Consistent Measures of Risk*

Jón Daníelsson
London School of Economics
j.danielsson@lse.ac.uk

Bjørn N. Jorgensen
Columbia Business School
bnj2101@columbia.edu

Mandira Sarma
Indian Statistical Institute
sarma_mandira@vsnl.net

Casper G. de Vries
Erasmus University
cdevries@few.eur.nl

Jean-Pierre Zigrand
London School of Economics
j.p.zigrand@lse.ac.uk

May 25, 2006

Abstract

In this paper we compare overall as well as downside risk measures with respect to the criteria of first and second order stochastic dominance. While the downside risk measures, with the exception of tail conditional expectation, are consistent with first order stochastic dominance, overall risk measures are not, even if we restrict ourselves to two-parameter distributions. Most common risk measures preserve consistent preference orderings between prospects under the second order stochastic dominance rule, although for some of the downside risk measures such consistency holds deep enough in the tail only. In fact, the partial order induced by many risk measures is equivalent to SOSD. Tail conditional expectation is not consistent with respect to second order stochastic dominance.

KEY WORDS: stochastic dominance, risk measures, preference ordering, utility theory

JEL Classification: D81, G00, G11

---

*Corresponding author Jean-Pierre Zigrand, London School of Economics, Houghton Street, London WC2A 2AE, United Kingdom. Our papers can be downloaded from www.RiskResearch.org. J. Danielsson and J.-P. Zigrand would like to acknowledge financial support under the EPSRC Grant GR/S83975/01 at the Financial Markets Group, London School of Economics.

1 Introduction

The stylized fact that financial returns exhibit fat tails, implies that variance (or "volatility") is not an ideal measure of risk for many applications. Volatility is an overall risk measure, with outperformance risk treated the same as underperformance risk. In applications depending on risk, such as pension portfolio choice, risk management and regulatory capital, more sophisticated risk measures than variance are needed since the upward potential is of lesser relevance. An alternative approach would be to model the distribution returns parametrically, but as a practical matter this is often not feasible because of the difficulties in identifying the appropriate distribution. For this reason risk measures are often used. Of such risk measures, Value–at–risk (VaR) has, for better or worse, become a central plank in bank regulations and internal bank risk management. While VaR is in some aspects superior to volatility as a measure of risk, VaR has come under considerable criticism, not the least because of its lack of subadditivity. It is therefore not a coherent risk measure as defined by Artzner et al. (1999), but as Daníelsson et al. (2005) argue, VaR is still subadditive in the tail region. Other risk measures have been advocated instead.

In general, one can view risk measures as belonging to one or more categories, such as *coherent* (see Artzner et al. (1999)), *consistent* (with respect to stochastic dominance), *practical* (as in easy to implement) and so forth. An unresolved issue is the complete characterization of the respective intersections. In this paper we focus on the consistency property. It is known for instance that coherent risk measures need not be consistent and vice versa.[1]

Our objective in this paper is to examine whether the most commonly used risk measures are consistent from a decision point of view, i.e. consistent with respect to first and second order stochastic dominance (FOSD, SOSD, respectively). Wherever possible, we shall provide the converses, i.e. statements of the sort: if a risk measure always attaches less risk to X than to Y, then X stochastically dominates Y (to some order). In classical utility theory, the notion of stochastic dominance is intrinsically related to the concept of maximisation of non–decreasing or concave expected Von Neumann-Morgenstern (VNM) utility functions. In particular, the first order stochastic dominance of a risky asset X over a risky asset Y is equivalent to saying that any expected utility maximising investor preferring more to less always weakly

[1]See the example in De Giorgi (2005) for an example of a probability space on which two random variables and a coherent risk measure are defined, but where the risk measure is not consistent. As to the converse, examples can easily be constructed based on results in this paper whereby VaR for instance is consistent but not coherent.

prefers X to Y. Similarly, saying that X dominates Y in the sense of second order stochastic dominance is equivalent to saying that all risk averse investors with an expected concave utility function would always weakly prefer X to Y. The question we pose is therefore the following: if X stochastically dominates Y to some order, so that all investors within the class of either non decreasing or concave utility functions prefer X to Y, does the given risk measure provide the same ordering? If the answer is affirmative for all non decreasing utility functions, the risk measure is said to be *consistent with respect to* FOSD. If the answer is affirmative for all concave utility functions, we call the risk measure *consistent with respect to* SOSD, or simply *consistent.* We answer this question for arbitrary distribution functions. Since stochastic dominance is a partial order only, and consistency with stochastic dominance is therefore inherently a weak statement, we believe consistency is a minimal desirable requirement for any risk measure to satisfy.

Dhaene et al. (2003) classify risk measures based on whether they consider the entire set of outcomes, referred to as *overall risk measures*, or only the tails, the so-called *downside risk measures.* Of the former, second-moment based risk measures such as volatility and beta are the most common, but we also consider the interquartile range (which is for instance a useful measure in circumstances where the first two moments do not exist). Of the downside risk measures, we consider value–at–risk (VaR), tail conditional expectation (TCE), expected shortfall (ES), lower partial moments (LPM) of the zeroth, first and second order (see Bawa, 1975), as well as the Omega function (Keating and Shadwick, 2002).

Fishburn (1977) shows that LPM retains a preference ordering of assets consistent with the first order stochastic dominance rule (FOSD). Kaplanski and Kroll (2000) demonstrates under simplifying conditions this also holds for VaR. This implies that an investment choice based on LPM and VaR is consistent with the choice of an expected utility maximising rational investor. We extend these results by also considering the overall risk measures and some downside measures such as TCE, ES and Ω. We find that all the common downside risk measures considered are consistent with FOSD, with the notable exception of TCE. Furthermore, we show that overall risk measures do not preserve the same preference ordering of assets as under the FOSD rule, even under the additional simplifying assumption of belonging to a two-parameter family of distributions. The intuition is that even if a prospect Y has overall a higher risk than X, Y could still be preferred by some non satiated expected utility maximiser over X, since less overall risk typically means both less downside risk and less upside risk. Downside risk measures are not affected by this tradeoff by definition and are therefore bound to be more consistent with FOSD.

Regarding the SOSD criterion, wee show that most common risk measures studied herein are consistent. Fishburn (1977) has shown consistency of the LPMs with SOSD. For some, such as VaR and IQR, we show this is true over a range of quantiles defined by crossing properties of distribution functions, for others (variance, beta, Omega, ES) consistency holds overall.

In fact, we also derive the converse for FLPM, ES and Ω in the sense that for equal means, the ordering induced by those three risk measures implies SOSD. The consistency found in the tails implies that at least in the most relevant regions of the asset returns, all measures of risk provide investment guidance which is consistent with the utility based selection of a risk averse investor. For instance, VaR is both consistent and coherent in the tails. We further strengthen those consistency relationships if the distribution functions belong to the same two-parameter family of distributions. For instance, SOSD is shown in fact to be equivalent to the ordering induced by any of the overall risk measures considered.

The intuitive and common sense measure $\text{TCE}_X := -E[X|X \leq -\text{VaR}_X]$ is neither consistent with FOSD nor with SOSD in general, unless distribution functions satisfy certain continuity conditions. In other words, TCE may not be consistent when returns can have point masses such as those occurring naturally in case of assets that are subjected to defaults, or assets whose returns are modelled discretely (e.g. those based on binomial steps), or portfolio returns whose distributions are mixtures of continuous and discrete distributions due to derivatives such as options or path dependent derivatives. Expected Shortfall is defined as TCE plus a correction term for discontinuities in the distribution function. In view of the fact that TCE is what many practitioners call "expected shortfall," caution must be applied since this measure of risk is in general neither coherent (see Artzner et al. (1999)), nor consistent in our sense. The variant of TCE which is ES (following Acerbi et al. (2001)), is known to be coherent (Acerbi and Tasche (2002)). We show ES also is consistent.

2 Risk measures

This paragraph introduces the notation. Suppose that X and Y are two risky asset returns with distribution functions F_X and F_Y respectively. A point of discontinuity corresponds to a point with strictly positive mass. The upper $p-$quantile $q_X(p)$ is defined as $\sup\{x : F_X(x) \leq p\}$. The lower p-quantile is the generalized inverse function $\tilde{F}_X^{-1}(p) := \inf\{z : F_X(z) \geq p\}$. Thus if

F_X is strictly increasing at level p (has no flats, but can be discontinuous), then $q_X(p) = \tilde{F}_X^{-1}(p)$. While if F_X is continuous but has a flat stretch at p, then $\tilde{F}_X^{-1}(p) < q_X(p)$. It follows that whenever F_X is continuous, $F_X(q_X(p)) = F_X(\tilde{F}_X^{-1}(p)) = p$, and if F_X is strictly increasing and continuous, then $F_X^{-1}(p) = \tilde{F}_X^{-1}(p) = q_X(p)$.

Further, X and Y have expected values $\mu_X := E[X] := \int_{\mathbb{R}} z dF_X(z)$ and $\mu_Y := E[Y] := \int_{\mathbb{R}} z dF_Y(z)$ (if the integrals exist). If second moments exist, we denote by σ_X and σ_Y the standard deviations of X and Y respectively.

We consider two sets of risk measures, viz., overall risk measures and downside risk measures. We use the definitions of Dhaene et al. (2003):

- An *overall risk measure* is a measure of the "distance" between the risky situation and the corresponding risk-free situation when both favourable and unfavourable discrepancies are taken into account.
- A *downside risk measure* is a measure of the "distance" between the risky situation and the corresponding risk-free situation when only unfavourable discrepancies contribute to the "risk".

The various risk measures considered in each category are as follows.

2.1 Overall risk measures

1. Variance σ_X^2 (provided it exists), is given by

$$\sigma_X^2 := \int_{-\infty}^{\infty} (x - E[X])^2 dF_X(x)$$

2. Market risk β_X is given by (assume that $\sigma_X^2 < \infty$, $0 < \sigma_R^2 < \infty$)

$$\beta_X := \rho_{X,R} \; \frac{\sigma_X}{\sigma_R}$$

   where $\rho_{X,R}$ is the correlation coefficient between X and R, the market portfolio, and σ_R and σ_X are the standard deviations of R and X respectively.

3. The interquartile range (IQR) measure reads

$$\text{IQR} = q_X(3/4) - q_X(1/4)$$

The IQR measure is sometimes used as a measure of overall risk when the second moment is not bounded. For example, for symmetric α-stable distributions with $1 < \alpha < 2$ the standard deviation does not exist, but the scale can be captured by IQR (Fama and Roll, 1968).

2.2 Downside risk measures

Risk measures considered under this category are lower partial moments of second, first and zeroth orders, Value-at-Risk, tail conditional expectation and expected shortfall. We also include Omega. Lower partial moment of order n is computed at some fixed quantile q, and defined as the n^{th} moment below q. Bawa (1975) developed the lower partial moment concept. Subsequently these measures were studied rigorously by Fishburn (1977).

1. Second Lower Partial Moment or Semi-variance (SLPM) is defined as[2]

$$\mathrm{SLPM}(q) := \int_{-\infty}^{q} (q-x)^2 dF_X(x) \quad = \quad 2\int_{-\infty}^{q} (q-x) F_X(x) dx$$

   assuming that $\int_{-\infty}^{0} x^2 dF_X(x) < \infty$.

2. First Lower Partial Moment (FLPM)

$$\mathrm{FLPM}(q) := \int_{-\infty}^{q} (q-x) dF_X(x) \quad = \quad \int_{-\infty}^{q} F_X(x) dx$$

   The equality follows again from integration by parts, provided $\int_{-\infty}^{0} x dF_X > -\infty$.

3. Zeroth Lower Partial Moment (ZLPM)

$$\mathrm{ZLPM}(q) := \int_{-\infty}^{q} dF_X = F_X(q)$$

4. Value-at-Risk (VaR): If the ZLPM(q) is fixed at p, then the negative of the upper quantile gives the Value-at-Risk as

$$\mathrm{VaR}_X(p) := -q_X(p)$$

   VaR is defined as the maximum potential loss to an investment with a pre-specified confidence level $(1-p)$.

[2] The second equality follows from integrating by parts, using the fact that $\int_{-\infty}^{0} x^2 dF_X(x) < \infty$ implies $\lim_{x\to-\infty} x^2 F_X(x) = 0$ and the fact that $\int_{-\infty}^{0} x^2 dF_X(x) < \infty$ implies that $\int_{-\infty}^{0} x dF_X > -\infty$, which in turn implies that $\lim_{x\to-\infty} x F_X(x) = 0$.

5. Tail Conditional Expectation (or "TailVaR"): TCE at the confidence level $(1-p) < 1$ is defined as

$$\begin{aligned}\mathrm{TCE}_X(p) &:= -E[X|X \le -\mathrm{VaR}_X(p)] \\ &= -\int_{-\infty}^{q_X(p)} z \, \frac{dF_X(z)}{F_X(q_X(p))} = -q_X(p) + \frac{1}{F_X(q_X(p))}\int_{-\infty}^{q_X(p)} F_X(z)dz\end{aligned}$$

The last equality (provided $p > 0$ and $\int_{-\infty}^{0} x dF_X > -\infty$) follows from integration by parts. Some authors and most practitioners call this risk measure "expected shortfall" (ES). We follow Artzner et al. (1999) by calling it TCE and follow Acerbi et al. (2001) by reserving the term ES for the following variant.

6. Expected Shortfall: ES at the confidence level $(1-p) < 1$ is defined as

$$\begin{aligned}\mathrm{ES}_X(p) &:= -\frac{1}{p}\Big(\int_{-\infty}^{q_X(p)} z dF_X(z) - q_X(p)[F_X(q_X(p)) - p]\Big) \\ &= \mathrm{TCE}_X(p) + \frac{F_X(q_X(p)) - p}{p}[\mathrm{TCE}_X(p) - \mathrm{VaR}_X(p)] \\ &= -\frac{1}{p}\int_0^p \tilde{F}_X^{-1}(z)dz\end{aligned}$$

The last equality is due to Acerbi and Tasche (2002)[3] who also show that ES_X is in fact identical to the risk measure known as "Conditional Value-at-Risk." Notice that when F_X is continuous, then $F_X(q_X(p)) = p$ and ES and TCE coincide.[4] If on the other hand F_X is not continuous and $p \notin F_X(\mathbb{R})$, then by definition of $q_X(p)$ the event $\{X \le q_X(p)\}$ has probability $F_X(q_X(p)) > p$, in which case TCE might violate subadditivity. To ensure subadditivity (and consistency, as we show later), the relevant amount has been substracted, and some authors (e.g. Acerbi and Tasche (2002)) reserve the term ES for this subadditive transformation of TCE, as we do here. Also, notice that $\mathrm{ES}_X(p) \ge \mathrm{TCE}_X(p)$.

[3] These authors also show that ES is not quantile dependent in the sense that in the definition of ES, the upper quantile $q_X(p)$ can be replaced by any $\alpha \in [\tilde{F}_X^{-1}(p), q_X(p)]$ without affecting the function ES.

[4] More generally, ES and TCE coincide iff for each $p \in [0,1]$, either $F_X(q_X(p)) = p$ or $\mathbb{P}(X < q_X(p)) = 0$.

7. Omega: Ω_X is a risk measure[5] defined as

$$\Omega_X(q) := \frac{\int_{-\infty}^{q} F_X(z)dz}{\int_{q}^{\infty}(1 - F_X(z))dz}$$

3 Stochastic dominance and consistency

One of the basic properties any risk measure ought to satisfy is second order stochastic dominance, SOSD. Whether investors do or do not act according to the axioms of expected utility, it nevertheless does not seem satisfactory for a risk measure to indicate that Y is riskier than X if all risk averse expected utility maximizers believe otherwise. The case for a risk measure to satisfy first order stochastic dominance, FOSD, is much weaker since FOSD is a much stronger concept that goes beyond risk and is also a statement about expected returns. In fact, in Proposition 1 we show for instance that for X and Y belonging to a two parameter distribution family, if X dominates Y to the first order then we must have $\mu_X \geq \mu_Y$ and $\sigma_X = \sigma_Y$. If the prospects belong to a two-parameter family, FOSD cannot therefore be a meaningful risk ordering. Nevertheless we briefly review FOSD and some connections between FOSD and risk measures. We do this not least because below the first crossing point of the distribution functions of X and Y, one of the two must dominate the other to the first order, and tails do matter for risk analysis.

3.1 First Order Stochastic Dominance (FOSD)

We say that asset X First Order Stochastically Dominates asset Y, denoted by $X\text{FOSD}Y$ if $F_X(z) \leq F_Y(z) \quad \forall z$, where $z \in S \subseteq \mathbb{R}$, the common support of the random variables. This means that the probability of asset return X falling below a specified level z is smaller than that of asset return Y falling below the same level. Define the following families of utility functions. $U^{nd} := \{u : \mathbb{R} \to \mathbb{R} |\ u \text{ non decreasing } \}$ and $U^c := \{u : \mathbb{R} \to \mathbb{R} |\ u \text{ concave}\}$. Notice that elements in U^c need not be non decreasing.

If S is compact, first order stochastic dominance of X on Y gives rise to the following equivalent statements:

[5] Due to Keating and Shadwick (2002), Omega has originally been designed as the performance measure $\frac{\int_q^\infty (1-F_X(z))dz}{\int_{-\infty}^q F_X(z)dz}$, balancing upside potential and risks.

1. XFOSDY

2. $X \stackrel{d}{=} Y + \epsilon,\ \epsilon \geq 0$, where $\stackrel{d}{=}$ denotes distributional equivalence.

3. $E[u(X)] \geq E[u(Y)]$ for all $u \in U^{nd}$.

The equivalence 1 $\Leftrightarrow$ 3 is due to Quirk and Saposnik (1962) and Hanoch and Levy (1969).

Since this paper is concerned with risk measures and tail risk, we are also interested in the general case for which $S = \mathbb{R}$. The previous relationships still hold, provided some integrability conditions are met. The equivalence 1 $\Leftrightarrow$ 3 applies for utility functions in $\{u : \mathbb{R} \to \mathbb{R} |\ u$ non-decreasing, continuous and $\int u dF_X - \int u dF_Y$ well defined$\}$. Well defined means that it is not of the form $\infty - \infty$. This result is in Brumelle and Vickson (1975) and in Tesfatsion (1976). Strict results (i.e. where there is a point z_0 s.t. $F_X(z_0) < F_Y(z_0)$) can also be reported.

In the equivalence statement above, the second equality (2.) implies that $\mu_X \geq \mu_Y$, but the converse is not true in general. It follows that ranking random variables according to FOSD is not a pure risk ranking.

3.2 Second order stochastic dominance (SOSD)

We say that asset X Second Order Stochastically Dominates asset Y, denoted by X SOSD Y, if[6]

$$\mu_X = \mu_Y \tag{1}$$

$$\int_{-\infty}^{q} F_X(z)dz \leq \int_{-\infty}^{q} F_Y(z)dz \quad \forall q \tag{2}$$

Assume that S is compact (or that $S = \mathbb{R}$ and that X and Y are essentially bounded). The following equivalent statements arise from the definition of second order stochastic dominance of X on Y:

1. X SOSD Y

2. $X \stackrel{d}{=} Y + \nu$, where $E[\nu|X] = 0$

[6] We follow the standard textbook definition (e.g. Huang and Litzenberger (1988)). There is, however, no unanimous agreement in the literature. Sometimes our SOSD partial ordering goes under the name of "Increasing Risk" or "more risky than" following Rothschild and Stiglitz (1970) (who do not use the term "second order stochastic dominance"), with the term SOSD defining the inequality (2).

3. $E[u(X)] \geq E[u(Y)]$ for all $u \in U^c$.

The equivalence $1 \Leftrightarrow 3$ is due to Hadar and Russell (1969) (in fact, they also impose $u \in \mathcal{C}^1$) and Hanoch and Levy (1969). Equivalences $1 \Leftrightarrow 2$ and $2 \Leftrightarrow 3$ are due to Rothschild and Stiglitz (1970).

Consider the general case with possibly unbounded support $S = \mathbb{R}$. We have $1 \Leftrightarrow 3$ for all utility functions in $\{u : \mathbb{R} \to \mathbb{R}|\ u$ non decreasing and concave, $\int udF_X - \int udF_Y$ well defined, $\int_{-\infty}^0 udF_X > -\infty$ and $\int_{-\infty}^0 udF_Y > -\infty\}$, see Tesfatsion (1976). The equivalence $2 \Leftrightarrow 3$ holds for all functions in U^c, see Brumelle and Vickson (1975) for a statement based on a result by Strassen (1965). Strict results can also be reported.

If second moments exist for X, then a direct implication of statement (2) is that $\sigma_X^2 \leq \sigma_Y^2$. Thus if $\sigma_X^2 < \infty$ and X SOSD Y then $\mu_X = \mu_Y$, and $\sigma_X^2 \leq \sigma_Y^2$; however the converse is not necessarily true. Classes of distributions for which it is true will be outlined in the sequel (Proposition 3).

In this paper, we always assume that the integrability conditions that warrant these equivalences hold.

3.3 Consistency

Let $\mathcal{M}$ be any one of the risk measures defined before, and let $\mathcal{M}_X(\pi)$ be the risk measure for prospect X at the parameter value $\pi \in \Pi$ (π is typically a quantile or a probability).

Definition 1 *We say that $\mathcal{M}$ is consistent with stochastic dominance (*SD*) if*

$$X\text{SD}Y$$

then

$$\mathcal{M}_X \leq \mathcal{M}_Y \qquad (\ i.e.\ \mathcal{M}_X(\pi) \leq \mathcal{M}_Y(\pi), \quad all\ \pi \in \Pi)$$

Similarly, we say that $\mathcal{M}$ is partially consistent with SD *if XSDY implies $\mathcal{M}_X \leq \mathcal{M}_Y$ over a subset of* Π.

As we have argued before, consistency is a desirable property of a risk measure. For instance, a measure inconsistent with SOSD would indicate, for some pair of prospects X and Y, that X is riskier than Y while each and every risk-averter would rather have X than Y. This would make it an unsuitable risk measure. In fact, for many of the risk measures studied in this paper we shall prove equivalence with SOSD, XSOSDY iff $\mathcal{M}_X \leq \mathcal{M}_Y$.

4 Comparing risk measures when X dominates Y in the sense of first order stochastic dominance

For the overall risk measures considered in this paper, it is easy to observe that for unknown F_X and F_Y, the first order stochastic dominance of X over Y does not lead to an unambiguous ordering between assets with respect to any of the overall risk measures. However, as shown in Proposition 1 below, the special assumption that X and Y belong to the same family of two parameter distributions (e.g Gaussian) allows us to establish that FOSD only leads to equivalence relationships between assets with respect to the overall risk measures. In other words, FOSD is not a useful criterion by which to judge risk measures. The fact that overall risk measures are not consistent (even if we restrict distributions to two-parameter families) is not in itself so surprising, for XFOSDY is equivalent to the statement that any nonsatiated investor prefers X over Y, whether risk averse or not. One would, however expect to be able to say something about the consistency of downside measures because less downside risk tends to mean more upside risk, which is of relevance to a nonsatiated investor.

Some results on downside measures already exist in the literature. From Fishburn (1977) and Kaplanski and Kroll (2000), we know that regardless of the distribution of X and Y, assets can be unambiguously ordered with respect to SLPM, FLPM, ZLPM and VaR. Furthermore, the ordering is consistent with the FOSD rule. We complement these results by establishing new results on ES. We list all these results in Proposition 1. Asset X is said to dominate asset Y under a specific risk measure ρ if $\rho_X \leq \rho_Y$. Notice that the orderings below are pointwise orderings of functions. For instance $\mathrm{ES}_X \leq \mathrm{ES}_Y$ means $\mathrm{ES}_X(p) \leq \mathrm{ES}_Y(p)$ for all $p \in (0, 1]$ and $\Omega_X \leq \Omega_Y$ means $\Omega_X(q) \leq \Omega_Y(q)$ for all $q \in \mathbb{R}$. The statement TCE_X ? TCE_Y means that for arbitrary distribution functions we are neither assured that $\mathrm{TCE}_X \leq \mathrm{TCE}_Y$ nor the other way around, and that there are cases of F_X and F_Y whereby $\mathrm{TCE}_X(p) < \mathrm{TCE}_Y(p)$ for some p and $\mathrm{TCE}_X(p) > \mathrm{TCE}_Y(p)$ for other p.

Proposition 1 *If X FOSD Y, then regardless of the distributions of X and*

Y, the following relationships hold:

$$
\begin{aligned}
\mathrm{SLPM}_X &\leq \mathrm{SLPM}_Y && (3)\\
\mathrm{FLPM}_X &\leq \mathrm{FLPM}_Y && (4)\\
\mathrm{ZLPM}_X &\leq \mathrm{ZLPM}_Y && (5)\\
\mathrm{VaR}_X &\leq \mathrm{VaR}_Y && (6)\\
\mathrm{ES}_X &\leq \mathrm{ES}_Y && (7)\\
\Omega_X &\leq \Omega_Y && (8)\\
\mathrm{TCE}_X &\ ?\ \mathrm{TCE}_Y && (9)
\end{aligned}
$$

Assume moreover that X and Y are such that $F_X(x) = F_Y(y)$ whenever $\frac{x-\mu_X}{\sigma_X} = \frac{y-\mu_Y}{\sigma_Y}$. If X FOSD Y, then the following relationships hold as well:

$$
\begin{aligned}
\sigma_X &= \sigma_Y, \quad \mu_Y \leq \mu_X && (10)\\
\mathrm{IQR}_X &= \mathrm{IQR}_Y && (11)
\end{aligned}
$$

Proof :
Relationships (3), (4) and (5) follow from Fishburn (1977). Relationship (6) follows from the observation that because $F_X(q) \leq F_Y(q)$, all $q \in \mathbb{R}$, we have $\{z : F_Y(z) \leq p\} \subseteq \{z : F_X(z) \leq p\}$, and therefore that the suprema satisfy $q_X(z) \geq q_Y(z)$, $\forall z \in \mathbb{R}$.

Notice that if $F_Y(z) \geq F_X(z)$ all z, then $\{z : F_X(z) \geq p\} \subseteq \{z : F_Y(z) \geq p\}$, so that the infima satisfy $\tilde{F}_X^{-1}(p) \geq \tilde{F}_Y^{-1}(p)$. Therefore $\mathrm{ES}_Y(p) - \mathrm{ES}_X(p) = -\frac{1}{p}\int_0^p [\tilde{F}_Y^{-1}(z) - \tilde{F}_X^{-1}(z)]dz \geq 0$, as required for (7).

As to (8), Omega is consistent since $\Omega_X \leq \Omega_Y$ iff

$$\left[\int_q^\infty [1 - F_X(z)]dz\right]\left[\int_{-\infty}^q F_Y(z)dz\right] \geq \left[\int_q^\infty [1 - F_Y(z)]dz\right]\left[\int_{-\infty}^q F_X(z)dz\right].$$

By FOSD, the first term on LHS is larger than the first term on the RHS, and the same holds for the second term.

In order to prove (9), it is sufficient to provide an example whereby XFOSDY and yet $\mathrm{TCE}_X(p) > \mathrm{TCE}_Y(p)$ for a range of p's and $\mathrm{TCE}_X(p) < \mathrm{TCE}_Y(p)$ over some other range. Obviously, the example must be based upon discontinuous distributions, so we choose trinomial random variables with realizations in $\{0, q, 1\}$. Consider the parameters $0 < \eta < \pi < 1$ and $0 < q < 1$. $F_Y(z) = \mathbf{1}_{0 \leq z < q}\eta + \mathbf{1}_{z \geq q}$ and $F_X(z) = \mathbf{1}_{0 \leq z < q}\eta + \mathbf{1}_{z \in [q,1)}\pi + \mathbf{1}_{z \geq 1}$, so XFOSDY (strictly). First, pick $p \in (\eta, \pi)$. We have $q_X(p) = q_Y(p) = q$, $F_X(q_X(p)) = \pi$ and $F_Y(q_Y(p)) = 1$. $\mathrm{TCE}_Y(p) - \mathrm{TCE}_X(p) = -q + q + q\eta - \frac{1}{\pi}q\eta = (1 - \pi^{-1})q\eta < 0$.

Now choose $p \in (\pi, 1)$. We have $q_X(p) = 1$, $q_Y(p) = q$, $F_X(q_X(p)) = 1$ and $F_Y(q_Y(p)) = 1$. $\text{TCE}_Y(p) - \text{TCE}_X(p) = -q + 1 + q\eta - (q\eta + (1-q)\pi) = (1-\pi)(1-q) > 0$.

Finally, we prove (10) and (11). Due to the first order stochastic dominance of X over Y, for any $z \in \mathbb{R}$,

$$F_X(z) \le F_Y(z) = F_X(z') \;\; ; \text{ with } z' := \mu_X + \frac{\sigma_X(z - \mu_Y)}{\sigma_Y}$$

Thus for any $z \in \mathbb{R}$ it holds that $z \le z'$ which is equivalent to

$$z\Big[1 - \frac{\sigma_X}{\sigma_Y}\Big] \le \mu_X - \frac{\sigma_X}{\sigma_Y}\mu_Y$$

Since this must hold for all $z \in \mathbb{R}$, we have $\sigma_X = \sigma_Y$ and $\mu_Y \le \mu_X$. From here, equality (11) follows. ■

Proposition (1) indicates that the preference ordering of assets with respect to the downside risk measures (other than TCE) is consistent with that under FOSD in the sense indicated. Thus, the choices made using the lower partial moments of second, first and zeroth order and VaR are consistent with the choice made under the utility theory framework.[7] The fact that the ES ordering follows from the FOSD ordering is interesting. One might have thought that while Y could have a larger expected shortfall than X given a tail event of probability p, there could be Y such that a non-satiated investor might nevertheless prefer Y ex-ante to X, which is FOSD. It is known since Artzner et al. (1999) that TCE may violate coherence. We further show the drawbacks of this popular measure by showing that when distributions are not continuous (for instance if there are point masses attributed to default events or to derivatives payoffs), then TCE may not be consistent with FOSD either, and therefore cannot be consistent with respect to SOSD.

5 Comparing risk measures when X dominates Y in the sense of second order stochastic dominance

If asset X dominates asset Y in the sense of second order stochastic dominance, then for unknown F_X and F_Y, the risk measure IQR does not provide

[7] It is also clear that we have the following converse: if $\text{VaR}_X(p) \le \text{VaR}_Y(p)$ for all $p \in [0, 1]$, then $X \text{FOSD} Y$.

any unambiguous ordering of assets. As far as the other measures are concerned, the ordering of assets by these measures is possible under the SOSD rule. Such an ordering, however , may only be partial for some of the measures. We present these observations in Proposition 2 below.

Comparing two betas is useful only if the market return plays a meaningful role. We have two interpretations in mind. First, assume that the market with return R is held by a well-diversified risk averse investor who then ranks the prospects $R+X$ versus $R+Y$. We say that $X\text{SOSD}_\beta Y$ if $(X+R)\text{SOSD}(Y+R)$ and if $\sigma_X = \sigma_Y$. For the second interpretation, assume that prospects X and Y have equally volatile idiosyncratic components ϵ_X and ϵ_Y respectively, where $X = \alpha_X + \beta_X R + \epsilon_X$, $E[\epsilon_X R] = E[\epsilon_X] = 0$, and similarly for Y. Then we say that $X\text{SOSD}'_\beta Y$ if $X\text{SOSD}Y$ and if $\sigma_{\epsilon_X} = \sigma_{\epsilon_Y}$.

Denote the first crossing quantile of the two distribution functions by $\bar{q}$. More precisely, $\bar{q}$ satisfies $F_X(z) \leq F_Y(z)$ for $z < \bar{q}$, $F_X(\bar{q}) \geq F_Y(\bar{q})$ and $F_X(z) > F_Y(z)$ for $z \in (\bar{q}, \bar{q}+\epsilon)$ for some $\epsilon > 0$, and there is no smaller such crossing quantile. If there is no crossing, the results of FOSD apply and we set $\bar{q} = \infty$. If there are multiple crossings but no first crossing,[8] set $\bar{q} = -\infty$. Define $\bar{p} := F_Y(\bar{q})$.

Proposition 2 *Suppose that $X\text{SOSD}Y$. Then regardless of the distribution of X and Y, the following relationships hold (if the respective integrals exist):*

$$\begin{aligned}
\sigma_X &\leq \sigma_Y && (12)\\
\text{SLPM}_X &\leq \text{SLPM}_Y && (13)\\
\text{FLPM}_X &\leq \text{FLPM}_Y && (14)\\
\text{ZLPM}_X(q) &\leq \text{ZLPM}_Y(q), \quad \forall q < \bar{q} && (15)\\
\text{VAR}_X(p) &\leq \text{VAR}_Y(p), \quad \forall p < \bar{p} && (16)\\
\text{ES}_X &\leq \text{ES}_Y && (17)\\
\Omega_X &\leq \Omega_Y && (18)\\
\text{TCE}_X &\; ? \; \text{TCE}_Y && (19)
\end{aligned}$$

Assume that $X\text{SOSD}Y$, that $\bar{p} \in [1/4, 3/4]$ and that no other crossing point is in $[1/4, 3/4]$. Then

$$\text{IQR}_X \leq \text{IQR}_Y \quad (20)$$

Assume either that $X\text{SOSD}_\beta Y$ or that $X\text{SOSD}'_\beta Y$, then

$$\beta_X \leq \beta_Y \quad (21)$$

[8] We thank Simon Polbennikov for pointing out that there might be infinitely many crossing quantiles, in which case $\bar{q}$ might not be defined.

Assume X and Y are such that $F_X(x) = F_Y(y)$ whenever $\frac{x-\mu_X}{\sigma_X} = \frac{y-\mu_Y}{\sigma_Y}$. If XSOSDY, then $\bar{q} = \mu := \mu_X = \mu_Y$ and $\bar{p} = F_Y(\mu)$. Also, always

$$\text{IQR}_X \leq \text{IQR}_Y \tag{22}$$

Proof :
Relationship (12) is an implication of the definition of SOSD. Fishburn (1977) has established relationships (13) and (14).

Inequalities (15) and (16) follow from the fact that below the first crossing quantile $F_X(q) \leq F_Y(q)$, and hence X FOSD Y below the first crossing quantile so that Proposition 1 can be applied below $\bar{q}$. For instance we show (16). For $p < \bar{p}$, $\{z : F_Y(z) \leq p\} = \{z < \bar{q} : F_Y(z) \leq p\}$. The inclusion $\supseteq$ is obvious, while the inclusion $\subseteq$ follows from $F_Y(\bar{q}) = \bar{p} > p \geq F_Y(z)$, so by non-decreasingness of F_Y we must have $z < \bar{q}$. Similarly, we show $\{z : F_X(z) \leq p\} = \{z < \bar{q} : F_X(z) \leq p\}$. The inclusion $\supseteq$ is again obvious, while the inclusion $\subseteq$ follows from the fact that any $\tilde{z}$ s.t. $F_X(\tilde{z}) \leq p < \bar{p} = F_Y(\bar{q}) \leq F_X(\bar{q})$, and so $\tilde{z} < \bar{q}$. Finally, notice that $\{z < \bar{q} : F_Y(z) \leq p\} \subseteq \{z < \bar{q} : F_X(z) \leq p\}$: pick $\tilde{z}$ such that $p \geq F_Y(\tilde{z}) > F_X(\tilde{z})$ since $\tilde{z} < \bar{q}$. The suprema must therefore satisfy $q_Y(p) \leq q_X(p)$.

As to (17), recall the result (see Levy (1998) for instance) that if XSOSDY then $\int_0^p \tilde{F}_X^{-1}(z)dz \geq \int_0^p \tilde{F}_Y^{-1}(z)dz$, all $p \in (0,1]$, and so $-\frac{1}{p}\int_0^p \tilde{F}_Y^{-1}(z)dz \geq -\frac{1}{p}\int_0^p \tilde{F}_X^{-1}(z)dz$, for all $p \in (0,1]$.

Omega (18) is consistent, since $\Omega_X \leq \Omega_Y$ iff

$$\left[\int_q^\infty [1 - F_X(z)]dz\right]\left[\int_{-\infty}^q F_Y(z)dz\right] \geq \left[\int_q^\infty [1 - F_Y(z)]dz\right]\left[\int_{-\infty}^q F_X(z)dz\right].$$

The second term on the LHS dominates the second term on the RHS by definition of SOSD. The same is true as to the first terms on each side. Indeed, a repeated application of integration by parts shows that $E[X] = q + \int_q^\infty [1 - F_X(z)]dz - \int_{-\infty}^q F_X(z)dz$ for any $q \in \mathbb{R}$. Since by definition of SOSD $E[X] = E[Y]$, $\int_{-\infty}^q F_X(z)dz \leq \int_{-\infty}^q F_Y(z)dz$ iff $\int_q^\infty [1 - F_X(z)]dz \leq \int_q^\infty [1 - F_Y(z)]dz$, which shows that the first term on LHS dominates the first term on RHS and establishes the consistency of Omega with SOSD.

Relationship (19) can be shown in a similar vein to (9). We augment the example given in that proof in order to ensure $\mu_X = \mu_Y$. The two random variables are now quadrinomial with distribution functions $F_Y(z) = \mathbf{1}_{0 \leq z < q}\eta + \mathbf{1}_{z \in [q,q')}\kappa + \mathbf{1}_{z \geq q'}$ and $F_X(z) = \mathbf{1}_{0 \leq z < q}\eta + \mathbf{1}_{z \in [q,q')}\pi + \mathbf{1}_{z \geq q'}$, with parameters satisfying $0 < \eta < \pi < \kappa < 1$ and realizations $0 < q < q' < 1$. First, we choose κ to ensure that $\mu_X = \mu_Y$, equivalently that $\int_0^1 [F_Y(z) - F_X(z)]dz = 0$, i.e. $\kappa = \frac{(1-q)-(1-\pi)(q'-q)}{1-q}$. It can be checked that always $\kappa \in (\pi, 1)$. Clearly, XSOSDY. Now we show that for $p \in (\eta, \pi)$ we have $\text{TCE}_Y(p) - \text{TCE}_X(p) < 0$ while for $p \in (\pi, \kappa)$ we have $\text{TCE}_Y(p) - \text{TCE}_X(p) > 0$. Pick any $p \in (\eta, \pi)$. Then $q_X(p) = q_Y(p) = q$, $F_X(q_X(p)) = \pi$ and $F_Y(q_Y(p)) = \kappa$. It follows that $\text{TCE}_Y(p) - \text{TCE}_X(p) = (\kappa^{-1} - \pi^{-1})q\eta < 0$. Finally, pick $p \in (\pi, \kappa)$. Then $q_X(p) = q'$, $q_Y(p) = q$, $F_X(q_X(p)) = F_Y(q_Y(p)) = 1$. It follows that $\text{TCE}_Y(p) - \text{TCE}_X(p) = (q' - q)(1 - \pi) > 0$.

Inequality (20) is immediate. Inequality (21) for the first interpretation follows from the fact that if $(X+R)$SOSD$(Y+R)$, then $\sigma_{X+R} \leq \sigma_{Y+R}$, which in turn is equivalent to $\beta_X \leq \beta_Y + \frac{\sigma_Y^2 - \sigma_X^2}{2\sigma_R^2} = \beta_Y$. As to the second interpretation, XSOSDY implies that $\sigma_X^2 \leq \sigma_Y^2$, i.e. $\beta_X^2\sigma_R^2 + \sigma_{\epsilon_X}^2 \leq \beta_Y^2\sigma_R^2 + \sigma_{\epsilon_Y}^2$. With equal idiosyncratic variances, the result follows.

Inequality (22) is shown as follows. For $q \leq \bar{q}$, $F_X(q) \leq F_Y(q) = F_X\big(\frac{\sigma_X(q-\mu_Y)}{\sigma_Y} + \mu_X\big)$, i.e. iff $q\big(1 - \frac{\sigma_X}{\sigma_Y}\big) \leq \mu\big(1 - \frac{\sigma_X}{\sigma_Y}\big)$. If $\sigma_X = \sigma_Y$, then both distribution functions coincide, and the result follows since all inequalities hold with equality. If they do not coincide ($\sigma_X < \sigma_Y$), then the inequality becomes $q \leq \mu := \mu_Y = \mu_X$. Then $\bar{p} = F_Y(\mu)$.

Recall that $\text{IQR}_X - \text{IQR}_Y = q_X(3/4) - q_X(1/4) - q_Y(3/4) + q_Y(1/4)$. Also, by the assumption of belonging to the same two-parameter family, $q_Y(p) = \mu_Y + (q_X(p) - \mu_X)\frac{\sigma_Y}{\sigma_X}$. Substituting this into the difference equation, we get $\text{IQR}_X - \text{IQR}_Y = \big(1 - \frac{\sigma_Y}{\sigma_X}\big)\big[q_X(3/4) - q_X(1/4)\big] \leq 0$ since $\sigma_X \leq \sigma_Y$ by SOSD.
■

Thus, under the second order stochastic dominance of X over Y, we can order random variables under the overall risk measure σ and the downside risk measures SLPM, FLPM, ES and Ω consistently without any explicit distributional assumption.

Market beta risk also leads to an unambiguous ordering. As to the first interpretation, we show that for two equally volatile prospects, if any well-diversified risk averse investor prefers X to Y, then X has less systematic risk than Y.[9] As to the second interpretation, if any risk averse investor (not

[9]The role of $\sigma_X = \sigma_Y$ is as follows. Assume first that we do not impose $\sigma_X = \sigma_Y$. Then the ordering implied by $(X+R)$SOSD$(Y+R)$ is $\beta_X \leq \beta_Y + \frac{\sigma_Y^2 - \sigma_X^2}{2\sigma_R^2}$. A risk averter might prefer the prospect X to Y even if the diversification benefits as encapsulated in β_X are less than the ones provided by β_Y if in return X is by itself sufficiently less volatile than Y. So the real test should be the following: for two prospects X and Y with equal means and equal standard deviations, if XSOSD$_\beta Y$, i.e. if any representative risk averter prefers

necessarily well-diversified) prefers prospect X to Y, with both prospects having equally volatile idiosyncratic shocks, then X has lower systematic risk than Y.

As far as the other measures are concerned, the downside risk measures ZLPM and VaR retain partial SOSD-consistent ordering of assets under arbitrary (not known) asset returns distributions, partial meaning below the first crossing quantile of the two distributions. VaR is therefore both coherent and consistent in the tail, provided there is a first crossing point. For ZLPM and VaR, we have a reversal of the ordering immediately to the right of the first crossing points. In that sense, the results reported in Proposition 2 are tight. As observed before, TCE is not in general consistent with SOSD, unless more is known about distributions, such as continuity.

If asset returns are distributed according to the same two-parameter family (with different parameters), then we can show that on top of the preservation of the orderings as spelled out in Proposition 2, IQR also preserves the same preference ordering as under the SOSD rule for the entire range of the distributions.

Furthermore, the following converses and equivalences can be shown:

Proposition 3 *The following are equivalent:*

1. X SOSD Y
2. $\mathrm{ES}_X \leq \mathrm{ES}_Y$ *and* $\mathrm{ES}_X(1) = \mathrm{ES}_Y(1)$.
3. $\Omega_X \leq \Omega_Y$ *and* $\Omega_X^{-1}(1) = \Omega_Y^{-1}(1)$.
4. $\mathrm{FLPM}_X \leq \mathrm{FLPM}_Y$ *and* $\lim_{q\to\infty}[\mathrm{FLPM}_Y(q) - \mathrm{FLPM}_X(q)] = 0$

Assume for the remainder of the proposition that X and Y are such that $F_X(x) = F_Y(y)$ whenever $\frac{x-\mu_X}{\sigma_X} = \frac{y-\mu_Y}{\sigma_Y}$, and that $|\bar{q}| < \infty$. The following are also equivalent:

a. X SOSD Y

b. $\sigma_X \leq \sigma_Y$ *and* $\mu_X = \mu_Y$

c. $\mathrm{IQR}_X \leq \mathrm{IQR}_Y$ *and* $\mu_X = \mu_Y$

Finally, are equivalent:

X to Y, then it must be that X provides better diversification than Y, i.e. $\beta_X \leq \beta_Y$.

i. $X\text{SOSD}_\beta Y$

ii. $\beta_X \leq \beta_Y$, $\mu_X = \mu_Y$ *and* $\sigma_X = \sigma_Y$

as well as

i'. $X\text{SOSD}'_\beta Y$

ii'. $\beta_X \leq \beta_Y$, $\mu_X = \mu_Y$ *and* $\sigma_{\epsilon_X} = \sigma_{\epsilon_Y}$

Proof :
First we show 1 $\Leftrightarrow$ 2. The implication $\Rightarrow$ has been shown in Proposition 2. The implication $\Leftarrow$ can be shown as follows. By a change of variables one shows that $E[X] = \int_0^1 \tilde{F}_X^{-1}(p)dp = -\text{ES}_X(1)$. Finally, we know from Levy (1998) that $\int_{-\infty}^q F_X(z)dz \leq \int_{-\infty}^q F_Y(z)dz$ for all q iff $\int_0^p \tilde{F}_X^{-1}(z)dz \geq \int_0^p \tilde{F}_Y^{-1}(z)dz$, all $p \in (0,1]$. Both facts together establish the equivalence.

Now turn to 1 $\Leftrightarrow$ 3. The implication $\Rightarrow$ has been shown in Proposition 2. The implication $\Leftarrow$ can be shown as follows. It is easy to see that the Omega function is strictly increasing and satisfies $\Omega_X(E[X]) = 1$. So assuming that $\Omega_X^{-1}(1) = \Omega_Y^{-1}(1)$ is equivalent to saying $E[X] = E[Y]$. Second, argue by contradiction and assume that $\exists\tilde{q}$ s.t. $\int_{-\infty}^{\tilde{q}} F_X(z)dz > \int_{-\infty}^{\tilde{q}} F_Y(z)dz$, equivalently that $\int_{\tilde{q}}^{\infty}[1 - F_X(z)]dz > \int_{\tilde{q}}^{\infty}[1 - F_Y(z)]dz$. Then by definition $\Omega_X(\tilde{q}) > \Omega_Y(\tilde{q})$, which proves the assertion.

As to 1 $\Leftrightarrow$ 4, the only thing to check is that $\lim_{q\to\infty}[\text{FLPM}_Y(q) - \text{FLPM}_X(q)] = 0$ is equivalent to $E[X] = E[Y]$, which is true in view of the fact that $E[X] - E[Y] = \int_{-\infty}^{\infty}[F_Y(z) - F_X(z)]dz$.

The results for the two parameter families are shown as follows. The proof that $\sigma_X \leq \sigma_Y$ is both necessary and sufficient for SOSD if there is a crossing point is due to Hanoch and Levy (1969). This fact also implies the sufficiency of the IQR ranking, combined with the observation above that $\text{IQR}_X \leq \text{IQR}_Y$ iff $\sigma_X \leq \sigma_Y$. The fact that both $X\text{SOSD}_\beta Y$ and $X\text{SOSD}'_\beta Y$ iff $\beta_X \leq \beta_Y$ and respective variances are equal follows from this also. ■

The risk measures ES, Ω and FLPM generate an ordering of prospects equivalent to SOSD, provided the means are set equal. The relative usefulness of the various risk measures then lies in their orderings of prospects where neither SOSD the other and where a chosen cutoff point reflects the behaviour of some relevant utility function. For instance in the case of Ω, this involves choosing a critical threshold point q above which the investor considers outcomes to be gains and below which outcomes are considered to be losses.

Since in practice p is way smaller than $F_Y(\mu)$, for two prospects from the same two-parameter distribution, the ordering based on SOSD implies a consistent ordering of all the risk measures studied herein in all practical circumstances. Conversely, the ordering of SOSD is, within such two parameter families, equivalent to the one induced by any of the overall risk measures as well as by any of the downside measures with the exception of ZLPM, VaR and TCE.

6 Conclusion

In this paper we compare the partial orderings induced by a set of commonly used risk measures with the one given by stochastic dominance of first and second orders. We show that the overall risk measures do not always display a consistent preference ordering under the FOSD condition, even after imposing the simplifying assumption of two-parameter distributions. However, regardless of the asset return distributions, the ordering displayed by all the downside risk measures is consistent with FOSD, with the exception of TCE.

We observe that regardless of the asset return distributions, all risk measures display consistent preference ordering of assets under the SOSD rule, at least over the relevant quantile regions, again with the exception of TCE. For FLPM, ES and Ω the converse holds as well. Under the assumptions of two-parameter distributions and the existence of a crossing quantile, we show the converse for the overall risk-measures, so that overall risk measures generate the same ordering as SOSD.

References

Acerbi, C., Nordio, C., and Sirtori, C. (2001). Expected shortfall as a tool for financial risk management. Mimeo, http://www.gloriamundi.org/var/wps.html.

Acerbi, C. and Tasche, D. (2002). On the coherence of expected shortfall. *Journal of Banking and Finance*, 26:1487–1503.

Artzner, P., Delbaen, F., Eber, J., and Heath, D. (1999). Coherent measures of risk. *Mathematical Finance*, 9 (3):203–228.

Bawa, V. (1975). Optimal rules for ordering uncertain prospects. *Journal of Financial Economics*, 2:95–121.

Brumelle, S. and Vickson, R. (1975). A unified approach to stochastic dominance. In Ziemba, W. T. and Vickson, R., editors, *Stochastic Optimization Models in Finance*. Academic Press.

Daníelsson, J., Jorgensen, B. N., Sarma, M., Samorodnitsky, G., and de Vries, C. G. (2005). Subadditivity re–examined: the case for value–at–risk. *www.RiskResearch.org*.

De Giorgi, E. (2005). Reward-risk portfolio selection and stochastic dominance. *Journal of Banking and Finance*, 29:895–926.

Dhaene, J., Goovaerts, M. J., and Kaas, R. (2003). Economic Capital Allocation Derived from Risk Measures. *North American Actuarial Journal*, 7(2):44–56.

Fama, E. and Roll, R. (1968). Some properties of symmetric stable distributions. *Journal of the American Statistical Association*, 63:817–836.

Fishburn, P. (1977). Mean–risk analysis with risk associated with below–target returns. *American Economic Review*, 67 (2):116–126.

Hadar, J. and Russell, W. (1969). Rules for ordering uncertain prospects. *American Economic Review*, 59:25–34.

Hanoch, G. and Levy, H. (1969). The efficiency analysis of choices involving risk. *Review of Economic Studies*, 36:335–346.

Huang, C.-F. and Litzenberger, R. (1988). *Foundations for Financial Economics*. North-Holland.

Kaplanski, G. and Kroll, Y. (2000). Var risk measures versus traditional risk measures. *Journal of Risk*, 4.

Keating, C. and Shadwick, W. F. (2002). A universal performance measure. *Journal of Performance Measurement*, 6:59–84.

Levy, H. (1998). *Stochastic Dominance*. Kluwer Academic.

Quirk, J. and Saposnik, R. (1962). Admissibility and measurable utility functions. *Review of Economic Studies*, 29:140–146.

Rothschild, M. and Stiglitz, J. (1970). Increasing risk: A definition. *Journal of Economic Theory*, 2:225–243.

Strassen, V. (1965). The existence of probability measures with given marginals. *Annals of Mathematical Statistics*, 36:423–439.

Tesfatsion, L. (1976). Stochastic dominance and the maximization of expected utility. *Review of Economic Studies*, 43:301–315.