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Abstract

We study the market for a risky asset with heterogeneous valuations. Agents seek to learn about their own valuation by acquiring private information and making inferences from the equilibrium price. As agents of one type gather more information, they pull the equilibrium price closer to their valuation and further away from the valuations of other types. Thus they exert a negative learning externality on other types. This, in turn, implies that a lower cost of information for one type induces all agents to produce more information. When evaluating agents' welfare, the learning externality has to be offset against a gains from trade externality, since agents who learn less because their valuation is further away from the price also stand to profit more from trading. In equilibrium, agents' information acquisition decisions are clustered together more than is socially optimal.

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1 Introduction

We study the market for a risky asset in which agents differ in their valuations for the asset. Heterogeneity in valuations can be due to different uses that agents have for the asset, motivated by speculation, hedging or liquidity considerations, for example, or for purely behavioral reasons.¹ Each agent collects private information about his own valuation, and the equilibrium price reflects some of this information. Our aim is to study the externalities that arise in this setting, and in particular how they affect the equilibrium allocation of private information and the welfare of market participants.

We analyze competitive rational expectations equilibria in a linear-normal model. To understand the mechanics of this model, suppose there are two types of agents, with valuations θ_1 and θ_2 . Agents of type i ($i = 1, 2$) choose the precision τ_i of a private signal about their valuation θ_i , at a cost that is increasing in the precision. For any given choice of precisions, τ_1 and τ_2 , the price function takes the form $p = \mu(\tau_1\theta_1 + \tau_2\theta_2)$, for some constant μ . The optimal choice of τ_i depends on how much agents of type i learn about θ_i from the price. Assuming that the correlation between θ_1 and θ_2 is nonnegative, agents of type 1 learn more, and agents of type 2 learn less, about their respective valuations the greater is the ratio τ_1/τ_2 .

Now consider an equilibrium (τ_1, τ_2) , and suppose there is a decrease in the cost of information for type 1. This induces type 1 agents to collect more information (increasing τ_1), thus reducing price informativeness for type 2. As a result, type 2 agents collect more information as well (increasing τ_2). This in turn reinforces the incentive of type 1 agents to accumulate more information, increasing τ_1 even further. The resulting feedback loop leads to an equilibrium in which both types gather more information. The effect is more pronounced for type 1 agents: τ_1/τ_2 is higher at the new equilibrium. Consequently, type 1 agents learn more from the price and type 2 agents learn less.

With more than two types, this *monotone comparative statics* property, whereby a lower cost of information for one type results in more information production by all types, holds if the economy exhibits *strategic complementarities* in information acquisition, by which we mean that more information collection by any one type leads to lower price informativeness for all other types. The strategic complementarities condition is satisfied in the two-type case if and only if the correlation between the valuations of the two types exceeds a (negative) lower bound ρ . With arbitrarily many types, the condition is satisfied if pairwise correlations exceed ρ and, in addition, do not vary too much. In such an economy, a lower cost of information for one type results in all types gathering more information. At the new equilibrium, the type whose cost is reduced learns more from the price while all other types learn

¹Rostek and Weretka (2012) provide examples of heterogeneous valuations based on group affiliations or on the geographic location of traders. Rahi and Zigrand (2018) show how diversity in valuations can be microfounded by adding hedgers to a model along the lines of Grossman and Stiglitz (1980) or Hellwig (1980).

less.

Next, we turn to the question of social optimality of private information acquisition. In particular, we investigate the welfare effects of a change in the precision vector $\tau := (\tau_1, \dots, \tau_N)$ in the neighborhood of an equilibrium. Agents' welfare depends on how much they learn about their valuation from the equilibrium price. All else equal, they are better off if they are better informed. However, the price is more informative about valuation θ_i only if it tracks θ_i more closely, which has a countervailing effect on the gains from trade that type i agents can exploit (there are no gains from trade for type i if $p = \theta_i$, for example). For each type, the overall welfare effect of a local change in τ can be written as the sum of a learning effect and a gains from trade effect.

To understand how these two effects interact, a useful benchmark is that of a symmetric economy in which all types have the same cost of acquiring information and the correlation between valuations is the same for any pair of types. Such an economy has a unique equilibrium at which all types choose the same precision. At this equilibrium, the two effects are collinear but opposite in sign. Moreover, the gains from trade effect dominates the learning effect, so that the types that are better off are precisely those for whom price informativeness is lower. Price informativeness cannot be lower for all types, however, and hence a perturbation of τ cannot make all types better off.

The possibility of a Pareto improvement arises in the non-symmetric case. We consider an economy with two types who differ in their cost of information collection. For example, suppose type 2 has the lower cost. Then there is a unique equilibrium at which $\tau_1 < \tau_2$. The low cost type produces more private information as intuition would suggest. But a Pareto improving allocation of information can always be found. It entails more information production in the aggregate, a higher proportion of which is acquired by the low cost type, i.e. a higher $\tau_1 + \tau_2$ and a higher $\tau_2/(\tau_1 + \tau_2)$. The higher proportion of information produced by type 2 makes prices more informative for this type, while the welfare effect is negative; the opposite is true for type 1. The higher amount of aggregate information redistributes gains from trade from type 1 to type 2, so that the net effect is a welfare improvement for both. Note that the two types end up being further apart in terms of the proportion of information that they collect. This is in line with our comparative statics results, which suggest that negative information spillovers across types lead to information acquisition decisions that are clustered together more than is socially optimal.

Related Literature:

A growing strand of literature starts from the premise that agents have interdependent private valuations for a traded asset, and each agent has private information about his own valuation. [Vives \(2011\)](#) studies strategic supply function competition. [Rostek and Weretka \(2012, 2015\)](#) extend this setup to investigate the effect of market size on information aggregation and market power. [Glebkin \(2019\)](#) considers the case of two types, one of which consists of large strategic traders while the other is

perfectly competitive, to study the interplay between liquidity and price informativeness. [Heumann \(2018\)](#) analyzes the welfare properties of a competitive economy with multidimensional signals, in terms of the distance between the equilibrium of this economy and that of a full information benchmark. In [Babus and Kondor \(2018\)](#), dealers engage in bilateral trading in a network.

These papers employ a linear-Gaussian framework with exogenously specified valuations that vary across agents, just as in the present paper, but with more stringent assumptions on the correlations between these valuations. [Vives \(2011\)](#), [Heumann \(2018\)](#), and [Babus and Kondor \(2018\)](#) assume that the correlations are the same for any pair of agents or agent types ($\rho_{ij} = \rho$ for all $i \neq j$, in the notation of our paper); this is also true in [Glebkin \(2019\)](#) since he has only two types. [Rostek and Wernetka \(2012, 2015\)](#) present a convincing argument for a general correlation structure, but restrict their analysis to the “equicommonal” case, wherein the average correlation between the valuation of a trader and that of the remaining traders is the same for all traders. Moreover, the symmetry assumptions imposed in all these papers ensure that price informativeness is the same for all agents, with the exception of [Babus and Kondor \(2018\)](#) who use an aggregate measure of constrained informational efficiency.²

Private information is exogenous in the papers cited above. [Vives \(2014\)](#) analyzes information acquisition in a model with private valuations. But there are no learning externalities in this model. As a consequence, information acquisition is socially efficient provided the marginal cost of information is sufficiently low. Learning externalities take center stage in [Rahi and Zigrand \(2018\)](#) (henceforth RZ), which serves as our point of departure, and from which we borrow some results (Proposition 2.1 and Lemma 3.1) on the price function, price informativeness, and utilities for exogenously given signal precisions. RZ study a binary information acquisition decision, wherein agents either acquire a piece of information at some cost or remain uninformed. In the present paper, we allow agents to choose the precision of their signal at a cost that increases in the precision. We assume that there is no fixed cost so that all types acquire some information. As such, our results complement those of RZ. Our welfare results, in particular, provide a different perspective on the Pareto inefficiency of the equilibrium allocation of private information. RZ show that discouraging information acquisition can be Pareto improving if private signals are sufficiently noisy. In the present paper, the precision of private signals is endogenous, and a Pareto improvement involves an increase in the total amount of information.³

There is a large literature on the social value of public information in a pure ex-

²Other papers that use a correlated private values setting in which equilibrium prices convey information include [Bernhardt and Taub \(2015\)](#) and [Du and Zhu \(2017\)](#).

³Compared to RZ, our analysis is also more general. In our model, a change in the cost of information acquisition of one type affects the information acquisition decisions of all types, unlike RZ who limit themselves to “corner” equilibria in which perturbing the cost of information acquisition affects the decisions of only one type. Furthermore, for many of our results, we allow an arbitrary correlation structure for valuations across types unlike RZ who (in their results with endogenous information acquisition) assume that all pairwise correlations are the same.

change economy. More information can reduce risk-sharing opportunities (or, indeed, destroy them altogether, as in [Hirshleifer \(1971\)](#)). If markets are incomplete, it can also allow agents to construct better hedges. The overall impact on agents' welfare can be in any direction ([Gottardi and Rahi \(2014\)](#)). Much less is known about the welfare properties of asset markets in which information is endogenous and asymmetric. Most of the rational expectations literature relies on exogenous noise trade and hence does not provide a suitable framework for welfare analysis. There are a few papers with fully optimizing traders but, with the exception of [Vives \(2014\)](#) and [Rahi and Zigrand \(2018\)](#) discussed above, they do not ask if the amount of information produced by agents is socially optimal.

A number of papers feature a complementarity in information acquisition that arises because prices become less informative as more agents acquire information. The underlying mechanism differs across these papers. In [Barlevy and Veronesi \(2008\)](#) price informativeness falls with the incidence of informed trading because the asset payoff is negatively correlated with the noise trade, in [Ganguli and Yang \(2009\)](#) and [Manzano and Vives \(2011\)](#) because agents have two sources of information (about the asset payoff and the asset supply), in [Goldstein et al. \(2014\)](#) because agents with different investment opportunities trade on the same information in opposite directions, and in [Breon-Drish \(2012\)](#) due to non-normality of shocks. These papers use complementarity in information acquisition as a vehicle for generating multiple equilibria. In contrast, strategic complementarities in our paper actually describe the “well-behaved” case. Consider a two-type economy (which provides the closest analog to the papers discussed here) with strategic complementarities. This economy has a unique equilibrium with intuitive comparative statics. The issue here is partly a terminological one. We use strategic complementarities as a label to describe the case where higher information production by one type lowers price informativeness for the other type. But this is precisely the case in which higher information production by a given type *increases* price informativeness for that type itself. Accordingly, we could equally classify the economy as one that exhibits strategic substitutes if we take the perspective of the complementarity acting within rather than across types. It is worth noting that the multiplicity of equilibria in [Rahi and Zigrand \(2018\)](#) requires a within-type complementarity which is ruled out by the (across-type) strategic complementarity assumption that we make in this paper.

We now lay out a brief road-map for the rest of the paper. In the next section, we describe the basic setup and the price function for a given vector of precisions for each type. We endogenize these precisions in Section 3. In Section 4, we provide sufficient conditions on the primitives for the economy to exhibit strategic complementarities. This forms the basis of the existence and comparative statics results in Section 5. The welfare analysis is in Section 6. Proofs are in the Appendix.

2 The Economy

There is a single risky asset in zero net supply, and a riskfree asset with the interest rate normalized to zero. There are N types of agents, $N \geq 2$, and a continuum of agents of unit mass of each type. The private valuation for the risky asset of an agent of type i is given by θ_i . Prior to trade, type i agents can acquire a private signal about θ_i . For agent n of type i (agent in for short) this signal takes the form $s_{in} = \theta_i + \epsilon_{in}$, where the precision (the reciprocal of the variance) of ϵ_{in} is τ_{in} . The cost of this signal is $\mathcal{C}_i(\tau_{in})$. For now we will assume that all agents of type i choose the same precision τ_i , and that $\tau_i > 0$. Later, when we impose some conditions on the function \mathcal{C}_i and endogenize precision choice, we will see that this assumption is indeed satisfied.

The random variables $\{\theta_i, \{\epsilon_{in}\}_{n \in [0,1]}\}_{i=1,\dots,N}$ are joint normal with mean zero. Let $\theta := (\theta_i)_{i=1}^N$. For each type i , the signal shock ϵ_{in} is independent of θ , and the signal shocks across agents, $\{\epsilon_{in}\}_{n \in [0,1]}$, are independent. Given the assumption that τ_{in} is the same for all n , these signal shocks are in fact i.i.d. Then, the average signal of agents of type i , $\int_n s_{in} dn$, is equal to θ_i . To ensure that the problem is nontrivial, we assume that the covariance matrix of θ is positive definite. We also assume that the variance of θ_i is the same for all i . We denote the correlation matrix of θ by R , with ij 'th element $\rho_{ij} := \text{corr}(\theta_i, \theta_j)$, and the i 'th column of R by R_i . Due to the symmetry of R , the i 'th row of R is R_i^\top .

If agent in buys q_{in} units of the risky asset at price p , his “wealth” is

$$W_{in} = (\theta_i - p)q_{in} - \mathcal{C}_i(\tau_{in}).$$

He has CARA utility with risk aversion coefficient r . He solves

$$\max_{q_{in}} E[-\exp(-rW_{in}) | s_{in}, p].$$

Agents have rational expectations: they know the price function, a function of the private signals of all agents in the economy, and condition on the price when making their portfolio decisions. We assume that the trade of agent in is measurable with respect to his information (s_{in}, p) .

An equilibrium consists of a vector of precisions $\tau := (\tau_i)_{i=1}^N$ and a price function p such that agents optimize and markets clear. Agent optimization requires that each agent is happy with his choice of precision given the price function p , and subsequently, for any realization of p , he chooses an optimal portfolio given his information. Letting $q_i := \int_n q_{in} dn$, the aggregate trade of type i , the market-clearing condition is $\sum_i q_i = 0$.

We define price informativeness for agents of type i by

$$\mathcal{V}_i := \frac{\sigma_\theta^2 - \sigma_{\theta_i|p}^2}{\sigma_\theta^2}, \tag{1}$$

where $\sigma_\theta^2 := \text{Var}(\theta_i)$, assumed to be the same for all i , and $\sigma_{\theta_i|p}^2 := \text{Var}(\theta_i|p)$. We denote the corresponding precisions by τ_θ and $\tau_{\theta_i|p}$, respectively. The following result is taken from [Rahi and Zigrand \(2018\)](#):

Proposition 2.1 (Price function, price informativeness) *For a given vector of precisions τ , there is a unique linear equilibrium price function:*

$$p = \mu \tau^\top \theta, \quad \mu \neq 0.$$

Price informativeness for type i is given by

$$\mathcal{V}_i = \frac{(R_i^\top \tau)^2}{\tau^\top R \tau}. \quad (2)$$

The proposition describes the linear rational expectations equilibrium (REE) price function for an arbitrary, exogenously given, precision vector τ . The coefficient of θ_i in the price function is proportional to the precision of type i . The price function does not fully reveal θ_i for any i ; hence $\mathcal{V}_i \in [0, 1)$. This follows from the assumption that $\tau_i > 0$ for all i and that the correlation matrix R is positive definite. Price informativeness for any type is homogeneous of degree zero in τ . Thus scaling the vector τ leaves price informativeness unchanged for all types.

3 Information Acquisition

We now endogenize the choice of precision. Agent in pays the cost $\mathcal{C}_i(\tau_{in})$ for a signal of precision τ_{in} . The function \mathcal{C}_i takes the following form: $\mathcal{C}_i(\tau_{in}) := \alpha_i C_i(\tau_{in})$, for some $\alpha_i \in [\underline{\alpha}, \bar{\alpha}]$, $\bar{\alpha} > \underline{\alpha} > 0$. Let $\alpha := (\alpha_i)_{i=1}^N$ and let \mathcal{A} denote the N -fold Cartesian product of $[\underline{\alpha}, \bar{\alpha}]$. We specify cost functions in this way as we will be interested in comparative statics with respect to $\alpha \in \mathcal{A}$. We impose the following conditions on C_i :

The function $C_i : [0, \infty) \rightarrow [0, \infty)$ is twice-differentiable and satisfies

- i. $C_i(0) = 0$;
- ii. $C'_i(0) = 0$, and $C'_i(x) > 0$ for $x > 0$;
- iii. $C''_i > 0$.

In particular, we assume that there are no fixed costs and that obtaining a small amount of information is cheap. This ensures that each agent acquires some information.

It is convenient to use the following monotonic transformation of ex ante expected utility:

$$\mathcal{U}_{in} := (E[\exp(-r_i W_{in})])^{-2}.$$

Let $\sigma_{\theta_i-p}^2 := \text{Var}(\theta_i - p)$. Then, from Lemma 6.1 in [Rahi and Zigrand \(2018\)](#), we have:

Lemma 3.1 (Utilities) *For a given vector of precisions τ , the utility of agent n of type i is given by*

$$\mathcal{U}_{in} = \exp[-2r\alpha_i C_i(\tau_{in})] (\tau_{in} + \tau_{\theta_i|p}) \sigma_{\theta_i-p}^2.$$

This is the indirect utility of agent in for an exogenously specified precision vector τ , and the REE price function associated with this τ . It depends on the agent's cost of acquiring information, through the term $\exp[-2r\alpha_i C_i(\tau_{in})]$, on how much he learns about his valuation, given by $[\text{Var}(\theta_i|s_{in}, p)]^{-1} = \tau_{in} + \tau_{\theta_i|p}$, and on $\sigma_{\theta_i-p}^2$, which captures his "gains from trade". We defer a discussion of gains from trade to Section 6. It plays no role in the agent's choice of precision, which is governed solely by the tradeoff between learning and the cost of information.

Maximizing \mathcal{U}_{in} with respect to τ_{in} , we get the first-order condition:

$$2r\alpha_i C'_i(\tau_{in})(\tau_{in} + \tau_{\theta_i|p}) = 1. \quad (3)$$

The second-order condition is satisfied:

$$C'_i(\tau_{in}) + C''_i(\tau_{in})(\tau_{in} + \tau_{\theta_i|p}) > 0.$$

There is a unique solution τ_{in} to (3) which is positive and the same for all n ; we write it as $\tau_{in}(\tau)$. In equilibrium $\tau_i = \tau_{in}(\tau)$ for all i , so that all agents of type i have the same utility, which we denote by \mathcal{U}_i :

$$\mathcal{U}_i = \exp[-2r\alpha_i C_i(\tau_i)] (\tau_i + \tau_{\theta_i|p}) \sigma_{\theta_i-p}^2. \quad (4)$$

Moreover, from the first-order condition (3),

$$2r\alpha_i C'_i(\tau_i)(\tau_i + \tau_{\theta_i|p}) = 1. \quad (5)$$

From (1), we see that

$$\tau_{\theta_i|p} = \tau_\theta [1 - \mathcal{V}_i(\tau)]^{-1},$$

where $\mathcal{V}_i(\tau)$ is given by (2). Hence we can write (5) as

$$2r\alpha_i C'_i(\tau_i) [\tau_i + \tau_\theta [1 - \mathcal{V}_i(\tau)]^{-1}] = 1. \quad (6)$$

A vector of precisions τ is an equilibrium if and only if it is a solution to the equation system given by (6), $i = 1, \dots, N$.

4 Strategic Complementarities

Most of our results rely on the assumption of strategic complementarities in information acquisition. In this section we provide sufficient conditions on the correlation matrix R for this assumption to be satisfied.

In order to give a precise definition of strategic complementarities, we first need to bound R away from singularity. This ensures that precisions are bounded away from zero. Formally, let \mathcal{R} be the set of N -dimensional positive definite correlation matrices, an open convex set. The closure of \mathcal{R} , denoted by $\text{cl}(\mathcal{R})$, is the set of positive semidefinite correlation matrices. The boundary of \mathcal{R} is the set of correlation matrices in $\text{cl}(\mathcal{R})$ with zero determinant. Let \mathcal{R}_η be the subset of \mathcal{R} consisting of correlation matrices R for which $\det(R) \geq \eta$, for some $\eta \in (0, 1)$.

Lemma 4.1 (Precision bounds) *Suppose $R \in \mathcal{R}_\eta$. Then there are positive scalars $\underline{\tau}$ and $\bar{\tau}$, that are independent of R and α , such that $\tau_i \in [\underline{\tau}, \bar{\tau}]$ for all i . If C_i is the same for all types, then $\lim_{\tau_\theta \rightarrow 0} (\underline{\tau}/\bar{\tau}) > \underline{\alpha}/\bar{\alpha}$.*

Let \mathcal{T} be the N -fold Cartesian product of the interval $[\underline{\tau}, \bar{\tau}]$. Lemma 4.1 tells us that if $R \in \mathcal{R}_\eta$, then $\tau \in \mathcal{T}$. The assumption that $R \in \mathcal{R}_\eta$ is essentially without loss of generality since η can be chosen to be arbitrarily close to zero. If $\rho_{ij} = \rho$ for all $i \neq j$, R is positive definite if and only if $-(N-1)^{-1} < \rho < 1$ (see [Rahi and Zigrand \(2018\)](#), Lemma 6.5); in this case $R \in \mathcal{R}_\eta$ amounts to the assumption that $\rho \in [\underline{\kappa}, \bar{\kappa}]$, for some $\underline{\kappa}, \bar{\kappa}$ satisfying $-(N-1)^{-1} < \underline{\kappa} < \bar{\kappa} < 1$. Specializing further to the case of $N = 2$, the condition $R \in \mathcal{R}_\eta$ is equivalent to $|\rho| \leq \sqrt{1-\eta}$.

A change in the precision of type i affects price informativeness for all types. We refer to $\partial \mathcal{V}_i / \partial \tau_i$ as an ‘‘own-effect’’ and to $\partial \mathcal{V}_i / \partial \tau_j$ for $i \neq j$ as a ‘‘cross-effect’’. By strategic complementarities we mean that all cross-effects are negative on \mathcal{T} :

Definition 4.1 *The economy exhibits strategic complementarities if $\partial \mathcal{V}_i / \partial \tau_j < 0$ for all $i \neq j$, and all $\tau \in \mathcal{T}$.*

In an economy with strategic complementarities, more information collection by any one type leads to lower price informativeness for all other types, inducing the latter to acquire more information as well. Since \mathcal{V}_i is homogeneous of degree zero in τ , we have,

$$\sum_j \tau_j \frac{\partial \mathcal{V}_i}{\partial \tau_j} = 0,$$

by Euler’s theorem. It follows that, in an economy that exhibits strategic complementarities, $\partial \mathcal{V}_i / \partial \tau_i > 0$ for all i and all $\tau \in \mathcal{T}$. In other words, if all cross-effects are negative on \mathcal{T} , then all own-effects must be positive on \mathcal{T} . For $N = 2$, the converse is true as well.

Next, we show that own-effects are positive on \mathcal{T} if correlations exceed a threshold level $\underline{\rho}$ given by

$$\underline{\rho} := -\frac{\underline{\tau}}{(N-1)\bar{\tau}}.$$

Lemma 4.2 (Positive own-effects) *$\partial \mathcal{V}_i / \partial \tau_i > 0$ if and only if $R_i^\top \tau > 0$. Furthermore, if $R \in \mathcal{R}_\eta$, a sufficient condition for $R_i^\top \tau > 0$ for all i and all $\tau \in \mathcal{T}$ is $\rho_{ij} > \underline{\rho}$ for all i, j .*

The lower bound $\underline{\rho}$ depends on the cost parameters and satisfies

$$-\frac{1}{N-1} < \underline{\rho} < 0.$$

Note that if all pairwise correlations are the same, this common value must be greater than $-(N-1)^{-1}$ for R to be positive definite.

Positive own-effects are necessary for strategic complementarities. But they are also of some interest in their own right. Recall, from Proposition 2.1, that the price function is given by $p = \mu \tau^\top \theta$, for some nonzero scalar μ . Positive own-effects imply that μ is in fact positive:

Lemma 4.3 (Price function) *Consider a vector of precisions τ such that $R_i^\top \tau > 0$ for all i . Then the equilibrium price function takes the following form:*

$$p = \mu \tau^\top \theta, \quad \mu > 0.$$

If $\mu > 0$, it is easy to check that $R_i^\top \tau > 0$ if and only if $\text{Cov}(\theta_i, p) > 0$. Hence, positive own-effects are equivalent to positive correlations of all valuations with the REE price function.

Another consequence of positive own-effects is that price informativeness cannot change in the same direction for all types when we perturb τ . Before stating this result, we need some more notation. Given a function $f : X \rightarrow \mathbb{R}$, $X \subset \mathbb{R}^n$, we denote by $\partial_z f(x)$ the directional derivative of f at x in the direction $z \in \mathbb{R}^n$, i.e.

$$\partial_z f(x) := \sum_k z_k \frac{\partial f}{\partial x_k}(x).$$

We say that $A \propto B$ if A and B have the same sign ($A = kB$, for some $k > 0$).

Lemma 4.4 (Price informativeness) *Consider a vector of precisions τ such that $R_i^\top \tau > 0$ for all i . Then $\partial_z \mathcal{V}_i(\tau) \propto \partial_z \mathcal{V}_j(\tau)$, for all i, j , if and only if $\partial_z \mathcal{V}_k(\tau) = 0$, for all k .*

In other words, if own-effects are positive, a local change in τ cannot increase price informativeness for all types, nor can it reduce price informativeness for all.

We now turn to strategic complementarities. First, consider the two-type case. As we have already seen, an economy with two types exhibits strategic complementarities if and only if both own-effects are positive on \mathcal{T} . By Lemma 4.2, the condition that all correlations exceed $\underline{\rho}$ suffices for positive own-effects on \mathcal{T} . In the two-type case, this lower bound condition is in fact necessary (since there is only one pairwise correlation, we denote it by ρ rather than ρ_{12}):

Proposition 4.5 (Complementarities I) *Suppose $N = 2$ and $R \in \mathcal{R}_\eta$. Then the economy exhibits strategic complementarities if and only if $\rho > \underline{\rho}$.*

When there are three or more types, we need additional conditions on the correlation matrix. Let

$$\delta := \frac{\tau}{\bar{\tau}}.$$

Proposition 4.6 (Complementarities II) *Suppose $N \geq 3$ and $R \in \mathcal{R}_\eta$. Then the economy exhibits strategic complementarities if any of the following conditions is satisfied:*

i. For all $i \neq j$,

$$\underline{\rho} < \rho_{ij} \leq \frac{\underline{\rho}^2}{\underline{\rho}^2 + (1 - \underline{\rho}^2)(N - 2)^2}; \quad (7)$$

ii. For all $i \neq j$,

$$\check{\rho} \leq \rho_{ij} \leq \min \left\{ [1 + \check{\rho}(N - 2)]\delta, \frac{(1 + \check{\rho})\delta^2 + 2\check{\rho}(N - 2)\delta + \check{\rho}^2(N - 2)^2}{(1 + \check{\rho})\delta^2 + 2\check{\rho}(N - 2)\delta + (N - 2)^2} \right\} \quad (8)$$

for some $\check{\rho} \in [0, 1)$;

iii. For all $i \neq j$, $\rho_{ij} = \rho > \underline{\rho}$. Furthermore, if $\delta < 1/2$, then

$$\rho \leq \frac{\delta^2}{(1 - 2\delta)(N - 1)}. \quad (9)$$

Given $R \in \mathcal{R}_\eta$, the condition that $\rho_{ij} > \underline{\rho}$ for all i, j ensures that all own-effects are positive on \mathcal{T} , by Lemma 4.2. For cross-effects to be negative on \mathcal{T} (strategic complementarities), low or negative correlations suffice (condition (i)). An alternative condition is that all correlations are nonnegative and close to each other (condition (ii)); we provide examples below.

If $\rho_{ij} = \rho$ for all $i \neq j$, two cases arise. If $\delta \geq 1/2$, there is no further restriction. If $\delta < 1/2$, we need to impose an upper bound on ρ , given by (9). The value of δ depends on the cost parameters $\underline{\alpha}$ and $\bar{\alpha}$. By Lemma 4.1, if C_i is the same for all types, then $\lim_{\tau_\theta \rightarrow 0} \delta > \underline{\alpha}/\bar{\alpha}$. Therefore, $\delta \geq 1/2$ if $\underline{\alpha}/\bar{\alpha} \geq 1/2$, the function C_i is not very dissimilar across types, and the uncertainty regarding valuations is large (τ_θ is small).

Example 4.1 For this example, we will need to refer to the proof of Lemma 4.1 in the Appendix, in particular to equations (10)–(12).

Suppose $r = 1$, $C_i(\tau_i) = (1/12)\tau_i^2$ for all i , and $\underline{\alpha} = 3$. Suppose further that $\tau_\theta = 1 - \bar{\mathcal{V}}$, where $\bar{\mathcal{V}}$ is the maximal price informativeness for any type, as defined by (11). From (10) and (12), the bounds for τ_i are given by

$$\bar{\tau} = 1, \quad \underline{\tau} = \frac{1}{2} \left[-1 + \sqrt{1 + \frac{12}{\bar{\alpha}}} \right].$$

Therefore, $\delta = \tau/\bar{\tau} \geq 1/2$ if and only if $\bar{\alpha} \in (\underline{\alpha}, 4] = (3, 4]$.

Now suppose $N = 3$. If we take $\bar{\alpha} = 4$, we get $\delta = 1/2$ and $\underline{\rho} = -\delta/2 = -1/4$. Condition (7) reduces to

$$\underline{\rho} < \rho_{ij} \leq \underline{\rho}^2,$$

while condition (8) becomes

$$0 \leq \check{\rho} \leq \rho_{ij} \leq 0.2 + 0.8\check{\rho},$$

which places no restrictions on the value of $\check{\rho}$ in $[0, 1)$. The economy exhibits strategic complementarities if any of the following conditions holds for all $i \neq j$:

- i. $\rho_{ij} \in (-0.25, 0.0625]$;
- ii. $\rho_{ij} \in [0.25, 0.4]$, or $\rho_{ij} \in [0.75, 0.8]$;
- iii. $\rho_{ij} = \rho \in (-0.25, 1)$.

In (ii), we have chosen two values of $\check{\rho}$, 0.25 and 0.75, for illustration. The restriction $R \in \mathcal{R}_\eta$ does not have any bite since η can be taken to be arbitrarily close to zero. With regard to condition (iii) above, note that R is positive definite if and only if $\rho \in (-0.5, 1)$. Thus this condition rules out only a small subset of admissible values of ρ , those in the interval $(-0.5, -0.25]$. ||

5 Equilibrium Characterization

In this section we characterize the equilibrium allocation of private information, providing conditions for existence and uniqueness, and studying comparative statics with respect to the cost parameters $\alpha \in \mathcal{A}$. We show that, under appropriate conditions, an increase in α_j for any j reduces the equilibrium precision for all types. It also lowers price informativeness for type j , while increasing it for all other types. We then present further properties of equilibrium under the assumption of equal pairwise correlations, and provide a detailed analysis of the two-type case.

We say that an equilibrium τ satisfies the monotone comparative statics (MCS) property if τ is decreasing in α , $\alpha \in \mathcal{A}$. It satisfies the strong MCS property if $\partial\tau_i/\partial\alpha_j < 0$, for all $\alpha_j \in (\underline{\alpha}, \bar{\alpha})$, and for all i, j . An equilibrium $\hat{\tau}$ is the highest equilibrium if $\hat{\tau} \geq \tau$ for any equilibrium τ . Similarly, an equilibrium $\check{\tau}$ is the lowest equilibrium if $\check{\tau} \leq \tau$ for any equilibrium τ .

Proposition 5.1 (Existence, comparative statics) *Consider an economy that exhibits strategic complementarities. Then there exists a highest equilibrium $\hat{\tau}$ and a lowest equilibrium $\check{\tau}$. Both $\hat{\tau}$ and $\check{\tau}$ satisfy the MCS property.*

The comparative statics property is valid for any change in α in \mathcal{A} (not just for a local change). It relies on a theorem in [Milgrom and Shannon \(1994\)](#). In our model the MCS property can be strengthened to the strong MCS property:

Proposition 5.2 (Strong MCS) *Consider an economy that exhibits strategic complementarities. An equilibrium τ satisfies the MCS property if and only if it satisfies the strong MCS property.*

Thus $\partial\tau_i/\partial\alpha_j < 0$ for all i : an increase in α_j induces all agents to cut back on private information production. Type j agents reduce τ_j in response to an increase in α_j . This makes prices more informative for all other types, who in turn gather less information. This feeds back into more informative prices for type j agents, causing them to reduce τ_j further.

It is apparent from (6) that for types other than j this must be accompanied by an increase in price informativeness. Whether price informativeness for type j itself goes up or down is not pinned down in general (lower precisions for types other than j increase price informativeness for j , while a lower precision for j itself has the opposite effect). But locally at least we know that price informativeness cannot increase for all types (Lemma 4.4); hence it must go down for type j :

Proposition 5.3 (Comparative statics II) *Consider an economy that exhibits strategic complementarities. For any equilibrium that satisfies the MCS property, we have $\partial\mathcal{V}_j/\partial\alpha_j < 0$ and $\partial\mathcal{V}_i/\partial\alpha_j > 0$ for all $i \neq j$.*

If an economy that exhibits strategic complementarities has a unique equilibrium, it must satisfy the MCS property by Proposition 5.1 (in this case the highest and lowest equilibria coincide), and hence the strong MCS property by Proposition 5.2. The following result provides sufficient conditions for uniqueness:

Proposition 5.4 (Uniqueness) *Consider an economy that exhibits strategic complementarities. There is a unique equilibrium if one of the following conditions is satisfied:*

- i. $\rho_{ij} = 0$ for all $i \neq j$;*
- ii. $\rho_{ij} = \rho$ for all $i \neq j$, and N_1 types have the same cost function \mathcal{C}_{N_1} , while the remaining N_2 types have the same cost function \mathcal{C}_{N_2} .*

These conditions are fairly stringent. Note that the two-type case is a special case of condition (ii).

In our next result, we provide sufficient conditions under which lower cost types acquire more information (and learn more from prices as well):

Proposition 5.5 (Cost and precision) *Suppose that the functions C_i are the same for all i , and $\rho_{ij} = \rho$ for all $i \neq j$. Consider an equilibrium τ at which $R_i^\top \tau > 0$ for all i . Then,*

- i. $\alpha_i < \alpha_j \Leftrightarrow \tau_i > \tau_j \Leftrightarrow \mathcal{V}_i > \mathcal{V}_j$;*
- ii. $\alpha_i = \alpha_j \Leftrightarrow \tau_i = \tau_j \Leftrightarrow \mathcal{V}_i = \mathcal{V}_j$.*

This result does not require that the economy exhibit strategic complementarities. Under the stated assumptions on correlations and cost functions, it applies to any equilibrium at which own-effects are positive (by Lemma 4.2, own-effects are positive at τ if and only if $R_i^\top \tau > 0$ for all i).

We say that the economy is *symmetric* if all types have the same cost function (\mathcal{C}_i is the same for all i) and all pairwise correlations are the same ($\rho_{ij} = \rho$ for all $i \neq j$). This will serve as a useful benchmark for our welfare analysis in the next section. Positive own-effects on \mathcal{T} suffice for existence (and uniqueness) of equilibrium for such an economy:

Proposition 5.6 (Symmetric economy) *Consider a symmetric economy with correlation parameter ρ . Suppose $R \in \mathcal{R}_\eta$ and $\rho > \underline{\rho}$. Then there is a unique equilibrium. At this equilibrium all types choose the same precision.*

The condition that $\rho > \underline{\rho}$ ensures that own-effects are positive on \mathcal{T} , by Lemma 4.2. Then, by Proposition 5.5 (part (ii)), precision choices are the same for all types at any candidate equilibrium. Proposition 5.6 asserts that such an equilibrium exists and is unique.

We conclude this section with a discussion of the two-type case. Suppose own-effects are positive on \mathcal{T} . Then the economy exhibits strategic complementarities (Proposition 4.5), and there is a unique equilibrium (Proposition 5.4, part (ii)) satisfying the strong MCS property (Propositions 5.1 and 5.2). Furthermore, if $C_1 = C_2$, precisions are ranked according to the cost parameters α_1 and α_2 (Proposition 5.5). We summarize these observations, and some additional comparative statics properties, in the next proposition:

Proposition 5.7 (Equilibrium: two types) *Suppose $N = 2$, $R \in \mathcal{R}_\eta$ and $\rho > \underline{\rho}$. Then there is a unique equilibrium (τ_1, τ_2) with the following properties:*

- i. If $C_1 = C_2$, then $\tau_1 = \tau_2$ if and only if $\alpha_1 = \alpha_2$, and $\tau_1 < \tau_2$ if and only if $\alpha_1 > \alpha_2$.*
- ii. An increase in α_1 reduces τ_1, τ_2 , and τ_1/τ_2 . This is accompanied by a lower \mathcal{V}_1 and a higher \mathcal{V}_2 .*

The additional results in Proposition 5.7 that we have not encountered previously are as follows. First, while an increase in α_1 induces both types to gather less information (the strong MCS property), the impact on type 1 is greater. Second, the effect on price informativeness is to reduce it for type 1 and increase it for type 2; unlike Proposition 5.3, this result applies for arbitrary changes in α_1 , not just for local changes. Comparative statics with respect to α_2 are analogous.

6 Welfare

We now turn to a welfare analysis of the equilibrium allocation of private information. More precisely, we consider an equilibrium vector of precisions τ , and investigate the

welfare effects that arise when we perturb this vector, and thereby perturb the REE associated with it (recall that for any given τ , there is a unique linear REE, by Proposition 2.1).

We define the gains from trade for type i as

$$G_i := \frac{\sigma_{\theta_i - p}^2}{\sigma_\theta^2}.$$

Agents of type i have more profitable trading opportunities the greater the distance between their own valuation θ_i and the overall market valuation, given by the equilibrium price.⁴ We can decompose the effect of a local change in τ on the welfare of type i into two components, one arising from a change in price informativeness \mathcal{V}_i (the *learning effect*) and the other from a change in G_i (the *gains from trade effect*):

Lemma 6.1 (Welfare effects) *At an equilibrium τ , we have*

$$\partial_z \log \mathcal{U}_i(\tau) = \underbrace{\frac{\tau_{\theta_i|p}}{(1 - \mathcal{V}_i)(\tau_i + \tau_{\theta_i|p})} \partial_z \mathcal{V}_i(\tau)}_{\text{learning effect}} + \underbrace{\frac{1}{G_i} \partial_z G_i(\tau)}_{\text{gains from trade effect}}.$$

(Recall that $\partial_z f(x) := \sum_k z_k \frac{\partial f}{\partial x_k}(x)$ is the directional derivative of f at x in the direction z). All else equal, agents are better off if they are better informed. They are also better off if they can reap higher gains from trade.

The key question, however, is how these two effects interact. Indeed, there is a fundamental tension between them. If the equilibrium price tracks the valuation of an agent closely, it will reveal more information about that valuation. But the agent has more to gain from trade the further his valuation is from the price. To take a stark example, suppose $p = \theta_i$. Then price informativeness is maximal for type i agents ($\mathcal{V}_i = 1$), but there are no gains from trade for these agents ($G_i = 0$); their optimal trade is zero. Such a price function cannot arise in our model, of course (the price does not fully reveal θ_i for any i , and gains from trade are positive for all i), but it serves to illustrate the tradeoff between learning from prices and gains from trade.

A useful benchmark for investigating this tradeoff is a symmetric economy in which cost functions and pairwise correlations are the same across types. Such an economy has a unique equilibrium at which all types choose the same precision (Proposition 5.6). We have the following result:

Proposition 6.2 (Welfare: symmetric economy) *Consider a symmetric economy with correlation parameter ρ . Suppose $R \in \mathcal{R}_\eta$ and $\rho > \rho$. Then, at the unique equilibrium τ of this economy, $\partial_z G_i(\tau) = -\partial_z \mathcal{V}_i(\tau)$ and $\partial_z \mathcal{U}_i(\tau) \propto -\partial_z \mathcal{V}_i(\tau)$, for all i .*

⁴In our model, the distance between two random variables is measured by the variance of the difference between these random variables.

Thus, at the equilibrium of a symmetric economy, the learning and gains from trade effects are collinear but of the opposite sign,⁵ with the latter dominating the first. A local change in τ makes type i agents better off if and only if these agents learn *less* from the price. This also means that at least one type must be worse off, since price informativeness cannot go down for all types, by Lemma 4.4. In other words, different types exert externalities on each other in an exactly offsetting way, so that the allocation of private information is locally Pareto efficient.

The welfare changes described in Proposition 6.2 are reminiscent of the so-called Hirshleifer effect, insofar as better informed agents are worse off. However, in the symmetric economy considered here, this effect does not go in the same direction for all types.

Once we depart from symmetry, the possibility of a local (and hence global) Pareto improvement arises. We show that the allocation of information is Pareto inefficient in the non-symmetric two-type case (with $\tau_1 \neq \tau_2$; the assumption in the proposition that $\tau_1 < \tau_2$ is without loss of generality):

Proposition 6.3 (Welfare: two types) *Suppose $N = 2$, $R \in \mathcal{R}_\eta$ and $\rho > \underline{\rho}$. Consider an equilibrium at which $\tau_1 < \tau_2$. Then there exists a strict local Pareto improvement which entails an increase in both $\tau_1 + \tau_2$ and $\tau_2/(\tau_1 + \tau_2)$.*

In the two-type case, the assumption that $R \in \mathcal{R}_\eta$ and $\rho > \underline{\rho}$ ensures that the economy exhibits strategic complementarities. For ease of interpretation, let us assume in addition that $C_1 = C_2$. Then there is a unique equilibrium, by Proposition 5.7. Moreover, we have $\tau_1 < \tau_2$ at this equilibrium if and only if $\alpha_1 > \alpha_2$. Thus type 2 can unambiguously be identified as the low cost type. Proposition 6.3 says that a local Pareto improvement exists, and it entails more information production in the aggregate, with a higher proportion produced by the low cost type. An increase in the relative precision for type 2 leads to higher price informativeness for type 2, while lowering it for type 1. For fixed total precision $\tau_1 + \tau_2$, the welfare effects are in the opposite direction. A Pareto improvement must therefore compensate the better informed type 2. An increase in the aggregate information does precisely this, redistributing gains from trade from type 1 to type 2. Such a redistribution is possible only if τ_1 and τ_2 differ at the initial equilibrium. In a symmetric economy, with equal precision choices, scaling these precisions up or down has no effect on welfare.

In an economy that exhibits strategic complementarities, there are negative learning externalities across types. If one type acquires more information, other types have an incentive to acquire more information as well. Conversely, if one type cuts back on information production, other types will also want to do that. This suggests that, in equilibrium, information acquisition decisions are more closely aligned than is socially optimal. Proposition 6.3 tells us that this is indeed true for the two-type case: a Pareto improvement entails a greater differentiation between the two types in

⁵An alternative way to express this relationship is $\nabla G_i(\tau) = -\nabla \mathcal{V}_i(\tau)$, where ∇ denotes the gradient vector with respect to τ .

terms of the proportion of information they collect, with the low cost type acquiring a greater proportion of the total than in equilibrium.

7 Concluding Remarks

In an economy with heterogeneous valuations, agents make inferences about their own valuation from the equilibrium price. Under natural conditions, more information acquisition by one type leads to lower price informativeness for all other types. One consequence of this externality is that a lower cost of information for one type induces all agents to produce more information.

Lower price informativeness tends to raise welfare. This is because gains from trade for an agent are higher the further away his valuation is from the equilibrium price. In general, an equilibrium allocation of private information is Pareto inefficient. In the case of two types who differ in their cost of information production, a Pareto improvement entails an increase in the aggregate amount of information, with a higher proportion produced by the low cost type.

Our existence and comparative statics results are fairly general (at least for the linear-normal setting). But the welfare analysis is harder and leaves a number of questions unanswered. For a symmetric economy, the information collected in equilibrium is locally Pareto efficient, but we do not know if it is also globally Pareto efficient. For the non-symmetric case with two types, we describe a Pareto improvement in the neighborhood of an equilibrium. Characterizing the Pareto frontier remains an open question, however, as does the case of more than two types.

A Appendix

In the proofs it will often be useful to work with relative precisions, given by

$$\overset{\circ}{\tau}_i := \frac{\tau_i}{\sum_k \tau_k},$$

for type i . We denote the vector of relative precisions by $\hat{\tau} := (\overset{\circ}{\tau}_i)_{i=1}^N$.

Proof of Proposition 2.1 The price function is obtained from Proposition 3.2 of [Rahi and Zigrand \(2018\)](#) (RZ), applying this result for the case in which all types are differentially informed and all agents of all types acquire information (in RZ's notation, we set $\sigma_{\eta_i}^2 = 0$, $\lambda_i = 1$, and $r_i = r$, for all i). Price informativeness \mathcal{V}_i is the same as in RZ, with τ playing the role of RZ's λ . \square

Proof of Lemma 4.1 An equilibrium value of τ_i satisfies (6). In particular, taking \mathcal{V}_i to be exogenous, τ_i is decreasing in α_i , \mathcal{V}_i and τ_θ . Thus $\tau_i \leq \bar{\tau}_i$, where $\bar{\tau}_i$ is the solution to

$$2r\alpha C'_i(\tau_i)\tau_i = 1. \quad (10)$$

Let $\bar{\tau} := \max_i \bar{\tau}_i$.

We now show that there exists $\zeta_i > 0$ such that $\tau_i \geq \zeta_i$ for all $(R, \alpha) \in \mathcal{R}_\eta \times \mathcal{A}$. Suppose not. Then we can find a sequence of economies $\{(R(k), \alpha(k))\}$ in $\mathcal{R}_\eta \times \mathcal{A}$, and a corresponding sequence of precisions for type i , $\{\tau_{i,k}\}$, such that $\lim_{k \rightarrow \infty} \tau_{i,k} = 0$. Let $\mathcal{V}_{i,k}$ be the price informativeness for type i in the economy $(R(k), \alpha(k))$, and $\overset{\circ}{\tau}_{i,k} := \tau_{i,k} / \sum_j \tau_{j,k}$. Using (6), and the assumption that $R(k) \in \mathcal{R}_\eta$ for all k ,

$$\begin{aligned} \lim_{k \rightarrow \infty} \tau_{i,k} = 0 &\Rightarrow \lim_{k \rightarrow \infty} \mathcal{V}_{i,k} = 1 \\ &\Rightarrow \lim_{k \rightarrow \infty} \overset{\circ}{\tau}_{i,k} = 1 \\ &\Rightarrow \lim_{k \rightarrow \infty} \tau_{j,k} = 0, \quad \forall j \neq i \\ &\Rightarrow \lim_{k \rightarrow \infty} \mathcal{V}_{j,k} = 1, \quad \forall j \neq i. \end{aligned}$$

Thus in the limit the price function is fully revealing for all types, a contradiction.

While the above argument establishes a lower bound for τ_i , we will go one step further and choose a lower bound that can be explicitly characterized, in order to prove the limiting result in the lemma. Let $\hat{\mathcal{T}} := \times_{i=1}^N [\zeta_i, \bar{\tau}]$ and let

$$\bar{\mathcal{V}} := \max_{\tau \in \hat{\mathcal{T}}, R \in \mathcal{R}_\eta} \mathcal{V}_i. \quad (11)$$

This maximum exists since both $\hat{\mathcal{T}}$ and \mathcal{R}_η are compact, and is strictly less than one. Since τ_i solves (6) and is decreasing in α_i and \mathcal{V}_i , it follows that $\tau_i \geq \underline{\tau}_i$, where $\underline{\tau}_i$ solves

$$2r\bar{\alpha} C'_i(\tau_i) [\tau_i + \tau_\theta(1 - \bar{\mathcal{V}})^{-1}] = 1. \quad (12)$$

Let $\underline{\tau} := \min_i \tau_i$. Now suppose that $C_i = C$ for all i . Let $\lim_{\tau_\theta \rightarrow 0} \underline{\tau} := t$ (note that $\bar{\tau}$ does not depend on τ_θ). Then, from (10) and (12), we have,

$$\begin{aligned} 2r_{\underline{\alpha}} C'(\bar{\tau}) \bar{\tau} &= 1, \\ 2r_{\bar{\alpha}} C'(t) t &= 1. \end{aligned}$$

Clearly, $t < \bar{\tau}$. Therefore,

$$\lim_{\tau_\theta \rightarrow 0} \left(\frac{\underline{\tau}}{\bar{\tau}} \right) = \frac{t}{\bar{\tau}} = \frac{\underline{\alpha} C'(\bar{\tau})}{\bar{\alpha} C'(t)} > \frac{\underline{\alpha}}{\bar{\alpha}}.$$

This proves the result. Note that the assumption that $R \in \mathcal{R}_\eta$ is only needed to obtain the lower bound $\underline{\tau}$, not the upper bound $\bar{\tau}$. \square

Proof of Lemma 4.2 Differentiating (2), we obtain:

$$\begin{aligned} \frac{\partial \mathcal{V}_i}{\partial \tau_j} &= \frac{2\rho_{ij}(\tau^\top R \tau)(R_i^\top \tau) - 2(R_i^\top \tau)^2(R_j^\top \tau)}{(\tau^\top R \tau)^2} \\ &= \frac{2}{\tau^\top R \tau} [\rho_{ij} R_i^\top \tau - \mathcal{V}_i(R_j^\top \tau)]. \end{aligned} \quad (13)$$

In particular, when $j = i$,

$$\frac{\partial \mathcal{V}_i}{\partial \tau_i} = \frac{2R_i^\top \tau(1 - \mathcal{V}_i)}{\tau^\top R \tau}.$$

Hence, $\partial \mathcal{V}_i / \partial \tau_i > 0$ if and only if $R_i^\top \tau > 0$, or equivalently $R_i^\top \hat{\tau} > 0$.

Now suppose $R \in \mathcal{R}_\eta$. Using the bounds for τ_i given by Lemma 4.1, we have

$$\hat{\tau}_i \geq \hat{\underline{\tau}} := \frac{\underline{\tau}}{\underline{\tau} + (N-1)\bar{\tau}}.$$

We can write $\underline{\rho}$ in terms of $\hat{\underline{\tau}}$ as follows:

$$\underline{\rho} := -\frac{\underline{\tau}}{(N-1)\bar{\tau}} = -\frac{\hat{\underline{\tau}}}{1 - \hat{\underline{\tau}}}.$$

Let $\check{\rho} := \min_{i,j} \rho_{ij}$. Then,

$$\begin{aligned} R_i^\top \hat{\tau} &= \hat{\tau}_i + \sum_{k \neq i} \rho_{ik} \hat{\tau}_k \\ &\geq \hat{\tau}_i + \check{\rho}(1 - \hat{\tau}_i) \\ &= (1 - \hat{\tau}_i) \left[\check{\rho} + \frac{\hat{\tau}_i}{1 - \hat{\tau}_i} \right] \\ &\geq (1 - \hat{\tau}_i) \left[\check{\rho} + \frac{\hat{\underline{\tau}}}{1 - \hat{\underline{\tau}}} \right] \\ &= (1 - \hat{\tau}_i)(\check{\rho} - \underline{\rho}). \end{aligned}$$

Therefore, $R_i^\top \tau > 0$ for all i and all $\tau \in \mathcal{T}$ if $\check{\rho} > \underline{\rho}$, or equivalently if $\rho_{ij} > \underline{\rho}$ for all i, j . \square

Proof of Lemma 4.3 From equation (4) in [Rahi and Zigrand \(2018\)](#), the aggregate trade of type i is given by

$$q_i = r^{-1}(\tau_i \theta_i - \mu_i p),$$

where

$$\begin{aligned} \mu_i &= \tau_i + \tau_{\theta_i|p} \left[1 - \frac{\text{Cov}(\theta_i, p)}{\text{Var}(p)} \right] \\ &= \tau_i + \tau_{\theta_i|p} \left[1 - \mu^{-1} \cdot \frac{R_i^\top \tau}{\tau^\top R \tau} \right]. \end{aligned}$$

Using the market-clearing condition $\sum_i q_i = 0$, we get $p = \mu \tau^\top \theta$, where $\mu = (\sum_i \mu_i)^{-1}$. Furthermore,

$$\mu^{-1} = \sum_i \mu_i = \sum_i (\tau_i + \tau_{\theta_i|p}) - \mu^{-1} \sum_i \tau_{\theta_i|p} \cdot \frac{R_i^\top \tau}{\tau^\top R \tau}.$$

It follows that

$$\begin{aligned} \mu &= \frac{1 + \sum_k \tau_{\theta_k|p} \cdot \frac{R_k^\top \tau}{\tau^\top R \tau}}{\sum_j (\tau_j + \tau_{\theta_j|p})} \\ &= \frac{\sum_k (\tau_k + \tau_{\theta_k|p}) \frac{R_k^\top \tau}{\tau^\top R \tau}}{\sum_j (\tau_j + \tau_{\theta_j|p})} \\ &= \sum_k \beta_k \frac{R_k^\top \tau}{\tau^\top R \tau}, \end{aligned} \tag{14}$$

where

$$\beta_k := \frac{\tau_k + \tau_{\theta_k|p}}{\sum_j (\tau_j + \tau_{\theta_j|p})}.$$

Hence $\mu > 0$ if $R_k^\top \tau > 0$ for all k . \square

Proof of Lemma 4.4 Using (13), we can write $\partial_z \mathcal{V}_i$ as follows:

$$\begin{aligned} \partial_z \mathcal{V}_i &:= \sum_j z_j \frac{\partial \mathcal{V}_i}{\partial \tau_j} \\ &= \frac{2}{\tau^\top R \tau} [(R_i^\top \tau)(R_i^\top z) - \mathcal{V}_i(\tau^\top R z)] \\ &= \frac{2(R_i^\top \tau)(\tau^\top R z)}{\tau^\top R \tau} \left[\frac{R_i^\top z}{\tau^\top R z} - \frac{R_i^\top \tau}{\tau^\top R \tau} \right]. \end{aligned}$$

Invoking the assumption that $R_i^\top \tau > 0$ for all i , we have

$$\sum_i \frac{\tau_i}{R_i^\top \tau} (\partial_z \mathcal{V}_i) = 0.$$

The result follows. \square

Proof of Proposition 4.5 Suppose $N = 2$. Then, by Euler's theorem, the economy exhibits strategic complementarities if and only if both own-effects are positive on \mathcal{T} , which is equivalent to $R_i^\top \tau > 0$ for both values of i and for all $\tau \in \mathcal{T}$ (Lemma 4.2). We have, for $j \neq i$,

$$\begin{aligned} R_i^\top \tau &= \tau_i + \rho \tau_j \\ &\propto \rho + \frac{\tau_j}{\tau_i} \\ &\geq \rho + \frac{\underline{\tau}}{\bar{\tau}} \\ &= \rho - \underline{\rho} \end{aligned}$$

which is positive for both values of i and all $\tau \in \mathcal{T}$ if and only if $\rho > \underline{\rho}$. \square

Proof of Proposition 4.6 In all three conditions in the statement of the proposition, $R \in \mathcal{R}_\eta$ and $\rho_{ij} > \underline{\rho}$ for all i, j . By Lemma 4.2, $R_i^\top \tau > 0$ and $\partial \mathcal{V}_i / \partial \tau_i > 0$ for all i and for all $\tau \in \mathcal{T}$. From (13),

$$\begin{aligned} \frac{\partial \mathcal{V}_i}{\partial \tau_j} &= \frac{2R_i^\top \tau}{\tau^\top R \tau} \left[\rho_{ij} - \mathcal{V}_i \cdot \frac{R_j^\top \tau}{R_i^\top \tau} \right] \\ &= \frac{2R_i^\top \tau}{\tau^\top R \tau} \left[\rho_{ij} - \mathcal{V}_i \cdot \sqrt{\frac{\mathcal{V}_j}{\mathcal{V}_i}} \right] \\ &= \frac{2R_i^\top \tau}{\tau^\top R \tau} \left[\rho_{ij} - \sqrt{\mathcal{V}_i \mathcal{V}_j} \right]. \end{aligned}$$

Since $R_i^\top \tau > 0$, $\partial \mathcal{V}_i / \partial \tau_j < 0$ if and only if $\rho_{ij} < \sqrt{\mathcal{V}_i \mathcal{V}_j}$, or equivalently

$$D := \left[\sqrt{\mathcal{V}_i \mathcal{V}_j} - \rho_{ij} \right] \tau^\top R \tau > 0.$$

This inequality is clearly satisfied if $\rho_{ij} \leq 0$; hence we only need to consider the case

where $\rho_{ij} > 0$ ($i \neq j$). We have

$$\begin{aligned}
D &= (R_i^\top \tau)(R_j^\top \tau) - \rho_{ij}(\tau^\top R \tau) \\
&= \left[\tau_i + \rho_{ij}\tau_j + \sum_{k \neq i,j} \rho_{ik}\tau_k \right] \left[\tau_j + \rho_{ij}\tau_i + \sum_{k \neq i,j} \rho_{jk}\tau_k \right] \\
&\quad - \rho_{ij}\tau_i \left[\tau_i + \rho_{ij}\tau_j + \sum_{k \neq i,j} \rho_{ik}\tau_k \right] - \rho_{ij}\tau_j \left[\tau_j + \rho_{ij}\tau_i + \sum_{k \neq i,j} \rho_{jk}\tau_k \right] \\
&\quad - \rho_{ij} \sum_{\ell \neq i,j} \tau_\ell \left[\rho_{\ell i}\tau_i + \rho_{\ell j}\tau_j + \sum_{k \neq i,j} \rho_{\ell k}\tau_k \right] \\
&= (\tau_i + \rho_{ij}\tau_j)(\tau_j + \rho_{ij}\tau_i) + (\tau_i + \rho_{ij}\tau_j) \sum_{k \neq i,j} \rho_{jk}\tau_k + (\tau_j + \rho_{ij}\tau_i) \sum_{k \neq i,j} \rho_{ik}\tau_k \\
&\quad + \left[\sum_{k \neq i,j} \rho_{ik}\tau_k \right] \left[\sum_{k \neq i,j} \rho_{jk}\tau_k \right] \\
&\quad - \rho_{ij}\tau_i(\tau_i + \rho_{ij}\tau_j) - \rho_{ij}\tau_i \sum_{k \neq i,j} \rho_{ik}\tau_k - \rho_{ij}\tau_j(\tau_j + \rho_{ij}\tau_i) - \rho_{ij}\tau_j \sum_{k \neq i,j} \rho_{jk}\tau_k \\
&\quad - \rho_{ij}\tau_i \sum_{\ell \neq i,j} \rho_{\ell i}\tau_\ell - \rho_{ij}\tau_j \sum_{\ell \neq i,j} \rho_{\ell j}\tau_\ell - \rho_{ij} \sum_{\ell \neq i,j} \tau_\ell \left[\sum_{k \neq i,j} \rho_{\ell k}\tau_k \right] \\
&= (1 - \rho_{ij}^2)\tau_i\tau_j + (\tau_i - \rho_{ij}\tau_j) \sum_{k \neq i,j} \rho_{jk}\tau_k + (\tau_j - \rho_{ij}\tau_i) \sum_{k \neq i,j} \rho_{ik}\tau_k \\
&\quad + \left[\sum_{k \neq i,j} \rho_{ik}\tau_k \right] \left[\sum_{k \neq i,j} \rho_{jk}\tau_k \right] - \rho_{ij} \sum_{\ell \neq i,j} \tau_\ell \left[\sum_{k \neq i,j} \rho_{\ell k}\tau_k \right] \\
&= (1 - \rho_{ij}^2)\tau_i\tau_j + (\tau_i - \rho_{ij}\tau_j)S_j + (\tau_j - \rho_{ij}\tau_i)S_i + S_iS_j - \rho_{ij} \sum_{\ell \neq i,j} \tau_\ell S_\ell, \tag{15}
\end{aligned}$$

where

$$S_m := \sum_{k \neq i,j} \rho_{mk}\tau_k.$$

Note that $S_m < \sum_{k \neq i,j} \tau_k$, for all m . We consider condition (ii) of the proposition first, followed by (i) and (iii).

Proof of (ii): Let $\check{\rho} := \min_{k,\ell} \rho_{k\ell}$, and suppose that $\check{\rho} \geq 0$. Then, for $m = i, j$,

$$S_m \geq \check{\rho} \sum_{k \neq i,j} \tau_k \geq \check{\rho}(N-2)\underline{\tau}.$$

Hence, from (15),

$$\begin{aligned}
\frac{\partial D}{\partial S_i} &= \tau_j - \rho_{ij}\tau_i + S_j \\
&\geq \underline{\tau} - \rho_{ij}\bar{\tau} + \check{\rho}(N-2)\underline{\tau} \\
&\propto [1 + \check{\rho}(N-2)]\delta - \rho_{ij},
\end{aligned}$$

which is nonnegative if

$$\rho_{ij} \leq [1 + \check{\rho}(N - 2)]\delta. \quad (16)$$

Assuming that (16) holds, D is increasing in S_i , and by symmetry in S_j as well. Moreover, D is decreasing in S_ℓ , $\ell \neq i, j$. Hence, from (15),

$$\begin{aligned} D > D_1 &:= (1 - \rho_{ij}^2)\tau_i\tau_j + \check{\rho}(\tau_i - \rho_{ij}\tau_j) \sum_{k \neq i, j} \tau_k + \check{\rho}(\tau_j - \rho_{ij}\tau_i) \sum_{k \neq i, j} \tau_k \\ &\quad + \check{\rho}^2 \left[\sum_{k \neq i, j} \tau_k \right]^2 - \rho_{ij} \left[\sum_{k \neq i, j} \tau_k \right]^2 \\ &= (1 - \rho_{ij}^2)\tau_i\tau_j + \check{\rho}(1 - \rho_{ij})(\tau_i + \tau_j) \sum_{k \neq i, j} \tau_k + (\check{\rho}^2 - \rho_{ij}) \left[\sum_{k \neq i, j} \tau_k \right]^2. \end{aligned} \quad (17)$$

Moreover, since D_1 is increasing in τ_i and τ_j ,

$$D_1 \geq D_2 := (1 - \rho_{ij}^2)\mathcal{I}^2 + 2\check{\rho}(1 - \rho_{ij})\mathcal{I} \sum_{k \neq i, j} \tau_k + (\check{\rho}^2 - \rho_{ij}) \left[\sum_{k \neq i, j} \tau_k \right]^2. \quad (18)$$

Now, for $m \neq i, j$,

$$\begin{aligned} \frac{\partial D_2}{\partial \tau_m} &= 2\check{\rho}(1 - \rho_{ij})\mathcal{I} + 2(\check{\rho}^2 - \rho_{ij}) \sum_{k \neq i, j} \tau_k \\ &\leq 2\check{\rho}(1 - \rho_{ij})\mathcal{I} + 2(\check{\rho}^2 - \rho_{ij})(N - 2)\mathcal{I} \\ &\propto \check{\rho}(1 - \rho_{ij}) + (\check{\rho}^2 - \rho_{ij})(N - 2) \\ &\leq \check{\rho}(1 - \check{\rho}) + (\check{\rho}^2 - \check{\rho})(N - 2) \\ &= -\check{\rho}(1 - \check{\rho})(N - 3) \\ &\leq 0. \end{aligned} \quad (19)$$

Therefore, from (18),

$$D_2 \geq D_3 := (1 - \rho_{ij}^2)\mathcal{I}^2 + 2\check{\rho}(1 - \rho_{ij})\mathcal{I}(N - 2)\bar{\tau} + (\check{\rho}^2 - \rho_{ij})(N - 2)^2\bar{\tau}^2. \quad (20)$$

Combining (17), (18), and (20), we have

$$\begin{aligned} D > D_3 &\propto (1 - \rho_{ij}^2)\delta^2 + 2\check{\rho}(1 - \rho_{ij})(N - 2)\delta + (\check{\rho}^2 - \rho_{ij})(N - 2)^2 \\ &\geq (1 - \rho_{ij})(1 + \check{\rho})\delta^2 + 2\check{\rho}(1 - \rho_{ij})(N - 2)\delta + (\check{\rho}^2 - \rho_{ij})(N - 2)^2, \end{aligned}$$

which is greater than equal to zero if

$$\rho_{ij} \leq \frac{(1 + \check{\rho})\delta^2 + 2\check{\rho}(N - 2)\delta + \check{\rho}^2(N - 2)^2}{(1 + \check{\rho})\delta^2 + 2\check{\rho}(N - 2)\delta + (N - 2)^2}. \quad (21)$$

Thus $D > 0$, for all $i \neq j$ and all $\tau \in \mathcal{T}$, if (16) and (21) hold. This is condition (8) in the proposition.

Proof of (i): Now we assume only that $\rho_{k\ell} > \underline{\rho}$ (which ensures that all own-effects are positive, by Lemma 4.2). Recall that $\underline{\rho} = -\delta(N-1)^{-1}$. For $m = i, j$, we have

$$S_m \geq \underline{\rho} \sum_{k \neq i, j} \tau_k \geq \underline{\rho}(N-2)\bar{\tau}.$$

Hence, from (15),

$$\begin{aligned} \frac{\partial D}{\partial S_i} &= \tau_j - \rho_{ij}\tau_i + S_j \\ &\geq \underline{\tau} - \rho_{ij}\bar{\tau} + \underline{\rho}(N-2)\bar{\tau} \\ &\propto \delta + \underline{\rho}(N-2) - \rho_{ij} \\ &= -\underline{\rho} - \rho_{ij}. \end{aligned}$$

Assuming that $\rho_{ij} \leq -\underline{\rho}$, D is increasing in S_i and, by symmetry, in S_j as well. Moreover, D is decreasing in S_ℓ , $\ell \neq i, j$. Hence, $D > D_1$ as in (17). We have,

$$\begin{aligned} \frac{\partial D_1}{\partial \tau_i} &= (1 - \rho_{ij}^2)\tau_j + \underline{\rho}(1 - \rho_{ij}) \sum_{k \neq i, j} \tau_k \\ &\geq (1 - \rho_{ij}^2)\underline{\tau} + \underline{\rho}(1 - \rho_{ij})(N-2)\bar{\tau} \\ &\propto (1 + \rho_{ij})\delta + \underline{\rho}(N-2) \\ &= \delta\rho_{ij} - \underline{\rho} \\ &> 0. \end{aligned}$$

Thus D_1 is increasing in τ_i and, by symmetry, in τ_j as well. Hence, $D_1 \geq D_2$ as in (18).

Now we show that D_2 is decreasing in τ_m , for $m \neq i, j$. If $\rho_{ij} \geq \underline{\rho}^2$, then $\partial D_2 / \partial \tau_m < 0$ from (19). If, on the other hand, $\rho_{ij} < \underline{\rho}^2$, we have

$$\begin{aligned} \frac{\partial D_2}{\partial \tau_m} &\leq 2\underline{\rho}(1 - \rho_{ij})\underline{\tau} + 2(\underline{\rho}^2 - \rho_{ij})(N-2)\bar{\tau} \\ &\propto \underline{\rho}(1 - \rho_{ij})\delta + (\underline{\rho}^2 - \rho_{ij})(N-2) \\ &= \underline{\rho}[\delta + \underline{\rho}(N-2)] - \rho_{ij}[(1 + \underline{\rho}\delta) + (N-3)] \\ &= -\underline{\rho}^2 - \rho_{ij}[(1 + \underline{\rho}\delta) + (N-3)] \\ &< 0. \end{aligned}$$

Therefore, $D_2 \geq D_3$ as in (20). Combining (17), (18), and (20), we have

$$\begin{aligned} D > D_3 &\propto (1 - \rho_{ij}^2)\delta^2 + 2\underline{\rho}(1 - \rho_{ij})(N-2)\delta + (\underline{\rho}^2 - \rho_{ij})(N-2)^2 \\ &> (1 - \rho_{ij})\delta^2 + 2\underline{\rho}(1 - \rho_{ij})(N-2)\delta + (\underline{\rho}^2 - \rho_{ij})(N-2)^2, \end{aligned}$$

which is greater than equal to zero if

$$\begin{aligned}
\rho_{ij} &\leq \frac{\delta^2 + 2\rho(N-2)\delta + \rho^2(N-2)^2}{\delta^2 + 2\rho(N-2)\delta + (N-2)^2} \\
&= \frac{[\delta + \rho(N-2)]^2}{[\delta + \rho(N-2)]^2 + (1-\rho^2)(N-2)^2} \\
&= \frac{\rho^2}{\rho^2 + (1-\rho^2)(N-2)^2}.
\end{aligned} \tag{22}$$

Notice that (22) strengthens our earlier assumption that $\rho_{ij} \leq -\rho$. Thus $D > 0$, for all $i \neq j$ and all $\tau \in \mathcal{T}$, if (22) holds. This gives us condition (7) in the proposition.

Proof of (iii): Suppose $\rho_{ij} = \rho$ for all $i \neq j$. Then,

$$S_i = S_j = \rho \sum_{k \neq i, j} \tau_k,$$

and for $\ell \neq i, j$,

$$S_\ell = (1-\rho)\tau_\ell + \rho \sum_{k \neq i, j} \tau_k.$$

There is nothing to prove if $\rho \leq 0$. If $\rho > 0$, we have, from (15),

$$\begin{aligned}
D &= (1-\rho^2)\tau_i\tau_j + \rho(1-\rho)(\tau_i + \tau_j) \sum_{k \neq i, j} \tau_k - \rho(1-\rho) \sum_{k \neq i, j} \tau_k^2 \\
&= (1+\rho)\tau_i\tau_j + \rho(\tau_i + \tau_j) \sum_{k \neq i, j} \tau_k - \rho \sum_{k \neq i, j} \tau_k^2
\end{aligned} \tag{23}$$

$$\geq (1+\rho)\bar{\tau}^2 + 2\rho\bar{\tau} \sum_{k \neq i, j} \tau_k - \rho \sum_{k \neq i, j} \tau_k^2 \tag{24}$$

$$\begin{aligned}
&\geq (1+\rho)\bar{\tau}^2 + 2\rho\bar{\tau}(N-2)\bar{\tau} - \rho(N-2)\bar{\tau}^2 \\
&\propto \delta^2 + [\delta^2 + (2\delta-1)(N-2)]\rho \\
&> \delta^2 + (2\delta-1)(N-1)\rho,
\end{aligned} \tag{25}$$

where we have used the fact that (23) is increasing in τ_i and τ_j , while (24) is decreasing in $\tau_k, k \neq i, j$. Hence, $D > 0$ if $\delta \geq \frac{1}{2}$. If $\delta < \frac{1}{2}$, $D > 0$ if (9) holds. Note that a less stringent condition on ρ can be derived from (25), but (9) is easier to interpret. \square

Proof of Proposition 5.1 Substituting $\tau_{\theta|p} = \tau_\theta[1 - \mathcal{V}_i(\tau)]^{-1}$ into agent i 's first-order condition (3) we obtain:

$$2r\alpha_i C'_i(\tau_{in}) [\tau_{in} + \tau_\theta[1 - \mathcal{V}_i(\tau)]^{-1}] = 1.$$

Since the economy exhibits strategic complementarities, τ_{in} is decreasing in τ_i and increasing in $\tau_{-i} := (\tau_j)_{j \neq i}$, for $\tau \in \mathcal{T}$. Moreover, τ_{in} is decreasing in α_i .

The following argument is similar to the one used by [Milgrom and Shannon \(1994\)](#) for general equilibrium with gross substitutes. For given α , consider the fictitious game $\Gamma(\alpha)$ with N players in which player i chooses $\tau_i \in [\underline{\tau}, \bar{\tau}]$ and has payoff

$$\pi_i(\tau_i, \tau_{-i}, \alpha) = -|\tau_i - \tau_{in}(\tau_i, \tau_{-i}, \alpha)|.$$

Let $t_i := (\tau_{-i}, -\alpha)$ and $f_i(\tau_i, t_i) := \tau_i - \tau_{in}(\tau_i, \tau_{-i}, \alpha)$. Player i 's payoff can then be written as

$$\pi_i(\tau_i, t_i) = -|f_i(\tau_i, t_i)|.$$

For any given t_i , f_i is continuous in τ_i , with $f(\underline{\tau}, t_i) \leq 0$ and $f(\bar{\tau}, t_i) \geq 0$. Hence, there exists a τ_i such that $\pi_i(\tau_i, t_i) = 0$. It follows that a profile of precisions τ is an equilibrium of our economy if and only if it is a pure strategy Nash equilibrium of $\Gamma(\alpha)$. Note that the function f_i is strictly increasing in τ_i , and decreasing in t_i .

We claim that π_i satisfies the single-crossing property in (τ_i, t_i) , i.e. for all $\check{\tau}_i > \tau_i, \check{t}_i > t_i$:

$$\pi_i(\check{\tau}_i, t_i) - \pi_i(\tau_i, t_i) \geq (>) 0 \quad \Rightarrow \quad \pi_i(\check{\tau}_i, \check{t}_i) - \pi_i(\tau_i, \check{t}_i) \geq (>) 0,$$

or, equivalently,

$$-|f_i(\check{\tau}_i, t_i)| + |f_i(\tau_i, t_i)| \geq (>) 0 \quad \Rightarrow \quad -|f_i(\check{\tau}_i, \check{t}_i)| + |f_i(\tau_i, \check{t}_i)| \geq (>) 0. \quad (26)$$

Since f_i is strictly increasing in τ_i , $f_i(\check{\tau}_i, t_i) > f_i(\tau_i, t_i)$. Hence, in order for the supposition in (26) to be true, we must have $f_i(\tau_i, t_i) < 0$. In fact, since f_i is decreasing in t_i , $f_i(\tau_i, \check{t}_i) \leq f_i(\tau_i, t_i) < 0$. Therefore, we can write (26) as follows:

$$|f_i(\check{\tau}_i, t_i)| + f_i(\tau_i, t_i) \leq (<) 0 \quad \Rightarrow \quad |f_i(\check{\tau}_i, \check{t}_i)| + f_i(\tau_i, \check{t}_i) \leq (<) 0. \quad (27)$$

Now note that $f_i(\tau_i, \check{t}_i) \leq f_i(\check{\tau}_i, \check{t}_i)$. Thus if $f_i(\check{\tau}_i, \check{t}_i) \geq 0$, (27) is satisfied (both terms in the implication are lower than the corresponding terms in the supposition). If $f_i(\check{\tau}_i, \check{t}_i) < 0$, we must have $f_i(\tau_i, \check{t}_i) < f_i(\check{\tau}_i, \check{t}_i) < 0$, so the implication in (27) holds. This verifies the single-crossing property.

Thus $\{\Gamma(\alpha)\}_\alpha$ is a family of games with strategic complementarities satisfying the single-crossing property, as defined by [Milgrom and Shannon \(1994\)](#). Hence there is a highest equilibrium $\hat{\tau}(\alpha)$ and a lowest equilibrium $\check{\tau}(\alpha)$, and both satisfy the MCS property. \square

Proof of Proposition 5.2 Consider an economy that exhibits strategic complementarities, and an equilibrium τ that satisfies the MCS property. Thus we have, for all i, j ,

$$\frac{\partial \mathcal{V}_i}{\partial \tau_i} > 0; \quad \frac{\partial \mathcal{V}_i}{\partial \tau_k} < 0, \quad k \neq i; \quad \frac{\partial \tau_i}{\partial \alpha_j} \leq 0, \quad \alpha_j \in (\underline{\alpha}, \bar{\alpha}). \quad (28)$$

We show that the last inequality is strict. Differentiating (6) with respect to α_j , we obtain:

$$C'''(\tau_i) \frac{\partial \tau_i}{\partial \alpha_j} [\tau_i + \tau_\theta (1 - \mathcal{V}_i)^{-1}] + C''(\tau_i) \left[\frac{\partial \tau_i}{\partial \alpha_j} + \tau_\theta (1 - \mathcal{V}_i)^{-2} \frac{\partial \mathcal{V}_i}{\partial \alpha_j} \right] = 0, \quad i \neq j, \quad (29)$$

and

$$\begin{aligned} & \left[\alpha_j C_j''(\tau_j) \frac{\partial \tau_j}{\partial \alpha_j} + C_j'(\tau_j) \right] [\tau_j + \tau_\theta (1 - \mathcal{V}_j)^{-1}] \\ & + \alpha_j C_j'(\tau_j) \left[\frac{\partial \tau_j}{\partial \alpha_j} + \tau_\theta (1 - \mathcal{V}_j)^{-2} \frac{\partial \mathcal{V}_j}{\partial \alpha_j} \right] = 0. \end{aligned} \quad (30)$$

Suppose $\partial \tau_j / \partial \alpha_j = 0$. Then,

$$\frac{\partial \mathcal{V}_j}{\partial \alpha_j} = \sum_{k \neq j} \frac{\partial \mathcal{V}_j}{\partial \tau_k} \frac{\partial \tau_k}{\partial \alpha_j},$$

which is nonnegative due to (28). But then (30) is not satisfied, a contradiction. It follows that $\partial \tau_j / \partial \alpha_j < 0$. Now suppose there is an $i \neq j$ such that $\partial \tau_i / \partial \alpha_j = 0$. Then,

$$\frac{\partial \mathcal{V}_i}{\partial \alpha_j} = \sum_{k \neq i} \frac{\partial \mathcal{V}_i}{\partial \tau_k} \frac{\partial \tau_k}{\partial \alpha_j},$$

which is positive due to (28) and the fact that $\partial \tau_j / \partial \alpha_j < 0$. This implies that (29) is not satisfied, a contradiction. Therefore, $\partial \tau_i / \partial \alpha_j < 0$ for all i . \square

Proof of Proposition 5.3 Suppose the economy exhibits strategic complementarities. Consider an equilibrium τ that satisfies the MCS property, and hence the strong MCS property by Proposition 5.2. From (29), it follows that $\partial \mathcal{V}_i / \partial \alpha_j > 0$ for all $i \neq j$. Hence, $\partial \mathcal{V}_j / \partial \alpha_j < 0$ due to Lemma 4.4 (note that, since \mathcal{V}_i depends on α_j only through τ , $\partial \mathcal{V}_i / \partial \alpha_j = \partial_z \mathcal{V}_i$ for some direction z , for all i). \square

Proof of Proposition 5.4 By Proposition 5.1, there is a highest equilibrium $\hat{\tau}$ and a lowest equilibrium $\check{\tau}$. Let $\hat{\mathcal{V}}_i$ and $\check{\mathcal{V}}_i$ denote the price informativeness of type i at $\hat{\tau}$ and $\check{\tau}$ respectively. Suppose $\hat{\tau} \neq \check{\tau}$, i.e. $\hat{\tau}_j > \check{\tau}_j$, for some j . We claim that we must in fact have $\hat{\tau}_i > \check{\tau}_i$ for all i . Suppose not, say $\hat{\tau}_k = \check{\tau}_k$, for some $k \neq j$. Then, because all cross-effects are negative, $\hat{\mathcal{V}}_k < \check{\mathcal{V}}_k$. But then (6) implies that $\hat{\tau}_k > \check{\tau}_k$, a contradiction. Thus $\hat{\tau}_i > \check{\tau}_i$, and by (6), $\hat{\mathcal{V}}_i < \check{\mathcal{V}}_i$, for all i . In other words, if the highest and lowest equilibria are distinct, price informativeness must be higher at the lowest equilibrium for all types.

Now, suppose $\rho_{ij} = \rho$ for all $i \neq j$. From (2):

$$\mathcal{V}_i = \frac{[(1 - \rho)\tau_i + \rho \sum_k \tau_k]^2}{(1 - \rho) \sum_k \tau_k^2 + \rho (\sum_k \tau_k)^2}. \quad (31)$$

If $\rho = 0$, $\sum \mathcal{V}_i = 1$. Hence it is not possible for all the \mathcal{V}_i 's to be higher at one equilibrium compared to another. It follows that there is a unique equilibrium.

For arbitrary ρ , at any equilibrium, $\tau_k > \tau_\ell$ if and only if $R_k^\top \tau > R_\ell^\top \tau$. Since $R_i^\top \tau > 0$ for all i (positive own-effects), $\tau_k > \tau_\ell$ if and only if $\mathcal{V}_k > \mathcal{V}_\ell$. On the

other hand, if types k and ℓ share the same cost function, (6) implies that $\tau_k > \tau_\ell$ if and only if $\mathcal{V}_k < \mathcal{V}_\ell$. It follows that all types with the same cost function \mathcal{C}_{N_k} have the same precision and price informativeness, which we denote by τ_{N_k} and \mathcal{V}_{N_k} , respectively. Letting $\gamma = \tau_{N_1}/\tau_{N_2}$, we have (using (31)):

$$\begin{aligned}\mathcal{V}_{N_1} &= \frac{\left([1 + \rho(N_1 - 1)]\tau_{N_1} + \rho N_2 \tau_{N_2}\right)^2}{\left[1 + \rho(N_1 - 1)\right]N_1\tau_{N_1}^2 + 2\rho N_1 N_2 \tau_{N_1} \tau_{N_2} + \left[1 + \rho(N_2 - 1)\right]N_2\tau_{N_2}^2} \\ &= \frac{\left([1 + \rho(N_1 - 1)]\gamma + \rho N_2\right)^2}{\left[1 + \rho(N_1 - 1)\right]N_1\gamma^2 + 2\rho N_1 N_2 \gamma + \left[1 + \rho(N_2 - 1)\right]N_2}, \\ \mathcal{V}_{N_2} &= \frac{\left([1 + \rho(N_2 - 1)]\tau_{N_2} + \rho N_1 \tau_{N_1}\right)^2}{\left[1 + \rho(N_1 - 1)\right]N_1\tau_{N_1}^2 + 2\rho N_1 N_2 \tau_{N_1} \tau_{N_2} + \left[1 + \rho(N_2 - 1)\right]N_2\tau_{N_2}^2} \\ &= \frac{\left([1 + \rho(N_2 - 1)] + \rho N_1 \gamma\right)^2}{\left[1 + \rho(N_1 - 1)\right]N_1\gamma^2 + 2\rho N_1 N_2 \gamma + \left[1 + \rho(N_2 - 1)\right]N_2}.\end{aligned}$$

Differentiating with respect to γ , we have

$$\begin{aligned}\frac{\partial \mathcal{V}_{N_1}}{\partial \gamma} &\propto \left([1 + \rho(N_1 - 1)]N_1\gamma^2 + 2\rho N_1 N_2 \gamma + [1 + \rho(N_2 - 1)]N_2\right)[1 + \rho(N_1 - 1)] \\ &\quad - N_1 \left([1 + \rho(N_1 - 1)]\gamma + \rho N_2\right)^2 \\ &= (1 - \rho)[1 + \rho(N - 1)]N_2, \\ \frac{\partial \mathcal{V}_{N_2}}{\partial \gamma} &\propto \left([1 + \rho(N_1 - 1)]N_1\gamma^2 + 2\rho N_1 N_2 \gamma + [1 + \rho(N_2 - 1)]N_2\right)\rho N_1 \\ &\quad - N_1 \left([1 + \rho(N_2 - 1)] + \rho N_1 \gamma\right) \left([1 + \rho(N_1 - 1)]\gamma + \rho N_2\right) \\ &= -(1 - \rho)[1 + \rho(N - 1)]N_1 \gamma,\end{aligned}$$

where we have used the fact that $R_i^\top \tau > 0$ for all i . Since $\rho > -(N - 1)^{-1}$ (this is required for R to be positive definite), we see that \mathcal{V}_{N_1} is increasing in γ while \mathcal{V}_{N_2} is decreasing in γ . Hence it is not possible for both \mathcal{V}_{N_1} and \mathcal{V}_{N_2} to be higher at one equilibrium than at another. Once again, equilibrium must be unique. \square

Proof of Proposition 5.5 In both (i) and (ii), the equivalence between the statements about precisions and price informativeness follows from (31), using the assumption that $R_i^\top \tau > 0$ for all i . The equivalence of these with the statement about the cost parameters α follows from (6), using the assumption that C_i is the same for all i . \square

Proof of Proposition 5.6 By Proposition 5.5 (part (ii)), at any candidate equilibrium (at which own-effects are positive), τ_i and \mathcal{V}_i are the same for all i , which we

denote by τ^* and \mathcal{V}^* , respectively. From (31),

$$\mathcal{V}^* = \frac{1 + \rho(N - 1)}{N},$$

which does not depend on τ^* . From (6), τ^* is the unique solution to

$$2r\mathcal{C}'(\tau^*)[\tau^* + \tau_\theta(1 - \mathcal{V}^*)^{-1}] = 1,$$

where \mathcal{C} is the cost function (assumed to be the same for all types). This proves the result. \square

Proof of Proposition 5.7 As discussed in the main text, the only part of this proposition that does not follow from our previous results is that an increase in α_i reduces τ_i/τ_j , $j \neq i$, leading to a lower \mathcal{V}_i and a higher \mathcal{V}_j .

Specializing the proof of part (ii) of Proposition 5.4 to the case of two types, we see that \mathcal{V}_1 is strictly increasing in τ_1/τ_2 , while \mathcal{V}_2 is strictly decreasing in τ_1/τ_2 . As noted in the text, an increase in α_i leads to a decrease in both τ_1 and τ_2 by the strong MCS property. For $j \neq i$, (6) implies that \mathcal{V}_j is higher. Hence, τ_i/τ_j must be lower, and consequently \mathcal{V}_i must be lower as well. \square

Proof of Lemma 6.1 The indirect utility of type i , for any τ , is given by (4). Using the definition of G_i and taking logs, we have

$$\log \mathcal{U}_i = \log(\tau_i + \tau_{\theta_i|p}) + \log G_i - 2r\alpha_i C_i(\tau_i) + \log \sigma_\theta^2.$$

Differentiating this expression with respect to τ_k , and using (5), we obtain (the indicator function $\mathbb{1}_{i=k}$ takes value 1 when $i = k$, and is 0 otherwise):

$$\begin{aligned} \frac{\partial \log \mathcal{U}_i}{\partial \tau_k} &= (\tau_i + \tau_{\theta_i|p})^{-1} \left[\mathbb{1}_{i=k} + \frac{\partial \tau_{\theta_i|p}}{\partial \tau_k} \right] + G_i^{-1} \frac{\partial G_i}{\partial \tau_k} - 2r\alpha_i C'_i(\tau_i) \mathbb{1}_{i=k} \\ &= (\tau_i + \tau_{\theta_i|p})^{-1} \frac{\partial \tau_{\theta_i|p}}{\partial \tau_k} + G_i^{-1} \frac{\partial G_i}{\partial \tau_k}. \end{aligned}$$

Recalling that $\tau_{\theta_i|p} = \tau_\theta(1 - \mathcal{V}_i)^{-1}$, we have

$$\begin{aligned} \frac{\partial \tau_{\theta_i|p}}{\partial \tau_k} &= \tau_\theta(1 - \mathcal{V}_i)^{-2} \frac{\partial \mathcal{V}_i}{\partial \tau_k} \\ &= \tau_{\theta_i|p}(1 - \mathcal{V}_i)^{-1} \frac{\partial \mathcal{V}_i}{\partial \tau_k}. \end{aligned}$$

Hence,

$$\frac{\partial \log \mathcal{U}_i}{\partial \tau_k} = \frac{\tau_{\theta_i|p}}{(1 - \mathcal{V}_i)(\tau_i + \tau_{\theta_i|p})} \cdot \frac{\partial \mathcal{V}_i}{\partial \tau_k} + \frac{1}{G_i} \cdot \frac{\partial G_i}{\partial \tau_k}.$$

The result follows. \square

Proof of Proposition 6.2 We first calculate G_i for an arbitrary precision vector τ . For this purpose, it is convenient to write the price function as $p = \xi \hat{\tau}^\top \theta$, where $\xi := \mu \sum_k \tau_k$. We have

$$\begin{aligned}\sigma_{\theta_i-p}^2 &= \sigma_\theta^2 + \sigma_p^2 - 2\text{Cov}(\theta_i, p) \\ &= \sigma_\theta^2 + \sigma_\theta^2 \xi^2 \hat{\tau}^\top R \hat{\tau} - 2\sigma_\theta^2 \xi R_i^\top \hat{\tau},\end{aligned}$$

so that

$$\begin{aligned}G_i &:= \frac{\sigma_{\theta_i-p}^2}{\sigma_\theta^2} = 1 + \hat{\tau}^\top R \hat{\tau} \left[\xi^2 - 2\xi \frac{R_i^\top \hat{\tau}}{\hat{\tau}^\top R \hat{\tau}} \right] \\ &= 1 - \frac{(R_i^\top \hat{\tau})^2}{\hat{\tau}^\top R \hat{\tau}} + \hat{\tau}^\top R \hat{\tau} \left[\xi^2 - 2\xi \frac{R_i^\top \hat{\tau}}{\hat{\tau}^\top R \hat{\tau}} + \left(\frac{R_i^\top \hat{\tau}}{\hat{\tau}^\top R \hat{\tau}} \right)^2 \right] \\ &= 1 - \mathcal{V}_i + (\hat{\tau}^\top R \hat{\tau}) \phi_i^2,\end{aligned}\tag{32}$$

where

$$\phi_i := \xi - \frac{R_i^\top \hat{\tau}}{\hat{\tau}^\top R \hat{\tau}}.$$

From (14),

$$\xi := \mu \sum_k \tau_k = \sum_k \beta_k \frac{R_k^\top \hat{\tau}}{\hat{\tau}^\top R \hat{\tau}}.$$

Hence we can write ϕ_i as follows:

$$\phi_i = \frac{\sum_k \beta_k (R_k - R_i)^\top \hat{\tau}}{\hat{\tau}^\top R \hat{\tau}}.\tag{33}$$

Now consider a symmetric economy with equilibrium τ . Since all pairwise correlations are the same, and all types choose the same precision, $R_k^\top \hat{\tau}$ is the same for all k . From (33), $\phi_i = 0$, and hence $\partial G_i / \partial \tau_j = -\partial \mathcal{V}_i / \partial \tau_j$ for all j , or equivalently $\partial_z G_i = -\partial_z \mathcal{V}_i$ for all directions $z \in \mathbb{R}^N$. Using Lemma 6.1, we have

$$\partial_z \log \mathcal{U}_i(\tau) = \left[\frac{\tau_{\theta_i|p}}{(1 - \mathcal{V}_i)(\tau_i + \tau_{\theta_i|p})} - \frac{1}{G_i} \right] \partial_z \mathcal{V}_i(\tau).$$

Since $G_i = 1 - \mathcal{V}_i$, from (32), $\partial_z \log \mathcal{U}_i(\tau) \propto -\partial_z \mathcal{V}_i(\tau)$. This proves the result. \square

Proof of Proposition 6.3 For this proof it is convenient to parametrize agents' welfare by $\hat{\tau}_1 := \tau_1 / (\tau_1 + \tau_2)$ and $\psi := \tau_1 + \tau_2$ (instead of τ_1 and τ_2), and define $\partial_z f := z_1 \frac{\partial f}{\partial \hat{\tau}_1} + z_2 \frac{\partial f}{\partial \psi}$. We have

$$\begin{aligned}R_i^\top \hat{\tau} &= (1 - \rho) \hat{\tau}_i + \rho, \\ \hat{\tau}^\top R \hat{\tau} &= (1 - \rho)(\hat{\tau}_1^2 + \hat{\tau}_2^2) + \rho,\end{aligned}$$

so that

$$1 - \mathcal{V}_i = 1 - \frac{(R_i^\top \dot{\tau})^2}{\dot{\tau}^\top R \dot{\tau}} = \frac{(1 - \rho^2) \dot{\tau}_j^2}{\dot{\tau}^\top R \dot{\tau}}, \quad j \neq i, \quad (34)$$

which does not depend on ψ . Differentiating with respect to $\dot{\tau}_1$, we obtain

$$\partial_z \mathcal{V}_1 = \frac{\partial \mathcal{V}_1}{\partial \dot{\tau}_1} z_1 = \frac{2(1 - \rho^2)(R_1^\top \dot{\tau}) \dot{\tau}_2}{(\dot{\tau}^\top R \dot{\tau})^2} \cdot z_1 = \frac{2(1 - \mathcal{V}_1)(R_1^\top \dot{\tau})}{\dot{\tau}_2(\dot{\tau}^\top R \dot{\tau})} \cdot z_1, \quad (35)$$

and, similarly,

$$\partial_z \mathcal{V}_2 = \frac{\partial \mathcal{V}_2}{\partial \dot{\tau}_1} z_1 = -\frac{2(1 - \mathcal{V}_2)(R_2^\top \dot{\tau})}{\dot{\tau}_1(\dot{\tau}^\top R \dot{\tau})} \cdot z_1. \quad (36)$$

From (33),

$$\begin{aligned} \phi_1 &= -\frac{(1 - \rho)(\dot{\tau}_1 - \dot{\tau}_2)\beta_2}{\dot{\tau}^\top R \dot{\tau}}, \\ \phi_2 &= \frac{(1 - \rho)(\dot{\tau}_1 - \dot{\tau}_2)\beta_1}{\dot{\tau}^\top R \dot{\tau}}. \end{aligned}$$

Hence, from (32),

$$G_i = 1 - \mathcal{V}_i + H\beta_j^2, \quad j \neq i,$$

where

$$H := \frac{(1 - \rho)^2(\dot{\tau}_1 - \dot{\tau}_2)^2}{\dot{\tau}^\top R \dot{\tau}}.$$

Since H does not depend on ψ , we have

$$\partial_z H = \frac{\partial H}{\partial \dot{\tau}_1} z_1 = \frac{2(1 - \rho)^2(1 + \rho)(\dot{\tau}_1 - \dot{\tau}_2)}{(\dot{\tau}^\top R \dot{\tau})^2} \cdot z_1,$$

and hence (for $j \neq i$),

$$\begin{aligned} \partial_z G_i &= -\partial_z \mathcal{V}_i + \beta_j^2 \partial_z H + 2H\beta_j \partial_z \beta_j \\ &= -\partial_z \mathcal{V}_i + \frac{2(1 - \rho)^2(\dot{\tau}_1 - \dot{\tau}_2)\beta_j [(1 + \rho)\beta_j z_1 + (\dot{\tau}_1 - \dot{\tau}_2)(\dot{\tau}^\top R \dot{\tau}) \partial_z \beta_j]}{(\dot{\tau}^\top R \dot{\tau})^2}. \end{aligned}$$

Using Lemma 6.1 (for $j \neq i$),

$$\begin{aligned} \partial_z \mathcal{U}_i &\propto \tau_{\theta_i|p} G_i \partial_z \mathcal{V}_i + (1 - \mathcal{V}_i)(\tau_i + \tau_{\theta_i|p}) \partial_z G_i \\ &= [\tau_{\theta_i|p}(1 - \mathcal{V}_i + H\beta_j^2) - (1 - \mathcal{V}_i)(\tau_i + \tau_{\theta_i|p})] \partial_z \mathcal{V}_i + (1 - \mathcal{V}_i)(\tau_i + \tau_{\theta_i|p}) [\partial_z \mathcal{V}_i + \partial_z G_i] \\ &= [H\beta_j^2 \tau_{\theta_i|p} - (1 - \mathcal{V}_i)\tau_i] \partial_z \mathcal{V}_i + (1 - \mathcal{V}_i)(\tau_i + \tau_{\theta_i|p}) [\partial_z \mathcal{V}_i + \partial_z G_i] \\ &\propto [(1 - \rho)(\dot{\tau}_1 - \dot{\tau}_2)^2 \beta_j^2 \tau_{\theta_i|p} - (1 + \rho)\tau_i \dot{\tau}_j^2] (\dot{\tau}^\top R \dot{\tau}) \partial_z \mathcal{V}_i \\ &\quad + 2(1 - \rho)(1 - \mathcal{V}_i)(\tau_i + \tau_{\theta_i|p})(\dot{\tau}_1 - \dot{\tau}_2)\beta_j [(1 + \rho)\beta_j z_1 + (\dot{\tau}_1 - \dot{\tau}_2)(\dot{\tau}^\top R \dot{\tau}) \partial_z \beta_j] \\ &\propto [(1 - \rho)(\tau_1 - \tau_2)^2 \beta_j^2 \tau_{\theta_i|p} - (1 + \rho)\tau_i \tau_j^2] (\dot{\tau}^\top R \dot{\tau}) \partial_z \mathcal{V}_i \\ &\quad + 2(1 - \rho)(1 - \mathcal{V}_i)(\tau_i + \tau_{\theta_i|p})(\tau_1^2 - \tau_2^2)\beta_j [(1 + \rho)\beta_j z_1 + (\dot{\tau}_1 - \dot{\tau}_2)(\dot{\tau}^\top R \dot{\tau}) \partial_z \beta_j]. \end{aligned}$$

Substituting from (35) and (36), we obtain $\partial_z \mathcal{U}_i \propto L_i$, where

$$\begin{aligned}
L_1 &:= [(1 - \rho)(\tau_1 - \tau_2)^2 \beta_2^2 \tau_{\theta_1|p} - (1 + \rho)\tau_1 \tau_2^2] (R_1^\top \tau) \tau_1 z_1 \\
&\quad + (1 - \rho)(\tau_1 + \tau_{\theta_1|p})(\tau_1^2 - \tau_2^2) \tau_1 \tau_2 \beta_2 [(1 + \rho)\beta_2 z_1 + (\dot{\tau}_1 - \dot{\tau}_2)(\dot{\tau}^\top R \dot{\tau}) \partial_z \beta_2] \\
&= \left((1 - \rho)(\tau_1 - \tau_2) \tau_1 \beta_2^2 [(\tau^\top R \tau) \tau_{\theta_1|p} + (1 + \rho)(\tau_1 + \tau_2) \tau_1 \tau_2] - (1 + \rho)(R_1^\top \tau) \tau_1^2 \tau_2^2 \right) \cdot z_1 \\
&\quad + (1 - \rho)(\tau_1 + \tau_{\theta_1|p})(\tau_1 - \tau_2)^2 (\dot{\tau}^\top R \dot{\tau}) \tau_1 \tau_2 \beta_2 \cdot \partial_z \beta_2, \\
L_2 &:= -[(1 - \rho)(\tau_1 - \tau_2)^2 \beta_1^2 \tau_{\theta_2|p} - (1 + \rho)\tau_1^2 \tau_2] (R_2^\top \tau) \tau_2 z_1 \\
&\quad + (1 - \rho)(\tau_2 + \tau_{\theta_2|p})(\tau_1^2 - \tau_2^2) \tau_1 \tau_2 \beta_1 [(1 + \rho)\beta_1 z_1 + (\dot{\tau}_1 - \dot{\tau}_2)(\dot{\tau}^\top R \dot{\tau}) \partial_z \beta_1] \\
&= \left((1 - \rho)(\tau_1 - \tau_2) \tau_2 \beta_1^2 [(\tau^\top R \tau) \tau_{\theta_2|p} + (1 + \rho)(\tau_1 + \tau_2) \tau_1 \tau_2] + (1 + \rho)(R_2^\top \tau) \tau_1^2 \tau_2^2 \right) \cdot z_1 \\
&\quad + (1 - \rho)(\tau_2 + \tau_{\theta_2|p})(\tau_1 - \tau_2)^2 (\dot{\tau}^\top R \dot{\tau}) \tau_1 \tau_2 \beta_1 \cdot \partial_z \beta_1 \\
&= \left((1 - \rho)(\tau_1 - \tau_2) \tau_2 \beta_1^2 [(\tau^\top R \tau) \tau_{\theta_2|p} + (1 + \rho)(\tau_1 + \tau_2) \tau_1 \tau_2] + (1 + \rho)(R_2^\top \tau) \tau_1^2 \tau_2^2 \right) \cdot z_1 \\
&\quad - (1 - \rho)(\tau_1 + \tau_{\theta_1|p})(\tau_1 - \tau_2)^2 (\dot{\tau}^\top R \dot{\tau}) \tau_1 \tau_2 \beta_2 \cdot \partial_z \beta_2.
\end{aligned}$$

The last equality follows from the observation that $(\tau_1 + \tau_{\theta_1|p})\beta_2 = (\tau_2 + \tau_{\theta_2|p})\beta_1$, and $\partial_z \beta_1 + \partial_z \beta_2 = 0$. We can write the equations for L_1 and L_2 compactly as follows:

$$\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} a_1 & b \\ a_2 & -b \end{bmatrix} \begin{bmatrix} z_1 \\ \partial_z \beta_2 \end{bmatrix} \quad (37)$$

$$= \begin{bmatrix} a_1 & b \\ a_2 & -b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{\partial \beta_2}{\partial \dot{\tau}_1} & \frac{\partial \beta_2}{\partial \psi} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad (38)$$

where

$$\begin{aligned}
a_1 &= (1 - \rho)(\tau_1 - \tau_2) \tau_1 \beta_2^2 [(\tau^\top R \tau) \tau_{\theta_1|p} + (1 + \rho)(\tau_1 + \tau_2) \tau_1 \tau_2] - (1 + \rho)(R_1^\top \tau) \tau_1^2 \tau_2^2, \\
a_2 &= (1 - \rho)(\tau_1 - \tau_2) \tau_2 \beta_1^2 [(\tau^\top R \tau) \tau_{\theta_2|p} + (1 + \rho)(\tau_1 + \tau_2) \tau_1 \tau_2] + (1 + \rho)(R_2^\top \tau) \tau_1^2 \tau_2^2, \\
b &= (1 - \rho)(\tau_1 + \tau_{\theta_1|p})(\tau_1 - \tau_2)^2 (\dot{\tau}^\top R \dot{\tau}) \tau_1 \tau_2 \beta_2.
\end{aligned}$$

A direction z is strictly Pareto improving if and only if both L_1 and L_2 are positive.

We claim that both the 2×2 matrices in (38) are nonsingular. As to the first matrix, we have $b \neq 0$, so it suffices to show that $a_1 + a_2 \neq 0$. Indeed,

$$\begin{aligned}
a_1 + a_2 &= (1 - \rho)(\tau_1 - \tau_2) \\
&\quad \cdot \left((\tau^\top R \tau) [\tau_1 \beta_2^2 \tau_{\theta_1|p} + \tau_2 \beta_1^2 \tau_{\theta_2|p}] + (1 + \rho) \tau_1 \tau_2 [(\tau_1 + \tau_2)(\tau_1 \beta_2^2 + \tau_2 \beta_1^2) - \tau_1 \tau_2] \right).
\end{aligned}$$

Moreover,

$$\begin{aligned}
(\tau_1 + \tau_2)(\tau_1 \beta_2^2 + \tau_2 \beta_1^2) - \tau_1 \tau_2 &\propto \dot{\tau}_1 \beta_2^2 + \dot{\tau}_2 \beta_1^2 - \dot{\tau}_1 \dot{\tau}_2 \\
&= \dot{\tau}_1 \beta_2^2 + (1 - \dot{\tau}_1) \beta_1^2 - \dot{\tau}_1 (1 - \dot{\tau}_1) \\
&= \dot{\tau}_1 (\beta_2^2 - \beta_1^2 - 1 + \dot{\tau}_1) + \beta_1^2 \\
&= \dot{\tau}_1 (\dot{\tau}_1 - 2\beta_1) + \beta_1^2 \\
&= (\dot{\tau}_1 - \beta_1)^2,
\end{aligned}$$

which is nonnegative. Therefore, $a_1 + a_2 \propto \tau_1 - \tau_2 < 0$. Turning now to the second matrix, we have

$$\begin{aligned}\beta_2 &= \frac{\tau_2 + \tau_{\theta_2|p}}{\tau_1 + \tau_2 + \tau_{\theta_1|p} + \tau_{\theta_2|p}} \\ &= \frac{\psi(1 - \dot{\tau}_1) + \tau_{\theta_2|p}}{\psi + \tau_{\theta_1|p} + \tau_{\theta_2|p}}.\end{aligned}\tag{39}$$

Using the fact that $\tau_{\theta_i|p} = \tau_\theta(1 - \mathcal{V}_i)^{-1}$, and equation (34),

$$\begin{aligned}\frac{\partial\beta_2}{\partial\psi} &= \frac{\beta_1 - \dot{\tau}_1}{\psi + \tau_{\theta_1|p} + \tau_{\theta_2|p}} \\ &\propto \frac{\tau_1 + \tau_{\theta_1|p}}{\psi + \tau_{\theta_1|p} + \tau_{\theta_2|p}} - \frac{\tau_1}{\psi} \\ &\propto \tau_2\tau_{\theta_1|p} - \tau_1\tau_{\theta_2|p} \\ &\propto \tau_2(1 - \mathcal{V}_1)^{-1} - \tau_1(1 - \mathcal{V}_2)^{-1} \\ &\propto \tau_1 - \tau_2,\end{aligned}\tag{40}$$

which is nonzero. This completes the verification of the claim that both the 2×2 matrices in (38) are nonsingular. Hence, there exists a vector z such that L_1 and L_2 are both positive. This is a strictly Pareto improving direction. Moreover, for any such direction z , $L_1 + L_2$ must be positive. Since

$$L_1 + L_2 = (a_1 + a_2)z_1 \propto (\tau_1 - \tau_2)z_1,$$

it follows that $z_1 < 0$.

In order to determine the sign of z_2 , we invoke the assumption that $\rho > \underline{\rho}$, and hence $R_1^\top \dot{\tau}$ and $R_2^\top \dot{\tau}$ are positive. Then $a_1 < 0$. We also have $b > 0$ (this true even without the assumption that $\rho > \underline{\rho}$), while the sign of a_2 is not pinned down. From (39), we have

$$\frac{\partial\beta_2}{\partial\dot{\tau}_1} = \frac{-\psi + \beta_1 \frac{\partial\tau_{\theta_2|p}}{\partial\dot{\tau}_1} - \beta_2 \frac{\partial\tau_{\theta_1|p}}{\partial\dot{\tau}_1}}{\psi + \tau_{\theta_1|p} + \tau_{\theta_2|p}}.$$

Noting that $\tau_{\theta_i|p} = \tau_\theta(1 - \mathcal{V}_i)^{-1}$, and using (35) and (36),

$$\begin{aligned}\frac{\partial\tau_{\theta_1|p}}{\partial\dot{\tau}_1} &\propto \frac{\partial\mathcal{V}_1}{\partial\dot{\tau}_1} > 0, \\ \frac{\partial\tau_{\theta_2|p}}{\partial\dot{\tau}_1} &\propto \frac{\partial\mathcal{V}_2}{\partial\dot{\tau}_1} < 0.\end{aligned}$$

It follows that $\partial\beta_2/\partial\dot{\tau}_1 < 0$. From (40), $\partial\beta_2/\partial\psi < 0$ as well.

Now consider a strictly Pareto improving direction z . We have already established that such a direction exists, and it has the property that $z_1 < 0$. Recall that $a_1 < 0$

and $b > 0$. Two cases arise, depending on the sign of a_2 . If $a_2 < 0$, we can choose z_2 such that $\partial_z \beta_2 = 0$. From (37), $L_i = a_i z_1$, which is positive for both types. If, on the other hand, $a_2 \geq 0$, we have $a_2 z_1 \leq 0$. Hence, in this case, a necessary condition for z to be strictly Pareto improving is $\partial_z \beta_2 < 0$.

Thus we have shown that there is a Pareto improving direction z with the property that $z_1 < 0$ and $\partial_z \beta_2 \leq 0$. Since

$$\partial_z \beta_2 = z_1 \frac{\partial \beta_2}{\partial \tau_1} + z_2 \frac{\partial \beta_2}{\partial \psi},$$

and both the partial derivatives are negative, $z_1 < 0$ and $\partial_z \beta_2 \leq 0$ together imply that $z_2 > 0$. \square

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