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Parameterized Games, Minimal Nash Correspondences, and Connectedness

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Abstract
Economics and game theory are replete with examples of parameterized games. We show that all minimal Nash payoff USCOs belonging to the Nash equilibrium correspondence of a parameterized game with payoff functions that are uniformly equicontinuous in players' action choices with respect to parameters have minimal Nash USCOs that are essentially-valued as well as connected-valued. We also show that in general for any uniformly equicontinuous parameterized game, the Nash equilibrium correspondence is the composition of two correspondences: the graph correspondence of the collective security mapping and the Ky Fan Correspondence. The graph correspondence, a mapping from the parameter space into Ky Fan sets, encodes the specifics of the parameterized game being consider, while the Ky Fan Correspondence (i.e., the KFC), a mapping from Ky Fan sets into Nash equilibria, is universal and common to all parameterized games. We also show that the range of the graph correspondence, contained in the hyperspace of Ky Fan sets is a hyperspace Peano continuum - and is therefore locally connected. This means that for any two distinct Ky Fan sets contained in the range of graph correspondence there is a continuous segment in the range of the graph correspondence containing these two distinct Ky Fan sets as endpoints.

Keywords: minimal USCO, uniformly equicontinuous sets of payoff functions, essential Nash equilibria, connected sets of Nash equilibria, hyperspaces of Ky Fan sets, Nikaido and Isoda functions, quasi-minimal USCOs, 3M mappings, KFC correspondences, dense selections, Peano continua, locally connected continua.

JEL Classification: C7

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Parameterized Games, Minimal Nash Correspondences, and Connectedness

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Abstract

Economics and game theory are replete with examples of parameterized games. We show that all minimal Nash payoff USCOs belonging to the Nash equilibrium correspondence of a parameterized game with payoff functions that are uniformly equicontinuous in players' action choices with respect to parameters have minimal Nash USCOs that are essentially-valued as well as connected-valued. We also show that in general for any uniformly equicontinuous parameterized game, the Nash equilibrium correspondence is the composition of two correspondences: the graph correspondence of the collective security mapping and the Ky Fan Correspondence. The graph correspondence, a mapping from the parameter space into Ky Fan sets, encodes the specifics of the parameterized game being consider, while the Ky Fan Correspondence (i.e., the KFC), a mapping from Ky Fan sets into Nash equilibria, is universal and common to all parameterized games. We also show that the range of the graph correspondence, contained in the hyperspace of Ky Fan sets is a hyperspace Peano continuum - and is therefore locally connected. This means that for any two distinct Ky Fan sets contained in the range of graph correspondence there is a continuous segment in the range of the graph correspondence containing these two distinct Ky Fan sets as endpoints. *Key words and phrases:* minimal USCO, uniformly equicontinuous sets of payoff functions, essential Nash equilibria, connected sets of Nash equilibria, hyperspaces of Ky Fan sets, Nikaido and Isoda functions, quasi-minimal USCOs, 3M mappings, KFC correspondences, dense selections, Peano continua, locally connected continua.

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1 Introduction

Economics and game theory are replete with examples of parameterized games. One of the most interesting examples can be found in the theory of discounted stochastic games with uncountable state spaces and compact metric action spaces. By Blackwell's Theorem (1965) extended to discounted stochastic games, we know that the key to showing that a discounted stochastic game has a stationary Markov equilibrium is to show that the parameterized collection of state-contingent, one-shot games underlying the discounted stochastic game contains an equilibrium state-contingent, one-shot game - that is, a one shot-game parameterized by an equilibrium vector of state-contingent prices - prices that players use to value their continued use of a particular stationary Markov strategy. Once an equilibrium one-shot game has been found, the equilibrium stationary Markov strategy profile is gotten by measurably stringing together, state-by-state, the Nash equilibria of the one-shot games corresponding to this equilibrium vector of valuation functions. The hard problem is finding the equilibrium vector of valuation functions - or equivalently, the hard problem is identifying the equilibrium state-contingent one-shot game. This problem is a fixed point problem involving the Nash payoff selection correspondence. The problem is very difficult because the Nash payoff selection mapping is neither closed valued nor convex valued. However, the problem can be solved by approximate fixed point methods provided the upper semicontinuous part of the underlying upper Caratheodory Nash payoff correspondence contains a contractible-valued minimal USCO. This will be the case if these minimal USCOs take connected, locally connected, and hereditarily unicoherent values. Here for non-state-contingent parameterized games, we show that all minimal Nash payoff USCOs belonging to a game where the parameterized collection of payoff functions is uniformly equicontinuous in players' actions have minimal Nash USCOs that are not only connected-valued, but also essential-valued (in the sense of Fort, 1950, and Jiang, 1962). Thus we show that in the case of uniformly equicontinuous parameterized games, one of the three conditions required for approximability is automatically satisfied - and for the one-shot games underlying discounted stochastic games, the uniform equicontinuity condition is satisfied automatically (see, Nowak and Raghavan, 1992, or Salon, 1998).

We also show that, in general, for any parameterized games, the Nash equilibrium correspondence is the composition of two correspondences: the graph correspondence of the collective security mapping and the Ky Fan Correspondence. The graph correspondence, a mapping from the parameter space into Ky Fan sets, encodes the specifics of the parameterized game being consider, while the Ky Fan Correspondence (i.e., the KFC), a mapping from Ky Fan sets into Nash equilibria, is universal and common to all parameterized games. We also show that the range of the graph correspondence, contained in the hyperspace of Ky Fan sets, is a hyperspace Peano continuum - and is therefore locally connected. This means that for any two distinct Ky Fan sets contained in the range of graph correspondence there is a continuous segment (a sub-continuum) in the range of the graph correspondence containing these two distinct Ky Fan sets as endpoints.

2 Parameterized m-Person Games: Primitives and Assumptions

An m-person parameterized game is defined by the following primitives:

$$(Z, X_d, u_d(z, (\cdot, \cdot)))_{d \in D} \tag{1}$$

where

(1) D is a finite set of players consisting of $|D| = m$ players and each player $d \in D$ has action or strategy set X_d and for each parameter, z, payoff function

$$u_d(z, (\cdot, \cdot)) : X_1 \times \cdots \times X_m \longrightarrow [-M, M],$$

where $M > 0$.

(2) Z is the parameter space, a compact metric space that is connected, locally connected, and (without loss of generality) equipped with the M-convex metric ρ_Z;

(3) X_d is the space of actions available to player d with typical element x_d where X_d is a compact, convex subset of a locally convex Hausdorff topological vector space E_d, metrizable for the relative topology inherited from E_d;

Letting $X := \prod_{d \in D} X_d$, X is the compact, convex subset of all possible action profiles with typical element $x = (x_d, x_{-d}) \in X$;

(4) $U(z, (y, x)) := \sum_d u_d(z, (y_d, x_{-d}))$ is the sum of players' z-payoffs where each player $d's$ payoff is evaluated at action profile (y_d, x_{-d}) where (i) for each player d, $u_d(z, (\cdot, \cdot))$ is player $d's$ z-payoff function, a continuous function on $X \times X$ with values in $[-M, M]$, $M > 0$, and where (ii) for each z and x, $U(z, (\cdot, x))$ is quasiconcave;

Label these assumptions collectively, [A-1].

Let $\rho_X := \sum_{d \in D} \rho_{X_d}$ be the sum metric on space of action profiles, $X := \prod_{d \in D} X_d$, where each metric ρ_{X_d} is M-convex and compatible with the relative topology on X_d inherited from E_d.[1] Thus, ρ_X is a M-convex metric on the compact, convex subset of all possible action profiles, X, compatible with the relative product topology inherited from $E := \prod_{d \in D} E_d$. We will denote by $\underset{\rho_X}{\longrightarrow}$ sequential convergence in X with respect to the metric ρ_X.

Also, let $\rho_Y := \sum_{d \in D} \rho_{Y_d}$ be a metric on space of payoff profiles, $Y := \prod_{d \in D} Y_d$, where $Y_d := [-M, M]$ for all d and each metric ρ_{Y_d} is given by $\rho_{Y_d}(y, y') := |y - y'|_R$ (i.e., the absolute value of payoff differences). Each metric, ρ_{Y_d} is M-convex and hence the metric ρ_Y is M-convex.

Finally, let $\rho_{Z \times X} := \rho_Z + \rho_X$ and $\rho_{Z \times Y} := \rho_Z + \rho_Y$ be the metrics on $Z \times X$ and $Z \times Y$ respectively. These metrics are also M-convex, implying that the spaces, $(Z \times X, \rho_{Z \times X})$ and $(Z \times Y, \rho_{Z \times Y})$, are M-convex, compact metric spaces.

We will call the game for each z a z-game. Thus, a z-game is specified by the following objects:

$$\mathcal{G}_z := \left(\underbrace{X_d}_{\substack{\text{player d's} \\ \text{strategy set}}} \quad , \quad \underbrace{u_d(z, (\cdot, \cdot))}_{\substack{\text{player d's} \\ \text{z-payoff function}}} \right)_{d \in D} .$$

[1] Because each action space, X_d, is convex (in the classical sense), each X_d is automatically connected. In addition, because each action space, X_d, is locally convex, each X_d is locally connected. Thus, each action set is a Peano continuum, and as a consequence, we can equip each X_d with an M-convex metric, ρ_{X_d}, compatible with the relative topology on X_d inherited from E_d (see Megginson, 1998, and Illanes and Nadler, 1999).

3 The Nash USCO for a Collection of z-Games

Given parameter z and given the profile of strategy choices made by other players,

$$x_{-d} \in \prod_{d' \neq d} X_{d'}, \tag{2}$$

player $i's$ choice problem is given by

$$\max_{x_d \in X_d} u_d(z, (x_d, x_{-d})). \tag{3}$$

3.1 Best Response Mappings

Under assumptions [A-1], player d's optimization problem (3) has a nonempty compact set of solutions. Let

$$u_d^*(z, x_{-d}) := \max_{x_d \in X_d} u_d(z, (x_d, x_{-d})), \tag{4}$$

be player $d's$ optimal payoff function, and let

$$\Gamma_d(z, x_{-d}) := \{x_d \in X_d : u_d(z, (x_d, x_{-d})) \geq u_d^*(z, x_{-d})\}, \tag{5}$$

be player $d's$ best response correspondence. It follows from The Berge Maximum Theorem (1962), that $u_d^*(\cdot, \cdot)$ is continuous on $Z \times X_{-d}$ and that for each z the joint best response correspondence,

$$x \longrightarrow \Gamma(z, x) := \prod_d \Gamma_d(z, x_{-d}), \tag{6}$$

is a ρ_Z-ρ_X-upper semicontinuous mapping with nonempty, ρ_X-compact values.[2] As in the literature (e.g., Hola-Holy, 2009), we call such a mapping an *USCO*. We will denote by

$$\mathcal{U}(X, P_{\rho_X f}(X)) \tag{7}$$

the collection of *all* such USCOs. Here, $P_{\rho_X f}(X)$ denotes the collection of all nonempty, ρ_X-closed, and convex subsets of $X := X_1 \times \cdots \times X_m$. Thus, for each $z \in Z$, the best response correspondence $\Gamma(z, \cdot)$ is an USCO, i.e.,

$$\Gamma(z, \cdot) \in \mathcal{U}(X, P_{\rho_X f}(X)) \text{ for all } z \in Z.$$

3.2 Nash USCOs

Our focus will be on the Nash USCO. A Nash equilibrium for the z-game,

$$\mathcal{G}_z := \{X_d, u_d(z, (\cdot, \cdot))\}_{d \in D}, \tag{8}$$

is a profile of strategy choices, $x^* \in X$, such that for each player d,

$$u_d(z, (x_d^*, x_{-d}^*)) = \max_{x_d \in X_d} u_d(z, (x_d, x_{-d}^*)),$$

[2] A correspondence, $\Lambda(\cdot)$, from Z into X is ρ_Z-ρ_X-upper semicontinuous at z if for every ρ_X-open subset V of X such that

$$\Lambda(z) \subseteq V,$$

there exists a ρ_Z-neighborhood U_z of z such that

$$\Lambda(z') \subseteq V \text{ for all } z' \in U_z.$$

$\Lambda(\cdot)$ is ρ_Z-ρ_X-upper semicontinuous (ρ_Z-ρ_X-usc) if it is ρ_Z-ρ_X-usc at all $z \in Z$. $\Lambda(\cdot)$ is an USCO if it is (i) ρ_Z-ρ_X-usc and if (ii) for all $z \in Z$, $\Lambda(z)$ is a nonempty, ρ_X–compact subset of X.

or equivalently, a Nash equilibrium is a profile of strategy choices, $x^* \in X$, is a fixed point of the best response correspondence,

$$x^* \in \Gamma(z, x^*).$$

The set of all Nash equilibria, $\mathcal{N}(z)$, for z-game $\mathcal{G}_z$ is therefore given by the set of all fixed points of the best response correspondence at z,

$$\mathcal{N}(z) := \{x^* \in X : x^* \in \Gamma(z, x^*)\}. \tag{9}$$

It is well known, and easily shown under assumptions [A-1], that the set-valued mapping, $z \longrightarrow \mathcal{N}(z)$, is ρ_Z-ρ_X-upper semicontinuous with nonempty, ρ_X-compact values. Thus, the Nash correspondence (or the Nash mapping), $\mathcal{N}(\cdot)$, is also an USCO, but one from the parameter space Z with values in $P_{\rho_X f}(X)$, i.e.,

$$\mathcal{N}(\cdot) \in \mathcal{U}_{\rho_Z\text{-}\rho_X} := \mathcal{U}(Z, P_{\rho_X f}(X)). \tag{10}$$

Our objective is to show that the Nash USCO for any parameterized game satisfying [A-1] always contains a minimal USCO taking connected values.

4 A Brief Tour of USCO Mappings

4.1 Basic Definitions

Consider the ρ_Z-ρ_Z-upper semicontinuous set-valued mapping

$$\mathcal{N}(\cdot) : Z \longrightarrow P_{\rho_X f}(X)$$

defined on Z taking nonempty, ρ_X-closed (and hence ρ_X-compact) values in X. Here, $P_{\rho_X f}(X)$ denotes the collection of nonempty, ρ_X-closed subsets of X. Following the literature (e.g., Crannell, Franz, and LeMasurier, 2005 and Hola and Holy, 2009) call such a set-valued mapping an USCO and denote by $\mathcal{U}_{\rho_Z\text{-}\rho_X} := \mathcal{U}(Z, P_{\rho_X f}(X))$ the collection of all USCOs. Also, denote by $Gr\mathcal{N}$ the graph of $\mathcal{N}(\cdot) \in \mathcal{U}_{\rho_Z\text{-}\rho_X}$ given by

$$Gr\mathcal{N} := \{(z, x) \in Z \times X : x \in \mathcal{N}(z)\}.$$

An USCO $\eta(\cdot) \in \mathcal{U}_{\rho_Z-\rho_X}$ is minimal if $\varphi(\cdot) \in \mathcal{U}_{\rho_Z-\rho_X}$ and $Gr\varphi \subseteq Gr\eta$ implies that $Gr\varphi = Gr\eta$. Denote by $\mathcal{M}_{\rho_Z-\rho_X} := \mathcal{M}(Z, P_{\rho_X f}(X))$ the collection of all minimal USCOs. Each USCO contains at least one minimal USCO (e.g., see Proposition 4.3 in Drewnowski and Labuda, 1990). Let $[\mathcal{N}(\cdot)]$ denote the collection of all minimal USCOs belonging to $\mathcal{N}(\cdot) \in \mathcal{U}_{\rho_Z-\rho_X}$. Thus,

$$[\mathcal{N}(\cdot)] := \left\{\eta(\cdot) \in \mathcal{M}_{\rho_Z-\rho_X} : Gr\eta \subseteq Gr\mathcal{N}\right\}.$$

An USCO, $\mathcal{N}(\cdot) \in \mathcal{U}_{\rho_Z-w^*}$, such that

$$[\mathcal{N}(\cdot)] = \{\eta(\cdot)\} \text{ for some } \eta(\cdot) \in \mathcal{U}_{\rho_Z-\rho_X},$$

is called a quasi-minimal USCO. Let $\mathcal{QM}_{\rho_Z-\rho_X}$ denote the collection of all quasi-minimal USCOs. Note that for any $\mathcal{N}(\cdot) \in \mathcal{U}_{\rho_Z-\rho_X}$, each minimal USCO belong to $\mathcal{N}(\cdot)$ is quasi-minimal. Thus,

$$[\mathcal{N}(\cdot)] \subseteq \mathcal{QM}_{\rho_Z-\rho_X}.$$

Finally, given any USCO $\mathcal{N}(\cdot) \in \mathcal{U}_{\rho_Z-\rho_X}$ let

$$S(\mathcal{N}) := \{z \in Z : \mathcal{N}(z) = \{x\} \text{ for some } x \in X\}$$

denote the set of points in Z where $\mathcal{N}(\cdot)$ is single valued. Because Z is a compact metric Baire space and X is metrizable (in this case with convex metric ρ_X), if $\mathcal{N}(\cdot) \in \mathcal{QM}_{\rho_Z-\rho_X}$, then $S(\mathcal{N})$ is a dense G_δ set (see Lemma 7 in Anguelov and Kalenda, 2009).

The following characterization of minimal USCOs will be useful later.

Theorem 1 *(A characterization of minimal USCOs, Anguelov and Kalenda, 2009)*
Suppose assumptions [A-1] hold. The following statements are equivalent:
(1) $\eta(\cdot) \in U_{\rho_Z-\rho_X}$ is a minimal USCO.
(2) If $U \subset Z$ and $V \subset X$ are open sets such that $\eta(U) \cap V \neq \emptyset$, then there is a nonempty open subset W of U such that $\eta(W) \subset V$.
(3) If $U \subset Z$ is an open set and $F \subset X$ is a closed set such that $\eta(z) \cap F \neq \emptyset$ for each $z \in U$, then $\eta(U) \subset F$.
(4) There exists a quasi-continuous selection f^* of $\eta(\cdot)$ such that $\overline{Grf}^* = Gr\eta$.[3]

4.2 USCOs in the Connected Class

Denote by $\mathcal{CU}_{\rho_Z-\rho_X} := \mathcal{U}(Z, C_{\rho_X f}(X))$ the collection of all USCOs with connected values (call these USCOs, CUSCOs). Here, $C_{\rho_X f}(X)$ denotes the hyperspace of all nonempty, ρ_X-closed, and *connected* subsets of X. We say that the USCO $\mathcal{N}(\cdot) \in \mathcal{U}_{\rho_Z-\rho_X}$ is in the connected class if

$$[\mathcal{N}(\cdot)] \cap \mathcal{CU}_{\rho_Z-\rho_X} \neq \emptyset.$$

Thus, if $\mathcal{N}(\cdot)$ is in the connected class, the correspondence $\mathcal{N}(\cdot)$ has a minimal USCO, $\eta(\cdot) \in [\mathcal{N}(\cdot)]$ such that for all z, $\eta(z)$ is nonempty, ρ_X-closed (an hence ρ_X-compact), and connected. Thus for all z, $\eta(z)$ is a subcontinuum of X.

4.3 Dense Selections and the Limit Point Characterization of Minimal Nash CUSCOs

Recall that in a topological space a point z is isolated if $\{z\} \cap U_z = \{z\}$ for all neighborhoods U_z of z. The point z is a limit point if for each neighborhood U_z of z contains a point z' ($\neq z$). The following result, characterizing USCOs with dense selections, is an immediate consequence of Theorem 1 in Beer (1983). In our statement of Beer's result we take as given the fact that Z and X are M-convex, compact metric spaces, equipped with convex metrics ρ_Z and ρ_X respectively. In fact, Beer's result requires only that Z be a complete separable metric space (i.e., a Polish space) and that X be a sigma compact complete, separable metric space.

Theorem 2 *(Beer, 1983)*
Suppose [A-1] holds. Let $\mathcal{N} \in \mathcal{U}_{\rho_Z\text{-}\rho_X}$. The following statements are equivalent.
(a) N has a dense selection.
(b) *$\mathcal{N}$ has the following properties:*
(b-1) For each $z \in Z$ the set

$$\{(z, \mathcal{N}(z))\} := \{(z, x) : x \in \mathcal{N}(z)\}$$

[3] A function $f^* : Z \to X$ is quasicontinuous at z^0 if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that inside the open ball, $B_{\rho_Z}(\delta, z^0)$, there is contained an open set, U, such that for all $z \in U$,

$$f^*(z) \in B_{\rho_X}(\varepsilon, f^*(z^0)).$$

includes at most one isolated point of $Gr\mathcal{N}$;
(b-2) For each $(z,x) \in \{(z,\mathcal{N}(z))\}$, (z,x) is not a limit point of

$$Gr\mathcal{N}\backslash\{(z,\mathcal{N}(z))\}$$

if and only if (z,x) is an isolated point of $Gr\mathcal{N}$.

Let $Z = X = [0,1]$ and consider the USCO, $\eta \in \mathcal{U}(Z, P_f(X))$, given by

$$\eta(z) = \begin{cases} \{0\} & 0 \leq z < \frac{1}{2} \\ \{0,1\} & z = \frac{1}{2} \\ \{1\} & \frac{1}{2} \leq z \leq 1. \end{cases}$$

The quasi-continuous function, $f \in \mathcal{QC}$, given by

$$f(z) = \begin{cases} \{0\} & 0 \leq z < \frac{1}{2} \\ \{1\} & \frac{1}{2} \leq z \leq 1, \end{cases}$$

is a dense selection of η. Note that $Gr\eta$ has no isolated points. Thus, by Beer's Dense Selection Theorem above, for all $(z,x) \in Gr\eta$, (z,x) is a limit point of $Gr\eta\backslash\{(z,\eta(z))\}$. For example, the point $(\frac{1}{2},0) \in Gr\eta$ is the limit of the sequence $\{(\frac{1}{2}-\frac{1}{n},0)\}_n$ where for all n, $(\frac{1}{2}-\frac{1}{n},0) \in Gr\eta\backslash\{(\frac{1}{2},\eta(\frac{1}{2}))\} = Gr\eta\backslash\{(\frac{1}{2},0),(\frac{1}{2},1)\}$. Thus, η is a minimal USCO with a dense selection. But note that the selection is not continuous - there is no continuous selection.

Next, consider the USCO, $\mathcal{N} \in \mathcal{U}(Z, P_f(X))$, given by

$$\mathcal{N}(z) = \begin{cases} \{0\} & 0 \leq z < \frac{1}{2} \\ [0,1] & z = \frac{1}{2} \\ \{1\} & \frac{1}{2} \leq z \leq 1. \end{cases}$$

The quasi-continuous function, $f \in \mathcal{QC}$, given by

$$f(z) = \begin{cases} \{0\} & 0 \leq z < \frac{1}{2} \\ \{1\} & \frac{1}{2} \leq x \leq 1, \end{cases}$$

is a selection of $\mathcal{N}$, but it is not a dense selection of $\mathcal{N}$, because $\overline{Grf} \neq Gr\mathcal{N}$. Note that $Gr\mathcal{N}$ has no isolated points. However, points in the interior of the vertical segment of the graph of $\mathcal{N}$ at $x = \frac{1}{2}$, for example the point $(\frac{1}{2},\frac{1}{2}) \in Gr\mathcal{N}$, are limit points of $\{(1,\mathcal{N}(1))\}$ and cannot be gotten as the limits of sequences in $Gr\mathcal{N}\backslash\{(1,\mathcal{N}(1))\}$. Thus, by Beer's Theorem, there is no dense selection of $\mathcal{N}$. We note that $\mathcal{N}$ is a quasi-minimal USCO with unique minimal USCO, η (i.e., $\{\eta\} = [\mathcal{N}]$).

By Theorem 1, under assumptions [A-1], all minimal USCOs have dense selections (see (1) $\Leftrightarrow$ (4) in Theorem 1 above). Suppose now that $\eta(\cdot)$ is a connected-valued, minimal USCO belonging to the Nash USCO, $\mathcal{N}(\cdot)$, an USCO in the connected class. By Theorem 3.2 and Corollary 3.3 in Hiriart-Urruty (1985), $Gr\eta$ as well as $\eta(Z) := \cup_{z\in Z}\eta(z)$ are connected sets. Let f^* be a quasicontinuous dense selection of the minimal Nash CUSCO, $\eta(\cdot)$. Thus, we have $\overline{Grf}^* = Gr\eta$. Because minimal USCO $\eta(\cdot)$ is connected-valued, $Gr\eta$ is connected, implying that $Gr\eta$ contains no isolated points. Thus, the set

$$(z^0, \eta(z^0)) := \left\{(z^0, x^0) \in Z \times X : x^0 \in \eta(z^0)\right\},$$

called a stalk by Beer (1983), contains no isolated points of $Gr\eta$, and therefore by Theorem 2 above (also, see Beer, 1983, Theorem 1), $(z^0,x^0) \in (z^0,\eta(z^0))$ if and only if (z^0,x^0) is a limit point of $Gr\eta\backslash(z^0,\eta(z^0))$. Thus, any point (z^0,x^0) contained in the stalk $(z^0,\eta(z^0))$ is a limit of some sequence of points contained in $Gr\eta\backslash(z^0,\eta(z^0))$ (i.e., each $(z^0,x^0) \in (z^0,\eta(z^0))$ can be approached from the sides - that is, from a sequence in $Gr\eta\backslash(z^0,\eta(z^0))$).

5 Ky Fan Analysis

5.1 Ky Fan sets and Ky Fan correspondences

Let E be a subset of $X \times X$ satisfying the following three properties:

(a) For each $x \in X$, $(x, x) \in E$.
(b) For each $y \in X$, $\{x \in X : (y, x) \in E\}$ is closed.
(c) For each $x \in X$, $\{y \in X : (y, x) \notin E\}$ is convex or empty.

We will call any subset E of $X \times X$ satisfying properties (a), (b), and (c) a *Ky Fan set*, and we will denote by $\mathbb{S}$ the collection of all Ky Fan sets in $X \times X$. Thus,

$$\mathbb{S} := \{E \subset X \times X : E \text{ has properties (a), (b), and (c)}\}.$$

Given $E \in \mathbb{S}$, let

$$E(y) := \{x \in X : (y, x) \in E\}$$
and
$$E(x) := \{y \in X : (y, x) \in E\}.$$

The section of E at y, $E(y)$, is the set of positions, x, that deter noncooperative defection y, while the section of E at x, $E(x)$, is the set of noncooperative defections, y, deterred by position x. Note that the set in property (c) above is given by

$$\{y \in X : (y, x) \notin E\} := X \backslash E(x).$$

Also, note that the deterrence mapping, $y \longrightarrow E(y)$, is an USCO defined on X and taking values in the hyperspace, $P_{\rho_X f}(X)$, of nonempty, closed subsets of X. We will use this fact to define a metric, $\varrho_{\mathbb{S}}$, on the hyperspace, $\mathbb{S}$, of Ky Fan sets. In particular, for E^1 and E^2 in $\mathbb{S}$, let

$$\varrho_{\mathbb{S}}(E^1, E^2) := \sup_{y \in X} h_X(E^1(y), E^2(y)), \tag{11}$$

where h_{ρ_X} is the Hausdorff metric on $P_{\rho_X f}(X)$ induced by the metric ρ_X. The basic Theorems about Ky Fan sets are the following:

Theorem 3 *(Lemma 3.1, Zhou, Xiang, Yang, 2005 - ZXY05)*
Under [A-1](1)-(3) the hyperspace of Ky Fan sets, $\mathbb{S}$, equipped with the metric $\varrho_{\mathbb{S}}$ is a complete metric space.

Theorem 4 *(Ky Fan, 1961)*
Under [A-1](1)-(3), if $E \in \mathbb{S}$, then

$$N(E) := \cap_{y \in X} E(y) \neq \emptyset.$$

Consider the mapping or correspondence,

$$E \longrightarrow N(E)$$

defined on $\mathbb{S}$ with values in $P_{\rho_X f}(X)$. We will call this mapping the KFC.

Theorem 5 *(Lemma 3.2, Zhou, Xiang, Yang, 2005 - ZXY05)*
Under [A-1](1)-(3), the KFC, $E \longrightarrow N(E)$, is an $\varrho_{\mathbb{S}}$-ρ_X-USCO from $\mathbb{S}$ into $P_{\rho_X f}(X)$.

Let

$$\mathcal{U}_{\varrho_{\mathbb{S}}\text{-}\rho_X} := \mathcal{U}(\mathbb{S}, P_{\rho_X f}(X)).$$

Let $\mathcal{M}_{\varrho_{\mathbb{S}}\text{-}\rho_X}$ denote the collection of all minimal USCOs, $\mathcal{QM}_{\varrho_{\mathbb{S}}\text{-}\rho_X}$, the collection of all quasi-minimal USCOs, and $[N(\cdot)]$ or $[N]$ the collection of all minimal KFCs belonging to KFC, $N(\cdot)$.

5.2 Minimal Essential Sets and the 3M Property

Let $N \in \mathcal{U}(\mathbb{S}, P_{\rho_X f}(X))$ be any KFC. We begin with the definitions.

Definition 1 *(Essential Sets and Minimal Essential Sets in the Sense of Fort, 1950, and Jiang, 1962)*
We say that a subset, $e(E^0) \in P_{\rho_X f}(X)$ is essential for N at $E^0 \in \mathbb{S}$ provided that for any $\varepsilon > 0$, there is a $\delta_\varepsilon > 0$ such that the $\varrho_{\mathbb{S}}$-open ball of radius δ_ε about $E^0 \in \mathbb{S}$ is such that if $E \in B_{\varrho_{\mathbb{S}}}(\delta_\varepsilon, E^0) \cap \mathbb{S}$ then

$$N(E) \cap B_{\rho_X}(\varepsilon, e(E^0)) \neq \emptyset.$$

Denote by $\mathcal{E}(N(E^0))$ the collection of all essential sets of N at E^0.

We say that $m(E^0) \in P_{\rho_X f}(X)$ is minimally essential for N at $E^0 \in \mathbb{S}$ provided no proper subset of $m(E^0)$ is essential for N at E^0. Denote by $\mathcal{E}^*(N(E^0))$ the collection of all minimally essential sets of N at E^0.

Definition 2 *(The 3M Property of USCOs)*
We say that a KFC, $N \in \mathcal{U}_{\varrho_{\mathbb{S}}\text{-}\rho_X}$ has the 3M property if, given any Ky Fan set $E^0 \in \mathbb{S}$, any pair of disjoint closed sets, F^1 and F^2, and any open ball, $B_{\varrho_{\mathbb{S}}}(\delta, E^0) \cap \mathbb{S}$, of radius $\delta > 0$ about E^0 contained in $\mathbb{S}$, the open ball of Ky Fan sets, $B_{\varrho_{\mathbb{S}}}(\delta, E^0) \cap \mathbb{S}$, contains two Ky Fan sets, E^1 and E^2 such that

$$N(E^1) \cap F^1 = \emptyset,$$
and
$$N(E^2) \cap F^2 = \emptyset,$$

then the larger open ball, $B_{\varrho_{\mathbb{S}}}(3\delta, E^0) \cap \mathbb{S}$ contains a third Ky Fan set, E^3, such that

$$N(E^3) \cap [F^1 \cup F^2] = \emptyset.$$

Conversely, a KFC, $N(\cdot)$, fails to satisfy the 3M property, if for some Ky Fan set E^0, there exists two disjoint closed sets, F^1 and F^2, and an open ball, $B_{\varrho_{\mathbb{S}}}(\delta, E^0) \cap \mathbb{S}$, containing two Ky Fan sets, E^1 and E^2 such that

$$N(E^1) \cap F^1 = \emptyset,$$
and
$$N(E^2) \cap F^2 = \emptyset,$$

but such that

$$N(E^3) \cap [F^1 \cup F^2] \neq \emptyset$$

for all $E \in B_{\varrho_{\mathbb{S}}}(3\delta, E^0) \cap \mathbb{S}$.

Theorem 6 *(All KFCs Have the 3M Property)*
Suppose [A-1](1)-(3) holds. The following statements are true:
(1) All KFCs,

$$N(\cdot) : \mathbb{S} \longrightarrow P_{\rho_X f}(X)$$

have the 3M property.
(2) All minimal USCOs belonging to a KFC inherit the 3M property.

Proof: (1): Suppose not. Then for some $E^0 \in \mathbb{S}$, the KFC

$$N(\cdot) : \mathbb{S} \longrightarrow P_{\rho_X f}(X)$$

is such that there exists a pair of disjoint closed sets, F^1 and F^2 in X, and an open ball, $B_{\varrho_{\mathbb{S}}}(\delta^0, E^0) \cap \mathbb{S}$, $\delta^0 > 0$, containing two Ky Fan sets, E^1 and E^2, such that

$$N(E^1) \cap F^1 = \emptyset \text{ and } N(E^2) \cap F^2 = \emptyset,$$

but such that for all $E^3 \in B_{\varrho_{\mathbb{S}}}(3\delta^0, E^0) \cap \mathbb{S}$

$$N(E^3) \cap [F^1 \cup F^2] \neq \emptyset.$$

First, given that $N(\cdot)$ is an USCO, under [A-1](1)-(3) there are disjoint open sets U^i such that $F^i \subset U^i$ and $N(E^i) \cap U^i = \emptyset$, $i = 1, 2$. Thus,

$$\left.\begin{array}{c} N(E^3) \cap [F^1 \cup F^2] \neq \emptyset \text{ for all } E^3 \in B_{\varrho_{\mathbb{S}}}(3\delta_\varepsilon, E^0) \cap \mathbb{S}, \\ \text{implies that} \\ N(E^3) \cap [U^1 \cup U^2] \neq \emptyset \text{ for all } E^3 \in B_{\varrho_{\mathbb{S}}}(3\delta_\varepsilon, E^0) \cap \mathbb{S}. \end{array}\right\} \tag{12}$$

We will show that (12) leads to a contradiction by constructing a Ky Fan set, $E^* \in \mathbb{S}$ with $E^* \in B_{\varrho_{\mathbb{S}}}(3\delta_\varepsilon, E^0)$ such that

$$N(E^*) \cap [U^1 \cup U^2] \neq \emptyset \ (*),$$

and such that (*) implies that $N(E^i) \cap U^i \neq \emptyset$ for some $i = 1$ and/or 2. Our candidate for such a set is given by

$$\begin{aligned} E^* &:= \left\{(y, x) \in X \times X : x \in [E^1(y) \cap (U^2)^c] \cup [E^2(y) \cap (U^1)^c]\right\} \\ &= [E^1 \cap (X \times U^2)^c] \cup [E^2 \cap (X \times U^1)^c], \end{aligned} \tag{13}$$

where

$$(X \times U^i)^c := \left\{(y, x) \in X \times X : x \notin U^i\right\}.$$

We must show that, (a) $E^* \in \mathbb{S}$, (b) $E^* \in B_{\varrho_{\mathbb{S}}}(3\delta_\varepsilon, E^0)$, and (c) $N(E^*) \cap \left[U^1 \cup U^2\right] \neq \emptyset \Rightarrow N(E^i) \cap U^i \neq \emptyset$ for some $i = 1$ and/or 2.

<u>(a) $E^* \in \mathbb{S}$</u>: Because $E^i \in \mathbb{S}$, $i = 1, 2$, it is easy to see that for each $x \in X$, $(x, x) \in E^*$

and that for each $y \in X$, $\{x \in X : (y,x) \in E\}$ is closed. It remains to show that for each $x \in X$,

$$\{y \in X : (y,x) \notin E^*\}$$

is convex or empty.

Let $x \in U^1$, then because U^1 and U^2 are disjoint,

$$\{y \in X : (y,x) \notin E^*\} = \left\{y \in X : (y,x) \notin E^1\right\},$$

a convex or empty set because $E^1 \in \mathbb{S}$.

Let $x \in U^2$, then because U^1 and U^2 are disjoint,

$$\{y \in X : (y,x) \notin E^*\} = \left\{y \in X : (y,x) \notin E^2\right\},$$

a convex or empty set because $E^2 \in \mathbb{S}$.

Let $x \in X \backslash U^1 \cup U^2$. Then

$$\{y \in X : (y,x) \notin E^*\}$$

$$= \left\{y \in X : (y,x) \notin E^1\right\} \cap \left\{y \in X : (y,x) \notin E^2\right\},$$

the later being the intersection of convex or empty sets. Therefore,

$$\{y \in X : (y,x) \notin E^*\}$$

is convex or empty.

(b) $E^* \in B_{\varrho_{\mathbb{S}}}(3\delta_\varepsilon, E^0)$: We have

$$E^* = [E^1 \cap (X \times U^2)^c] \cup [E^2 \cap (X \times U^1)^c] \tag{14}$$

and by the triangle inequality, for each $y \in X$

$$\begin{gathered} h_X(E^1(y), E^2(y)) \leq h_X(E^1(y), E^0(y)) + h_X(E^2(y), E^0(y)) < 2\delta^0, \\ \text{and} \\ h_X(E^*(y), E^0(y)) \leq h_X(E^*(y), E^1(y)) + h_X(E^1(y), E^0(y)). \end{gathered} \tag{15}$$

We know already that $h_X(E^1(y), E^0(y)) < \delta^0$. Consider $h_X(E^*(y), E^1(y))$. We have

$$h_X(E^*(y), E^1(y)) := \max\left\{e_X(E^*(y), E^1(y)), e_X(E^1(y), E^*(y))\right\}.$$

It is easy to check that,

$$e_X(E^*(y), E^1(y)) = \sup\nolimits_{x \in E^*(y)} dist_X(x, E^1(y))$$

$$= \sup\nolimits_{x \in [E^2(y) \cap (U^1)^c]} dist_X(x, E^1(y))$$

$$\leq \sup\nolimits_{x \in E^2(y)} dist_X(x, E^1(y)) = e_X(E^2(y), E^1(y)).$$

To show that $e_X(E^1(y), E^*(y)) \leq e_X(E^1(y), E^2(y))$ observe that

$$e_X(E^1(y), E^*(y)) = \sup\nolimits_{x \in E^1(y)} dist_X(x, E^*(y))$$

$$= \sup\nolimits_{x \in E^1(y)} dist_X(x, [E^1(y) \backslash (U^2)] \cup [E^2(y) \backslash (U^1)]).$$

Letting $E^1(y) = [E^1(y)\backslash U^2] \cup [E^1(y) \cap U^2]$, we have for all

$$x \in E^1(y)\backslash U^2,$$

$$\begin{aligned}
&dist_X(x, E^*(y)) \\
&= dist_X(x, [E^1(y)\backslash U^2] \cup [E^2(y)\backslash U^1]) \\
&\leq dist_X(x, [E^2(y)\backslash U^1] \cup [E^2(y) \cap U^1]) \\
&= dist_X(x, E^2(y)).
\end{aligned}$$

Moreover, we have for all

$$x \in E^1(y) \cap U^2,$$

$$\begin{aligned}
&dist_X(x, E^*) \\
&= dist_X(x, [E^1(y)\backslash U^2] \cup [E^2(y)\backslash U^1]) \\
&= dist_X(x, [E^2(y)\backslash U^1]),
\end{aligned}$$

and

$$\begin{aligned}
&dist_X(x, E^2(y)) \\
&= dist_X(x, [E^2(y)\backslash U^1] \cup [E^2(y) \cap U^1]) \\
&= dist_X(x, [E^2(y)\backslash U^1]).
\end{aligned}$$

Thus, for all $x \in E^1(y)$,

$$dist_X(x, E^*(y)) \leq dist_X(x, E^2(y)),$$

implying that $e_X(E^1(y), E^*(y)) \leq e_X(E^1(y), E^2(y))$. Together,

$$\begin{gathered}
e_X(E^1(y), E^*(y)) \leq e_X(E^1(y), E^2(y)) \\
\text{and} \\
e_X(E^*(y), E^1(y)) \leq e_X(E^2(y), E^1(y))
\end{gathered}$$

imply that

$$h_X(E^*(y), E^1(y)) \leq h_X(E^2(y), E^1(y)) < 2\delta^0.$$

Thus, we have for each $y \in X$

$$\begin{aligned}
&h_X(E^*(y), E^0(y)) \leq h_X(E^*(y), E^1(y)) + h_X(E^1(y), E^0(y)) \\
&\leq h_X(E^2(y), E^1(y)) + h_X(E^1(y), E^0(y)) \\
&< 2\delta^0 + \delta^0 = 3\delta^0.
\end{aligned}$$

(c) $N(E^*) \cap \left[U^1 \cup U^2\right] \neq \emptyset \Rightarrow N(E^i) \cap U^i \neq \emptyset$ for some $i = 1$ and/or 2:

WLOG suppose that $x \in N(E^*) \cap U^1$. Given the definition of the KFC, $N(\cdot)$, we have for each $y \in X$,

$$x \in \left(E^1(y) \cap \left(U^2\right)^c\right) \cup \left(E^2(y) \cap \left(U^1\right)^c\right),$$

and because $x \in U^1$, this implies that for each $y \in X$,

$$x \in E^1(y) \cap \left(U^2\right)^c,$$

and specifically, that for each $y \in X$,

$$x \in E^1(y) \cap U^1. \ (*)$$

Thus, given the definition of the KFC, $N(\cdot)$, (*) implies that

$$x \in N(E^1) \cap U^1,$$

contradicting the fact that $N(E^1) \cap U^1 = \emptyset$. Thus we must conclude that $N(\cdot)$ has the 3M property.

Proof of (2): Let $n(\cdot) : \mathbb{S} \longrightarrow 2^X$ be a minimal USCO belong to some KFC $N(\cdot)$ and suppose $n(\cdot)$ does not have the 3M property. Then for some $E^0 \in \mathbb{S}$, the minimal USCO

$$n(\cdot) : \mathbb{S} \longrightarrow P_{\rho_X f}(X)$$

is such that there exists a pair of disjoint closed sets, F^1 and F^2 in X, and an open ball, $B_{\varrho_{\mathbb{S}}}(\delta^0, E^0) \cap \mathbb{S}$, $\delta^0 > 0$, containing two Ky Fan sets, E^1 and E^2 in $\mathbb{S}$, such that

$$n(E^1) \cap F^1 = \emptyset \text{ and } n(E^2) \cap F^2 = \emptyset,$$

but such that for all $E^3 \in B_{\varrho_{\mathbb{S}}}(3\delta^0, E^0) \cap \mathbb{S}$

$$n(E^3) \cap [F^1 \cup F^2] \neq \emptyset.$$

As in the proof of part (a) above there are disjoint open sets U^i such that $F^i \subset U^i$ and $n(E^i) \cap U^i = \emptyset$, $i = 1, 2$. Thus,

$$\left.\begin{array}{c} n(E^3) \cap [F^1 \cup F^2] \neq \emptyset \text{ for all } E^3 \in B_{\varrho_{\mathbb{S}}}(3\delta^0, E^0) \cap \mathbb{S}, \\ \text{implies that} \\ n_{E^0}(E^3) \cap [U^1 \cup U^2] \neq \emptyset \text{ for all } E^3 \in B_{\varrho_{\mathbb{S}}}(3\delta^0, E^0) \cap \mathbb{S}. \end{array}\right\} \quad (16)$$

And as in the proof of part (a) above, we will show that (16) leads to a contradiction by showing that the Ky Fan set,

$$E^* := [E^1 \cap (X \times U^2)^c] \cup [E^2 \cap (X \times U^1)^c] \in \mathbb{S}$$

with $E^* \in B_{\varrho_{\mathbb{S}}}(3\delta^0, E^0)$ and

$$n(E^*) \cap [U^1 \cup U^2] \neq \emptyset,$$

implies that $n(E^i) \cap U^i \neq \emptyset$ for some $i = 1$ and/or 2.

Suppose then that $n(E^*) \cap \left[U^1 \cup U^2\right] \neq \emptyset$ implying WLOG that $n(E^*) \cap U^1 \neq \emptyset$. Let $x \in n(E^*) \cap U^1$. Given the definition of the minimal KFC, $n(\cdot)$, we have for each $y \in X$,

$$(y, x) \in \left(E^1 \cap \left(X \times U^2\right)^c\right) \cup \left(E^2 \cap \left(X \times U^1\right)^c\right),$$

and because $x \in U^1$, this implies that for each $y \in X$,

$$(y, x) \in E^1 \cap \left(X \times U^2\right)^c,$$

and specifically, that for each $y \in X$,

$$(y, x) \in E^1 \cap \left(X \times U^1\right). \quad (**)$$

Thus, given that $x \in n(E^*) \cap U^1$ we know that for all y, $x \in \left(E^1(y) \cap \left(U^2\right)^c\right) \cup \left(E^2(y) \cap \left(U^1\right)^c\right)$. Moreover, given that in addition, $x \in U^1$ we can conclude that in fact, $x \in E^1(y) \cap U^1$ for all y. Finally, because $x \in n(E^*) \cap U^1$, the fact that $x \in E^1(y) \cap U^1$ for all y implies that

$$x \in n(E^1) \cap U^1,$$

contradicting the fact that $n(E^1) \cap U^1 = \emptyset$. Thus we must conclude that $n(\cdot)$ has the 3M property. Q.E.D.

5.3 All KFCs Are in the Connected Class

We begin with a Lemma which establishes a fundamental fact about minimal USCOs: any minimal USCO belonging to a quasi-minimal USCO is minimally essential valued. Here we state the Lemma for KFCs.

Lemma 7 *(Minimal KFCs Belonging to Quasi-Minimal KFCs Are Minimally Essential Valued)*
Suppose assumptions [A-1](1)-(3) hold. Let $N(\cdot) \in \mathcal{QM}_{\varrho_{\mathbb{S}}\text{-}\rho_X}$ with $[N(\cdot)] = \{n(\cdot)\}$ for some $n(\cdot) \in \mathcal{U}_{\varrho_{\mathbb{S}}\text{-}\rho_X}$. Then for each $E \in \mathbb{S}$, $n(E) \in \mathcal{E}^*[N(E)]$.

Proof: Suppose that for some E^0 there is some nonempty, closed and proper subset $e(E^0)$ of $n(E^0)$ with $e(E^0) \in \mathcal{E}[N(E^0)]$. Fix some $x^0 \in \overline{n(E^0) \diagdown e(E^0)}$ and let $B_{\rho_X}(\varepsilon^0, e(E^0)) \subset X$ be an open enlargement of $e(E^0)$ such that $x^0 \notin \overline{B_{\rho_X}(\varepsilon^0, e(E^0))}$. Since $e(E^0) \in \mathcal{E}[N(E^0)]$ there is a $\delta^0 > 0$ such that for all $E \in B_{\varrho_{\mathbb{S}}}(\delta^0, E^0)$, $N(E) \cap B_{\rho_X}(\varepsilon^0, e(E^0)) \neq \emptyset$. Define the mapping $\varphi(\cdot)$ as follows:

$$\varphi(E) := \begin{cases} N(E) \cap \overline{B_{\rho_X}(\varepsilon^0, e(E^0))} & E \in B_{\varrho_{\mathbb{S}}}(\delta^0, E^0) \\ N(E) & z \in \mathbb{S} \diagdown B_{\varrho_{\mathbb{S}}}(\delta^0, E^0). \end{cases}$$

By Lemma 2(ii) in Anguelov and Kalenda (2009), $\varphi(\cdot)$ is an USCO with $Gr\varphi \subset GrN$ and hence $Grn \subset Gr\varphi$. In particular, $x^0 \in \varphi(E^0)$, a contradiction. Q.E.D.

As the following example makes clear, the quasi-minimality of the USCO is critical to the above result.

Example 1 *(Quasi-Minimality is Critical)*
Let $Z = X = [-1, 1]$ and define $N \in \mathcal{U}_{\rho_Z\text{-}\rho_X}$ as follows:

$$N(z) := \begin{cases} \{-1\} & z \in [-1, -\frac{1}{2}) \\ \{-1, 1\} & z \in [-\frac{1}{2}, \frac{1}{2}] \\ \{1\} & z \in (\frac{1}{2}, 1]. \end{cases}$$

While the mapping N is an USCO is not quasi-minimal.

Next consider the following USCO:

$$n(z) := \begin{cases} \{-1\} & z \in [-1, 0) \\ \{-1, 1\} & z = 0 \\ \{1\} & z \in (0, 1]. \end{cases}$$

We have $n \in [N]$ but $n(0)$ is not minimally essential for N at $z = 0$ because $n(0) = \{-1, 1\}$ but the smaller sets $\{-1\}$ and $\{1\}$ are each minimally essential for N at $z = 0$.

The following Theorem establishes a fundamental fact about minimal USCOs: any minimal USCO corresponding to any KFC mapping is minimally essentially valued - and therefore all KFCs are in the connected class.

Theorem 8 *(The Connection Between a KFC's Minimal USCOs and Minimal Essential Sets)*
Suppose assumptions [A-1] hold and let $N(\cdot)$ be a KFC. If $n(\cdot)$ is a minimal USCO belonging to $N(\cdot)$, then the following statements are true:
(1) For each $E \in \mathbb{S}$, $n(E) \in \mathcal{E}^*[n(E)]$.
(2) *For each $E \in \mathbb{S}$, $\mathcal{E}^*[n(E)]$ consists of connected sets.*

Proof: (1): Because $n(\cdot)$ is a minimal USCO belonging to $N(\cdot)$, $n(\cdot)$ is quasi-minimal. Thus (1) follows from the Lemma above.

(2): Suppose not. In particular, suppose that for some $\widetilde{E} \in \mathbb{S}$, $n(\widetilde{E}) \in \mathcal{E}^*[n(\widetilde{E})]$ is not connected. Then there are two nonempty, compact sets, $n^1(\widetilde{E})$ and $n^2(\widetilde{E})$, and two nonempty, *disjoint* open subsets, W^1 and W^2, in X such that (i) $n^1(\widetilde{E}) \subset W^1$ and $n^2(\widetilde{E}) \subset W^2$, and (ii) $n(\widetilde{E}) = n^1(\widetilde{E}) \cup n^2(\widetilde{E})$.

Therefore, neither $n^1(\widetilde{E})$ nor $n^2(\widetilde{E})$ are essential implying that there are two nonempty, open sets G^1 and G^2 with

$$n^1(\widetilde{E}) \subset G^1 \text{and } n^2(\widetilde{E}) \subset G^2$$

such that *for all $\delta > 0$, there exists Ky Fan* $E^{\delta 1}$ and $E^{\delta 2}$ in $B_{\varrho_{\mathbb{S}}}(\delta, \widetilde{E})$ such that

$$n(E^{\delta 1}) \cap G^1 = \emptyset \text{ and } n(E^{\delta 2}) \cap G^2 = \emptyset.$$

Let $U^1 = W^1 \cap G^1$ and $U^2 = W^2 \cap G^2$. We have $U^1$and U^2 disjoint open sets such that $n^1(\widetilde{E}) \subset U^1$ and $n^2(\widetilde{E}) \subset U^2$ and for all $\delta > 0$, there exist

$$E^{\delta 1} \in B_{\varrho_{\mathbb{S}}}(\delta, \widetilde{E}) \cap \mathbb{S} \text{ and } E^{\delta 2} \in B_{\varrho_{\mathbb{S}}}(\delta, \widetilde{E}) \cap \mathbb{S}$$

such that

$$n(E^{\delta 1}) \cap U^1 = \emptyset \text{ and } n(E^{\delta 2}) \cap U^2 = \emptyset. \tag{17}$$

Given that the sets $n(E^{\delta i})$ are compact, under [A-1](1)-(3), there exists open sets V^1 and V^2 such that for $i = 1, 2$,

$$n^i(\widetilde{E}) \subset V^i \subset \overline{V}^i \subset U^i.$$

Thus, we have for all $\delta > 0$, $E^{\delta i} \in B_{\varrho_{\mathbb{S}}}(\delta, \widetilde{E}) \cap \mathbb{S}$ such that

$$n(E^{\delta 1}) \cap \overline{V}^1 = \emptyset \text{ and } n(E^{\delta 2}) \cap \overline{V}^2 = \emptyset. \tag{18}$$

Now we have a contradiction: First, because $n(\widetilde{E})$ is a minimal essential set of $n(\widetilde{E})$ and because $n(\widetilde{E}) \subset \left[V^1 \cup V^2\right]$, there exists a positive number $\delta^* > 0$ such that *for all Ky Fan sets* $E \in B_{\varrho_{\mathbb{S}}}(\delta^*, \widetilde{E}) \cap \mathbb{S}$,

$$n(E) \cap \left[V^1 \cup V^2\right] \neq \emptyset. \tag{19}$$

But because $\delta > 0$ can be chosen arbitrarily, choosing $\delta = \frac{\delta^*}{3}$, we have by (18) and the 3M property, the existence of a

$$\overline{E} \in B_{\varrho_{\mathbb{S}}}(3\tfrac{\delta^*}{3}, \widetilde{E}) \cap \mathbb{S} = B_{\varrho_{\mathbb{S}}}(\delta^*, \widetilde{E}) \cap \mathbb{S},$$

such that

$$n(\overline{E}) \cap \left[\overline{V}^1 \cup \overline{V}^2\right] = \emptyset.$$

Q.E.D.

5.4 Quasi-minimal KFCs and Minimal KFCs

We close this subsection with the following result concerning the relation between quasi-minimal KFCs and minimal KFCs.

Theorem 9 *(Quasi-minimal KFCs and minimal KFCs)*
Suppose assumptions [A-1](1)-(3) hold. Let $N(\cdot) \in \mathcal{QM}_{\varrho_{\mathbb{S}}\text{-}\rho_X}$ *with* $[N(\cdot)] = \{n(\cdot)\}$ *for some* $n(\cdot) \in \mathcal{U}_{\varrho_{\mathbb{S}}\text{-}\rho_X}$. *If* $\phi \in \mathcal{U}_{\varrho_{\mathbb{S}}\text{-}\rho_X}$ *is such that* $Gr\phi$ *is a proper subset of* GrN, *then* $Grn = Gr\phi$.

Proof: First, suppose that $(E^0, x^0) \in Gr\phi$, but $(E^0, x^0) \notin Grn$. Thus, we have $x^0 \notin n(E^0)$. Because $n(E^0)$ is ρ_X-closed, there is a closed ball, $\overline{B}_{\rho_X}(\varepsilon_{x^0}, x^0)$ of sufficiently small radius $\varepsilon_{x^0} > 0$, such that $\overline{B}_{\rho_X}(\varepsilon_{x^0}, x^0) \cap n(E^0) = \emptyset$.

Consider the correspondence, $\phi^0_\delta : \mathbb{S} \longrightarrow X$, given by

$$\phi^0_\delta(E) := \begin{cases} \phi(E) \cap \overline{B}_{\rho_X}(\varepsilon_{x^0}, x^0) & E \in B_{\varrho_{\mathbb{S}}}(\delta, E^0) \\ \phi(E) & E \in \mathbb{S} \backslash B_{\varrho_{\mathbb{S}}}(\delta, E^0). \end{cases}$$

By Lemma 2(ii) in Anguelov and Kalenda (2009), $\phi^0_\delta(\cdot)$ is an USCO provided $\phi^0_\delta(E) \neq \emptyset$ for all $E \in \mathbb{S}$. To show that this is true, it suffices to show that for some $\delta^0 > 0$,

$$\phi(E) \cap \overline{B}_{\rho_X}(\varepsilon_{x^0}, x^0) \neq \emptyset \text{ for all } E \in B_{\varrho_{\mathbb{S}}}(\delta^0, E^0). \tag{20}$$

Suppose that (20) is false. Thus for each n, there exists $E^n \in B_{\varrho_{\mathbb{S}}}(\frac{1}{n}, E^0)$ such that

$$dist_{\rho_X}(x^0, \phi(E^n)) := \min_{x \in \phi(E^n)} \rho_X(x^0, x) > \varepsilon_{x^0}.$$

For each n, the closest point $x \in \phi(E^n)$ to x^0 is at a ρ_X-distance from x^0 greater than ε_{x^0}. Thus, no point in $\phi(E^n)$ is contained in the closed ball $\overline{B}_{\rho_X}(\varepsilon_{x^0}, x^0)$. Thus, for E^n not equal to E^0 but arbitrarily close to E^0, $dist_{\rho_X}(x^0, \phi(E^n)) > \varepsilon_{x^0}$, but at E^0, $dist_{\rho_X}(x^0, \phi(E^0)) = 0$. We will show that this jump discontinuity leads to a contradiction. First note that because the function $\rho_X(x^0, \cdot)$ is ρ_X-continuous on X, for each $x' \in \phi(E^0)$ there exists $\varepsilon_{x'} > 0$ and an ρ_X-open ball, $B_{\rho_X}(\varepsilon_{x'}, x')$ such that for all $x \in B_{\rho_X}(\varepsilon_{x'}, x')$,

$$\rho_X(x^0, x) \leq \rho_X(x^0, x') + \varepsilon_{x^0}.$$

Thus, we have $\phi(E^0) \subset \cup_{x' \in \phi(z^0)} B_{\rho_X}(\varepsilon_{x'}, x')$ implying via the ρ_X-compactness of $\phi(E^0)$ that there are finitely many balls,

$$\left\{ B_{\rho_X}(\varepsilon_{x^0}, x^0), B_{\rho_X}(\varepsilon_{x^1}, x^1), \ldots, B_{\rho_X}(\varepsilon_{x^n}, x^n) \right\},$$

covering $\phi(E^0)$, where $\{x^0, x^1, \ldots, x^n\} \subset \phi(E^0)$. Given that $\phi(\cdot)$ is USCO, there exists $\delta_{E^0} > 0$ such that for all $E \in B_{\varrho_{\mathbb{S}}}(\delta_{E^0}, E^0)$,

$$\phi(E) \subset \cup_{i=0}^n B_{\rho_X}(\varepsilon_{x^i}, x^i).$$

Thus, if $E \in B_{\varrho_{\mathbb{S}}}(\delta_{E^0}, E^0)$ and $x \in \phi(E)$, then $x \in B_{\rho_X}(\varepsilon_{x^i}, x^i)$ for some $i = 0, 1, \ldots, n$, and therefore, we have for all $x \in B_{\rho_X}(\varepsilon_{x^i}, x^i)$

$$dist_{\rho_X}(x^0, \phi(E)) := \min_{x \in \phi(E)} \rho_X(x^0, x) \leq \rho_X(x^0, x) \leq \rho_X(x^0, x^i) + \varepsilon_{x^0}.$$

Because

$$dist_{\rho_X}(x^0, \phi(E^0)) := \min_{x \in \phi(E^0)} \rho_X(x^0, x) = 0,$$

we have for $E \in B_{\varrho_{\mathbb{S}}}(\delta_{E^0}, E^0)$,

$$dist_{\rho_X}(x^0, \phi(E)) \leq \min_{0 \leq i \leq n} \rho_X(x^0, x^i) + \varepsilon_{x^0} \leq dist_{\rho_X}(x^0, \phi(E^0)) + \varepsilon_{x^0} = \varepsilon_{x^0}.$$

Thus, we have a contradiction and we must conclude that (20) is true for $\delta_{x^0} > 0$. Therefore, $\phi^0_{\delta_{E^0}}(\cdot) \in \mathcal{U}_{\varrho_{\mathbb{S}}\text{-}\rho_X}$. Letting φ be any minimal USCO contained in $[\phi^0_{\delta_{E^0}}]$, we have a contradiction: $[\psi]$ contains at least two different minimal USCO maps, n and φ - but $[\psi] = \{n\}$. Therefore, we must conclude that $Grn = Gr\phi$. Q.E.D.

6 Parameterized Games, Ky Fan Sets, and The Ky Fan Correspondence

In this section, we will bring together the specifics of the underlying parameterized game and the Ky Fan correspondence. In particular, we will show that the Nash USCO, $z \longrightarrow \mathcal{N}(z)$, corresponding to a parameterized game, $\mathcal{G} := \{\mathcal{G}_z : z \in Z\}$, satisfying [A-1] is a composition of two mappings: the GCS mapping, a ρ_Z-$\varrho_{X\times X}$-continuous mapping, $z \longrightarrow K(z)$, from parameters Z into Ky Fan sets $\mathbb{S}$, and the KFC mapping, an USCO, $E \longrightarrow N(E)$, from Ky Fan sets $\mathbb{S}$ into Nash equilibria. The GCS mapping encodes all the information specific to a particular parameterized game in the game's Ky Fan sets, while the KFC - a mapping common to all games - maps these Ky Fan sets into specific sets of Nash equilibria. The KFC is a mapping common to all strategic form games, while the GCS mapping is specific to a particular parameterized game.

We begin with a discussion of Nikaido-Isoda functions and the graph of collective security mappings (i.e., the GCS mapping).

6.1 Nikaido-Isoda Functions

With each z-game,

$$\mathcal{G}_z := \{X_d, u_d(z, (\cdot,\cdot))\}_{d\in D}\,, \tag{21}$$

we can associate a *Nikaido-Isoda function* (Nikaido and Isoda, 1955) given by

$$\left.\begin{aligned} &\varphi(z,(y,x)) := U(z,(y,x)) - U(z,(x,x)) \\ &:= \textstyle\sum_{d\in D} u_d(z,(y_d,x_{-d})) - \sum_{d\in D} u_d(z,(x_d,x_{-d})). \end{aligned}\right\} \tag{22}$$

Let

$$\mathbb{F} := \{\varphi(z,(\cdot,\cdot)) : z \in Z\}\,, \tag{23}$$

denote the collection of Nikaido-Isoda functions associated with the parameterized game,

$$\mathcal{G} := \{\mathcal{G}_z : z \in Z\}\,.$$

Under assumptions [A-1], each function, $\varphi(z,(\cdot,\cdot)) \in \mathbb{F}$ has the following properties:

(F1) $\varphi(z,(\cdot,\cdot))$ is continuous on the compact metric space, $X \times X$;

(F2) $\varphi(z,(\cdot,x))$ is quasiconcave in y on X.

We will add to our list of assumptions [A-1] the following assumption concerning the collection of Nikaido-Isoda functions, $\mathbb{F}$:

A-1 (5) $\mathbb{F}$ is equicontinuous.

For each parameter z^0, the corresponding Nikaido-Isoda function, $\varphi(z^0,(\cdot,\cdot))$, is uniformly continuous on the compact metric space $X \times X$. Thus for this z^0, we have that for any $\varepsilon > 0$ a $\delta_\varepsilon > 0$ such that for any pair of points, (y,x) and (y',x') in $X \times X$ at $\rho_{X\times X}$-distance apart less than δ_ε, $\left|\varphi(z^0,(y,x)) - \varphi(z^0,(y',x'))\right| < \varepsilon$. If this is true for all parameter values z in Z, then $\mathbb{F}$ is equicontinuous. We note that if

$$\varphi(\cdot,(\cdot,\cdot)) \text{ is continuous on the compact metric space, } Z \times (X \times X)$$

then, because $\varphi(\cdot,(\cdot,\cdot))$, is uniformly continuous on the compact metric space $Z \times (X \times X)$, $\mathbb{F}$ is automatically equicontinuous.

We will denote our augmented list of assumptions, [A-1](1)-(5) by [A-1]*.

6.2 The Game's Collective Security Mappings and the Set of Nash Equilibria

Corresponding to each z-game's Nikaido-Isoda function, $\varphi(z,(\cdot,\cdot))$, there is the subset of $X \times X$ given by

$$K(z) := \{(y,x) \in X \times X : \varphi(z,(y,x)) \leq 0\}. \tag{24}$$

It is easy to verify that $K(z)$ is a Ky Fan set for each z. Moreover, by the Ky Fan Section Theorem (1961), if we let

$$K(z)(y) := \{x \in X : \varphi(z,(y,x)) \leq 0\},$$

then

$$\cap_{y \in X} K(z)(y) \neq \emptyset.$$

Thus,

$$\cap_{y \in X} K(z)(y) = \{x \in X : \varphi(z,(y,x)) \leq 0 \text{ for all } y \in X\}.$$

For each defection profile $y \in X$, we can construct a set of m noncooperative defections of the form $y = (y_d, x_{-d})$ - i.e., one for each player d. The set $K(z)(y)$ is the ρ_X-closed set of choice profiles, $x = (x_d, x_{-d})$, in X that are *collectively secure* against the list of potential noncooperative defection profiles

$$y = (y_d, x_{-d}) \in X := \textstyle\prod_d X_d. \tag{25}$$

Note that if, given parameter z, x is contained in $K(z)(y)$ for *all* possible defection profiles $y \in X$, that is, if

$$x \in \cap_{y \in X} K(z)(y) \tag{26}$$

then for each player d, $x = (x_d, x_{-d})$ is secure against any noncooperative defections. Thus, $x \in \cap_{y \in X} K(z)(y)$ implies that

$$u_d(z,(y_d,x_{-d})) \leq u_d(z,(x_d,x_{-d})),$$

for all pairs $y = (y_d, x_{-d})$ and $x = (x_d, x_{-d})$ - and conversely. Thus, the set of Nash equilibria given parameter z can be fully characterized as follows:

$$x \in \mathcal{N}(z) \text{ if and only if } x \in \cap_{y \in X} K(z)(y), \tag{27}$$

and therefore, the Nash USCO is given by,

$$\left.\begin{aligned} & z \longrightarrow \mathcal{N}(z) = \cap_{y \in X} K(z)(y). \\ & = \{x \in X : x \in \Gamma(z,x)\}. \end{aligned}\right\} \tag{28}$$

Our main results regarding the GCS mapping, $z \longrightarrow K(z)$, are the following:

Theorem 10 *(The GCS mapping, $K(\cdot)$, is a continuous, Ky Fan Valued mapping)*
Let $K(\cdot)$ be the GCS mapping for a parameterized game, $\mathcal{G} := \{\mathcal{G}_z : z \in Z\}$, satisfying [A-1]*. Then the following statements are true:
(1) $K(\cdot)$ is a ρ_Z-$\varrho_{\mathbb{S}}$-continuous mapping
(i.e., $z^n \underset{\rho_Z}{\longrightarrow} z^0$ implies that $\varrho_{\mathbb{S}}(K(z^n), K(z^0)) \longrightarrow 0$).
(2) For each parameter value $z \in Z$, $K(z)$ is a Ky Fan set, i.e.,

$$K(z) \in \mathbb{S} \text{ for all } z \in Z.$$

Proof: (1): First, note that

$$\varrho_{\mathbb{S}}(K(z^n), K(z^0)) := \sup_{y \in X} h_X(K(z^n)(y), K(z^0)(y))$$
where
$$K(z)(y) := \{x \in X : \varphi(z,(y,x)) \leq 0\}.$$

Thus, it follows immediately from the assumptions [A-1]* and the equicontinuity of the functions $\mathbb{F} := \{\varphi(z,(\cdot,\cdot)) : z \in Z\}$ on the compact metric space $X \times X$ that $K(\cdot)$ is ρ_Z-$\varrho_{\mathbb{S}}$-continuous.

(2): We have
$$K(z) := \{(y,x) \in X \times X : \varphi(z,(y,x)) \leq 0\}.$$

Note that for all $y \in X$, $\varphi(z,(y,y)) = 0$. Thus, property (a) of Ky Fan sets holds for $K(z)$. By the continuity of $\varphi(z,(y,\cdot))$ on X, it is easy to see that property (b) holds for $K(z)$. Finally, to see that property (c) holds observe that because $\varphi(z,(\cdot,x))$ is quasiconcave in y, $y \in X$ such that
$$(y,x) \notin K(z)$$
is given by the set,
$$\{y \in X : \varphi(z,(y,x)) > 0\},$$
and this set is convex (or empty). Q.E.D.

Let
$$\mathcal{C}_{\rho_Z\text{-}\varrho_{\mathbb{S}}}(Z,\mathbb{S}) \tag{29}$$
denote the collection of all continuous functions defined on Z taking values in $\mathbb{S}$.

In summary, for the collection of z-games (i.e., the parameterized game), $\mathcal{G} := \{\mathcal{G}_z : z \in Z\}$, satisfying [A-1]*, the Nash correspondence is given by

$$\mathcal{N}(z) = N(K(z)) \text{ for all } z \in Z, \tag{30}$$

where the Ky Fan valued GCS mapping, $K(\cdot) : Z \longrightarrow \mathbb{S}$, is a ρ_Z-$\varrho_{\mathbb{S}}$-continuous on Z with values given by

$$K(z) := \{(y,x) \in X \times X : \varphi(z,(y,x)) \leq 0\} \in \mathbb{S} \text{ for each } z \in Z, \tag{31}$$

and where the KFC, $N(\cdot) : \mathbb{S} \longrightarrow P_{\rho_X f}(X)$, is an USCO with values given by

$$N(E) = \cap_{y \in X} \{x \in X : (y,x) \in E\}, \text{ for each } E \in \mathbb{S}. \tag{32}$$

Thus, given the GCS function $K(\cdot)$, we have for all $z \in Z$

$$\mathcal{N}(z) = N(K(z)) = \cap_{y \in X} \{x \in X : (y,x) \in K(z)\},$$

where $K(z) \in \mathbb{S}$.

Because all minimal USCOs, $n(\cdot)$, belonging to a KFC, $N(\cdot)$, are connected-valued (i.e., are CUSCOs) and because the GCS mapping, $K(\cdot)$, belonging to the parameterized game, $\mathcal{G} := \{\mathcal{G}_z : z \in Z\}$, is a Ky Fan valued continuous function, the mapping

$$z \longrightarrow n(K(z))$$

is a minimal CUSCO for the Nash USCO, $\mathcal{N}(\cdot)$ (i.e., $n(K(\cdot)) \in [\mathcal{N}(\cdot)]$ with connected values). Thus, we have shown that for any parameterized game, $\mathcal{G} := \{\mathcal{G}_z : z \in Z\}$, satisfying [A-1]* with Nash mapping, $\mathcal{N}(\cdot)$, GCS mapping, $K(\cdot)$, and KFC, $N(\cdot)$,

$$n(K(\cdot)) \in [\mathcal{N}(\cdot)] \cap \mathcal{CU}(Z, P_{\rho_X f}(X)),$$

But we can say much more: We have shown that each KFC and all of its minimal KFCs are 3M and that, as a consequence, each minimal USCO contained in a KFC has connected values that are minimally essential for that minimal USCO. Because the GCS mapping, $K(\cdot)$, is continuous with Ky Fan values, $K(z) \in \mathbb{S}$ for all z, the induced USCO, $z \longrightarrow n(K(z))$, also has minimally essential, connected values - $n(K(\cdot))$ is minimally essential valued for itself because it is minimal (and hence quasi-minimal) on Z. What we conclude from all of this is that if $n(\cdot) \in [N(\cdot)]$ (i.e., if $n(\cdot)$ is a minimal USCO for the KFC $N(\cdot)$), then $n(K(\cdot)) \in [\mathcal{N}(\cdot)]$ (i.e., $n(K(\cdot))$ is a minimal USCO for the Nash USCO, $\mathcal{N}(\cdot)$, via the continuous GCS mapping, $K(\cdot)$, belonging to the parameterized game $\mathcal{G} := \{\mathcal{G}_z : z \in Z\}$). Formally, we have the following result.

Theorem 11 *(Minimal KFCs and Minimal Nash USCOs)*
Suppose the parameterized game, $\mathcal{G} := \{\mathcal{G}_z : z \in Z\}$, satisfies assumption [A-1]* with corresponding Nash USCO $\mathcal{N}(\cdot) = N(K(\cdot))$ where

$$N(\cdot) \in \mathcal{U}_{3M}(\mathbb{S}, P_{\rho_X f}(X))$$

is the KFC and

$$K(\cdot) \in \mathcal{C}_{\rho_Z\text{-}\varrho_{\mathbb{S}}}(Z, \mathbb{S})$$

is the GCS function. Then, for each $n(\cdot) \in [N(\cdot)]$,

$$n(K(\cdot)) \in [\mathcal{N}(\cdot)].$$

6.3 Observations

Let

$$K(Z) := \cup_z K(z)$$

denote the range of the GCS function, $z \longrightarrow K(z)$, from parameters, Z, into Ky Fan sets, $\mathbb{S}$. Because Z is ρ_Z-compact and $K(\cdot)$ is ρ_Z-$\varrho_{\mathbb{S}}$-continuous, $K(Z)$ is $\varrho_{\mathbb{S}}$-compact. Moreover, because Z is locally connected, $K(Z)$ is locally connected - and because $K(Z)$ is also connected, $K(Z)$ is a Peano continuum. Therefore, we can assume without loss of generality that the sub-hyperspace of Ky Fan sets, $K(Z)$, specific to a particular parameterized game,

$$\mathcal{G} := \{\mathcal{G}_z : z \in Z\},$$

can be equipped with an M-convex metric, $\rho_{K(Z)}$ equivalent to the metric $\varrho_{\mathbb{S}}$ restricted to $K(Z)$. Thus, for any two distinct Ky Fan sets E^1 and E^2 in $K(Z)$ there is a third Ky Fan set $\overline{E} \in K(Z)$ such that

$$\rho_{K(Z)}(E^1, E^2) = \rho_{K(Z)}(E^1, \overline{E}) + \rho_{K(Z)}(\overline{E}, E^2).$$

Moreover, by Theorem 2.7 in Nadler (1977) for any two distinct Ky Fan sets E^1 and E^2 in $K(Z)$ there is a subset $\gamma \subset K(Z) \subset \mathbb{S}$ such that $E^1 \in \gamma$ and $E^2 \in \gamma$ where γ is isometric to the interval $[0, \rho_{K(Z)}(E^1, E^2)]$ and such that if $E^1 = E^2$ then $\gamma = \{E^1\} = \{E^2\}$ and if $E^1 \neq E^2$, then γ is an arc with end points E^1 and E^2.

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