | 0.00 |
| | Stocks | 0.19 | 0.20 | 0.21 | 0.22 | 0.23 | 0.24 |
| 20 years | 5 year bond | 0.49 | 0.56 | 0.61 | 0.66 | 0.71 | 0.76 |
| | 20 year bond | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | Cash | 0.33 | 0.24 | 0.18 | 0.12 | 0.06 | 0.00 |

mean-variance tangency portfolio, see Table III), capturing the risk premium on stocks apparently has high importance when faced with the limitation to hold nonnegative positions for moderate values for $h$.[16] Consequently, in this range the observed increase in stock allocation with $h$ and the decrease with horizon can be largely explained by the leveraging effect and the effective to total wealth ratio respectively. For higher values for $h$ stock holdings begin to be crowded out by the short-sale constraints in combination with the desire to hold bonds.

Another interesting point we learn from Table V (Panel A) is that for low $h$ the investor invests in short-term bonds, while for a larger ratio he switches to long-term bonds, i.e. he increases the weighted average duration of his bond holdings. The explanation is that bonds with longer maturities have larger negative loadings on $dz_r$ and $dz_\pi$. For large housing to total wealth ratios one would like large leveraged exposures to $dz_r$ and $dz_\pi$. An amplifying effect is that for large values for $h$ large stockholdings leave little financial wealth to invest in bonds.

Now focus on Panel B of Table V, which presents results for $\gamma = 7$. As one would expect from the expression for unconstrained portfolio choice presented in Theorem II, we see that the stock allocation is much lower for the fairly risk-averse investor ($\gamma = 7$) compared to the moderately risk-averse investor ($\gamma = 3$). The allocation to the 20-year bond is zero, except for very large $h$ in combination with a 1-month horizon.

Because of the more modest positions in the mean-variance tangency portfolio, the nonnegativity constraint on cash is less constraining, and sometimes not binding. Now both the stock and 5-year bond allocations are hump-shaped in $h$ for short and medium horizon investors and peak at the moment the nonnegativity constraint starts to bind. A final interesting point is that when the nonnegativity constraint on cash is not binding (e.g. for $h = 0.2$), the position in the 5-year bond initially increases with horizon due to the increasing hedge against real interest rate changes. At longer horizons the position in the 5-year bond decreases again with horizon because the effective to total wealth ratio decreases.

We next calculate the utility loss that an investor incurs if he neglects the impact of a house position on the optimal financial portfolio. Figure II presents the wealth equivalent

---

[16]Implicit in this argument is the assumption that the risky assets compete for the limited investment possibilities in the presence of nonnegativity constraints (especially on cash). Indeed this seems to be the case. First of all, in Table V the cash position is zero for all housing to total wealth ratios and all horizons. Second, when we set the risk premium on stocks to zero, bond positions increase and cash positions still equal zero.

**Figure II: Wealth equivalent loss for ignoring homeownership in constrained financial asset allocation ( $\gamma = 3$ and $\gamma = 7$ ).**

The figure shows the wealth equivalent loss when portfolio choice is suboptimally based on the case with no homeownership ($h = 0$).

Panel A: the investor has risk aversion $\gamma = 3$
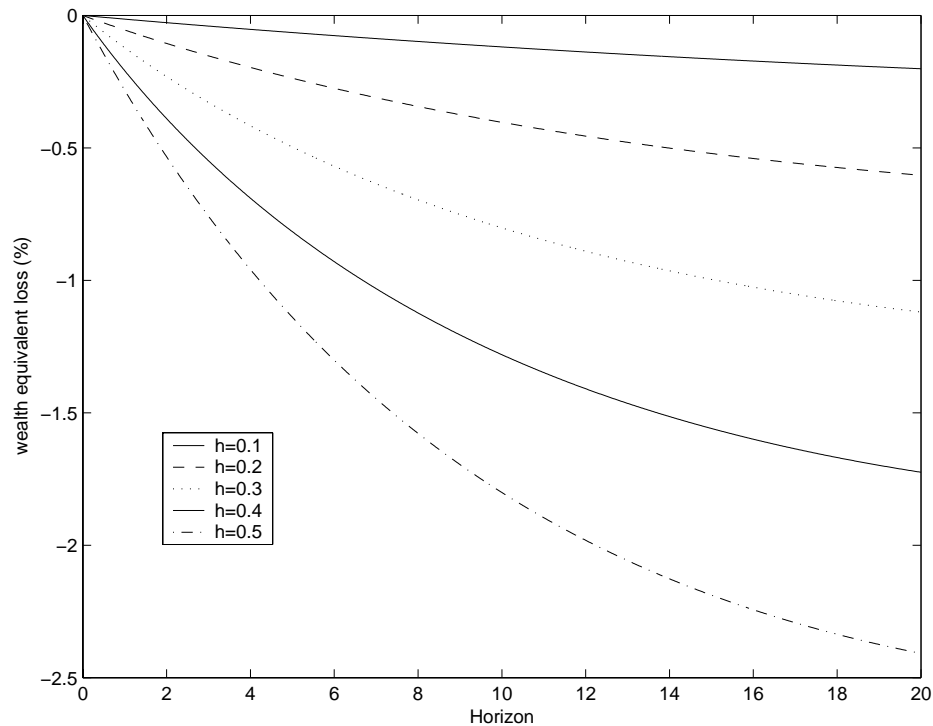


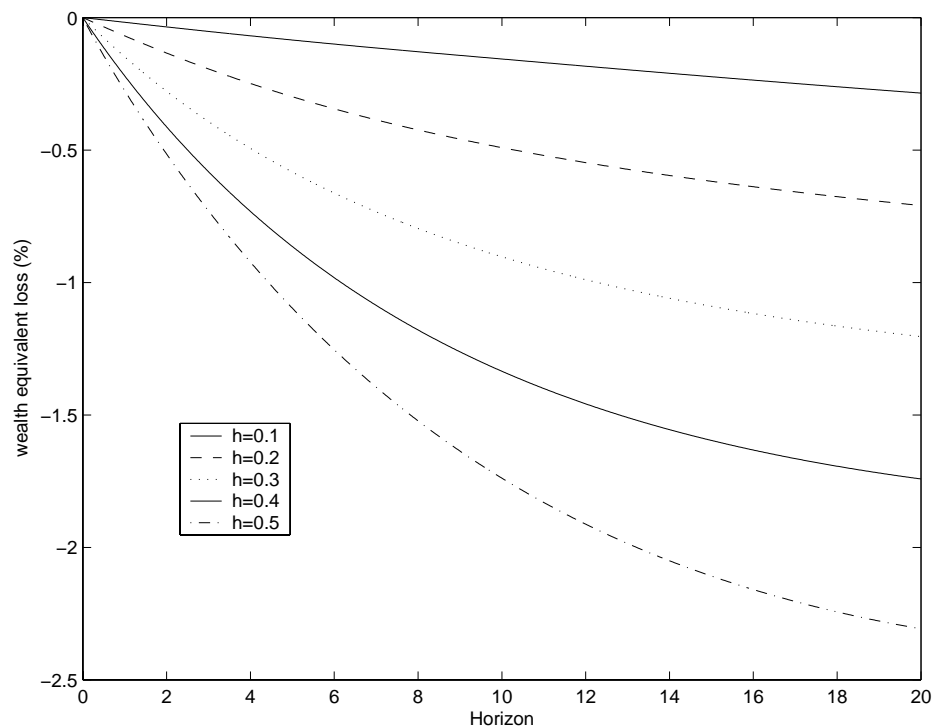Panel B: the investor has risk aversion $\gamma = 7$

**Table VI. Constrained portfolio choice (alternative parameter values, $\gamma = 3$).**
The table presents optimal financial portfolio weights for stocks, bonds with maturities of 5 and 20 years, and cash, in the presence of short-sale constraints, using (i) zero correlations between house price shocks and other assets, (ii) the base case parameter set in Table I. The investor has risk aversion $\gamma = 3$. The bond maturities are assumed to be constant over investment period.

| Horizon | Asset | $\theta_{S,r,\pi} = 0$ ($h = 0.2$) | base case ($h = 0.2$) | $\theta_{S,r,\pi} = 0$ ($h = 0.4$) | base case ($h = 0.4$) |
|---|---|---|---|---|---|
| | | **$h = 0.2$** | | **$h = 0.4$** | |
| | | $\theta_{S,r,\pi} = 0$ | base case | $\theta_{S,r,\pi} = 0$ | base case |
| 1 month | Stocks | 0.44 | 0.43 | 0.57 | 0.51 |
| | 5 year bond | 0.35 | 0.32 | 0.00 | 0.00 |
| | 20 year bond | 0.21 | 0.25 | 0.43 | 0.49 |
| | Cash | 0.00 | 0.00 | 0.00 | 0.00 |
| 5 years | Stocks | 0.43 | 0.42 | 0.53 | 0.51 |
| | 5 year bond | 0.41 | 0.38 | 0.13 | 0.07 |
| | 20 year bond | 0.16 | 0.20 | 0.33 | 0.41 |
| | Cash | 0.00 | 0.00 | 0.00 | 0.00 |
| 20 years | Stocks | 0.40 | 0.40 | 0.46 | 0.45 |
| | 5 year bond | 0.47 | 0.45 | 0.33 | 0.28 |
| | 20 year bond | 0.12 | 0.15 | 0.21 | 0.27 |
| | Cash | 0.00 | 0.00 | 0.00 | 0.00 |

loss, defined as the decrease in total wealth, keeping $h$ constant, that would make an investor equally well off with the fully optimal portfolio choice compared to the case where he holds the suboptimal portfolio ($h = 0$). The wealth equivalent loss is presented for $\gamma = 3$ (Panel A) and $\gamma = 7$ (Panel B). As one would expect, the loss increases with horizon and $h$. For both $\gamma = 3$ and $\gamma = 7$, at a 20-year horizon and for a housing to total wealth ratio of 0.5, the wealth equivalent loss is over 2%. This illustrates the importance of taking into account the owner-occupied house in financial portfolio choice.

## 4.3 Hedging house price risk

In the unconstrained case the effective housing wealth hedge is captured by the third component in Table III. The positions in this third component were modest compared to the first component, the position in the nominal mean-variance tangency portfolio. To assess the portfolio implications of the desire to hedge the effective housing wealth risk with short-sale constraints, we provide in Table VI the portfolio choice for a set of alternative parameter values. In this alternative set we assume that the excess return on housing is uncorrelated with the excess return on financial assets ($\theta_S = \theta_r = \theta_\pi = 0$). The expected housing return remains unchanged, i.e. at the mean return of our housing index over the sample period.

We see that the influence of the housing hedge clearly does *not* vanish in the constrained case. Neglecting the house price hedge leads to a bond duration and bond positions that

are all too low compared to the base case. In addition, the equity weight is too high if one neglects the house price hedge: for a housing to total wealth ratio of 0.4 and a 1-month horizon, the stock allocation substantially increases from 0.51 in the base case to 0.57 in the alternative case. We think that an important direction for future empirical research is to obtain more insight in the interaction of owner-occupied housing with financial assets.

## 5  Introduction of a mortgage loan

In this section we explore how the introduction of mortgages changes optimal portfolio choice. We compare two mortgage types, a fixed-rate mortgage (FRM) loan and an adjustable-rate mortgage (ARM). We model an FRM as the possibility to take a short position in the 20-year bond up to the time-$t$ market value of the house, i.e. the short position may not exceed $-h/(1-h)$ at each point in time. This uses that an FRM has a market value comparable to a bond with long duration. The size of the interest payments plays no direct role since our model does not have interim consumption. We model an ARM similarly as a short position in cash.

Table VII reports the optimal portfolio for a moderately risk-averse ($\gamma = 3$) and a fairly risk-averse investor ($\gamma = 7$) for the different mortgage types. It also reports the wealth equivalent gain, defined as the increase in total wealth, keeping $h$ constant, that would make an investor equally well off without a mortgage compared to the case where he does have the possibility to take a mortgage.

We first focus on the $\gamma = 3$ investor, presented in Panel A. In the no-mortgage case, the nonnegativity constraint on a 20-year bond is not binding for the presented parameter values. It is therefore not surprising that an FRM, modelled as the possibility to take a short position in the 20-year bond up to the value of the house, is not utility increasing. In the no-mortgage case the nonnegativity constraint on cash *is* binding. An ARM relaxes this constraint. It turns out that the maximum ARM mortgage size is optimal. The position in the bond portfolio is increased and the duration reduced. The wealth equivalent gain increases in horizon and $h$, up to a substantial 6.46% for a 20-year horizon and $h = 0.4$.

Now consider the $\gamma = 7$ investor in Panel B. In contrast to the $\gamma = 3$ investor the nonnegativity constraint on the 20-year bond is binding in the no-mortgage case. As a result an FRM is now utility enhancing. The wealth equivalent gain increases in horizon and decreases in $h$. For a 20-year horizon and $h = 0.2$ it equals 3.84%. An ARM is also

**Table VII. Constrained portfolio choice with a mortgage ($\gamma = 3$ and $\gamma = 7$).**
The table presents optimal financial portfolio weights for stocks, bonds with maturities of 5 and 20 years, and cash, in the presence of short-sale constraints. The bond maturities are assumed to be constant over investment period. We consider the case of no mortgage, a fixed-rate mortgage (FRM), an adjustable-rate mortgage (ARM) and a hybrid mortgage. An FRM (ARM) allows a short position in the 20-year bond (cash) up to $-h/(1-h)$. With a hybrid mortgage a short position in both is possible. The table also presents the wealth equivalent gain of having access to the mortgage loan.

Panel A: the investor has risk aversion $\gamma = 3$

| Horizon | Asset | h = 0.2 | | | | h = 0.4 | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | no | FRM | ARM | hybrid | no | FRM | ARM | hybrid |
| 1 month | Stocks | 0.43 | 0.43 | 0.43 | 0.43 | 0.51 | 0.51 | 0.56 | 0.56 |
| | 5 year bond | 0.32 | 0.32 | 0.70 | 0.70 | 0.00 | 0.00 | 0.90 | 0.90 |
| | 20 year bond | 0.25 | 0.25 | 0.12 | 0.12 | 0.49 | 0.49 | 0.21 | 0.21 |
| | Cash | 0.00 | 0.00 | −0.25 | −0.25 | 0.00 | 0.00 | −0.67 | −0.67 |
| | weq gain | | 0.00% | 0.01% | 0.01% | | 0.00% | 0.03% | 0.03% |
| 5 years | Stocks | 0.42 | 0.42 | 0.42 | 0.42 | 0.51 | 0.51 | 0.53 | 0.53 |
| | 5 year bond | 0.38 | 0.38 | 0.75 | 0.75 | 0.07 | 0.07 | 1.05 | 1.05 |
| | 20 year bond | 0.20 | 0.20 | 0.07 | 0.07 | 0.41 | 0.41 | 0.09 | 0.09 |
| | Cash | 0.00 | 0.00 | −0.25 | −0.25 | 0.00 | 0.00 | −0.67 | −0.67 |
| | weq gain | | 0.00% | 0.79% | 0.79% | | 0.00% | 1.70% | 1.70% |
| 20 years | Stocks | 0.40 | 0.40 | 0.41 | 0.41 | 0.45 | 0.45 | 0.48 | 0.48 |
| | 5 year bond | 0.45 | 0.45 | 0.79 | 0.79 | 0.28 | 0.28 | 1.18 | 1.18 |
| | 20 year bond | 0.15 | 0.15 | 0.05 | 0.05 | 0.27 | 0.27 | 0.00 | 0.00 |
| | Cash | 0.00 | 0.00 | −0.25 | −0.25 | 0.00 | 0.00 | −0.67 | −0.67 |
| | weq gain | | 0.00% | 3.01% | 3.01% | | 0.00% | 6.46% | 6.46% |

Panel B: the investor has risk aversion $\gamma = 7$

| Horizon | Asset | h = 0.2 | | | | h = 0.4 | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | no | FRM | ARM | hybrid | no | FRM | ARM | hybrid |
| 1 month | Stocks | 0.20 | 0.16 | 0.20 | 0.17 | 0.21 | 0.20 | 0.25 | 0.22 |
| | 5 year bond | 0.66 | 1.02 | 0.66 | 1.08 | 0.79 | 0.81 | 1.01 | 1.45 |
| | 20 year bond | 0.00 | −0.18 | 0.00 | −0.18 | 0.00 | −0.02 | 0.00 | −0.21 |
| | Cash | 0.14 | 0.00 | 0.14 | −0.07 | 0.00 | 0.00 | −0.26 | −0.46 |
| | weq gain | | 0.01% | 0.00% | 0.01% | | 0.00% | 0.00% | 0.01% |
| 5 years | Stocks | 0.22 | 0.16 | 0.22 | 0.18 | 0.23 | 0.18 | 0.26 | 0.21 |
| | 5 year bond | 0.67 | 1.07 | 0.67 | 1.07 | 0.77 | 0.96 | 0.89 | 1.46 |
| | 20 year bond | 0.00 | −0.23 | 0.00 | −0.20 | 0.00 | −0.14 | 0.00 | −0.27 |
| | Cash | 0.11 | 0.00 | 0.11 | −0.05 | 0.00 | 0.00 | −0.16 | −0.39 |
| | weq gain | | 0.87% | 0.00% | 0.98% | | 0.27% | 0.14% | 1.16% |
| 20 years | Stocks | 0.21 | 0.16 | 0.21 | 0.17 | 0.23 | 0.16 | 0.23 | 0.19 |
| | 5 year bond | 0.61 | 1.09 | 0.61 | 1.08 | 0.71 | 1.06 | 0.72 | 1.48 |
| | 20 year bond | 0.00 | −0.25 | 0.00 | −0.22 | 0.00 | −0.22 | 0.00 | −0.33 |
| | Cash | 0.18 | 0.00 | 0.18 | −0.03 | 0.06 | 0.00 | 0.05 | −0.34 |
| | weq gain | | 3.84% | 0.01% | 4.18% | | 3.26% | 0.11% | 5.73% |

valuable, but only for $h = 0.4$. For both the FRM and the ARM the proceeds are used to increase the position in the bond portfolio. For both the FRM and the ARM the optimal mortgage size is in most cases below the maximum size allowed. For the 20-year horizon and $h = 0.4$, the wealth equivalent gain is equal to 3.26% and 0.11% for the FRM and ARM respectively.

In a search for potentially better (in utility terms) mortgage types we first experimented with mortgages that allow the investor to hold a short position in a bond with an optimally chosen maturity in the range $[0, 20]$. The optimal maturity turned out to be either 0 or 20 years. Remembering the unconstrained case where the investor has a long position in a 5-year bond and is short in a 20-year bond and cash, this is not completely surprising. Interestingly, a hybrid mortgage, which allows a short position in both the 20-year bond and cash (adding up to the value of the house) can further increase utility. For the $\gamma = 7$ investor, Panel B of Table VII shows that such a combination of an ARM and FRM is very attractive. The wealth equivalent gain with the hybrid mortgage is 5.73% for the 20-year horizon and $h = 0.4$, which is much larger than the gain from an ARM or FRM.

We can summarize the optimal mortgage choice for different degrees of risk aversion as follows. A moderately risk-averse investor ($\gamma = 3$) prefers an ARM. A fairly risk-averse investor ($\gamma = 7$) rather chooses an FRM. These results are similar to the results of Campbell and Cocco (2003), but follow for different reasons. Campbell and Cocco examine the trade-off between the wealth risk of an FRM and the income risk of an ARM. In our set-up, there is no income risk, but wealth risk is more complex and also involves real interest rate risk. An FRM allows the investor to improve the hedge against wealth risk. This is particularly valuable for a highly risk averse investor, who cares most about reducing wealth risk and less about expected returns. A less risk averse investor prefers the ARM, which allows him to leverage up the financial portfolio and capture the risk premia on stocks and long-term bonds. Finally we show that a hybrid mortgage, being a combination of an FRM and an ARM, would be an even better choice for a fairly risk-averse investor than the FRM.

# 6  Conclusion

In this paper we studied optimal financial portfolio choice for investors with house ownership. We provided an implicit expression for the optimal portfolio for an investor who can take unconstrained positions in stock, two bonds with different maturities and cash. This portfolio can be seen as the weighted average of three portfolios. These are the nomi-

nal mean-variance tangency portfolio, a portfolio that most closely resembles an inflation-indexed bond, and a portfolio that best hedges the risk of the owner-occupied house. House ownership affects the optimal financial portfolio in three ways. First, it gives rise to a portfolio hedging the house risk. Second, it determines a factor to leverage the financial portfolio weights in order to maintain an appropriate absolute financial risk exposure. Third, it determines the effective to total wealth ratio. Horizon effects in the portfolio choice arise due to a horizon dependent hedge against real interest changes and due to the effective to total wealth ratio that decreases with horizon.

We use numerical techniques to analyze the model with short sale constraints. The interpretation of the optimal portfolio choice at different horizons and housing to total wealth ratios is greatly enhanced by the lessons learned from the no-constraints case. Our results show that besides stocks, bonds play a crucial role in the investor's financial portfolio. The duration of the bond portfolio is found to increase with the housing to total wealth ratio, which is consistent with a desire for a leveraged exposure to real interest rate and expected inflation rate shocks.

We then allow for mortgage loans, and investigate the choice between fixed-rate and adjustable-rate mortgages. In our analysis optimal portfolio and mortgage choice constitutes an integrated financial planning problem. We find that a moderately risk-averse investor ($\gamma = 3$) prefers an ARM. A fairly risk-averse investor ($\gamma = 7$) rather chooses an FRM. An even better mortgage for this investor is a hybrid mortgage, being a combination of an FRM and an ARM.

The focus in this paper is on owner-occupied housing. The non-traded asset in our model has characteristics particular to an owner-occupied house, and we calibrate the model parameters to house price data. The imputed rent is a non-monetary dividend (in contrast to e.g. a private business or a fixed stock position). The non-monetary character gives rise to lower effective than total wealth which has substantial consequences for portfolio choice. The gains from this dividend are captured by a Cobb-Douglas utility function which is a common choice in the housing literature (but not necessary for other non-traded assets with non-monetary dividends). The investor can borrow the market value of the house with the house as collateral, which typically is not possible for other non-traded assets. However, by making case-specific adjustments, our approach is also likely to be capable of handling portfolio implications in the presence of other non-tradable assets. In light of our rich asset menu, in particular the two-factor model for bonds, this would advance the non-tradable asset literature (see e.g. Grossman-Laroque (1990) and Faig and Shum (2002)).

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# Appendix A: proof Theorem I

Using Ito's lemma we get

$$dh = h\left[\left(\mu_q^e - \mu_w^e\right)dt + \left(\sigma_q' - \sigma_w'\right)dz\right] - h\sigma_w'\rho\left(\sigma_q - \sigma_w\right)dt \tag{18}$$

The indirect utility function is given by

$$J\left(w, h, r, t\right) = \max_{x \in A} E_t\left[\frac{(w_T)^{1-\gamma}}{1-\gamma}\right]\nu_H \tag{19a}$$

$$\text{s.t.}$$

$$dw = rwdt + w\mu_w^e dt - w\xi_u\lambda_u^* dt + w\sigma_w' dz - w\xi_u dz_u \tag{19b}$$

$$dh = h\left[\left(\mu_q^e - \mu_w^e\right)dt + \left(\sigma_q' - \sigma_w'\right)dz\right] - h\sigma_w'\rho\left(\sigma_q - \sigma_w\right)dt \tag{19c}$$

$$dr = \kappa\left(\bar{r} - r\right)dt + \sigma_r dz_r \tag{19d}$$

Because $A$ is independent of $w_t$, we can write

$$\max_{x \in A} E_t\left[\frac{(w_T)^{1-\gamma}}{1-\gamma}\right] = \frac{w_t^{1-\gamma}}{1-\gamma}\min_{x \in A} E_t\left[\left(\frac{w_T}{w_t}\right)^{1-\gamma}\right] \tag{20}$$

For a given strategy for $x$ we have

$$\frac{w_T}{w_t} = \exp\left\{\int_t^T\left(r_s + \mu_w^e - \frac{1}{2}\sigma_w'\rho\sigma_w\right)ds + \int_t^T\sigma_w' dz\right\} \tag{21}$$

$$* \exp\left\{\int_t^T\left(-\xi_u\lambda_u^* - \frac{1}{2}\xi_u^2\right)ds + \int_t^T -\xi_u dz_u\right\}$$

We have $r_s = r_t + \int_t^s \kappa\left(\bar{r} - r_u\right)du + \int_t^s \sigma_r dz_r$. Now define $\tilde{r}_s$ as the process for the real interest rate with the same stochastic component as $r_s$, but with start value $\tilde{r}_t = \bar{r}$ at time $t$. This means that $\tilde{r}_s = \bar{r} + \int_t^s \kappa\left(\bar{r} - \tilde{r}_u\right)du + \int_t^s \sigma_r dz_r$. Now we have $\int_t^T r_s ds = \int_t^T \tilde{r}_s ds + \int_t^T\left(r_s - \tilde{r}_s\right)ds = \int_t^T \tilde{r}_s ds + \int_t^T e^{-\kappa(s-t)}\left(r_t - \bar{r}\right)ds = \int_t^T \tilde{r}_s ds + \left(r_t - \bar{r}\right)B_{tT}$. Using this we can write

$$\frac{w_T}{w_t} = \exp\left\{\left(r_t - \bar{r}\right)B_{tT}\right\} * \exp\left\{\int_t^T\left(\tilde{r}_s + \mu_w^e - \frac{1}{2}\sigma_w'\rho\sigma_w\right)ds + \int_t^T\sigma_w' dz\right\} \tag{22}$$

$$* \exp\left\{\int_t^T\left(-\xi_u\lambda_u^* - \frac{1}{2}\xi_u^2\right)ds + \int_t^T -\xi_u dz_u\right\}$$

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where $\tilde{r}$ has the same dynamics as $r$, but $\tilde{r}_t = \bar{r}$.

Now notice that $\exp\left\{\int_t^T \left(\tilde{r}_s + \mu_w^e - \frac{1}{2}\sigma_w'\rho\sigma_w\right) ds + \int_t^T \sigma_w' dz\right\}$ does not depend on $w$ and $r$, and by assumption neither does $A$. The expression does depend on $h$ however. This means that we can write

$$
\begin{aligned}
J\left(w,h,r,t\right) \quad = \quad & \frac{w_t^{1-\gamma}}{1-\gamma} * \nu_H * \exp\left\{(1-\gamma)(r_t - \bar{r}) B_{tT}\right\} * \qquad\qquad\qquad (23\text{a}) \\
& \exp\left\{(1-\gamma)\left(-\xi_u\lambda_u^* - \frac{\gamma}{2}\xi_u^2\right)(T-t)\right\} * \\
& \min_{x\in A} E_t\left[\left(\exp\left\{\int_t^T \left(\tilde{r}_s + \mu_w^e - \frac{1}{2}\sigma_w'\rho\sigma_w\right) ds + \int_t^T \sigma_w' dz\right\}\right)^{1-\gamma}\right] \\
& \text{s.t.}
\end{aligned}
$$

$$
\begin{aligned}
dh \quad &= \quad h\left[\left(\mu_q^e - \mu_w^e\right) dt + \left(\sigma_q' - \sigma_w'\right) dz\right] - h\sigma_w'\rho\left(\sigma_q - \sigma_w\right) dt & (23\text{b}) \\
d\tilde{r} \quad &= \quad \kappa\left(\bar{r} - \tilde{r}\right) dt + \sigma_r dz_r & (23\text{c}) \\
\tilde{r}_t \quad &= \quad \bar{r} & (23\text{d})
\end{aligned}
$$

or shorter

$$
\begin{aligned}
J\left(w,h,r,t\right) \quad \equiv \quad & \frac{w_t^{1-\gamma}}{1-\gamma} * \nu_H * \exp\left\{(1-\gamma)(r_t - \bar{r}) B_{tT}\right\} * \qquad\qquad (24) \\
& \exp\left\{(1-\gamma)\left(-\xi_u\lambda_u^* - \frac{\gamma}{2}\xi_u^2\right)(T-t)\right\} * I\left(h_t, t\right),
\end{aligned}
$$

where we use that

$$
E_t\left[\left(\exp\left\{\int_t^T \left(-\xi_u\lambda_u^* - \frac{1}{2}\xi_u^2\right) ds + \int_t^T -\xi_u dz_u\right\}\right)^{1-\gamma}\right] \qquad (25)
$$
$$
= \quad \exp\left\{(1-\gamma)\left(-\xi_u\lambda_u^* - \frac{\gamma}{2}\xi_u^2\right)(T-t)\right\}.
$$

# Appendix B: proof Theorem II and Corollary I

We find it convenient to first prove Corollary I and then use this to prove Theorem II. The expressions for $\mu_w^e$ and $\sigma_w$ are given by

$$
\begin{aligned}
\mu_w^e &= \left[ (1-h)\, x_S \sigma_S + h\theta_S - \xi_S \right] \lambda_S^* + \left[ (1-h)\, x_r \sigma_r + h\theta_r - \xi_r \right] \lambda_r^* \\
&\quad + \left[ (1-h)\, x_\pi + \sigma_\pi h\theta_\pi - \xi_\pi \right] \lambda_\pi^* + \left[ \sigma_v h\theta_v - \xi_v \right] \lambda_v^* - hr^{imp} \\
\sigma_w &= \left( (1-h)\, x_S \sigma_S + h\theta_S - \xi_S, (1-h)\, x_r \sigma_r + h\theta_r - \xi_r, \right. \\
&\quad \left. (1-h)\, x_\pi \sigma_\pi + h\theta_\pi - \xi_\pi, h\theta_v - \xi_v \right)'
\end{aligned}
$$

(26a)

(26b)

The HJB equation for $I$ is

$$
\begin{aligned}
\min_{x_S, x_r, x_\pi} &\left\{ \left( \frac{I_t}{I} + (1-\gamma)\,\bar{r} \right) + (1-\gamma)\,\mu_w^e - \frac{1}{2}\gamma\,(1-\gamma)\,\sigma_w' \rho \sigma_w \right. \\
&+ \frac{1}{2}(1-\gamma)^2 B^2 \sigma_r^2 + (1-\gamma)^2 B\sigma_w' \rho e_2 \sigma_r + h\frac{I_h}{I}\left( \mu_q^e - \mu_w^e - \sigma_w' \rho\,(\sigma_q - \sigma_w) \right) \\
&+ \frac{1}{2}h^2 \frac{I_{hh}}{I}\left( \sigma_q' - \sigma_w' \right)\rho\,(\sigma_q - \sigma_w) + (1-\gamma)\,h\frac{I_h}{I}\sigma_w' \rho\,(\sigma_q - \sigma_w) \\
&\left. + h\frac{I_h}{I}(1-\gamma)\,B\left( \sigma_q' - \sigma_w' \right)\rho e_2 \sigma_r \right\} \\
&= 0
\end{aligned}
$$

(27)

where $e_2 \equiv (0, 1, 0, 0)'$. Using functional forms for $\mu_q^e$, $\mu_w^e$, $\sigma_q$ and $\sigma_w$, the three first order conditions for $x_S$, $x_r$ and $x_\pi$ form a system of three linear equations in three unknowns. Solving this system gives the presented proportional asset allocations in the factor assets in Corollary I.

Applying the chain rule we can straightforwardly determine $J_{w^F}$, $J_{w^F w^F}$ and $J_{w^F w^H}$ in terms of partial derivates of $J$ to $w$ and $h$. For example

$$
J_{w^F} \equiv J_w \frac{dw}{dw^F} + J_h \frac{dh}{dw^F} = J_w - \frac{h}{w} J_h.
$$

(28)

Using the functional form for $J(w, h, r, t)$ as given in equation (24) we get

$$
-\frac{J_{w^F}}{w^F J_{w^F w^F}} = \frac{1}{1-h} \frac{(1-\gamma)\,I - hI_h}{\gamma\,(1-\gamma)\,I - 2\gamma hI_h - h^2 I_{hh}}
$$

(29)

and

$$
\frac{J_{w^F w^H}}{J_{w^F w^F}} = 1 + \frac{\gamma I_h + hI_{hh}}{\gamma\,(1-\gamma)\,I - 2\gamma hI_h - h^2 I_{hh}}.
$$

(30)

Using equations (29) and (30) the allocation to factor assets in Corollary I can be rewritten to the allocation to factor assets in Theorem II.

## Appendix C: proof Theorem III

The expressions for $\mu_w^e$ and $\sigma_w$ are given by

$$
\begin{aligned}
\mu_w^e &= [(1-h)\, x_S \sigma_S + h\theta_S - \xi_S]\, \lambda_S^* + [(1-h)\, x_r \sigma_r + h\theta_r - \xi_r]\, \lambda_r^* \qquad (31\text{a})\\
&\quad + [(1-h)\, x_\pi + \sigma_\pi h\theta_\pi - \xi_\pi]\, \lambda_\pi^* - hr^{imp}\\
\sigma_w &= ((1-h)\, x_S \sigma_S + h\theta_S - \xi_S, (1-h)\, x_r \sigma_r + h\theta_r - \xi_r, \qquad (31\text{b})\\
&\quad (1-h)\, x_\pi \sigma_\pi + h\theta_\pi - \xi_\pi)'
\end{aligned}
$$

Notice that we have $\xi_v = \lambda_v = 0$ and that all unhedgeable unexpected inflation is captured by $\xi_u$.

The HJB equation for $I$ is

$$
\begin{aligned}
\min_{x_S, x_r, x_\pi} \Big\{ & \left(\frac{I_t}{I} + (1-\gamma)\,\bar{r}\right) + (1-\gamma)\,\mu_w^e - \frac{1}{2}\gamma\,(1-\gamma)\,\sigma_w'\rho\sigma_w \qquad (32)\\
& + \frac{1}{2}(1-\gamma)^2 B^2 \sigma_r^2 + (1-\gamma)^2 B\sigma_w'\rho e_2 \sigma_r + h\frac{I_h}{I}\left(\mu_q^e - \mu_w^e - \sigma_w'\rho\,(\sigma_q - \sigma_w)\right)\\
& + \frac{1}{2}h^2\frac{I_{hh}}{I}\left(\sigma_q' - \sigma_w'\right)\rho\,(\sigma_q - \sigma_w) + (1-\gamma)\,h\frac{I_h}{I}\sigma_w'\rho\,(\sigma_q - \sigma_w)\\
& + h\frac{I_h}{I}(1-\gamma)\,B\left(\sigma_q' - \sigma_w'\right)\rho e_2 \sigma_r \Big\}\\
= & \ 0
\end{aligned}
$$

where $e_2 \equiv (0, 1, 0, 0)'$. Conjecturing the functional form $I(h, t) = \left[1 - \left(1 - e^{-r^{imp}\tau}\right) h\right]^{1-\gamma}\hat{I}(t)$, using functional forms for $\mu_q^e$, $\mu_w^e$, $\sigma_q$, $\sigma_w$, and solving the three first order conditions for $x_S$, $x_r$ and $x_\pi$, gives the presented proportional asset allocations in the factor assets.

Substituting these values in (32), changing variables from $t$ to $\tau = T - t$, and simplifying yields

$$
\frac{\hat{I}_\tau}{\hat{I}} = (1-\gamma)\left[\bar{r} + \frac{1}{2}\frac{1}{\gamma}\phi'\rho\phi + \left(1 - \frac{1}{\gamma}\right)B\sigma_r\phi'\rho e_2 - \frac{1}{2}\left(1 - \frac{1}{\gamma}\right)B^2\sigma_r^2\right]. \qquad (33)
$$

Since no terms involving $h$ remain, our conjecture is proven. Solving the differential equation, using that $\hat{I}(0) = 1$, gives the presented solution for $I$.

## Appendix D: Calibration of the term structure model

The continuous-time model equations for long-term interest rates, short term interest rates and inflation can be discretized as follows

$$y_t = a + b(r_t - \bar{r}) + c(\pi_t - \bar{\pi}) + u_{yt} \tag{34a}$$

$$R_t^f = d + (r_t - \bar{r}) + (\pi_t - \bar{\pi}) + u_{ft} \tag{34b}$$

$$\Delta \ln \Pi_{t+1} = \bar{\pi} + (\pi_t - \bar{\pi}) + \epsilon_{t+1} \tag{34c}$$

$$r_t - \bar{r} = \phi(r_{t-1} - \bar{r}) + \eta_t^r \tag{34d}$$

$$\pi_t - \bar{\pi} = \varphi(\pi_{t-1} - \bar{\pi}) + \eta_t^\pi \tag{34e}$$

where $y_t$ is a vector of long-term coupon bond yields (which we approximate by zero-coupon yields with constant durations of 3.4 and 10.4 years), $R_t^f$ the 3-month t-bill rate, $r_t$ the real interest rate, $\pi_t$ the expected inflation, and $\Delta \ln \Pi_{t+1}$ the actual inflation. The error terms $\epsilon$, $\eta^\pi$ and $\eta^r$ are discretized versions of $\sigma_\Pi dZ_\Pi$, $\sigma_\pi dZ_\pi$, and $\sigma_r dZ_r$ respectively. The terms $u_{yt}$ and $u_{ft}$ are measurement error terms, assumed to be i.i.d with mean zero and variance $\sigma^2$. The parameters $b$, $c$, $\phi$ and $\varphi$ are functions of the mean reversion parameters, as follows

$$b = \frac{1 - \exp(-\kappa T)}{\kappa T}, \quad c = \frac{1 - \exp(-\alpha T)}{\alpha T} \tag{35}$$

where $T$ is the maturity of the bond, and

$$\phi = \exp(-\kappa \Delta t), \quad \varphi = \exp(-\alpha \Delta t) \tag{36}$$

where $\Delta t$ is the period of the observations (0.25 for our quarterly observations).

In the estimation, we first remove the intercepts $a$, $d$, and $\bar{\pi}$ by fitting them to the sample mean of the observed yields, short rates, and actual inflation. There is no need to estimate $\bar{r}$ since we take $r_t - \bar{r}$ and $\pi_t - \bar{\pi}$ as zero-mean state variables. This leaves six parameters to be estimated: $(\alpha, \kappa, \sigma_\eta^\pi, \sigma_\eta^r, \sigma_\epsilon, \sigma)$. The estimation of these parameters is done using the Kalman filter based Quasi Maximum Likelihood method described in detail in De Jong (2000).

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