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Keywords: Systemic Risk, Shadow Banking Regulation, Discounted Stochastic Games, Endogenous Network Formation, Multi-layered Financial Network

JEL Classification: D85, G32

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Shadow Banks and Systemic Risks*

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Abstract

We answer the following question: Does regulating the banking network increase systemic risk in the entire financial network in the presence of unregulated shadow banks? In order to answer this question, we introduce a formal definition of systemic risk based on the equilibrium state dynamics generated by a stochastic game of network formation with banks and shadow banks. Based on the equilibrium dynamics, we discuss the properties of equilibrium financial networks. The predictions of the model are consistent with empirical evidence in the literature. Our analysis makes clear the systemic importance of connections between commercial banks and shadow banks in determining the health of the entire financial network. Lastly, but most importantly, we give a definition of systemic risk based on equilibrium dynamics, and therefore, a definition which takes into account the strategic interactions of financial institutions when facing different regulatory policies. Given our definition of systemic risk, we discuss various policies for regulating the financial system. Then, specializing our dynamic stochastic game model to a more basic model in which banks and shadow banks form a network including connections to the real economy, we show, via numerical simulations of our stochastic game model under various regulatory regimes, that imposing bank regulations not faced by shadow banks does indeed increase systemic risk throughout the financial network.

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1 Introduction

We make two main contributions to the literature on systemic risk in financial networks. First, we construct a dynamic network formation model to study the strategic interactions of banks and shadow banks.[1] Equilibrium state dynamics are found by formulating the problem of network formation as a dynamic stochastic game. From this game we obtain the stationary Markov network formation strategies that give rise to our equilibrium state dynamics. A key ingredient in our analysis is the Markov supernetwork representation of the equilibrium state dynamics. In the supernetwork, the nodes represent states and arcs pointing from one state (node) to another represent the equilibrium Markov transition probabilities of moving from one state to another. Viewing the supernetwork as a map of the transportation network over which the state process travels, we are led to define systemic risk in a given state as the probability that the state process departing the given state arrives at a failed state on or before a given time. Using a specialized version of our game-theoretic model, we carry out a simulation based policy study (i.e., a computational policy study) which confirms the widely held view that regulations aimed at making the banking system safer, but which exclude shadow banks, increase the systemic risk of the entire financial system. Moreover, we show that our conclusions are robust with respect to variations in the parameters used in our simulations.

Because our model allows financial institutions to be farsighted and behave strategically, it provides a potential path to policy design which escapes the Lucas critique (See Lucas (1976)). Moreover, as we demonstrate here, our model can easily be specialized (or simplified) to become a very basic model for carrying out computational policy studies. One of our objectives here is to examine, via computational policy studies, the proposition that regulating banks without similarly regulating shadow banks increases the systemic risk of the entire financial system. In the US, investment funds (e.g., shadow banks), held only 3 times as many bonds as banks in 2003. Now, after the implementation of regulations designed to make the banking system safer (excluding shadow banks), investment funds hold almost 20 times as many, according to data pattern analyzed by The Economist (2015). Also, according to The Economist (2015), in 2007 JPMorgan and its peers had \$2.7 trillion available to make markets. Now they just have \$1.7 trillion. One question naturally arises: would this have been the case had the new, post crisis regulations applied to shadow banks as well? Here, using a computationally friendly version of our model we will find robust support, via numerical simulations, for the proposition that such regulations, aimed at banks but excluding shadow banks, incentivizes shadow banks to adopt network formation strategies (i.e., to adopt strategic interactions) that increases system risk throughout the financial network. Another unique aspect of our model is that the type of networks we consider, specifically heterogeneous directed networks (see Page et al. (2005)), allow for multi-layered financial networks. Kivelä et al. (2014) give an overview of multilayed network

[1] Shadow banks are financial institutions outside the traditional banking regulation system. Shadow banks are not directly regulated by central banks, and they are not included in the safety net.

analysis. For example in our model networks can be layered by a partitioning of the players as well as by a partitioning by the types of connections.

The study of strategic interactions between banks and shadow banks is a missing part in theoretical works on financial networks. Several attempts have been made, using network based models, to explain the 2008 crisis and some policies designed to make the financial system more stable have been studied from a network perspective - with a fixed or randomly generated network (without regard to strategic interactions). One thread of research focuses on whether or not and how shadow banks may have triggered the financial crisis. Gennaioli et al. (2013) argue that financial intermediaries, boosting leverage through securitization, may have caused the financial crisis with neglected risk. How shadow banks are interconnected and function in a financial system is explained in detail in Pozsar et al. (2012). Various Regulatory policies related to shadow banks are discussed by Gorton and Metrick (2012). Another thread of research focuses on interbank lending networks and the role of central banks in these networks. For example, Elliot et al. (2014) study features of international debt cross-holdings networks that might trigger network failures. Farboodi (2014) builds a network formation model, which generates an interbank network with a core-periphery structure - a structure often observed in real-world banking networks. Gai and Kapadia (2010) develop a random network model to explore the impact of shock-induced contagion in banking networks. Their work suggests that financial networks exhibit a robust-yet-fragile tendency: while the probability of contagion is low, if contagion occurs, the damage is widespread. Our paper studies the strategic underpinnings of endogenous banking network formation and shows that the strategic interactions between banks and shadow banks play an important role in determining the nature and magnitude of the systemic risk.

We view our main theoretical contribution to be our definition of "systemic risk" for financial networks. In the classical terminology of Markov chains, we define systemic risk to be the first passage probability to some failed states from a given state. Thus, our notion of systemic risk is one inextricably linked to the underlying equilibrium dynamics of network formation as represented by the supernetwork. Following our approach, rather than there being a single measure of systemic risk, there is instead a schedule of systemic risk measures which lists the probabilities of various arrival times at various failed states in the supernetwork, departing from any given state in the supernetwork. It is the structure and stochastic properties of this transportation system which determine systemic risk. Moreover, our notion of system risk is computable, and allows for different failure time horizons as well as for different failure criteria and different levels of failure severity. Because our notion of systemic risk is easy to calculate, policies studies aimed at determining the impact on systemic risk of various regulatory policies can easily be carried out - as we do here.

The rest of the paper is as follows: in section 2, we set up the model as a discounted stochastic game and claim the equilibrium exists. In section 3, we study the equilibrium strategies and the network properties. In section 4, we study an example and visualize the transition of states. In section 5, we give a formal definition of systemic risk and discuss a post-crisis regulatory policy by central banks. In section 6, we conclude the paper and show some possible further research topics.

2 A Discounted Stochastic Game Model of Financial Networks

In this section we will present the primitives of our discounted stochastic game model of interbank contracting. In our model, banks and shadow banks seek to form a network of loan and equity arrangements so as to maximize the sum of all of their discounted future payoffs. Here are the details.

2.1 Players

- The set of players I consists of banks and shadow banks. The set of banks is denoted by M, with $M = \{1, \cdots, m\}$. And the set of shadow banks is N, with $N = \{m+1, \cdots, m+n\}$. $I = M \cup N$.

By banks we mean commercial banks overseen by a Central Bank (in the USA, the Federal Reserve Bank - the FED), and therefore commercial banks are banks operating within the safety net provided by the Central Bank (henceforth the CB). In our model, banks in the CB system never default because the CB automatically covers any short fall in their ability to cover their debt obligations (i.e., if a bank in the CB system cannot pay back its debt, then the CB will cover the bank's repayment obligations at the end of each period). In return for the security offered by the CB, banks are regulated by the CB - more on this later.

Shadow banks are different. Shadow banks are defined to be market-based financial institutions that operate outside of the regulatory jurisdiction of the CB, as discussed in Pozsar et al. (2012). While shadow banks face no regulatory constraints, they fall outside the CB's safety net. Therefore, shadow banks can default on their debt obligations. While banks are usually large institutions engaged in the full line of traditional banking activities, shadow banks are usually smaller and more focused on specific market based financial contracting activities such as markets for securitization vehicles, asset-backed commercial paper (ABCP) conduits, money market mutual funds, and repurchase agreements (repos). Shadow banks include investment banks, mortgage companies, and hedge funds, just to name a few.

2.2 States

- Ω is a compact set representing the states of nature, with a typical element $\omega \in \Omega$, $\omega = (C, F, s)$. (C, F) specifies the state of a financial system. $C = (C^i)_{i \in I} \in \times_{i \in I} \mathbb{C}^i$, and each C^i is the net cash flow of player i realized at the beginning of a period. $F \subset I$ is the set of players who have defaulted. For simplicity, let $C^i = 0, \forall i \in F$. $s \in \mathcal{S} = \{s^1, \cdots, s^k\}$ is the state of the real economy.

$C^i \in \mathbb{C}^i$ is the net cash flow of player i, where i could be either a bank or a shadow bank. If $C^i > 0$, then player i has C^i amount of money to invest in other banks or in the real economy. If $C^i < 0$, then player i has insufficient cash inflow to cover his debt obligations and has to borrow money to cover the short fall - or default. Let $\mathbb{C}^i := [\underline{C}^i, \overline{C}^i]$ be each player's set of all possible net cash flows with $0 \in \mathbb{C}^i$. At the beginning of each time period, the financial state and the state of the real economy are realized. It is possible that the incoming cash flow of a player is less than the player's liability amount - leading to a negative net cash flow (i.e., $C^i < 0$). Because any financial institution can borrow from other financial institutions having money to lend, the fact that $C^i < 0$ does not necessarily imply that player i defaults - even when player i suffers a loss and has negative net cash flow. Thus, we allow C^i to be negative. There are few empirical studies of default in financial systems. The reason for this is that very often insolvent institutions are rescued before they collapse[2]. Allowing C^i to be negative will allow us to model the rescue process. Following Eisenberg and Noe (2001), we will also assume players with insufficient cash to cover their debt obligations pay their debt obligations proportionally[3]. In particular, in our model we will deduce the vector of actually payments among players that clears the market.

The set F is an accumulated set of defaulted players in financial network - reflecting the persistence of default. In particular, once a player defaults and becomes a member of F, that player remains a member of F for all subsequent periods. We will assume that there is a state, ω, of the economy where all shadow banks default and all banks have 0 net cash flow.

The state of the real economy influences the financial system by determining the allocation of investment returns across states at the end of each period - and these investment returns affect the cash flow that financial institutions have available to cover debt obligations.

In particular, we will assume that the set of states of the real economy, $\{s^1, \cdots, s^k\}$, is ordered from good states to bad states. In particular, we will assume that if $k_2 > k_1$, then random returns from each investment project, as well as players' net cash flows, generated in state s^{k_1} stochastically dominate the random returns from each investment project and each player's net cash flows generated in state s^{k_2} - in the sense that returns in s^{k_1} have higher mean and lower variance than returns in s^{k_2}. We assume there are $\bar{K}$ projects in the real economy having for each state of the real economy, s, a return vector, given by

$$\widetilde{R}_s := (\widetilde{R}_s^1, \ldots, \widetilde{R}_s^{\overline{K}}).$$

Let $X := \{1, \ldots, \bar{K}\}$ be the index set with typical element k, and assume that $k = 1$ indexes the safe project.

Finally, we will assume that states $(C, F, s) \in \Omega$ are publicly observable at the beginning of each period.

[2] See Upper (2011).

[3] See Eisenberg and Noe (2001).

2.3 Network Formation

2.3.1 Basics

Each player's action takes the form of a network proposal. Let $L \subset \mathbb{R}^{m+n}$ with typical element l, be a *finite set of vectors,* (including $l = \vec{0}$), where the components of each vector represent the dollar amounts of borrowing or lending from one bank to another (with $l^{ij} > 0$ indicating the bank i, which makes the proposal (action), is proposing to lend to bank j an amount, $|l^{ij}|$, and $l^{ij} < 0$ indicating that bank i is proposing to borrow from j an amount, $|l^{ij}|$). For each $i \in I$, let

$$r^i := (r^{ij})_{j \in I} \in R_L \subset \mathbb{R}_+^{m+n}$$

be the vector of proposed interest rates corresponding to i's vector of proposed lending and borrowing amounts.

Finally, let *finite set* $Q \subseteq \mathbb{R}_+^{\overline{K}}$ be the feasible set of project investment vectors, q (including $q = \vec{0}$) where the k^{th} component, $q^k \geq 0$, of investment vector

$$q = (q^1, q^2, \ldots, q^{\overline{K}}) \in Q$$

represents the dollar amount invested in project $k \in X := \{1, 2, \ldots, \overline{K}\}$.

In addition to choosing a vector of investment levels (an investment portfolio), $q^i := (q^{ik})_{k \in X} \in Q$, each player $i \in I$ makes a borrowing-lending proposal, denoted

$$l^i := (l^{ij})_{j \in I} \in L \subset \mathbb{R}^{m+n}; r^i := (r^{ij})_{j \in I} \in R_L \subset \mathbb{R}^{m+n}.$$

Given investment portfolio, q^i, player $i's$ investment return in state s of the real economy is

$$\langle q^i, \widetilde{R}_s \rangle := \sum_{k \in X} q^{ik} \widetilde{R}_s^k.$$

If player i's borrowing and lending proposal (l^i, r^i), is accepted, then player $i's$ net contractually specified return is

$$\langle r^i, l^i \rangle := \sum_{j \in I} r^{ij} l^{ij}.$$

We will designate project $k = 1$ to be the project with a riskless zero rate of return. If a player puts money into project 1, the player gets back exactly what he puts in, with no risk (i.e., investment in project 1 is equivalent to keeping your money). We will let 0^i denote the zero network (i.e., $0^i = (0^i, 0^i, 0^i)$ or $l^{ij} = r^{ij} = q^{ik} = 0, \forall j \in I, \forall k \in X$). For any player i in F, the only feasible action this player can propose is $G^i = (l^i, r^i, q^i) = 0^i$. Moreover, any player i can choose to abstain safely from connecting to any other player by choosing

$$(l^i, r^i, q^i) = (\underbrace{0^i}_{l^i}, \underbrace{0^i}_{r^i}, (\underbrace{\max\{C^i, 0\}}_{q^{i1}}, \underbrace{0^{-ik}}_{q^{-i1}}))$$

- a network with no borrowing or lending and all money invested in project 1 - if there is money to invest. In other words, players with *nonnegative net cash flows* can always make no connections with other players (via borrowing and leanding contracts) and invest all their money in project 1, the safe project. The safe network for player i is given by

$$G_0^i := (l_0^{ij}, r_0^{ij}, q_0^{ik})_{j \in I, k \in X} = (0^i, 0^i, (\max\{C^i, 0\}, 0^{-i1})).$$

Players with negative net cash flow face a different fate. If the player experiencing negative flow is a shadow bank (i.e., if $i \in N$), then player i must either make-up the short fall by borrowing (bringing their available loanable and/or investible funds to a nonnegative level), or failing that, must become a life member of the set F of defaulted players. Alternatively, if the player experiencing negative net cash flow is a bank (i.e., if $i \in M$), then player i must either make-up the short fall by borrowing (bringing their available loanable and/or investible funds to a nonnegative level), or failing that, be bailed out by the Central Bank.

2.3.2 The Network Budget Constraint

Given the current state $\omega = (C, F, s)$ whose first component, C, is the profile of players' current net cash flows, $C := (C^i)_{i \in I}$, a network proposal, $G^i := (l^i, q^i)$ is affordable

$$\left.\begin{array}{c} \text{if for } i \in I \text{ with } C^i \geq 0, \\ \sum_{j \in I} l^{ij} + \sum_{k \in X} q^{ik} \leq C^i, \\ \text{and} \\ \text{if for } i \in I \text{ with } C^i < 0, \\ l^{ij} \leq 0 \text{ (borrowing is proposed) for all } j \in I, \text{ and } q^{ik} = 0. \end{array}\right\} \tag{1}$$

Banks may face regulations restricting their investments to certain projects. Let $X^M \subset X$ denote the index of projects that banks can invest in, and let Q^M be the feasible set of available projects to banks.

$$Q^M := \{q \in Q | q^k = 0, \forall k \notin X^M\}.$$

Let $Q^i = Q^M$ if $i \in M$, and $Q^i = Q$ otherwise. Define

$$\mathbb{B}^i(\omega) := \begin{cases} \{(l^i, r^i, q^i) \in L \times R_L \times Q^i | (l^i, r^i, q^i) \text{satisfies (1)}\} & i \notin F, \\ \{(l^i, r^i, q^i) \in L \times R_L \times Q^i | (l^i, r^i, q^i) = (0^i, 0^i, 0^i)\} & i \in F. \end{cases}$$

$\mathbb{B}^i(\omega)$ is the i^{th} player's state-contingent network budget constraint. For each state ω, $\mathbb{B}^i(\omega)$ contains those combinations of borrowing-lending networks and investments that are affordable given the player's net cash flow in state $\omega = (C, F, s)$. But there is one other constraint that must be satisfied if a proposed network is to become a network: matching. In order for a player's proposed borrowing-lending network (henceforth, contracting network) to be viable, it must be a match with the other players' proposed contracting networks.

2.3.3 Matching Network Proposals

We say that a profile of player network proposals

$$(l^i, r^i, q^i)_{i\in I} = \left\{ (l^{i1}, \cdots, l^{i(m+n)}, r^{i1}, \cdots, r^{i(m+n)}, q^{i1}, \cdots, q^{i\bar{K}}) \right\}_{i\in I},$$

where $l^i \in L$,$r^i \in R_L$ and $q^i \in Q$, $\forall i \in I$ is matching if

$$l^{ij} + l^{ji} = 0 \text{ and } r^{ij} = r^{ji}, \text{ for all } i \text{ and } j.$$

Let

$$\mathbb{M} := \left\{ (l^i, r^i, q^i)_{i\in I} : l^{jk} + l^{kj} = 0 \text{ and } r^{jk} = r^{jk} \text{ for all } j \text{ and } k \right\}.$$

2.3.4 The Feasible Networks: Affordable and Matching

Letting

$$\mathbb{B}(\omega) := \times_{i\in I}\mathbb{B}^i(\omega),$$

$\mathbb{B}(\omega)$ is set of affordable network proposals in state $\omega = (C, F, s)$. Thus, the set of affordable and matching network proposals is given by

$$\mathbb{G}(\omega) := \mathbb{B}(\omega) \cap \mathbb{M} := \left\{ (l^i, r^i, q^i)_{i\in I} \in \mathbb{B}(\omega) : l^{ij} + l^{ji} = 0 \text{ and } r^{ij} = r^{ji} \text{ for all } j \text{ and } k \right\}.$$

Thus, the feasible network correspondence,

$$\omega \longrightarrow \mathbb{G}(\omega)$$

is upper-semicontinous, with nonempty, and closed valued (note that $G_0 := (G_0^i)_{i\in I} \in \mathbb{G}(\omega)$ for all ω). The affordable network proposal correspondence is given by

$$\omega \longrightarrow \mathbb{B}(\omega)$$

is also upper-semicontinous, with nonempty, and closed valued - and with

$$\mathbb{G}(\omega) \subset \mathbb{B}(\omega).$$

- In state ω, each player's action choice (proposal choice) is given by,

$$G^i := (l^i, r^i, q^i) := (l^{ij}, r^{ij}, q^{ik})_{j\in I, k\in X} \in \mathbb{B}^i(\omega).$$

Let

$$G(\omega) = (l^i_\omega, r^i_\omega, q^i_\omega)_{i\in I} \in \mathbb{B}(\omega)$$

be the $|I|$-tuple of affordable player network proposals in state ω. If $G(\omega)$ is such that the contracting part is matching, that is, if $(l^i_\omega, r^i_\omega, q^i_\omega)_{i\in I} \in \mathbb{M}$, then in state ω the new status quo network, $\bar{G}(\omega)$, resulting from proposal $G(\omega) = (l^i_\omega, r^i_\omega, q^i_\omega)_{i\in I}$, will be $G(\omega)$. However, if $(l^i_\omega, r^i_\omega, q^i_\omega)_{i\in I} \notin \mathbb{M}$, then the new status quo network will be

$$G_0 = (0^i, 0^i, (\max\{C^i, 0\}, 0^{-ik}))_{i\in I} \in \mathbb{G}(\omega),$$

i.e. the network where no pair of players has any connection and all players invest all their money in project 1. Given state ω, let $\bar{G}(\omega)$ denote the new status quo network given the proposed network, $G(\omega) = (l^i_\omega, r^i_\omega, q^i_\omega)_{i\in I}$, we have

$$\bar{G}(\omega) = \begin{cases} G(\omega) = (l^i_\omega, r^i_\omega, q^i_\omega)_{i\in I} & \text{if } (l^i_\omega, r^i_\omega, q^i_\omega)_{i\in I} \in \mathbb{M}, \\ G_0(\omega) = (0^i, 0^i, (\max\{C^i, 0\}, 0^{-ik}))_{i\in I} & \text{if } (l^i_\omega, r^i_\omega, q^i_\omega)_{i\in I} \notin \mathbb{M}. \end{cases} \tag{2}$$

Thus, if the proposal $G(\omega)$ is a mis-matched, then the new status quo network will be

$$\bar{G}(\omega) = (0^i, 0^i, (\max\{C^i_s, 0\}, 0^{-ik}))_{i\in I},$$

and if $G(\omega)$ is matched, then the new status quo network will be the same as the proposal, i.e.,

$$\bar{G}(\omega) = (l^i_\omega, r^i_\omega, q^i_\omega)_{i\in I}.$$

Let $F : Gr\mathbb{G}(\cdot) \longrightarrow \mathbb{M}$ be the matching function implied by expression (2). We have,

$$F(\omega, G) := GI_{\mathbb{G}(\omega)}(G) + G_0(\omega)(1 - I_{\mathbb{G}(\omega)}(G)).$$

The matching function $F(\cdot,\cdot)$ is continuous in both variables, but not necessarily joint continuous.

2.3.5 Payoff Functions

We will assume that players are risk averse.

- Each player $i \in I$ has an immediate payoff function, $u^i : Gr\mathbb{G}(\cdot) \longrightarrow \mathbb{R}$, given by

  $$u^i(\omega, G) := \mathbb{E}[C'^i | F(\omega, G)] - \alpha Var[C'^i | F(\omega, G)] := V^\alpha_i(\omega, F(\omega, G))$$

  with risk aversion level, $\alpha > 0$.

Thus, we will assume that both banks and shadow banks are risk averse. It is generally the case that after the 2008 crisis, both banks and shadow banks became more risk averse. Banks, as large financial institutions, are very careful about their reputations - and therefore very risk averse. Shadow banks, although they behaved aggressively before the 2008 crisis, became much more cautious and risk averse after the crisis due to the realization, by many of the shadow banks, of the extent of the damage caused by the crisis, as well as coming to a full appreciation of how close many of them came to being trapped, without warning, on a path of financial instability.

- Each player $i \in I$ has discount factor, $\beta \in [0, 1)$.

2.3.6 Rules of Network Formation

We will consider our financial network formation problem from the perspective of two different sets of rules of financial network formation. Under Rule 1, all network deviations are noncooperative and correspond to Nash equilibrium (NE). In particular, under Rule 1, each network deviation is made by a single player. Under Rule 2, network deviations are allowed to be coalitional. In particular, under Rule 2, network deviations can be made by a group or coalition of players as long as the deviations do not make any player outside the deviating group worse off. Thus, under Rule 2, preferred coalitional deviations correspond to Pareto improvements. Networks immune to such Pareto improving deviations are weakly Pareto efficient. We will be interested here in weakly Pareto efficient Nash networks and Pareto efficient Nash equilibria (PENE) - both notions are closely related to the notions of strong equilibria and strong stability introduced by Jackson and van den Nouweland (2005). In fact, in two-player games of network formation, the set of strong equilibria is the same as the set of weakly Pareto efficient Nash equilibria[4]. We will write, $G \longrightarrow G'$ to denote that under the rules of network formation (either Rule 1 or Rule 2), G can be changed to G'. We give a simple example to explain the network formation process. The example is as follows:

Example: Let $I = M = \{1,2\}$, $\mathcal{S} = \{s\}$, $X = \{1,2\}$, $R_L = \{0,1\}$, $(\alpha, \beta) = (0,0)$, $\omega = ((1,1), \varnothing, s)$, $d^i = 0, \forall i \in I$, and the distribution of two assets are given as the following.

$$R'^2|s = \begin{cases} 0 & \text{with probability } \frac{1}{4}, \\ 2 & \text{with probability } \frac{3}{4}. \end{cases}$$

For simplicity, let $r = 1$ for all nonzero lending proposals. I.e., $r^{ij} = 1$ if $l^{ij} \neq 0$, $\forall i, j \in I$. The two cases below correspond to two different action sets. $G^i = (l^{i1}, l^{i2}, q^{i1}, q^{i2})$.
Case 1: $\mathbb{G}^1(\omega) = \{G_1^1 = (0,-1,2,0), G_2^1 = (0,-1,1,1), G_3^1 = (0,-1,0,2), G_4^1 = (0,0,1,0), G_5^1 = (0,1,0,0)\}$, and $\mathbb{G}^2(\omega) = \{G_1^2 = (-1,0,2,0), G_2^2 = (-1,0,1,1), G_3^2 = (-1,0,0,2), G_4^2 = (0,0,1,0), G_4^2 = (1,0,0,0)\}$.
Case 2: $\mathbb{G}^1(\omega) = \{G_1^1 = (0,-1,1,1), G_2^1 = (0,0,1,0), G_3^1 = (0,0,\frac{1}{2},\frac{1}{2}), G_4^1 = (0,0,0,1), G_5^1 = (0,1,0,0)\}$, and $\mathbb{G}^2(\omega) = \{G_1^2 = (-1,0,1,1), G_2^2 = (0,0,1,0), G_3^2 = (0,0,\frac{1}{2},\frac{1}{2}), G_4^2 = (0,0,0,1), G_5^2 = (1,0,0,0)\}$.

Then, the payoff matrices are listed as below.

In both cases, the payoffs in blue correspond to actions that proposals are matched. For example, in Case 1, If player 1 takes action G_5^1, and player 2 chooses G_2^2, then player 1 lends to player 2 one unit of money and player 2 hold an investment portfolio of $R'^1 + R'^2$. If their actions are (G_1^1, G_1^2), then no lending connection is formed. Therefore, the set of pure PENE is $\{(G_3^1, G_5^2), (G_5^1, G_3^2)\}$. Moreover, notice that the strategies are also strong Nash equilibria. Similarly, in Case 2, the set of pure PENE is $\{(G_4^1, G_4^2)\}$.

[4] See Definition 2.2 in subsection 2.6.2.

Table 1: Payoff Matrices

	G^2_1	G^2_2	G^2_3	G^2_4	G^2_5
G^1_1	1,1	1,1	1,1	1,1	1,1
G^1_2	1,1	1,1	1,1	1,1	$\frac{3}{2}$,1
G^1_3	1,1	1,1	1,1	1,1	2,1
G^1_4	1,1	1,1	1,1	1,1	1,1
G^1_5	1,1	1,$\frac{3}{2}$	1,2	1,1	1,1

Case 1

	G^2_1	G^2_2	G^2_3	G^2_4	G^2_5
G^1_1	1,1	1,1	1,1	1,1	$\frac{3}{2}$,1
G^1_2	1,1	1,1	1,$\frac{5}{4}$	1,$\frac{3}{2}$	1,1
G^1_3	1,1	$\frac{5}{4}$,1	$\frac{5}{4}$,$\frac{5}{4}$	$\frac{5}{4}$,$\frac{3}{2}$	1,1
G^1_4	1,1	$\frac{3}{2}$,1	$\frac{3}{2}$,$\frac{5}{4}$	$\frac{3}{2}$,$\frac{3}{2}$	1,1
G^1_5	1,$\frac{3}{2}$	1,1	1,1	1,1	1,1

Case 2

The general model we will analyze here is an infinite time horizon, discounted stochastic game of network formation. In our model, actions by the players not only affect current period payoffs (by determining the prevailing network), but also beliefs about next period's state via the law of motion. In the next subsection, we will discuss in more detail the law of motion how believes about future states are formed in the presence of defaulted states.

2.4 The Law of Motion, Default, and Contract Resolution

- Let ω denote a state and $G = (l^i, r^i, q^i)_{i \in I}$ a profile of proposals. A controlled Markov transition probability is a measurable mapping $(\omega, G) \to \eta(\cdot|F(\omega, G))$ from state-proposal pairs to probability measures on states.

If at time t in state ω_t player i's cash flow is negative, that is, if $C^i_t < 0$, then in order for player i to avoid default - and therefore avoid becoming a permanent member of the defaulted players club, F - player i must be able to borrow enough money to restore his cash flow to a positive amount. If player i is a shadow bank, such borrowing must be done in the open market (i.e., via the contracting network), and therefore, it may not be possible for player i (a shadow bank) to borrow sufficient funds. Alternatively, if player i is a bank, then even if player i is unable to borrow enough money in the open market to restore his cash flow to a positive amount, player i can stay in the game because player i's cash flow will be restored via bail out from the Central Bank. How do we identify defaulted players? Suppose the network formed at t is $\bar{G}(\omega_t) = (\bar{l}^i_{\omega_t}, \bar{r}^i_{\omega_t}, \bar{q}^i_{\omega_t})_{i \in I}$, and let D_{ω_t} be given by

$$D^i_{\omega_t} := 1\{C^i_{\omega_t} + \sum_j \bar{l}^{ji}_{\omega_t} < 0, \text{ or } i \in F_{\omega_t}\},$$

$$D_{\omega_t} := \sum_{i \in I} D^i_{\omega_t}.$$

Thus D_{ω_t} is the number of players who do not have enough money at t in state ω_t. The state of the real economy next period s_{t+1} is a function of D_{ω_t}. Given a state ω_t, and two proposals G_{ω_t}, G'_{ω_t}, let D_{ω_t} and D'_{ω_t} be the number of players without enough money given (ω_t, G_t) and (ω_t, G'_t) respectively and let s_{t+1} and s'_{t+1} be the corresponding states of the real economy next period. We will assume that

$$D_{\omega_t} < D'_{\omega_t} \Rightarrow s_{t+1} \succ s'_{t+1} \Rightarrow \widetilde{R}_{t+1}|s_{t+1} \succ_{FOSC} \widetilde{R}_{t+1}|s'_{t+1}.$$

If $\mathbb{E}[R_{t+1}|(\omega_t, G_t)] = \mathbb{E}[R_{t+1}|s_{t+1}]$, then

$$s_{t+1} \succ s'_{t+1} \Leftrightarrow \widetilde{R}_{t+1}|(\omega_{t+1}, G_t) \succ_{FOSC} \widetilde{R}_{t+1}|(\omega_{t+1}, G'_t).$$

The state of the economy s_{t+1}, in turn, determines the distribution of deposits[5], $F^i_{t+1}(d^i_{t+1}|s_{t+1})$ for player (bank) i, $i \in I \backslash F_{t+1}$ (a non-defaulted bank), in the coming time period. The state-contingent, probability distribution functions of deposits, $F^i_{t+1}(\cdot|s_{t+1})$, are continuous, non-decreasing functions with support $[0, \bar{d}]$, $\bar{d} > 0$, and $F^i_{t+1}(0|s_{t+1}) = 0$ and $F^i_{t+1}(\bar{d}|s_{t+1}) = 1$ for all i. Moreover, for any two states s_{t+1}, s'_{t+1} with corresponding distribution functions, $F^i_{t+1}(\cdot|s_{t+1})$ and $F^i_{t+1}(\cdot|s'_{t+1})$, we have

$$s_{t+1} \geq s'_{t+1} \Rightarrow F^i_{t+1}(\cdot|s_{t+1}) \succ_{FOSD} F^i_{t+1}(\cdot|s'_{t+1}), \forall i \in I,$$

that is, $s_{t+1} \geq s'_{t+1}$ implies that $F^i_{t+1}(\cdot|s_{t+1})$ stochastically dominates $F^i_{t+1}(\cdot|s'_{t+1})$ in the first order sense.[6] Thus, the greater the number of defaulted players, the lower is the state of the real economy, implying via first order stochastic dominance the higher is the the probability that deposits will be smaller. Let r denote interest rate of deposits.

The random vector, C_{t+1}, of player net cash flows has a probability distribution determined by current state ω_t and actions G_t. Given current state ω_t and current actions (network proposals) G_t, let $\bar{G}_t$ be the resulting network. For any player i, if $D^i_{\omega_t} = 1$, then $C^i_{t+1} = 0$. Moreover, if i is a bank (rather than a shadow bank), it will remain silent during the coming period starting at t but still remain in the network. If i is a shadow bank, it will default in current period t if it hasn't defaulted already. For banks with $D^i_{\omega_t}$ equal to 0, residing in a network with defaulted banks (i.e., banks i' with $D^{i'}_{\omega_t} = 1$), a clearing vector will have to be computed in order to determine the actual payments at $t+1$. Let realizations of random variables, $(\tilde{R}^k_{t+1})_{k \in X}$, be $(R^k_{t+1})_{k \in X}$. Actual debt repayment from i to j based on the debt contract agreed upon at time t will be denoted by $\tilde{p}^{ij}_{t+1}$, $\tilde{p}^{ij}_{t+1} \geq 0$. And let $\tilde{p}^i_{t+1} := \sum_{j \in I} \tilde{p}^{ij}_{t+1}$. Since debt clears before deposits come in, the incoming cash flow to player i is given by

$$\sum_{k \in X} \bar{q}^{ik}_{\omega_t} R^k_{t+1} + \sum_{j \in I} \tilde{p}^{ji}_{t+1}.$$

[5] Shadow banks in general do not take deposits. However, a large amount of funding for shadow banks comes from money market mutual funds. For simplicity, we use the same terminology "deposits" to represent deposits for banks and funding from money market mutual funds for shadow banks.

[6] $F^i_{t+1}(\cdot|s_{t+1}) \succ_{FOSD} F^i_{t+1}(\cdot|s'_{t+1})$ if and only if for all $d \in [0, \bar{d}]$,

$$F^i_{t+1}(d|s'_{t+1}) \geq F^i_{t+1}(d|s_{t+1}),$$

with

$$F^i_{t+1}(d'|s'_{t+1}) > F^i_{t+1}(d'|s_{t+1}),$$

for some $d' \in [0, \bar{d}]$.

Total obligation or liability of player i is given by

$$\tilde{L}^i_{t+1} := \sum_{j\in I} \bar{r}^{ji}_{\omega_t} \bar{l}^{ji}_{\omega_t} 1\{\bar{l}^{ji}_{\omega_t} > 0\} + r\tilde{d}^i_{\omega_t}.$$

Let node $m+n+1$ represent depositors. And let Π be a $m+n+1$ by $m+n+1$ matrix, with

$$\Pi_{ij} = \begin{cases} \frac{\bar{r}^{ji}_{\omega_t} \bar{l}^{ji}_{\omega_t} 1\{\bar{l}^{ji}_{\omega_t} > 0\}}{\tilde{L}^i_{t+1}} & \text{if } i \in I, \\ 1_{m+n+1}(j) & \text{otherwise.} \end{cases}$$

Where $1_{m+n+1}(j)$ is an indicator function, with value equal to 1 when $j = m+n+1$. The matrix Π captures the nominal liability of one node to another in the network as a proportion of the node's total liabilities. The same as Eisenberg and Noe (2001), we assume debt is paid proportionally if a borrower's cash flow is less than the borrower's total liability, and all value is paid to creditors. Then $\tilde{p}^{ij}_{t+1} = \Pi_{ij}\tilde{p}^i_{t+1}$. For simplicity, let $\tilde{p}^{m+n+1}_{t+1} = 0$, the clearing vector $\tilde{p}_{t+1} = (\tilde{p}^i_{t+1})_{i\in I\cup\{m+n+1\}}$ satisfies

$$\tilde{p}^i_{t+1} = \begin{cases} \min\{\tilde{L}^i_{t+1}, \sum_{k\in I} \bar{q}^{ik}_{\omega_t} \tilde{R}^k_{t+1} + \sum_{j\in I}, \tilde{p}^{ji}_{t+1}\} & \text{if } i \in I, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Let

$$\tilde{L}_{t+1} = (\tilde{L}^1_{t+1}, \cdots, \tilde{L}^{m+n}_{t+1}, 0),$$

and

$$\tilde{E}^i_{t+1} = \sum_{k\in X} \bar{q}^{ik}_{\omega_t} \tilde{R}^k_{t+1}, \forall i \in I, \text{and } \tilde{E}_{t+1} = (\tilde{E}^1_{t+1}, \cdots, \tilde{E}^{m+n}_{t+1}, 0).$$

Then equation (3) becomes

$$\tilde{p}_{t+1} = \tilde{L}_{t+1} \wedge (\tilde{E}_{t+1} + \tilde{p}_{t+1}\Pi).$$

Where $\forall x, y \in \mathbb{R}^n$,

$$x \wedge y := (\min\{x_1, y_1\}, \cdots, \min\{x_n, y_n\}),$$

and

$$x \vee y := (\max\{x_1, y_1\}, \cdots, \max\{x_n, y_n\}).$$

Notice that the existence of a clearing vector $\tilde{p}_{t+1}$ is a fixed point problem. Define

$$\Phi(\tilde{p}_{t+1}; \Pi, \tilde{L}_{t+1}, \tilde{E}_{t+1}) := \tilde{L}_{t+1} \wedge (\tilde{E}_{t+1} + \tilde{p}_{t+1}\Pi).$$

Since $\Phi(\cdot; \Pi, \tilde{L}_{t+1}, \tilde{E}_{t+1})$ is a continuous function from a compact convex set $\times_{i\in I\cup\{m+n+1\}}[0, \bar{L}^i]$ into itself, where $\bar{L}^{m+n+1} = 0$, $\Phi(\cdot; \Pi, \tilde{L}_{t+1}, \tilde{E}_{t+1})$ has a fixed point for all Π, $\tilde{L}_{t+1}$, and $\tilde{E}_{t+1}$ by Brouwer's fixed point theorem[7]. Here $\bar{L}^i$ is player i's maximum liability. The following property shows the uniqueness of the clearing vector.

[7]Notice that the proof of existence of fixed point is different from Eisenberg and Noe (2001). Since boundedness of the clearing vector is naturally inherited by compactness of states and finiteness of actions, the existence result could be got from Brouwer's fixed point theorem.

Proposition 1: There exists a unique clearing vector $\tilde{p}_{t+1}$ to clear debts. I.e., there exists a unique $\tilde{p}_{t+1}$, such that

$$\tilde{p}_{t+1} = \Phi(\tilde{p}_{t+1}; \Pi, \tilde{L}_{t+1}, \tilde{E}_{t+1}).$$

Define $\tilde{p}_{t+1}(\Pi, \tilde{L}_{t+1}, \tilde{E}_{t+1})$ to be the fixed point of $\Phi(\cdot; \Pi, \tilde{L}_{t+1}, \tilde{E}_{t+1})$. $\tilde{p}(\cdot)$ is jointly continuous.

The proof is given in Appendix A. The analysis above gives both the existence and uniqueness of the debt clearing vector. The uniqueness of net cash flow C^i_{t+1} for each player $i \in I$ is derived from the uniqueness of the clearing vector.

For each player i, define

$$\tilde{C}^i_{t+1} := \begin{cases} \sum_{k\in X} \bar{q}^{ik}_{\omega_t} \tilde{R}^k_{t+1} + \sum_{j\in I} \tilde{p}^{ji}_{t+1} + d^i_{t+1} - \tilde{L}^i_{t+1} & \text{if } D^i_{\omega_t} = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The realized net cash flow of player i is C^i_{t+1}. $C^i_{t+1} \in \mathbb{C}^i$ satisfies

$$|C^i_{t+1} - \tilde{C}^i_{t+1}| \le |C^i - \tilde{C}^i_{t+1}|, \forall C^i \in \mathbb{C}^i, \text{ and}$$

$$C^i_{t+1} > C^i, \forall C^i \in \mathbb{C}^i, \text{ such that } |C^i_{t+1} - \tilde{C}^i_{t+1}| = |C^i - \tilde{C}^i_{t+1}|.$$[8]

From above, we know that the underlying distributions of the returns, R_{t+1}, of projects in real economy determine the realization of net cash flows in the financial system. And the distribution of R_{t+1} is determined by the state of the real economy s_{t+1}. In a better state of the real economy, players expect a higher return with lower variance. From this assumption of transition probability between states, we know solvent banks and shadow banks have incentives to lend to insolvent players to avoid a bad state in the next time period.

Lastly, note that F_{t+1} is the set of players who have defaulted before period t and those who default during period t.

$$F_{t+1} := F_t \cup \{i \in N : C^i_t + \sum_{j\in I} \bar{l}^{ji}_{\omega_t} < 0\} = \{i \in N | D^i_{\omega_t} = 1\}.$$

Thus F_{t+1} is deterministic at the end of period t. Because defaulted players never come back into the financial system, F_{t+1} tells us which players will be silent from now on. Moreover, s_{t+1}, specifies the state of the real economy as a result of last period state and actions. Therefore, in the financial network setup, the financial system and real

[8] If $\mathbb{C}^i$ is a continuous set, then the net cash flow will be simply as below,

$$C^i_{t+1} = \begin{cases} \tilde{C}^i_{t+1} & \text{if } \tilde{C}^i_{t+1} \in \mathbb{C}^i, \\ \overline{C}^i & \text{if } \tilde{C}^i_{t+1} > \overline{C}^i, \\ \underline{C}^i & \text{otherwise.} \end{cases}$$

economy features a two-sided interaction. Given the state of the real economy s_{t+1}, s_{t+1} has a direct influence on financial system, and largely determines the expected payoffs at period $t+1$ corresponding to any action profile chosen by the players in the financial network. On the other hand, the actions of players in the financial network influence the real economy in the sense that the value of s_{t+1} is determined by the number of players who are in trouble in the financial network.

2.5 Timing of the Stochastic Game

In our financial network formation game, assume the initial state is ω_0. During each period t, $t \geq 0$, the game unfolds in several stages described as the following:

Stage 1: The current state ω_t is realized, and it is publicly observable.

In other words, $\omega_t = (C_t, F_t, s_t)$ is revealed to all players. After observing the net cash flows, the set of defaulted players, and the state of the real economy, players move to stage 2 to lend and borrow money, and invest in real economic projects.

Stage 2: Players propose connections and make investments.

In this stage, a network $\bar{G}_t$ is formed based on a proposal G_t. Therefore, the distribution of states next period is given by $\eta(\cdot|\omega_t, G_t)$. Notice that for all $t \geq 0$, the distribution of the net cash flow C_{t+1} at $t+1$ is determined by the state and actions at t (i.e., ω_t and G_t). The actions, G_t, that players take at t are optimal with respect to state ω_t.

Stage 3: Some shadow banks default and the safety net takes care of banks that default.

Let $\tilde{I}_t$ be the set of banks and shadow banks that have defaulted, or have net cash flow less than 0 in current period, i.e.,

$$\tilde{I}_t = \{i \in I : D^i_{\omega_t} = 1\}, \forall t \geq 1 \text{ and } \tilde{I}_0 = \varnothing.$$

Therefore, $\tilde{I}_t \cap M$ is the set of insolvent banks which do not have enough money to pay back debt. The central bank will take actions to save each insolvent bank $i \in \tilde{I}_t \cap M$.[9] Without the safety net provided by the central banks, in an economic downturn, people afraid of losing money when banks do not have enough liquidity would withdraw money simultaneously, i.e., there would be a bank run. The safety net provided by the central banks deters investors from losing faith in banking system and therefore makes bank

[9] Our assumption that the central bank provides the funds to save bank i is based on the fact that as a result of policies put in place during the Great Depression of the 1930s Hooks and Robinson (2002), one of the primary functions of the central bank is to do precisely that - provide liquidity to troubled banks.

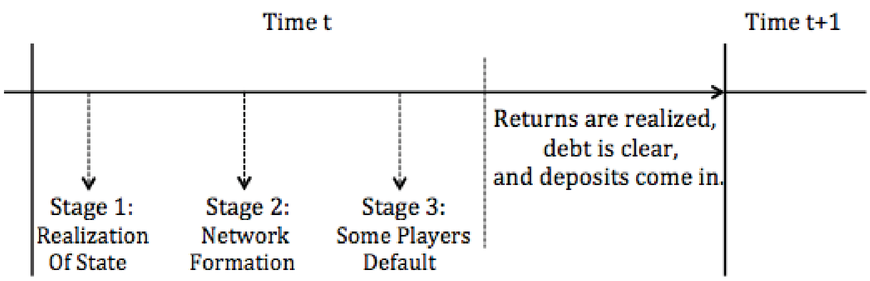
runs less likely. On the other hand, shadow banks have no such safety net - making runs on shadow banks much more likely during economic downturns. Therefore, it is the set of defaulted shadow banks $\tilde{I}_t \cap N$ (those who have just defaulted as well as those who were already defaulted), that determines F_{t+1} - and in particular,

$$F_{t+1} = \tilde{I}_t \cap N.$$

Those players who are permanent members of F_{t+1}will be regarded as silent nodes. In particular, they can only propose the zero vector, and active players make no connections with them in a financial network. Projects returns are realized then automatically, debt is cleared, and new deposits come in and stages 1 to 3 are repeated.

The following picture shows the timing of the stochastic game in each time period. At time t, regard activities after stage 3 as happening during the night. The returns of the projects are realized, so a clearing vector for debt is automatically determined. The value of incoming deposits depends on the state of time period $t+1$, which is determined at stage 3 of time t. When the night ends, the financial system goes into a new time period $t+1$, and players get the information of the new state ω_{t+1}. In traditional one-period models, the timing is similar to what we have assumed here in each time period of our discounted stochastic game. The major difference lies in the stage where players default. In Eisenberg and Noe (2001), given a lending network, default of banks happens before return of projects are realized. So as long as returns are realized, banks default automatically with no actions taken by players strategically. Similarly, in Gofman (2013), shock comes in between the stages of network formation and default of banks. Thus, players can not take any action to save the financial system. We model the stage of default after the stage of network formation because it is closer to what happens in reality. Short-term interbank lending is primarily motivated by avoiding the penalty of not having enough capital requirement before the end of the day. Thus, before default happens, strategic interactions take place to avoid default.

Figure 1: Timing of the Stochastic Game



2.6 Stationary Markov Strategies and Pareto Efficient Nash Equilibria

For each state ω, let $\mathcal{P}(\mathbb{G}^i(\omega))$ denote the set of all probability measures with support contained in $\mathbb{G}^i(\omega)$, where recall, $\mathbb{G}^i(\omega)$ is the set of feasible and affordable contracting and investment networks for player i in state ω. Because the mapping, $\mathbb{G}^i(\cdot)$, is measurable, with nonempty, closed, and convex valued, the mapping from states into probability measures with supports contained in $\mathbb{G}^i(\omega)$,

$$\omega \longrightarrow \mathcal{P}(\mathbb{G}^i(\omega))$$

is measurable, with nonempty, closed, and convex valued. Let $\Sigma^i := \Sigma(\mathcal{P}(\mathbb{G}^i(\cdot)))$ denote the collection of all (measurable) selections of $\mathcal{P}(\mathbb{G}^i(\cdot))$ and let $\Sigma := \Sigma^1 \times \cdots \times \Sigma^{m+n}$ denote the collection of all profiles of (measurable) selections, $\sigma := (\sigma^1, \ldots, \sigma^{m+n}) \in \Sigma$. mappings from ω into the set of all product probability measures,

$$\omega \longrightarrow \sigma(\cdot|\omega) := \times_{i \in I} \sigma^i(\cdot|\omega) \in \mathcal{P}(\mathbb{G}(\omega)) := \mathcal{P}(\mathbb{G}^1(\omega) \times \cdots \times \mathbb{G}^{m+n}(\omega)),$$

where for each i, $\sigma^i(\cdot|\cdot) \in \sigma^i$.[10] We begin with a definition.

2.6.1 Stationary Markov Equilibria in Behavioral and Pure Strategies

Definition 1.1: A *stationary Markov (behavioral) strategy for player i* is a constant sequence of selections, $\sigma^i_{\mathcal{M}} := (\sigma^i, \sigma^i, \cdots)$, with $\sigma^i \in \sigma^i$, such that at any time point, if the current state is ω, then player i chooses a network proposal in $\mathbb{G}^i(\omega)$ according to the probability measure $\sigma^i(\cdot|\omega) \in \mathcal{P}(\mathbb{G}^i(\omega))$.

A stationary Markov (pure) strategy for player i is a constant sequence of selections, $\sigma_{\mathcal{M}} := (\sigma_{G^i}, \sigma_{G^i}, \cdots)$, with $\sigma_{G^i} \in \sigma^i$, such that for some (measurable) selection $G^i(\cdot) \in \Sigma(\mathbb{G}^i(\cdot))$,

$$\sigma_{G^i}(G^i(\omega)|\omega) = 1 \text{ for all } \omega.$$

Given stationary Markov strategy profile, $\sigma_{\mathcal{M}} := \left(\underbrace{(\sigma^1, \ldots, \sigma^{m+n})}_{\sigma}, \underbrace{(\sigma^1, \ldots, \sigma^{m+n})}_{\sigma}, \ldots \right)$, where $\sigma \in \Sigma$, σ induces a (measurable) mapping

$$\omega \longrightarrow \sigma(\cdot|\omega) := \times_{i \in I} \sigma^i(\cdot|\omega) \in \mathcal{P}(\mathbb{G}(\omega)),$$

from the state space into the set of all product probability measures, a subset of the set of all probability measures, $\mathcal{P}(\mathbb{G}(\omega))$, on $\mathbb{G}(\omega) := \mathbb{G}^1(\omega) \times \cdots \times \mathbb{G}^{m+n}(\omega)$. Under stationary Markov strategy profile, $\sigma_{\mathcal{M}}$, if at any time point the state is ω, then profile of network proposals is chosen according to the product probability measure

$$\sigma(\cdot|\omega) := \times_{i \in I} \sigma^i(\cdot|\omega) \in \mathcal{P}(\mathbb{G}(\omega)).$$

[10] $\sigma \in \sigma^i$ if and only if

$$\sigma(\cdot|\omega) \in \mathcal{P}(\mathbb{G}^i(\omega)) \text{ for all } \omega.$$

Given

$$\sigma_{\mathcal{M}} := (\sigma^i(\cdot|\cdot))_{i\in I} \longrightarrow \sigma := \sigma(\cdot|\cdot) := \times_{i\in I}\sigma^i(\cdot|\cdot) \in \Sigma,$$

player i's payoff at any time point in state ω under strategy profile $\sigma_{\mathcal{M}}$, denoted by $\bar{u}^i$, $\bar{u}^i : Gr\mathcal{P}(\mathbb{G}(\cdot)) \longrightarrow \mathbb{R}$, is given by

$$\bar{u}^i(\omega,\sigma_{\mathcal{M}}) = \sum_{G\in\mathbb{G}(\omega)} \sigma(G|\omega)u^i(\omega,G). \tag{5}$$

Therefore, given initial state ω_0 and strategy $\sigma_{\mathcal{M}}$, the (expected) period-payoff for player i at period t is $\tilde{u}^i_t : \Omega\times\Sigma$.

$$\tilde{u}^i_t(\omega_0,\sigma_{\mathcal{M}}) := \begin{cases} \bar{u}^i(\omega_0,\sigma_{\mathcal{M}}) & \text{for } t=0,\\ \int_\Omega \bar{u}^i(\omega_t,\sigma_{\mathcal{M}})\eta_t(\omega_t|\omega_0,\sigma_{\mathcal{M}})d\omega_t & \text{for } t\geq 1. \end{cases} \tag{6}$$

The law of motion $\eta_t(\cdot|\omega_0,\sigma_{\mathcal{M}})$ is defined recursively by

$$\eta_t(\omega_t|\omega_0,\sigma_{\mathcal{M}}) = \int_\Omega \left(\sum_{G\in\mathbb{G}} \eta(\omega_t|\omega_{t-1},G)\sigma(G|\omega_{t-1})\right)\eta_{t-1}(\omega_{t-1}|\omega_0,\sigma_{\mathcal{M}})d\omega_{t-1}, \forall t\geq 2. \tag{7}$$

Let $\eta_0(\omega_0|\omega_0,\sigma_{\mathcal{M}}) = 1$ and denote the discounted expected payoff to player i, over an infinite time horizon under Markov strategy $\sigma_{\mathcal{M}}$, starting at state ω_0 by $U^i(\omega_0,\sigma_{\mathcal{M}})$, where

$$U^i(\omega_0,\sigma_{\mathcal{M}}) = \sum_{t=0}^{\infty}\beta^t\tilde{u}^i_t(\omega_0,\sigma_{\mathcal{M}}). \tag{8}$$

Definition 1.2: A stationary Markov strategy $\sigma^*_{\mathcal{M}}$ is a *stationary Markov equilibrium* of the discounted stochastic game Γ, given any initial state, ω_0, if no player can benefit by deviating from his strategy $\sigma^{*i}_{\mathcal{M}}$ to any other other stationary Markov strategy, i.e., for each ω_0,

$$U^i(\omega_0,\sigma^{*i}_{\mathcal{M}},\sigma^{*-i}_{\mathcal{M}}) \geq U^i(\omega_0,\sigma^i,\sigma^{*-i}_{\mathcal{M}}), \forall\sigma^i\in\sigma^i, \forall i\in I.$$

Theorem 1: The discounted stochastic network formation game Γ has a stationary Markov equilibrium in Behavioral Strategies.

This result is implied by Theorem 1 of Federgruen (1978). Notice that mismatched networks are also taken into consideration. Intuitively, if a network is mismatched and payoffs are low, then players will have the incentive to deviate and form a matched network. The intuition behind stationary Markov equilibrium is that, at any time period and any state, no player has the incentive to propose a different network. Therefore, if in some state, the financial network is formed and no bank or shadow bank make another proposal to change it, the network they form is an equilibrium network based on deviation Rule 1. Notice that there might be many equilibria because of the assumption that mismatched proposals give no connection among players. Here, we will consider Pareto efficient Nash equilibrium.

2.6.2 Pareto Efficient Nash Equilibria in Pure Strategies

Rule 2 (see subsection 2.3.6), which allows coalition deviations, is related to the concept of Pareto efficient Nash equilibrium (PENE). We generalize PENE to an infinite horizon strategy space, consider pure strategy equilibrium only. We begin with the definition of a stationary Markov pure strategy. A formal definition of Rule 2 in infinite-period discounted stochastic game is introduced in Definition 2.2.

Definition 2.1: A *pure stationary Markov strategy* $\sigma^i_{\mathcal{P}} = (\sigma^i, \sigma^i, \cdots) \in \Sigma^i$ for player i is a constant sequence of state-dependent measures on Ω, such that for some function $f^i(\cdot) : \Omega \to \mathbb{G}^i$ with $f^i(\omega) \in \mathbb{G}^i(\omega), \forall \omega \in \Omega$, $\sigma^i(f^i(\omega)|\omega) = 1, \forall \omega \in \Omega$. Let $\Sigma_p \subset \Sigma$ denote the set of pure stationary Markov strategies.

Utilities are the same as equations (5)-(8) after replaying $(\sigma_{\mathcal{M}}, \sigma^i_{\mathcal{M}})$ by $(\sigma_{\mathcal{P}}, \sigma^i_{\mathcal{P}})$. Network formation Rule 2 is defined as below.

Definition 2.2: Under Rule 2, a deviation from a pure Pareto efficient stationary Markov strategy $\sigma_{\mathcal{P}} = (\sigma^i_{\mathcal{P}})_{i \in I} \in \Sigma_p$ to another pure strategy $\sigma'_{\mathcal{P}} = (\sigma'^i_{\mathcal{P}})_{i \in I} \in \Sigma_p$ initiated by $K_0 \subset I$, where $K_0 = \{i \in I | \sigma^i_{\mathcal{P}} \neq \sigma'^i_{\mathcal{P}}\}$, can be carried out by a coalition $K \subset I$ if one of the following holds.

- Either $|K_0| \geq 2$, with $K = \{i : V^i(\omega_0, \sigma_{\mathcal{P}}) \neq V^i(\omega_0, \sigma'_{\mathcal{P}})\}$.
- Or $|K_0| = 1$, $K = K_0$.

If $|K_0| = 1$, then only one player deviates. If original proposal is matched in any state, then a unilateral deviation breaks the match, and in some states, there would be no connection in the new networks. As in Jackson and Wolinsky (1996), cutting a link only requires a unilateral move. On the other hand, if some players want to cooperate and move together, it requires permission from all players involved in the move. Thus, a set of players who initiate a change, and all players involved in the change form a coalition. The deviation then goes forward provided all players in the coalition prefer the change - not just the players who initiated the change. Under Rule 2, the definition of a pure Pareto efficient stationary Markov equilibrium is as follows:

Definition 2.3: Given initial state ω_0, a pure Pareto efficient stationary Markov strategy $\sigma^*_{\mathcal{P}} \in \sigma_p$ is a *pure Pareto efficient stationary Markov equilibrium* of the game Γ, if no coalition K, $K \subset I$, can achieve Pareto improvement by a deviation initiated by $K_0 \subset I$ from his (their) strategy (strategies) $(\sigma^{*i}_{\mathcal{P}})_{i \in K_0}$ to any other other pure stationary Markov strategy (strategies) $\sigma^{K_0}_{\mathcal{P}} := (\sigma^i_{\mathcal{P}})_{i \in K_0}$, with $\sigma^i_{\mathcal{P}} \in \sigma^i$, where σ^i is the set of pure stationary Markov strategies for player i, and $\Sigma^{K_0}_p := \times_{i \in K_0} \sigma^i_p$. I.e., $\forall K_0 \subset I$,

$$\exists i \in K, V^i(\omega_0, \sigma^{*K_0}_{\mathcal{P}}, \sigma^{*I \backslash K_0}_{\mathcal{P}}) > V^i(\omega_0, \sigma^{K_0}_{\mathcal{P}}, \sigma^{*I \backslash K_0}_{\mathcal{P}}), \forall \sigma^{K_0}_{\mathcal{P}} \in \Sigma^{K_0}_p.$$

We have the following result on the existence of a pure Pareto efficient stationary Markov equilibrium.

Theorem 2: The discounted stochastic game of financial network formation, Γ, has a pure Pareto efficient stationary Markov equilibrium $\sigma_{\mathcal{P}} \in \Sigma_p$.

Given deviation rule specified in Definition 2.2 and matching function, $F(\cdot,\cdot)$, each pure Pareto efficient stationary Markov equilibrium is a Nash equilibrium (for a proof see Appendix A). Thus, the existence of a stationary Markov equilibrium is an immediate consequence of Theorem 2.

The process of forming a pure Pareto efficient stationary Markov equilibrium is of course the result of some underlying bargaining process, which is not modeled in this paper. But it is not necessary to explicitly model the bargaining process in order to obtain the equilibrium dynamics. Many bargaining processes are allowed without losing the general properties of the game. Under Rule 2 (see Definition 2.2), if players start from one network and have no incentive to make any further move, the network is stable and the underling strategy is an equilibrium. How an equilibrium is reached need not be taken into consideration. For example, if banks have the bargaining power when forming connections with shadow banks, then among the Pareto efficient choices, efficient networks preferred by banks will be formed. Moreover, if the bargaining powers of all players can be ordered, then starting from no connection, the final payoffs under pure Pareto efficient stationary Markov equilibrium will be unique.

2.7 Equilibrium Dynamics

Given a pure Pareto efficient stationary Markov equilibrium $G^*(\cdot) \in \sigma_p$, the equilibrium Markov transition governing the Markov process of net cash flow, defaulted players, and real economy states is given by

$$\mathbb{P}(A|\omega) := \eta(\cdot|\omega, G^*(\omega)), \forall A \subset \Omega.$$

3 What Does the Network Look Like?

In this section, we discuss the predictions of pure Pareto efficient stationary Markov equilibrium with deviation Rule 2 in the discounted stochastic financial network formation game. The equilibrium is a Nash equilibrium strategy and gives Pareto efficient payoffs. We assume the initial state $\omega_0 = (C_0, F_0, s_0)$, with $C_0 = (C_0^i)_{i \in I}$, $\underline{C}^i << C_0^i << \bar{C}^i, \forall i \in I$, $F_0 = \varnothing$, and $s_0 \in \mathcal{S}$. It is similar to other finite-period models, where banks get deposits from investors. Moreover, we call players with $C_t^i < 0$ to be insolvent players. If those insolvent shadow banks could not borrow enough money from others and pay all debt, they will default. If they are banks, they either borrow enough money from others, or they receive help from the central bank. We make the following assumptions first. Notice that these assumptions are only valid in section 3.

-Assumption 1: $\beta = 0$.
-Assumption 2: $\mathbb{E}[\tilde{R}_s^k] > \max\{r, \max\{r' | r' \in R_L\}\}$, $\forall k \in X$, $\forall s' \in \mathcal{S}$.

Under Assumption 1 (A-1), the solution of the infinite time period stochastic game can be characterized. Intuitively, if players have discounted rate 0, they only take one period-payoff into account. Assumption 2 (A-2) implies that the expected return of investments in real economy is always higher than the expected return of debts. Moreover, although in our model, we take the set of actions to be finite, due to compactness this finite set of actions can be taken to be such that any action in the compact set is within ε of some action in the finite set. As explained in section 2, the money payoffs of contracts are rounded to the nearest penny. Thus, when the players make decisions in each period, optimal leading and investment levels are close to the solutions of utility maximizations with compact and continuous state and action spaces.

Recall that the set of project investment levels for each player - as well as the number of projects in the real economy - are finite. As a consequence, if q^i is the optimal portfolio of player i, then we can identify as a project, the project with return, $\sum_{k \in X} q^{ik} \widetilde{R}^k$. Although in the model, we do not restrict each player to invest in one project only, it is as if each player invests in one of only finitely many project - namely the project with return, $\sum_{k \in X} q^{ik} \widetilde{R}^k$. Such a construction allows us to transform the game into an equivalent game in which each player invest in one project only.

We divide states into 3 categories and study how players behave. The division is as follows: Good States, Median States, and Bad States. We call a state a *good state* if $C^i \geq 0, \forall i \in I$. A state is *median* if $C^i \geq 0, \forall i \in M$, but some shadow banks have negative net cash flows. Finally, a state is *bad* if there exists some banks with negative net cash flows.

3.1 In Good States

In good states, we will analyze the form of equilibrium network, especially the links between banks and shadow banks. Direction of funding flow is studied. We also discuss how “core-periphery” could endogenously arise in a financial network.

3.1.1 Determinants of Funding Flow - Efficiency V.S. Stability

Funding flow is of special interest in the study of shadow banking. During good times, shadow banks hold more assets. While in bad times, assets of shadow banks shrink dramatically. In general, it is not rigorous to claim the exact funding flow. In general, because the feedback between states of the economy and the structure of financial networks move stochastically across time, it is difficult to draw any hard and fast conclusions concerning the exact pattern of funding flows. However, our model predicts that whether funds go from shadow banks to banks or from banks to shadow banks the direction of these flows depend on the risk aversion characteristics of banks relative to shadow banks. More precisely, if lenders are more risk averse (and therefore more concerned with financial stability), they are more likely to lend to banks since banks invest in low risk assets in general. Thus, banks are less likely to become insolvent and fail to meet their payment obligations. Alternatively, if lenders are less risk averse (and therefore more concerned with efficiency), they are more likely to lend to shadow banks if shadow banks provide higher interest rate.

Let player $i \in I$ be a potential lender for the financial network (i has no lending contract and only invest in the safe project, but i could lend). Player i is considering whether to lend to bank j, or shadow bank j'. Assume $C^j = C^{j'}$, and j, j' are borrowers and positional equivalent in the network, i.e., $(l^{jo}, r^{jo}, q^{jk}) = (l^{j'o}, r^{j'o}, q^{j'k})$, $\forall o \in I \backslash \{i\}$, $\forall k \in X$, and $l^{jo} \le 0, q^{ik} = 0$. Further, assume that j can offer a loan contract to i with (l^{ij}, r^{ij}), where $l^{ij} = l > 0$ and $r^{ij} = r^M$, and invest l^{ij} in some asset in X^M. Similarly, assume that shadow bank j' can offer a loan contract to i with $(l^{ij'}, r^{ij'})$, where $l^{ij'} = l > 0$ and $r^{ij'} = r^N$, and invest $l^{ij'}$ in an asset in X.

Without loss of generality, let $X^B = \{1, \cdots, \bar{K}^B\} \subset X$. For simplicity, assume

$$\tilde{R}^k_{s_t} = \left\{ \begin{array}{cc} R^k & \text{with probability } p^k \\ 0 & \text{with probability } 1 - p^k \end{array} \right.$$

where $R^k > R^l$ and $p^k < p^l$ if and only if $k > l$. Since shadow banks have more investment projects to choose from, they can raise the interest rate on lending contract and invest in riskier projects. In fact, this is precisely what is observed in reality. Motivated by above observation, assume the bank j invest in project k_j and shadow bank j' invest in $k_{j'}$, with $k_j < k_{j'}$. Furthermore, we assume interest offers satisfy $r^M < \frac{p^{k_{j'}}}{p^{k_j}} r^N$. The case we have now constructed is the case where player i has to decide whether or not to put his money into a bank or a shadow bank, given that two borrowers, j and j' are positionally equivalent in the lending network. The following proposition reveals how i making the lending decision.

Proposition 2: There exists a level of risk aversion α^*, such that if $\alpha > \alpha^*$, any potential lender i prefers to lend to a bank. Otherwise, player i prefers to lend to a shadow bank. Moreover, if both network proposals are equilibria, and i has enough bargaining power to determine which equilibrium network will prevail, then in equilibrium, funding will flow to banks if lenders are risk averse and go to shadow banks otherwise.

If we further assume that interest rate r^N a bank can offer is increasing in the size $\bar{K}^B$ of available investment projects, then lenders' decisions also depend on the size $\bar{K}^B$. Intuitively, if banks can only offer a low interest rate, it is not profitable to lend to banks even though banks are more likely to pay back loans. The following proposition summarizes i's lending behavior when varying the set of investment projects that banks can invest in.

Proposition 3: If $\frac{\partial r^N}{\partial \bar{K}^B} > 0$, then there exists a threshold index $\bar{K}^{B*}$ for investment projects, such that any potential lender i prefer to lend to bank j whenever $\bar{K}^B > \bar{K}^{B*}$. And i prefer to lend to shadow bank j otherwise. Moreover, if both network proposals are equilibria, and i has enough bargaining power to determine which equilibrium network will prevail, then in equilibrium, funding will flow to banks if lenders are risk averse and go to shadow banks otherwise.

The above Proposition has important policy implications. After the 2008 crisis, more regulations were imposed on the banking system so that banks faced ore regulations when investing - thus making it more likely that shadow banks would get more fundings and make risky investments. Such a policy outcome would make the whole financial system more risky. Using a simulation example, based on our model, we will confirm that this precisely what happens. In order to give a computational example, we must know how to measure the risk of the financial system, and for this, we must have a formal definition of systemic risk. We will provide one in section 5, and our computational example will confirm the conclusions of Proposition 3 (i.e., our example will confirm that regulations in the banking sector lead to higher systemic risk for the whole financial system).

The two propositions above continue to hold even if we allow i to invest in risky projects and we assume that the risky projects, i and j (and i and j') invest in do not overlap. In next subsection, we continue discussing properties of financial networks and how they look in equilibrium.

3.1.2 Core-Periphery Structure with "α"-rule

If we make further simplifications and allow banks to share equity returns according to the "α" -rule[11], then the equilibrium financial network is similar to the network in Farboodi (2014)[12]. Figure 2 shows the network structure under equity sharing. When banks can invest and share the risks, more funding will flow to banks - a funding flow pattern consistent with the finding[13] that banks are mainly the loan originators, where funding flow ends and asset flow begins. Banks originate loans, and funding flows come from money market mutual funds to shadow banks and from shadow banks to banks or other big shadow banks (for example, investment banks). In other words, if one allows equity sharing, banks will become fully connected with each other as in Figure 2. In reality, financial institutions do share equity returns. On the other hand, because banks are not able to know the private information of shadow banks, banks tend to make equity investment connections with other banks only. The "α" -rule is

[11] See Farboodi (2014).

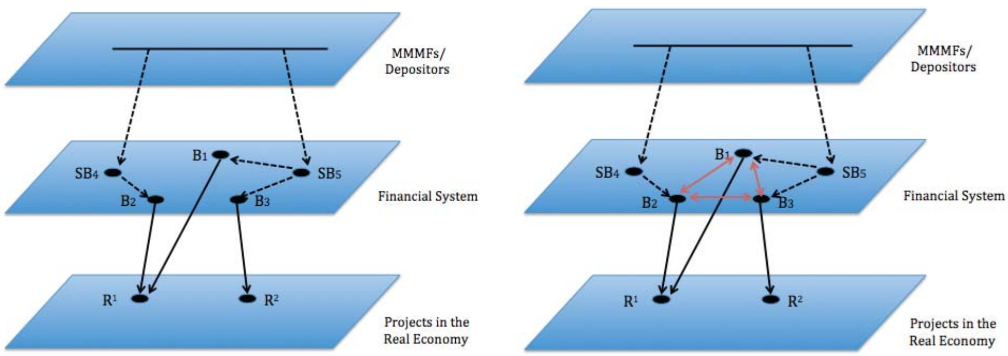
[12] In Farboodi (2014), a set of players have net cash flow equals 1 unit of money at the beginning of the one-shot game, while others have no endowment. And among the players who do not have endowment, some of them have investment opportunities. Then the network will be a core-periphery structure with funding flow from banks who have endowments to whom with investment opportunities. And banks with investment opportunities form cores.

[13] See Pozsar et al. (2012)

a simplification of the real mechanism that divides equity returns among banks. But it captures the fact that banks do share risks. With risks being reduced, banks invest more in the real economy and become net money borrowers in the system. Allen and Sauders (1986) provide a similar explanation for large banks being net borrowers. They argue that large banks may not be able to correctly judge the risk of lending to small, rural banks. And Allen et al. (1989) empirically confirm this result.

Moreover, in reality, shadow banking systems are vertically integrated. Figure 2 shows the case where banks, as loan originators, attract funding from shadow banks - and shadow banks get their funding from the depositors. In the second layer, players make lending connections with each other. Between layers, lending or investing connections are made. Dashed arrows are used to denote a lending contracts, with the direction of the arrow indicating the direction of the funding flow. In fact, Figure 2 is a simplification of the vertically integrated shadow banking system. A more detailed picture is shown in Figure 10 in Appendix B. The structure in Figure 10 strongly supports the methodology of using layered network structures in analyzing financial networks. Next, we study how the network looks in median and bad states of the economy.

Figure 2: Financial Networks in Good States.



(a) Without "α"-Rule (b) With "α"-Rule

3.2 In Median and Bad States

In median and bad states of the economy, some financial institutions (especially shadow banks) will have negative net cash flows. In fact any shock to the real economy will lead to some shadow banks having losses and insufficient money to operate. Besides shadow banks, banks may also have negative net cash flows as a result of shocks to the economy.

3.2.1 Systemically Important Links

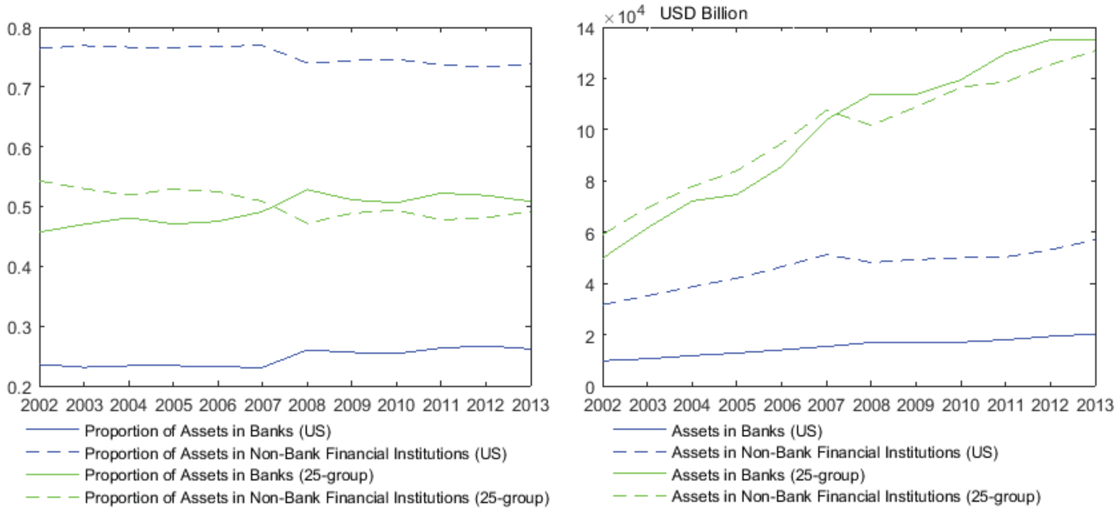
Notice that although players are myopic, the state of the real economy next period determines current period payoffs. Thus players care about the state next period. When a bank or a shadow bank has negative net cash flow, other banks are often willing to come to the rescue with lending. This is especially true for shadow banks which find themselves in trouble. Many shadow banks have commercial banks as parent banks. From the perspective of banks, such an arrangement gives them an advantage in competing for funds because a bank's shadow can be a source of such funds. Thus, when a bank's shadow is in trouble, the bank has a built in incentive to come to the rescue with lending - simply put, banks lend to their shadows to preserve their sources of funding. As discussed in Kodres (2013), "real banks were caught in the shadows", and "some shadow banks are controlled by commercial banks and for reputation reasons were salvaged by their stronger bank parent."

In good states, shadow banks buy commercial paper and other short-term debt from banks, as shown in Figure 2. These are the "direct linkages" discussed in Ghosh et al. (2012). Thus, in good states, the funding flow is from shadow banks to banks. However, if shadow banks lose money on their investment projects and default, these direct linkages from shadows to banks become impaired and sometimes break. The direction of funding flows is reversed, with some money going to the insolvent shadow banks, thereby raising the likelihood of defaults by banks.

Finally, the proportion of total investment by shadow banks is less, in both median and bad states, than in good states, when shadow banks have liquidity problems. This is consistent with what happened during 2008. Consider the following charts.[14] The left chart shows a decrease in proportion of assets held by shadow banks during the crisis both in US and in 25 financially important countries. Meanwhile, banks held a higher proportion of total assets than before. However, absolute amount of assets held by banks did not increase. In the right chart, both banks and shadow banks experience stagnation in asset accumulation. Especially for shadow banks, their assets decreased from the year 2007 to 2008. The following year (2008-2009), assets of banks had little change in spite of the fact that the assets of banks experienced stable growth for 6 years. The reason is two-folded. First, banks, which are inside the whole financial system, are influenced by the shadow banking system and the state of the real economy. Second, banks directly invest some shadow banks to back up some insolvent institutions - putting banks in a riskier position, and as observed in the data, causing their investments in the real economy to shrink.

[14] The data is got from Financial Stability Board (2014)

Chart: Assets of banks and non-bank financial institutions.



25-group: Argentina, Australia, Brazil, Canada, Switzerland, Chile, China, Germany, Spain, France, the United Kingdom, Hong Kong, Indonesia, India, Italy, Japan, Korea, Mexico, the Netherlands, Russia, Saudi Arabia, Singapore, Turkey, United States, South Africa.

Source: Financial Stability Board (2014)

4 A Numerical Example

In this section, we present an example of network formation game from the class of games upon which we have based our computation and conclusions. The basic ingredients are given in section 2. We specify the state space and action space and we analyze pure Pareto efficient stationary Markov equilibrium. The state transition matrix is calculated and how states evolve is shown. The model is described as below:

4.1 Setups

- There are 3 players B_1, SB_2, SB_3. $I = \{1,2,3\}$. Let $M = \{1\}$ be the set of banks[15], and $N = \{2,3\}$ be the set of shadow banks. $I = M \cup N$.
- The set of states $\Omega = \{\omega^1, \cdots, \omega^H\}$ is a finite set. Each state has three components - net cash flows C, the set of default players F, and the state of the real economy s. $C^i \in \mathbb{C}^i$, where $\mathbb{C}^i = [-1, 6]$, $\forall i$[16]. $F \in \{\varnothing, \{\text{SB}_1\}, \{\text{SB}_2\}, \{\text{SB}_1, \text{SB}_2\}\}$. $s \in \{s^1, s^2, s^3\}$, ordered from good to bad.

[15]Generalizing this setting into multiple banks is easy if one assume banks could apply "α' - rule. Since heterogeneity of banks would be reduced if they share investment projects. In Gennaioli et al. (2013), banks might have no difference in terms of return in investment through holding the same portfolio.

[16]All states are listed in online Appendix.

- $X = \{1, 2\}$, and $G^i \in \mathbb{G}^i(\omega)$ represent one action[17] of player i, where $G^i = (l^{ij}, r^{ij}, q^{ik})_{j \in I, k \in X}$. A proposal $G = (G^i)_{i \in I} = ((l^{ij}, r^{ij}, q^{ik})_{j \in I, k \in X})_{i \in I}$ specifies the lending and investment proposals by all players. Project 1 is the safe project with zero rate of return, while Project 2 is risky. Assume banks can determine the prevailing equilibrium network in each state in case of multiple equilibria exist. Under these assumptions, the Pareto-Nash equilibrium strategies that prevail in each period are those determined by the banks. Notice that Assumptions in section 3 are automatically satisfied given the assumptions we have made here.

In reality, shadow banking networks are more vertically integrated in a sense that investors (or depositors) buy assets from some shadow banks after securitization activities. In Figure 2, shadow banks can lend to different banks and hold different assets. But they will pool the assets together and tranche the pool according to their credit rating. Each tranche then becomes a portfolio and investors buy portfolios from shadow banks. It turns out that the assets making up the investors' portfolios were highly correlated, even though investors bought from different financial institutions. Even though shadow bank assets were claimed to be safe, this hidden mechanism, discussed in Coval et al. (2009), was was at work making investors' portfolios more risky - one of the factors underlying the 2008 crisis. Investors buying highly correlated or even homogenous assets does not violate our assumption that shadow banks are heterogeneous because the shadow banks in our model are upstream from the asset flow in the shadow banking system - and depositors do not engage in strategic interactions. Here, we will not provide an analysis of pooling and tranching processes, but such an analysis is not required for our model and our conclusions.

- As before, let $D^i = 1\{\sum_j \bar{l}^{ji} + C^i < 0, \text{ or } i \in F\}$. $D = \sum_i D^i$. The period-payoff function of a player is

$$u^i(\omega, g) = \begin{cases} \mathbb{E}[C'^i|s'] - \frac{1}{2}Var[C'^i|s'] & \text{if } D^i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

  Where C'^i is specified by equation (4).

- The state of the real economy next period s' depends on current net cash flows of all players C and action g.

$$s' = \begin{cases} s^1 & \text{if } D = 0, \\ s^2 & \text{if } D = 1, \\ s^3 & \text{otherwise.} \end{cases} \tag{9}$$

- Return from the risky project, R^2, for per dollar investment is a random variable. Assume

$$R^2 = \begin{cases} 2 & \text{with probability } p(s'), \\ 0 & \text{with probability } 1 - p(s'). \end{cases}$$

[17] The set of actions are listed in online Appendix.

Assume the return from a lending contract is $r = 1/\beta$ one period later for each unit of loan. Notice that after clearing debts, some realized net cash flow may not be in $\mathcal{C}^i$. We use the nearest integer C'^i in $\mathcal{C}^i$ to approximate. Moreover, equation (9) tells how the state of the real economy next period is determined. The distribution of the state in next period will be known by using the law of motion we specified in subsection 2.4.

Given any pure Pareto efficient stationary Markov strategy σ^*, the probability of moving from one state ω^x to ω^y is known. Thus, we can compute that $|\Omega|\times|\Omega|$ transition matrix Q, where $|\Omega|$ is the number of states.

4.2 Computing Equilibrium Supernetworks

(β, p) are the only parameters we need to know to simulate a supernetwork. For the pictures below, we use $(\beta, p(s^1), p(s^2), p(s^3)) = (0.8, 0.93, 0.92, 0.91)$. And the initial state is $\omega_0 = (C_0, s_0, F_0)$, where $C_0 = (2,2,2,2,2)$, $F_0 = \varnothing$, and the state of the real economy is $s_0 = s^1$. For simplicity, assume $d_t \equiv 2$.

Pure stationary strategy for each player is a function from states into actions. In the above stochastic game with finitely many states and actions, the strategy set is also a finite set. Let $\Sigma_{PE} \subset \Sigma_p$ be the set of all pure Pareto efficient stationary Markov strategies. By Theorem 2, the set is not empty. Assume the bargaining power of banks is much larger than shadow banks and shadow bank SB_2 has much more bargaining power than SB_3. Let

$$\mathbb{V} = \{V | \exists \sigma \in \Sigma_{PE}, V = V(\omega_0, \sigma)\}.$$

Then there is a unique equilibrium payoff vector satisfying the following properties.

$$\begin{aligned} \exists! V \in \mathbb{V}, \text{ such that } \quad & V^b \geq V'^b, \forall b \in M, \forall V' \in \mathbb{V}, \\ & \text{and } V^2 \geq V'^2, \forall V' \in \mathbb{V}, \text{ such that } V'^b = V^b, \forall b \in M, \\ & \text{and } V^3 \geq V'^3, \forall V' \in \mathbb{V}, \text{ such that } V'^b = V^b, \forall b \in M, \text{ and } V'^2 = V^2. \end{aligned} \tag{10}$$

Let $\Sigma^* \subset \Sigma_{PE}$ be the set of strategies that assign the unique equilibrium payoffs given in (10). Pick the strategy $\sigma^* \in \Sigma^*$ such that σ^* has smaller strategy index. Then the unique strategy has been contracted.

Assume the initial state is $\omega_0 = \omega^e$. Let ι_e be a row vector with the eth component to be 1, other components are zero. And let $v^i(\omega, \sigma^*)$ be a column vector with the hth component equals to $\bar{u}^i(\omega^h, \sigma^*)$, i.e.,

$$v^i(\omega, \sigma^*) = (\bar{u}^i(\omega^1, \sigma^*), \cdots, \bar{u}^i(\omega^{|\Omega|}, \sigma^*))'.$$

Therefore,

$$\begin{aligned} V^i(\omega_0, \sigma^*) =& \sum_{t=0}^{\infty} \beta^t \mathbb{E}[\bar{u}^i(\omega, \sigma^*)] \\ =& \bar{u}^i(\omega_0, \sigma^*) + \iota_e \beta Q v^i(\omega, \sigma^*) + \iota_e \beta^2 Q^2 v^i(\omega, \sigma^*) + \cdots \\ =& \iota_e [I - \beta Q]^{-1} v^i(\omega, \sigma^*). \end{aligned}$$

Starting from $\omega_0 = \omega^e$, for each state ω, there is an equilibrium strategy taken by players. Therefore, in each state, an optimal network is formed by players. The equilibrium transition probability matrix Q is the transition matrix corresponds to strategy σ^*. It gives the probability of moving from one state to another in one period. Suppose $\Omega_f \subset \Omega$ is a set of failure or default states the central bank want the system to avoid. Then one could calculate the probabilities of getting into the set of failure states Ω_f in T periods, $\forall T = 1, 2, \cdots$. We use the idea of "supernetwork", which is introduced by Page et al. (2005) to analyze the transition of financial states. A supernetwork is a network of networks and shows how networks evolve over time. Our supernetwork is a network of states and shows the transitions of all states.

Figure 3 and 4 are supernetworks. In order to reduce the dimensionality of the computational problem, we will let each node (rather than representing a single state) will denote a class of financial states. More precisely, each node represents a class of 3-tuple net cash flows and a set consisting of defaulted players. States in the same class are similar so that we regard them as one state in order to show the transition more clearly. Specifically, we make a partition for the set of possible net cash flow $\mathcal{C}^i$. So that a state with a player having 3 units of net cash flow is in the same class as another state with that player having 4 units of net cash flow, holding everything else the same. And a state with a player having 5 units of net cash flow is in the same class as another state with that player having 6 units of net cash flow, holding everything else the same. I.e., let $\mathcal{C}_p^i$ be a partition of $\mathcal{C}^i$,

$$\mathcal{C}_p^i = \{\{-1\}, \{0\}, \{1\}, \{2\}, \{3, 4\}, \{5, 6\}\}.$$

Denote two states ω, ω' in the same class as $\omega \sim \omega'$. Then

$$\omega \sim \omega' \text{ if and only if } C^i = C'^i \text{ or } \{C^i, C'^i\} \in \mathcal{C}_p^i, \forall i \in I.$$

Moreover, we do not differentiate the nodes by the state of the real economy s. In Figure 3, the color of a node specifies the level of total equity. The order from healthy to unhealthy is blue (total equity > 10), green (equity in $[6, 10]$), red (equity in $[1, 5]$), black(equity ≤ 0). Black nodes denote the worst financial states where players lose all their endowment. An arc represents a positive probability of moving from one state to another. Notice that the supernetwork itself is a weighted directed network. The weights correspond the probability. So the thicker the arc is, the higher probability it represents.

Similarly, for Figure 4, we use the number of defaulted players as the measurement of healthiness of a financial state. Since banks never default, number of failed shadow banks is the measure for healthiness of the financial system. Green nodes correspond to states that no shadow bank has failed. Red nodes are states where there is one shadow bank failed, while black nodes are cases where both shadow banks defaulted.

It is obvious to see in Figure 3 and Figure 4, that there are basins of attractions[18]

[18] See Page and Wooders (2009). We adopt the definition of basins of attraction from Page and Wooders (2009). The definition of "basins of attraction" in our model will be described in the next section.

in the supernetwork, and as the theory predicts, most of them include black nodes. A "basin of attraction", is an absorbing set once a state enters a basin, it will remain there forever. Because defaulted players stay in default, basins of attraction are the natural residence of defaulted states. For instance, the black node $\{x^{164}\}$ is a singleton basin of attraction, which corresponds to the state when two shadow banks default and all banks have 0 net cash flow. As long as the financial state enters state x^{164}, it will remain there forever.

4.3 Predictions

One interesting feature of the supernetwork pictures is that they indicate that sudden banking failure is possible. For example, in Figure 3, there is a link from state class x^{110} to x^{131}. Thus, the equilibrium banking supernetwork indicates that there is a positive probability of failure even though the even though the banking network is healthy. This is consistent with many empirical findings about the fast contagion of a crisis, like Bordo (2006), and the unpredictability of a crisis such as Canova (1994) and Kenny and Morgan (2011).

Our computational example also predicts that it is possible for financial networks "to freeze" even without a bank run on deposits or a run on funding from money market mutual funds. From Gorton and Metrick (2012), money market mutual funds did not lower their net lending level to the repo market, which is the key funding source for shadow banks, before or during the crisis. The crisis could be interpreted as a self-fulling systemic run or better a systemic freeze. In the computation setup, we assume that funding flow for banks and shadow banks always stays at a constant level. But in bad states, it is possible that no lending contract is signed among players.

Figure 3: A Markov Supernetwork.

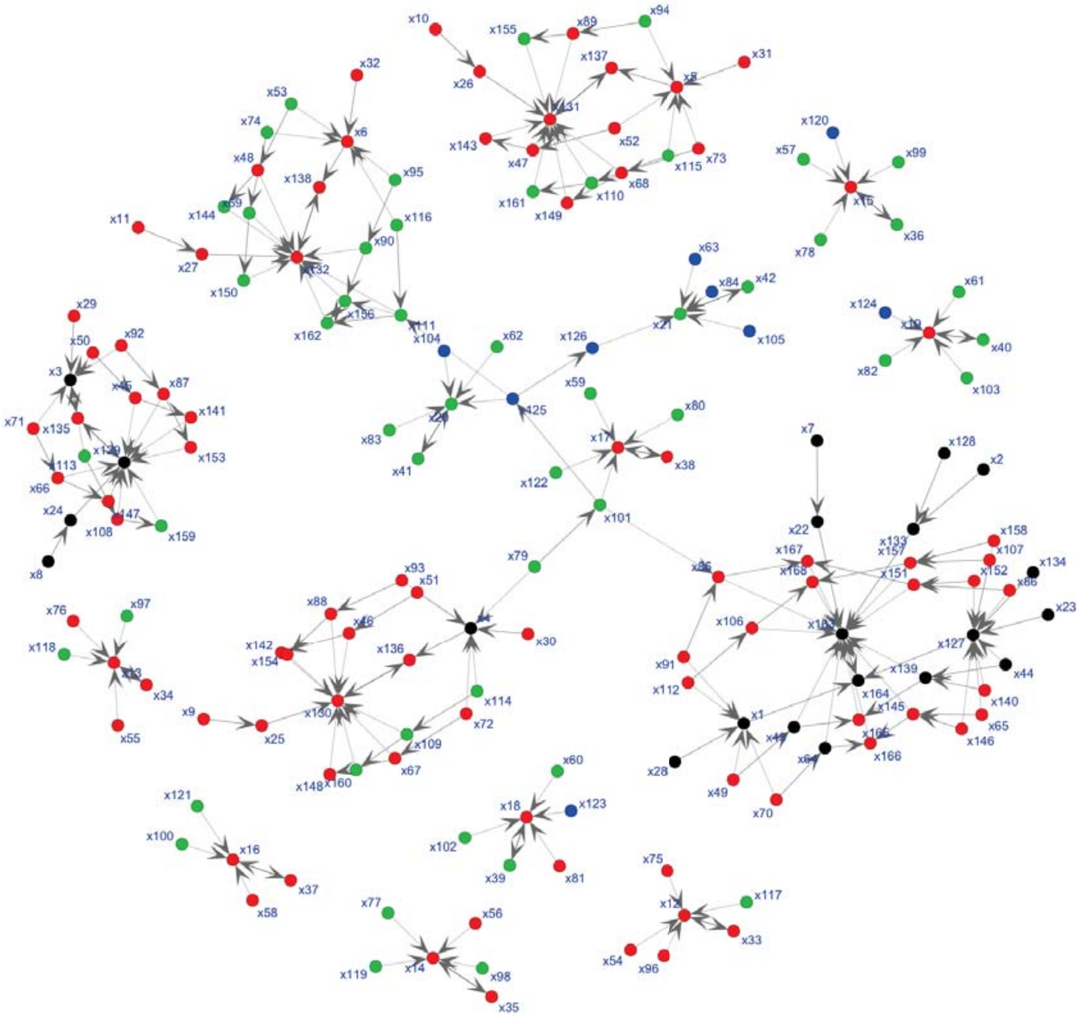
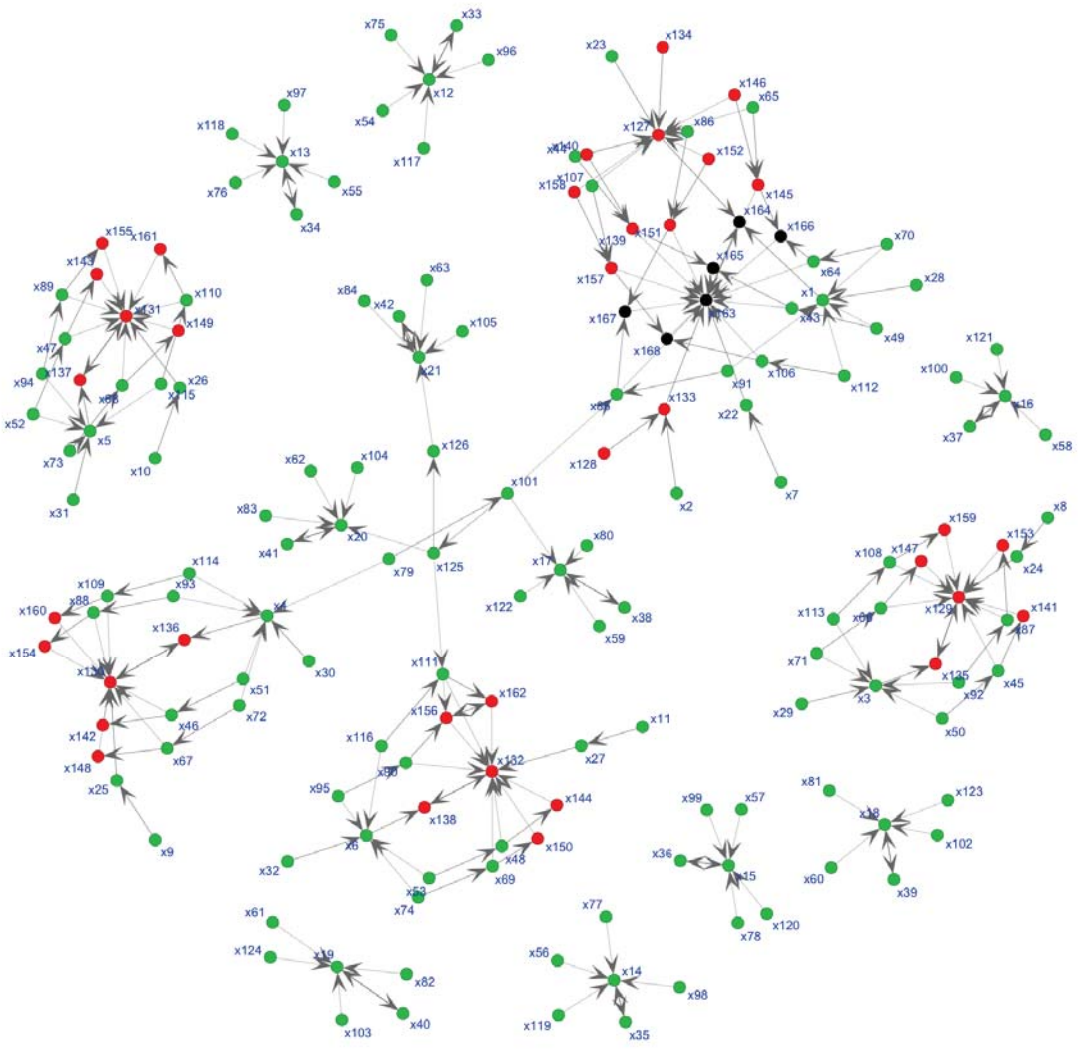


Figure 4: A Markov Supernetwork.



5 A Definition of Systemic Risks and Policy Implications

In this section, we give a formal definition of systemic risk and conduct one policy study. It is a general definition in the sense that policy makers can define freely what failure states are. In other words, the definition allows different thresholds of "system fail". With the definition given in the first subsection, one can calculate risks. The policy we are considering is imposing capital requirement for banks and for banks only. Other regulation policies, which restrict banks investment behavior without considering strategic interaction of all the financial intermediaries, have similar effects. We do not go into details of those different policies. Instead, we study the representative one stated above. Systemic risks with policies and without policies are calculated and compared. Our example will support the claim that regulating banks without also similarly regulating shadow bank leads to higher systemic risk for the entire financial system.

5.1 Systemic Risk Schedules

Much attention has been paid to systemic risk since the 2008 crisis. Here we give a formal definition of systemic risk motivated by the Markov supernetwork representation of the dynamics generating the the states of the financial system. Gong et al. (2015)[19] give a theoretical framework for endogenous systemic risk in arbitrary network formation games. In a financial setting, in words, *systemic risk* is usually taken to mean the conditional likelihood that the financial system in a particular state will fail (will enter a failed state) if the system experiences a particular event. Usually the event is taken to be a shock. This informal definition of systemic risk is silent on two important issues: (1) the severity of the failure (the number of banks that will fail) as well as on (2) the timing of the failure (when will the failure occur, immediately, in one time period, in n time periods). Our definition is not only based on equilibrium dynamics, but also takes into account timing and severity.

Viewing the Markov supernetwork as a map of the transportation network over which the financial state process must travel in moving from one state to another, we are naturally led to define *systemic risk as the probability that the stochastic state process, starting at a given state (i.e., at a given node in the supernetwork), will arrive at a failed state (i.e., another node in the supernetwork), at or before a given time.* In classical terminology, we define systemic risk as the *first passage probability* to a failed state from a given state. Under our definition, rather than there being a single measure of systemic risk, there is instead a schedule of systemic risk measures which lists the probabilities of various arrival times at various failed states in the supernetwork, departing from any given state in the supernetwork.

[19]Gong et al. (2015) give the existence result of a stochastic game with arbitrarily states and actions, and defines systemic risk.

Let $(\Omega, B_\omega, P^*_\omega)_{\omega\in\omega}$ be the probability space of states together with its equilibrium Markov transition, $\omega \to P^*_\omega := q_{(\omega,\sigma^*(\omega))}$, (i.e., for $E \in B_\omega$, $P^*_\omega(E) := q_{(\omega,\sigma^*(\omega))}(E)$ is the probability that, under equilibrium strategy profile, σ^*, the state process starting at state ω will enter set E next). Here, ω is a compact set and B_ω is the Borel σ-field of events. As before, let $\Omega_f \subset \omega$ the set of all failed states and let σ^* be the equilibrium Markov stationary strategy. Finally, let

$$P^*_\omega(T_{\omega_f} \le t)$$

be the probability that the state process enters the set of failed states, ω_f, on or before time $t \in \mathcal{T} = \{1, 2, \cdots\}$ starting at state $\omega \in \omega$. When ω is finite, then $P^*_\omega(T_{\omega'} \le t)$ is the probability of arriving at state $\omega' \in \omega_f$ from state ω within t periods. Notice that for each $\omega \in \omega$, $P^*_\omega(T_{\omega_f} \le 1) = P^*_\omega(\omega_f)$. Given current state ω, the risk that the process generates a failed state in exactly t periods is denoted as $SR_t(\omega, \omega_f)$, while the risk that the process generates a failed state within t periods is denoted as $SR_{(0,t]}(\omega, \omega_f)$.

Definition 3: Systemic Risk Schedule
Assume financial networks are endogenously formed, and financial institutions are far-sighted and behave strategically. Let $P^*_{(\cdot)}$ be the equilibrium Makov kernel governing the state process, where $\{P^*_\omega \in \Delta(\omega) : \omega \in \Omega\}$. *Systemic risks* are probabilities of entering failure states $\Omega_f \subset \Omega$ within $T, T \in \mathcal{T}$ time periods conditioning on some initial state $\omega \in \Omega$, I.e, systemic risks are a table of numbers - $SR(\omega, T)_{\omega\in\Omega, T\in\mathcal{T}}$, where

$$SR(\omega, T) := P^*_\omega(T_{\Omega_f} \le T).$$

Notice that $P^*_{\omega^x}(T_{\Omega_f} \le T), T \ge 2$ could be calculated by the iteration process given by the following.

$$P^*_{\omega^x}(T_{\Omega_f} \le T) = P^*_{\omega^x}(\Omega_f) + \int_{\Omega\backslash\Omega_f} P^*_{\omega^x}(\omega^z) \cdot P^*_{\omega^z}(T_{\Omega_f} \le T-1) d\omega^z.$$

Moreover, if the state set is finite, then the systemic risk will be simply as below.

$$SR(\omega, T) = P^*_{\omega^x}(T_{\Omega_f} \le T) = \sum_{\omega'\in\Omega_f} P^*_{\omega^x}(T_{\omega^y} \le T).$$

And $P^*_{\omega^x}(T_{\Omega_f} \le T), T \ge 2$ could be calculated by the iteration process given by the following, when Ω is finite.

$$P^*_{\omega^x}(T_{\Omega_f} \le T) = P^*_{\omega^x}(\Omega_f) + \sum_{\omega^z\notin\Omega_f} P^*_{\omega^x}(\omega^z) \cdot P^*_{\omega^z}(T_{\Omega_f} \le T-1).$$

Also, we can define the hitting time for a financial system to "hits" failure states. More formally, the *hitting time* $\tau : \Omega \to [0, \infty]$ is a random variable defined by

$$\tau(\omega^x) := \inf\{T \in \mathcal{T} | P^*_{\omega^x}(T_{\Omega_f} \le T) > 0\}.$$

The question is whether it is possible to control the behavior of banks and shadow banks so that the state process that emerges together with its equilibrium supernetwork and its equilibrium profile of systemic risk schedules are such that the state process is always more likely than not to avoid failed states or to be moving in the company of good states. The answer depends on the definition of "failed states". In our framework, how we define a "failed state" depends on specific aspects of the financial system that a researcher or a policy maker cares about. Figure 3 and 4 correspond to two different notions of "failure", and with each notion, we get a different answer to the question as to whether or not - and to what extent - systemic risk can be controlled. For example, state ω^{28} corresponds to a case where all players have 0 net cash flow. If we define a state to be failed if there are any defaulted players, then the financial system will never go into failure from ω^{28} since it will remain in that state forever. Therefore, using the above standard the system could remain healthy forever starting from state ω^{28} . However, if a state is defined to be a failed if players loose all the money they have in the initial period, then ω^{28} is a failed state. Under that standard, the system always has a positive probability of entering a set of failed states.

Next, for the finite state version of our model, we formally define the concept of "basins of attraction", which turns out to be useful in studying the properties of the financial system and in analyzing the systemic risk.

Definition 4: Basins of Attraction
Give a network formation game $\Gamma = (I, \Omega, (\mathbb{G}, \rightarrow, \Sigma, u^i, \beta), \eta)$, let $\sigma^* \in \Sigma$ be an equilibrium strategy. Let $\omega \rightarrow P^*_\omega$ be the equilibrium Markov transition induced by the stationary Markov equilibrium, $\sigma^* \in \Sigma$, of Γ. A nonempty set of states $\mathbb{A} \subset \Omega$ is said to be a *basin of attraction* $\mathcal{BOA}_{\sigma^*}$ for Γ if

- $\forall \omega \in \mathbb{A}$, $P^*_\omega(\mathbb{A}) = 1$, and
- $\forall \mathbb{A}' \subset \mathbb{A}$, if $P^*_\omega(\mathbb{A}) = 1, \forall \omega \in \mathbb{A}'$, then $\mathbb{A}' = \mathbb{A}$.

There are three reasons why basins are important to our understanding of systemic risk. First, if all basins of attraction contains failed state, then the financial system will experience a failure cascade with probability 1. This is essentially what we observe from history and in our simulations. An economic upturn cannot not last forever, and if all basins contain failed states, then failure of the financial system is unavoidable without a central bank to bail out insolvent banks. Moreover, some failed states are hard to identify. If we know a failed state is in a basin, we only need to keep tract of the state process relative to this basin - if the state process enters this basin this basin (containing a failed state), then the financial system will experience some level of failure with probability 1.

The second reason basins are important for understanding systemic risk is due to the fact that basins are homogeneous with respect to their failure characteristics - as the next proposition shows. As a consequence, it is the distribution of defaulted players across basins that that determines the severity of systemic risk. In Proposition 4 below, we show that all the states in the same basin have the same set of defaulted players. Thus, the severity of a system failure depends on the failure characteristics of the basin the state processes ends up in in the long run.

Proposition 4: Homogeneity Within Basins
Let Γ be a stochastic game with equilibrium stationary Markov strategy profile, σ^*, and equilibrium Markov transition, $\omega \longrightarrow P^*_\omega$, having unique, finite, disjoint basins of attractions given by

$$\mathcal{BOA}_{\sigma^*} = \{\mathbb{A}^*_1, \cdots, \mathbb{A}^*_h\}.$$

If basin, $\mathbb{A}^*_l$, contains a state, ω, having defaulted player set, F^l, then all states contained in $\mathbb{A}_l$ have the same set of defaulted players. - i.e.,

$$\text{if } \omega = (C, F, s) \text{ and } \omega' = (C', F', s') \text{ are contained in } \mathbb{A}_l,\ l \in \{1, \cdots, h\}, \text{ then } F = F' = F^l.$$

Moreover, any state, $\omega = (C, F, s)$, such that $C^i = 0, \forall i \in I$, $F = N$, and $s = s^k$, is a singleton basin of attraction - i.e. if there is a state where all shadow banks are defaulted and all banks have zero net cash flow, then it is a basin of attraction.

The proof is in Appendix A. Notice that the Proposition holds without specifying Ω_f, the set of failure states.

There is another reason basins of attraction are important for our understanding of systemic risk. The unique profile of basins of attraction,

$$\{\mathbb{A}^*_1, \cdots, \mathbb{A}^*_h\}$$

corresponding to an equilibrium Markov transition, $P^*_{(\cdot)}$, possesses a unique set of tipping points[20]. Tipping points (or tipping states) are the process's early warning system for bad basins. In particular, each tipping point is a gateway to an unavoidable sequence of states leading to a particular basin of attraction. If this basin is a severely failed basin (i.e., if it is a basin containing states with many failed banks), then knowledge of this tipping point is important. It opens the possibility of designing policies to incentivize bankers to take actions which minimize the likelihood that when such a tipping point is reached that the financial system tips onto a path that inexorably leads to such a severely failed basin - or more precisely, tips onto a default cascade leading into a severely failed basin.

Let

$$\{(\mathbb{A}^*_1, F^1), \cdots, (\mathbb{A}^*_h, F^h)\}$$

be the profile of basins-failed nodes pairs, $(\mathbb{A}^*_l, F^l)$, corresponding to the equilibrium Markov transition, $P^*_{(\cdot)}$. We say that basin $\mathbb{A}^*_l$ is more severely failed than basin $\mathbb{A}^*_{l'}$, if $|F^l| > |F^{l'}|$. If $|F^l| = 0$, the basin is default free, and if $|F^l| = N$, then all shadow banks are in default (recall that banks cannot default) because the unique profile of

[20] See Gong et al. (2015).

basins of attraction corresponding to an equilibrium Markov transition possesses a unique set of tipping points. Tipping points (or tipping states) are the state process's early warning system for bad basins. In particular, each tipping point is a gateway to an unavoidable sequence of states leading to a particular basin of attraction. If this basin is a severely failed basin (i.e., if it is a basin containing states with many failed banks), then knowledge of this tipping point is important. It opens the possibility of designing policies to incentivize bankers to take actions which minimize the likelihood that when such a tipping point is reached that the financial system tips onto a path that inexorably leads to such a severely failed basin - or more precisely, tips onto a default cascade leading into a severely failed basin.

Because it allows central banks to have know the tipping point when the system is heading to a dangerous position. A basin consists of a failure state is likely to contain other failure states. Moreover, as long as the system goes into a basin that consists of a failure state, it will never come out of the basin and will hit the failure states infinitely many times. Thus, an early warning mechanism that helps to avoid going into a bad basin is of significant importance. The construction of set of tipping points are described as follows.

Let $\Omega_f \subset \Omega$ be the set of failure states. Proposition 4 gives the existence and uniqueness of the collection of basins of attractions, and it gives a partition of the set of states Ω. I.e.,

$$\Omega = \mathbb{S} \cup \mathbb{A}_1 \cup \cdots \cup \mathbb{A}_h,$$

where $\mathbb{S}$ is the set of transient states. Define

$$\tilde{\Omega}_f := \Omega_f \cup \Big(\cup_{l:\Omega_f \cap \mathbb{A}_l \neq \varnothing} \mathbb{A}_l \Big).$$

Notice that $\tau(\omega) < \infty$, $\forall \omega \in \tilde{\Omega}_f$, and $\Omega_f \subset \tilde{\Omega}_f$. In other words, as long as the system goes into $\tilde{\Omega}_f$, the system will fail with probability 1. Thus, the actual set of states that deserves attention from central bank is actually larger than Ω_f. Let $\mathbb{T}_{\tilde{\Omega}_f} \subset \Omega$ be the set of tipping points of entering the set of states $\tilde{\Omega}_f$. I.e.,

$$\mathbb{T}_{\tilde{\Omega}_f} := \{\omega : P^*_\omega(\tilde{\Omega}_f) > 0\}.$$

The set of tipping points $\mathbb{T}_{\tilde{\Omega}_f}$ is an early warning for central banks. As long as the system is in a state which is a tipping point, it has positive probability of ending up in a failure state in Ω_f.

As a conclusion, we would like to discuss the some previous attempts to analytically or numerically capture systemic risk and study its dependence on network structures. A pioneering study by Allen and Gale (2000) shows that complete structure is more robust than incomplete structure. Although they did not use network terminologies and methodologies, the dependency of financial stability and network structure is pointed out. Duffie et al. (2014) discuss an insightful network-based approach to calculating systemic financial risk. Acemoglu et al. (2015) claim the systemic risk does not only depends on the network structure, it also relies on the magnitude of shocks. Specifically, policy makers are concern about the interconnectedness of financial institutions. Federal Reserve Bank Chairman Ben Bernanke told the Financial Crisis Inquiry Commission of Congress: “If the crisis has a single lesson, it is that the too-big-to-fail problem must be solved.” [21] Paul Volcker, former Chairman of the Federal Reserve, argued in 2011 that “[T]he risk of failure of large, interconnected firms must be reduced, whether by reducing their size, curtailing their interconnections, or limiting their activities.” [22] What we claim in our model it that the dependency of systemic risk and financial network structure has too much to do with the environment or primitives. Number and characters of players, action sets, rate of returns from real economy could all come into the model and play a role. Thus, it might not always the case that high interconnectedness brings higher risk, or vice versa. As argued in section 3, links between financial institutions could play an important role in maintaining financial stableness sometimes. In next subsection, we calculate systemic risks using computational example in section 4 and illustrate the nonlinear relationship of systemic risk and network structure again.

5.2 Post-crisis Policies

In this subsection we discuss our policy study carried out using numerical methods. In order to prevent a repeat of the bankruptcies and bailouts of 2008, central banks have imposed more regulations to restrict the behavior of banks, especially in the area of lending. But the regulatory environment has only become more tedious and complicated. For example, as discussed in The Economist (2015), since the crisis, JPMorgan Chase has employed an additional 950 people, approximately 400 of which are required to monitor and implement some 500 regulations focused on the liquidity of its assets. These regulations were put into place to prevent banks from toppling in the event of a liquidity freeze. A team of 300 employees is needed to monitor compliance with the 1000 page Volcker rule, a rule, a rule which restricts banks from trading on their own account. While these regulations might make banks safer, their effect on the overall stability of the entire financial system is unclear. Undoubtedly, stricter regulations make banks safer. But the effects of these policies on the whole financial system may very well be negative.

Suppose the state ω is good in the sense that every bank or shadow bank has excess amount of money. Assume that the central bank restricts bank investment by imposing a limit on the amount banks can invest in the real economy. In the network shown below, the red cross signs represent a decrease in lending due to these policies.

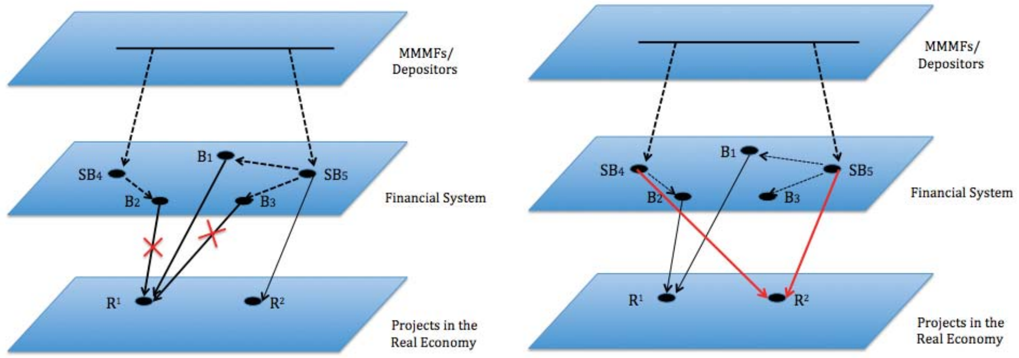
On one hand, banking sector regulations could have the side effect of weakening the ability of banks to bail out shadow banks. With such policies, banking sector becomes less profitable. In cases where shadow banks are insolvent, due to such regulations, banks have limited ability to lend to liquidity strapped shadow banks. In other words, banks limit their lending activities, and the channel that liquidity flows from banks to shadow banks could be destroyed.

[21] See Bernanke (2010).

[22] See Volcker (2012).

On the other hand, if banks' investment activities are restricted, then banks may also limit their borrowing activities. The reason is as follows: it does not make sense for banks to continue borrowing the same amount of money, when banks, due to restrictions on their investing activities (i.e., buying the debt of other banks), already have excessive amounts of money. Moreover, there is no incentive to borrow because they will have to pay back lenders next period with interest. As a consequence, shadow banks (via the borrowing activity of banks from shadow banks) cannot put the same amount of money into banks as before. Instead, shadow banks will invest in the real economy directly. The new network will be like Figure 5(b). Intuitively, if more investments

Figure 5: Regulating Investments Behavior of Banks



(a) Network before imposing a policy (b) Network after imposing a policy

in the real economy are made by shadow banks, the financial system will become less stable. Shadow banks are not in the safety net provided by the central bank. Moreover, unlike banks, shadow banks do not diversify their investments across different projects as do banks. Thus the investments of shadow banks are more risky than the investments of bank. Lastly, because shadow banks unregulated, they face no reserve requirements. The absence of capital reserves may potentially lead to liquidity problems.

The dual effects of restricting banks' investment activities, via regulations, tends to weaken or destroy the borrowing and lending links between banks and shadow banks. As discussed in section 3, links between banks and shadow banks play an important role in maintaining stability the system.

Based on our numerical example given in section 4, in this Subsection we will apply our definition of systemic risk and show that by only regulating banks and not shadow banks, the systemic risk of the whole financial system is increased. Recall that we define systemic risk as the probability that the banking system reaches a failed state, starting from a given state, within certain time period. We will carry out our analysis of systemic risk under there different definitions of what it means for a state to be failed. Our three different definitions of failure are as follows: First, we will define failed states to be states in which the banking system (not including shadow banks) has banks with realized net cash flow less than zero (i.e., states where the set of banks contains some insolvent banks). Under this definition of failed states and referring to the details in the example in section 4, the set of failed states given by

$$\Omega_f^B = \{\omega \in \Omega : C^1 < 0\}.$$

Second, we will define failed states to be states in which the aggregate net cash flow of all players (banks plus shadow banks) is negative. Under this negative net cash flow definition of failed states and referring to the details in the example in section 4, the set of failed states becomes

$$\Omega_f^2 = \{\omega \in \Omega : \sum_{i \in I} C^i < 0\}.$$

Finally, we will define failed states to be states in which some shadow banks default. Under this defaulted shadow bank definition of failed states and referring to the details in the example in section 4, the set of failed states is given by

$$\Omega_f^3 = \{\omega \in \Omega : F \neq \varnothing\}.$$

Failed states, Ω_f^2, is from the perspective of welfare loss (in that it is about the loss in aggregate net cash flows). Failed states, Ω_f^3, simply uses the number of defaulted players.

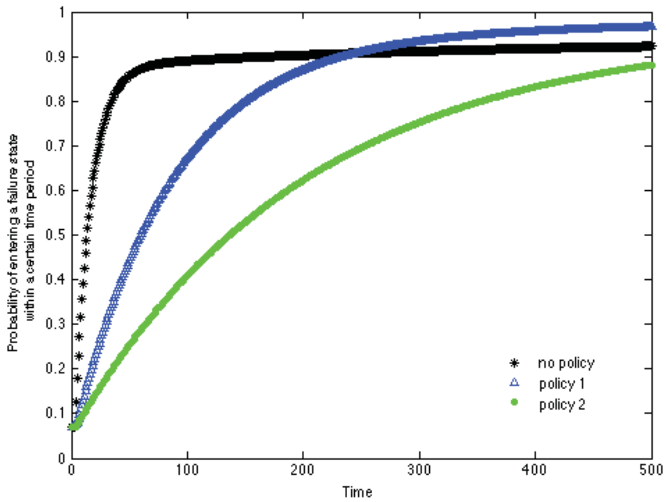
Figure 6, shows shows systemic risks, using failed states, ω_f^B, under policies 1 and 2 where these policies are aimed specifically at the banking system (excluding shadow banks). Policy 1 bans a bank from investing more than 4 units of money, and Policy 2 disallows a bank from investing more than 3 units. The computational results are consistent with the aim of the policies, which is to make the banking sector safer. With above policies restricting banks' investment behavior, the risks for banking system are greatly reduced within the first 200 periods. However, in the long run, such policies may make the banking system more unstable[23]. This is because such policies weaken or break the links between banks and shadow banks in the financial network. These connections should not be ignored when designing policies.

Figure 7 shows shows systemic risks, using failed states, Ω_f^2 and Ω_f^3, again under policies 1 and 2. Figure 7(a), (b) imply that under policies 1 and 2, systemic risks increase. In other words, imposing regulations on banks investment behavior makes the whole system even more risky. The result that systemic risks, under ω_f^2 and Ω_f^3, increase under such policies is robust with respect to different p parameters.

The financial system turns out to have more systemic risk because banks, with investment restriction, have difficulties in cumulating asset. When shadow banks are in distress, banks are unable to bail them out. Under policy 1, funding flow from banks to shadow banks is reduced by 25.8%, and it is reduced even further by 50% in

[23]The result that policies make banking system more stable is very robust with different p parameters. But the result that banking system with policies face higher systemic risk in the long run is not robust with different p values. Grids of p and systemic risks are shown in Appendix.

Figure 6: Banking Systemic Risk



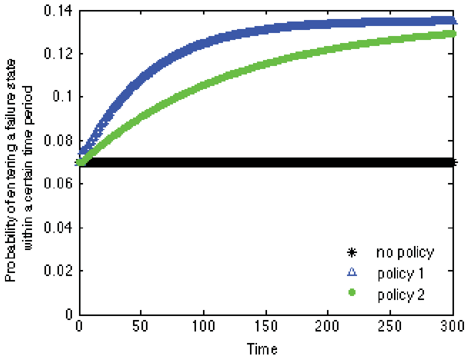
states where some shadow bank are insolvent, compared to the funding flow without any policies. Under policy 2, funding flows from banks to shadow banks is reduced by 50%, and is reduced to zero in states where some shadow banks are insolvent. Shadow banks can only borrow from banks when they face liquidity problem. If links between shadow banks and banks are destroyed, the financial system becomes riskier.

Our computational results above imply that imposing regulations on bank investment levels can lead to a decrease in the bank's lending amount and an increase in systemic risks. We will verify the causal effect of an decrease in the bank's lending on increases in systemic risks. Policy 3 sets a maximum amount of possible transitions between banks and shadow banks. The transactions cap is 1 unit of money. Under policy 4, the banks' set of lending actions is a subset of the lending action set without any policy. Figure 7(c), (d) imply that systemic risks increase when the funding channel from banks to shadow banks is weakened.

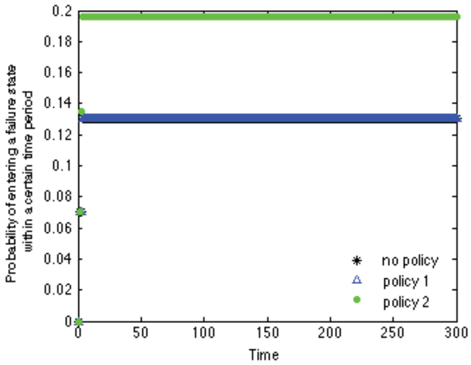
From our computations above, we show that imposing a policy restricting banks' investment activities can reduce banks' incentives to lend, causing an increase in systemic risks. On the other hand, banks may also reduce borrowing leading to an increase in systemic risks, as shown in Figure 5. We use another probability parameter $p = (0.98, 0.93, 0.88)$, and hold everything else the same. From Figure 8(a) and 8(b), It is obvious that imposing policies to restrict banks investment behavior raises systemic risk.

We will verify the causal effect of an decrease in the bank's borrowing amount on changes in systemic risks. Policy 3, as before, sets a maximum amount of possible transitions between banks and shadow banks. And Policy 5 disallows banks to borrow from shadow banks. Figure 8(c), 8(d) imply that systemic risks increase when the funding channel from shadow banks to banks is weakened.

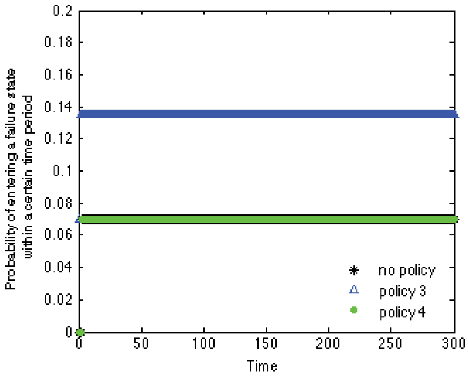
Figure 7: Systemic Risks



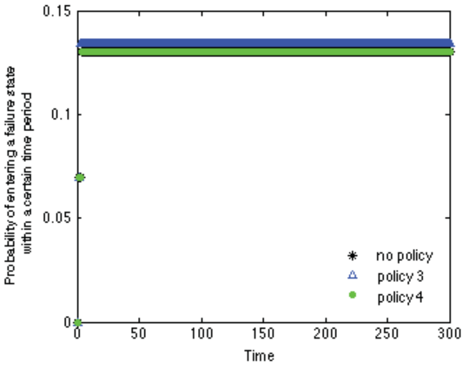
(a) Systemic Risk: Failure states = Ω_f^2



(b) Systemic Risk: Failure states = Ω_f^3



(c) Systemic Risk: Failure states = Ω_f^2

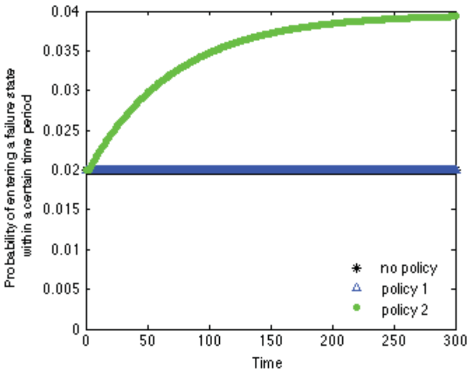


(d) Systemic Risk: Failure states = Ω_f^3

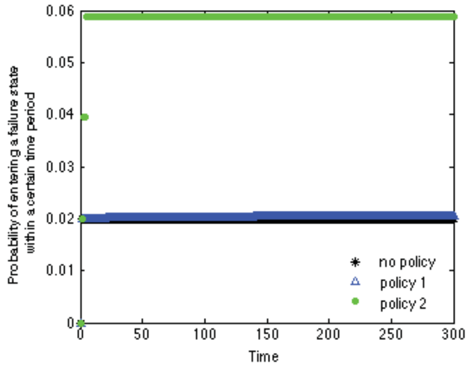
6 Conclusion and Further Works

This paper builds a computable and empirically testable game-theoretic model of endogenous network formation, which includes interactions between financial institutions and the real economy. In our model, financial intermediaries are farsighted, and behave strategically. Therefore, financial networks are endogenously formed. Our work here also points the way to several other areas for further investigation. First, systemic risk schedules and supernetworks are calculated from simulations, rather than from empirically based estimates of the underlying parameters which determine supernetworks - and hence systemic risk schedules. If the model is correct, given a time series of financial system evolution, the probability of moving from one state ω^i to another one ω^j generated from the true parameter values should be similar to the proportion of times that ω^j is observed among all states, starting from ω^i. With estimated parameter values, policy studies could be done more precisely. One could further study and compare several policies. Given on different policies, an empirically based version of

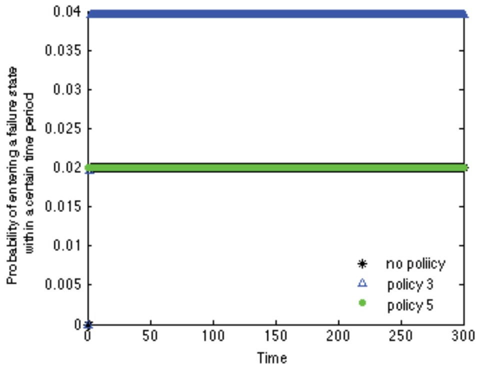
Figure 8: Systemic Risks



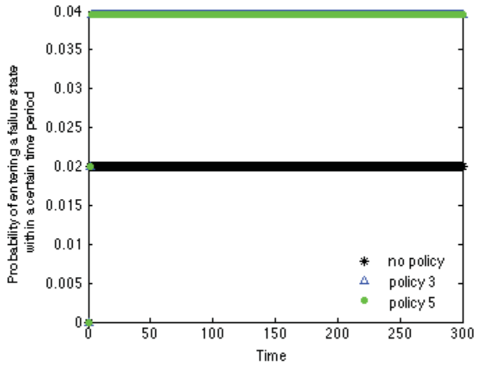
(a) Systemic Risk: Failure states = Ω_f^2



(b) Systemic Risk: Failure states = Ω_f^3



(c) Systemic Risk: Failure states = Ω_f^2



(d) Systemic Risk: Failure states = Ω_f^3

our model could be used to to predict players' state-dependent decisions. An empirically based study similar to those presented in sections 4 and 5 could be done. Since the 2008 crisis, questions of whether or not and how to regulate shadow banks have become increasingly important. Using an empirically-based, game-theoretic model of financial network formation and equilibrium dynamics, we could study the implications for systemic risk schedules of policies to restrict haircuts in the ABS market, or to impose strict guidelines on collateral, or to increase transactions costs - or of policies to regulate shadow bank liquidity - or the extension of the safety net to shadow banks.

Second, in our model, asymmetric information is not modeled. What role did asymmetric information play in the 2008 crisis? Depositors, commercial banks, and even central banks did not have access to the information of shadow banks. With hidden information, some insolvent shadow banks could pretend to be in a good state and borrow money from solvent players.

Third, what about the relationship between debt maturity and systemic risk. In this paper, we only considered short term debt. It would be an interesting extension to build a game-theoretic model of endogenous network formation which allowed us to understand the emergence of an equilibrium debt maturity structure.

Fourth, in section 2 we discussed briefly the relationship between network formation rules and equilibrium strategies. Here we focused on pure Pareto optimal stationary Markov equilibrium. In reality, financial networks may not be efficient. Further analysis of network formation rules and equilibrium concepts would be useful.

Lastly, in our model, network failure came through two channels: loan contracts and the real economy. The failure of a bank to meet its contractual loan obligations can cause other banks to fail to meet their contractual loan obligations as well. Moreover, the failure of projects in the real economy to produce returns can also lead to bank failures. But further study of the interactions of the financial network and the real economy in causing network failures is needed.

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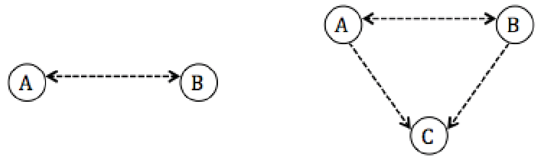
Appendices

A Proofs

Proof of Proposition 1

The intuition of the proof is as follows. Eisenberg and Noe (2001) show the uniqueness of "value of equity". They also shows the uniqueness of clearing vector when "regulatory condition"[24] is satisfied. Regularity is to rule out cases when network is too complete within a set of bankrupt nodes. In fact, regularity condition is sufficient but not necessary. In our framework, regularity condition does not hold while uniqueness is still valid. Examples in Figure 9 below illustrate the idea. The network on the left corre-

Figure 9: Examples



sponds to a lending network without depositors. In the example, Bank A and bank B owe 1 unit of money to each other and generate no money from the real economy. Let clearing vector p be defined as equation (3). Then $p = (0, 0)$, and $p = (1, 1)$ are both clearing vectors. The system does not have a unique clearing vector. However on the right hand side, C corresponds to a depositor, and both banks owe some money to the depositor. The unique clearing vector is $p = (0, 0, 0)$. The presence of a depositor guarantees the uniqueness of the clearing vector.

A formal proof of uniqueness is given as below:

First, claim that there exists a greatest and least clearing vector, $\tilde{p}^{+}_{t+1}$ and $\tilde{p}^{-}_{t+1}$. $\Phi(\cdot)$ is positive, increasing, concave, and nonexplosive. Thus the set of fixed point of Φ has a greatest and least element by Knaster-Tarski Fixed Point Theorem (see Aliprantis and Border (2006), Theorem 1.10.).

Second, define value of net cash flow NCF^i of node i before new deposits come in as

$$NCF^i = \begin{cases} (\tilde{E}^i_{t+1} + \sum_j \tilde{p}^{ji}_{t+1} - \tilde{L}^i_{t+1})^+ & \text{if } i \in I, \\ 0 & \text{otherwise.} \end{cases}$$

[24] See Eisenberg and Noe (2001).

We claim that the vector of NCF is the same under any fixed point $\tilde{p}_{t+1}$. I.e.,

$$(\tilde{E}_{t+1} + \tilde{p}_{t+1}\Pi - \tilde{L}_{t+1})^+ = (\tilde{E}_{t+1} + \tilde{p}'_{t+1}\Pi - \tilde{L}_{t+1})^+, \forall \tilde{p}_{t+1}, \tilde{p}'_{t+1} \text{being fixed points of } \Phi.$$

The only thing needs to be proved that

$$(\tilde{E}_{t+1} + \tilde{p}^+_{t+1}\Pi - \tilde{L}_{t+1})^+ = (\tilde{E}_{t+1} + \tilde{p}_{t+1}\Pi - \tilde{L}_{t+1})^+, \forall \tilde{p}_{t+1} \text{being fixed points of } \Phi.$$

Suppose there exists a clearing vector $\tilde{p}_{t+1}$, such that $(\tilde{E}_{t+1} + \tilde{p}_{t+1}\Pi - \tilde{L}_{t+1})^+ \neq (\tilde{E}_{t+1} + \tilde{p}_{t+1}\Pi - \tilde{L}_{t+1})^+$. Because of the non-negativity of Π, we have

$$(\tilde{E}_{t+1} + \tilde{p}^+_{t+1}\Pi - \tilde{L}_{t+1})^+ \underset{\neq}{\geq} (\tilde{E}_{t+1} + \tilde{p}_{t+1}\Pi - \tilde{L}_{t+1})^+. \tag{11}$$

Because $\tilde{p}_{t+1}$ and $\tilde{p}^+_{t+1}$ are clearing vectors,

$$\begin{aligned} (\tilde{E}_{t+1} + \tilde{p}^+_{t+1}\Pi - \tilde{L}_{t+1})^+ &= \tilde{E}_{t+1} + \tilde{p}^+_{t+1}\Pi - \tilde{p}^+_{t+1}, \\ (\tilde{E}_{t+1} + \tilde{p}_{t+1}\Pi - \tilde{L}_{t+1})^+ &= \tilde{E}_{t+1} + \tilde{p}_{t+1}\Pi - \tilde{p}_{t+1}. \end{aligned} \tag{12}$$

Equations (11) and (12) imply

$$\tilde{E}_{t+1} + \tilde{p}^+_{t+1}\Pi - \tilde{p}^+_{t+1} \underset{\neq}{\geq} \tilde{E}_{t+1} + \tilde{p}_{t+1}\Pi - \tilde{p}_{t+1}. \tag{13}$$

Notice that $\Pi\iota = \iota$, where ι is $m+n+1$ column vector with any component equals to 1. We have

$$\iota(\tilde{E}_{t+1} + \tilde{p}^+_{t+1}\Pi - \tilde{p}^+_{t+1}) = \iota(\tilde{E}_{t+1} + \tilde{p}_{t+1}\Pi - \tilde{p}_{t+1}),$$

contradicts with equation (13). Therefore, the vector of NCF is unique under any clearing vector.

Last, we will show $\tilde{p}^+_{t+1} = \tilde{p}^-_{t+1}$, which implies the uniqueness of a clearing vector.

Suppose $\tilde{p}^+_{t+1} \neq \tilde{p}^-_{t+1}$. I.e., $\exists i \in I$, such that

$$\tilde{p}^{+i}_{t+1} > \tilde{p}^{-i}_{t+1},$$

where $\tilde{p}^{+i}_{t+1}$ and $\tilde{p}^{-i}_{t+1}$ are the i-th components of $\tilde{p}^+_{t+1}$ and $\tilde{p}^-_{t+1}$. We say there is an arc from i to j, $\forall i, j \in I \cup \{m+n+1\}$ if $\Pi_{ij} > 0$. Denote an arc from i to j as $i \dashrightarrow j$. It is obvious that $i \dashrightarrow m+n+1$. Notice that NCF^{m+n+1} is unique under any clearing vector. Thus,

$$\tilde{E}^{m+n+1}_{t+1} + \sum_j \tilde{p}^{+j}_{t+1}\Pi_{j(m+n+1)} - \tilde{p}^{+(m+n+1)}_{t+1} = \tilde{E}^{m+n+1}_{t+1} + \sum_j \tilde{p}^{-j}_{t+1}\Pi_{j(m+n+1)} - \tilde{p}^{-(m+n+1)}_{t+1}.$$

By assumption, $\tilde{p}_{t+1}^{+(m+n+1)} = \tilde{p}_{t+1}^{-(m+n+1)} = 0$, and $\tilde{E}_{t+1}^{m+n+1} = 0$. Therefore,

$$\sum_j \tilde{p}_{t+1}^{+j}\Pi_{j(m+n+1)} = \sum_j \tilde{p}_{t+1}^{-j}\Pi_{j(m+n+1)}.$$

I.e.,

$$\sum_j (\tilde{p}_{t+1}^{+j} - \tilde{p}_{t+1}^{-j})\Pi_{j(m+n+1)} = 0. \tag{14}$$

$i \dashrightarrow m+n+1$ imples $\Pi_{i(m+n+1)} > 0$. $\tilde{p}_{t+1}^{+i} > \tilde{p}_{t+1}^{-i}$. Thus

$$(\tilde{p}_{t+1}^{+i} - \tilde{p}_{t+1}^{-i})\Pi_{i(m+n+1)} > 0.$$

$\forall k \in I \cup \{m+n+1\}$, $\tilde{p}_{t+1}^{+i} \geq \tilde{p}_{t+1}^{-i}$, and $\Pi_{k(m+n+1)} \geq 0$. Therefore,

$$\sum_j (\tilde{p}_{t+1}^{+j} - \tilde{p}_{t+1}^{-j})\Pi_{j(m+n+1)} > 0,$$

which contradicts with equation (14). Thus the clearing vector $\tilde{p}_{t+1}$ is unique. The uniqueness of net cash flow is easy to be obtained then.

Next, we will prove the joint continuity of fixed point $\tilde{p}_{t+1}$ in $\Pi, \tilde{L}_{t+1}, \tilde{E}_{t+1}$. For simplicity, we change notations. Let $\gamma^* := (\Pi, \tilde{L}_{t+1}, \tilde{E}_{t+1})$, and $p^* := \tilde{p}_{t+1}$, representing the fixed point of $\Phi(\cdot;\gamma^*)$.

Given any sequence of variables $\{\gamma_n\} = (\Pi_n, L_n, E_n)$, with $\gamma_n \to \gamma^*$, define p_n be the unique fixed point of $\Phi(\cdot;\gamma_n)$. I.e.,

$$p_n = \Phi(p_n;\gamma_n).$$

Want to show $p_n \to p^*$.

Because $\{\gamma_n\}$ is bounded and Φ is nonnegative and nonexplosive, $\{p_n\}$ is bounded. Thus, $\exists \{p_{n_k}\} \subset \{p_n\}, p_{n_k} \to p^{k*}$ for some p^{k*}.

On the other hand, by the construction of the sequence $\{p_n\}$, we have

$$p_{n_k} = L_{n_k} \wedge (E_{n_k} + p_{n_k}\Pi_{n_k}).$$

Therefore,

$$p^{k*} = L^* \wedge (E^* + p^{k*}\Pi^*).$$

Thus, $p^{k*} = p^*$, since p^* is the unique fixed point of $\Phi(\cdot;\gamma^*)$.

Therefore, every converging subsequence of p_n converges to p^*. Thus, $p_n \to p^*$, which completes the proof.

Q.E.D.

Proof of Theorem 2

Define

$$\mathcal{N}_{\mathcal{M}} := \{\sigma^*_{\mathcal{M}} \in \Sigma | V^i(\omega_0, \sigma^{*i}_{\mathcal{M}}, \sigma^{*-i}_{\mathcal{M}}) \geq V^i(\omega_0, \sigma^i_{\mathcal{M}}, \sigma^{*-i}_{\mathcal{M}}), \forall \sigma^i_{\mathcal{M}} \in \Sigma^i, \forall i \in I\},$$

$$\begin{aligned}\mathcal{N} := \quad & \{\sigma^*_{\mathcal{M}} = \times_{i\in I}\sigma^{*i}_{\mathcal{M}} \in \mathcal{N}_{\mathcal{M}} | \exists (f^i)_{i\in I}, f^i(\cdot) : \Omega \to \mathbb{G}^i, \text{ with } f^i(\omega) \in \mathbb{G}^i(\omega), \forall \omega \in \Omega, \forall i \in I, \\ & \text{such that } \sigma^i(f^i(\omega)|\omega) = 1, \forall \omega \in \Omega, \forall i \in I\},\end{aligned}$$

where $\sigma^{*i}_{\mathcal{M}} = (\sigma^i, \sigma^i, \cdots)$. Thus $\mathcal{N}_{\mathcal{M}}$ is the set of all stationary Markov equilibrium strategies, and $\mathcal{N}$ is the set of all pure stationary Markov equilibrium strategies. Moreover, let $\mathcal{N}_{\mathcal{P}}$ be the set of all pure Pareto optimal stationary Markov equilibrium strategies defined in Definition 2.3. Define

$$\mathcal{N}_{\mathcal{S}} := \{\sigma^*_{\mathcal{M}} \in \mathcal{N} | \forall \sigma'_{\mathcal{M}} \in \mathcal{N}, V(\omega_0, \sigma^*_{\mathcal{M}}) \neq V(\omega_0, \sigma'_{\mathcal{M}}) \Rightarrow \exists i \in I, V^i(\omega_0, \sigma'_{\mathcal{M}})) > V^i(\omega_0, \sigma^*_{\mathcal{M}})\}, \},$$

where $V(\omega_0, \sigma_{\mathcal{M}}) = (V^i(\omega_0, \sigma_{\mathcal{M}}))_{i\in I}$ is the vector of expected utility. $\mathcal{N}_{\mathcal{S}}$ is the set of pure stationary Markov equilibrium strategies with expected utility not Pareto dominated by any other pure stationary Markov equilibrium strategy. Notice that strategies in $\mathcal{N}_{\mathcal{S}}$ may not be Pareto optimal, since they only dominate Nash equilibria by definition. Next, given $\tilde{G}$ defined in (2), define

$$\tilde{\sigma}_{\mathcal{E}} := \times_{i\in I}\tilde{\sigma}^i_{\mathcal{E}} \in \Sigma_p, \tilde{\sigma}^i_{\mathcal{E}} = (\tilde{\sigma}^i, \tilde{\sigma}^i, \cdots), \text{ such that } \tilde{\sigma}^i(\tilde{G}^i|\omega) = 1, \forall \omega \in \Omega, \forall i \in I.$$

By assumption in section 2, $\tilde{G}(\omega) \in \mathbb{G}(\omega), \forall \omega \in \Omega$. Notice that $\tilde{G}$ denotes a network that there is no connection between players, and they invest all they have into the first project. Among all proposals that players form no connection, allocation of money in real economy corresponds to $\tilde{G}$ may not be optimal. Due to the finiteness of action set, there is a proposal that all players allocate their endowment optimally. Define

$$\begin{gathered}\tilde{G}^*(\omega) := (\tilde{G}^{*1}, \cdots, \tilde{G}^{*m+n}) \in \mathbb{G}(\omega), \\ \text{such that, } l^{ij} = 0, \forall j \in I, \\ \text{and } \forall G'(\omega) = (G'^1, \cdots, G'^{m+n}) \in \mathbb{G}(\omega), \text{with } l^{ij} = 0 \ \forall i, j \in I, \text{we have} \\ V^i(\omega_0, \sigma_{\mathcal{E}}) \geq V^i(\omega_0, \sigma'_{\mathcal{E}}), \forall i \in I,\end{gathered} \tag{15}$$

where

$$\sigma_{\mathcal{E}} := \times_{i\in I}\sigma^i_{\mathcal{E}} \in \Sigma_p, \sigma^i_{\mathcal{E}} = (\sigma^i, \sigma^i, \cdots), \text{ such that } \sigma^i(\tilde{G}^{*i}|\omega) = 1, \forall \omega \in \Omega, \forall i \in I,$$

and

$$\sigma'_{\mathcal{E}} := \times_{i\in I}\sigma'^i_{\mathcal{E}} \in \Sigma_p, \sigma'^i_{\mathcal{E}} = (\sigma'^i, \sigma'^i, \cdots), \text{ such that } \sigma'^i(G'^i|\omega) = 1, \forall \omega \in \Omega, \forall i \in I.$$

Since a mismatched network proposal gives no final connections among players, any unilateral deviation from strategy $\sigma_{\mathcal{E}}$ gives the players outcomes the same as $V(\omega_0, \sigma^0_{\mathcal{E}})$, which is less than $V(\omega_0, \sigma_{\mathcal{E}})$ by (15). Therefore $\sigma_{\mathcal{E}} \in \mathcal{N}$. Therefore, $\mathcal{N} \neq \varnothing$, and the set of Pareto efficient strategies $\mathcal{N}_{\mathcal{S}} \neq \varnothing$.

Claim: $\mathcal{N}_{\mathcal{S}} = \mathcal{N}_{\mathcal{P}}$. Therefore, the discounted stochastic game Γ has a pure Pareto optimal stationary Markov equilibrium.

The proof is as the following. It is easy to show $\mathcal{N}_{\mathcal{P}} \subset \mathcal{N}_{\mathcal{S}}$. This is because pure Pareto optimal stationary Markov equilibrium strategy dominates all other pure Nash equilibria. The only part needs to be proved is $\mathcal{N}_{\mathcal{S}} \subset \mathcal{N}_{\mathcal{P}}$. Suppose $\mathcal{N}_{\mathcal{S}} \not\subset \mathcal{N}_{\mathcal{P}}$. $\therefore \exists \sigma^*_{\mathcal{M}} \in \mathcal{N}_{\mathcal{S}}$, and $\sigma^*_{\mathcal{M}} \notin \mathcal{N}_{\mathcal{P}}$. I.e., returns of some strategy Pareto dominate the returns of $\sigma^*_{\mathcal{M}}$.

$$\therefore \exists K_0 \subset I, \text{ and } \exists \sigma'_{\mathcal{M}} \in \Sigma_p, \text{ and } \sigma'_{\mathcal{M}} = \times_{i \in K_0} \sigma'^i_{\mathcal{M}} \times_{i \in I \backslash K_0} \sigma^{*i}_{\mathcal{M}}, \text{ with } \sigma'^i_{\mathcal{M}} \in \Sigma^i_p, \forall i \in I,$$

such that

$$V^i(\omega_0, \sigma'_{\mathcal{M}}) > V^i(\omega_0, \sigma^*_{\mathcal{M}}), \forall i \in K, \tag{16}$$

K is defined in Definition 2.2. Moreover,

$$\sigma^*_{\mathcal{M}} \in \mathcal{N} \Rightarrow V^i(\omega_0, \sigma^*_{\mathcal{M}}) \geq V^i(\omega_0, \sigma_{\mathcal{E}}), \forall i \in I. \tag{17}$$

Case I: $|K_0| = 1$.

Then $K = K_0 = \{i^*\}$, for some $i^* \in I$, with $\sigma^*_{\mathcal{M}}, \sigma'_{\mathcal{M}} \in \Sigma$, and

$$V^{i^*}(\omega_0, \sigma'_{\mathcal{M}}) > V^{i^*}(\omega_0, \sigma^*_{\mathcal{M}})$$

Therefore, $\sigma^*_{\mathcal{M}}$ violates the definition of a Nash equilibrium. $\sigma^*_{\mathcal{M}} \notin \mathcal{N}$ contradicts with $\mathcal{N}_{\mathcal{S}} \subset \mathcal{N}$ and $\sigma^*_{\mathcal{M}} \in \mathcal{N}_{\mathcal{S}}$.

Case II: $|K_0| \geq 2$.

By definition of K when $|K_0| \geq 2$, $\forall i \in I \backslash K$,

$$V^i(\omega_0, \sigma'_{\mathcal{M}}) = V^i(\omega_0, \sigma^*_{\mathcal{M}}). \tag{18}$$

By inequalities (16) and (18),

$$\begin{aligned} V^i(\omega_0, \sigma'_{\mathcal{M}}) &\geq V^i(\omega_0, \sigma^*_{\mathcal{M}}), \forall i \in I, \\ V^i(\omega_0, \sigma'_{\mathcal{M}}) &> V^i(\omega_0, \sigma^*_{\mathcal{M}}), \forall i \in K. \end{aligned}$$

Together with inequality (17),

$$\begin{aligned} V^i(\omega_0, \sigma'_{\mathcal{M}}) &\geq V^i(\omega_0, \sigma_{\mathcal{E}}), \forall i \in I, \\ V^i(\omega_0, \sigma'_{\mathcal{M}}) &> V^i(\omega_0, \sigma_{\mathcal{E}}), \forall i \in K. \end{aligned}$$

Therefore, $\sigma'_{\mathcal{M}}$ is a pure Nash equilibrium strategy. $\sigma'_{\mathcal{M}} \in \mathcal{N}$, and Pareto dominates $\sigma^*_{\mathcal{M}}$. It leads to a contradiction that $\sigma^*_{\mathcal{M}} \in \mathcal{N}_{\mathcal{S}}$.

Therefore, $\mathcal{N}_{\mathcal{S}} \not\subset \mathcal{N}_{\mathcal{P}}$ does not hold. We have $\mathcal{N}_{\mathcal{S}} \subset \mathcal{N}_{\mathcal{P}}$, and thus $\mathcal{N}_{\mathcal{P}} \neq \emptyset$.

Moreover, $\mathcal{N}_{\mathcal{P}} \subset \mathcal{N}$ by definition. Therefore, a pure Pareto optimal stationary Markov equilibrium exists, and it is a Nash equilibrium in the stationary Markov strategy set.

Q.E.D.

Proof of Proposition 2

Player i will compare its utility when lending to j and to j'. Either j or j' that invest into one risky project would fail if the realization of the asset is 0. The actually payment j or j' could pay back is 0. Thus the utility of player i when lending to j is the following

$$V^i = u^i = C^i - l + p^{k_j} r^M l - \alpha p^{k_j}(1 - p^{k_j})(r^M l)^2.$$

And utility of player i when lending to j' is:

$$V^i = u^i = C^i - l + p^{k_{j'}} r^N l - \alpha p^{k_{j'}}(1 - p^{k_{j'}})(r^N l)^2.$$

Claim that $p^{k_j} r^M < p^{k_{j'}} r^N$, and $p^{k_j}(1 - p^{k_j})(r^M)^2 < p^{k_{j'}}(1 - p^{k_{j'}})(r^N)^2$.

The first part is directly from assumption $r^M < \frac{p^{k_{j'}}}{p^{k_j}} r^N$. Together with $p^{k_{j'}} < p^{k_j}$, $p^{k_j}(1 - p^{k_j})(r^M)^2 < p^{k_{j'}}(1 - p^{k_{j'}})(r^N)^2$ could be got.

Q.E.D.

Proof of Proposition 3

The derivation is the same as Proposition 2.

Q.E.D.

Proof of Proposition 4

By Theorem 6 and 7 in Gong et al. (2015)[25], there exists a unique, finite, and disjoint collection basins of attractions $\{\mathbb{A}_1, \cdots, \mathbb{A}_h\}$. Notice that each basin is a recurrent set. And state not in any basin is transient.

What remains to be proved is the following statement. $\forall l \in \{1, \cdots, h\}$ and $\forall \omega^x = (C^x, F^x, s^x), \omega^y = (C^y, F^y, s^y) \in \mathbb{A}_l$, we have $F^x = F^y$.

[25] More previous discussion could be found in Page and Wooders (2009), and classical graph theory of network formation games, for example, Chapter 2 of Berge (2001). Moreover, decomposition theorem with respect transient states and recurrent states is described in Cox and Miller (1977).

Suppose $\exists l \in \{1, \cdots, h\}$, such that $\exists \omega^x = (C^x, F^x, s^x), \omega^y = (C^y, F^y, s^y) \in \mathbb{A}_l$, and $F^x \neq F^y$.

$$\therefore \exists i \in F^x, i \notin F^y, \text{ or } \exists i \in F^y, i \notin F^x.$$

Without loss of generality, assume $\exists i \in F^x$ and $i \notin F^y$.

Next, we construct two disjoint subset of $\mathbb{A}_l$ so that both sets are absorbing. Let $\mathbb{A}' = \{\omega = (C, F, s) \in \mathbb{A}_l | i \in F\}$ be the subset of $\mathbb{A}_l$ containing states where i defaults, $\omega^x \in \mathbb{A}'$. Thus, $\mathbb{A}_l \backslash \mathbb{A}'$ will be a nonempty subset of $\mathbb{A}_l$ containing all states that i does not default.

By assumption in section 2.2, default players stay in default set forever. Starting from any state in $\mathbb{A}'$, there is no chance to get to a state in $\mathbb{A}_l \backslash \mathbb{A}'$. I.e., $\mathbb{P}(\mathbb{A}' | \omega) = 1, \forall \omega \in \mathbb{A}'$. This violates $\mathbb{A}_l$ being a basin of attractions.

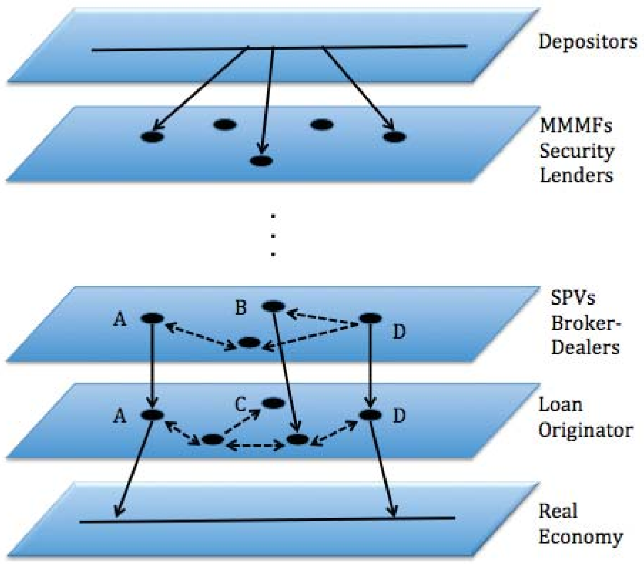
Therefore, there is no such an $\mathbb{A}_l$. In other words, all states in the same basin of attractions should have the same number of default players.

The existence of the singleton basin which contains the state where all players have zero net cash flow and all shadow banks default is easily derived from the assumptions of the law of motion.

Q.E.D.

B Multilayered Financial Networks

Figure 10: Multi-layered Network



The above figure illustrates the reason why financial network including shadow banks is a multilayered network. Connections within layer are draw by dashed lines and connections between layer are shown by solid lines. A formal definition of multi-layered network[26] is as the following.

Definition 5: Multi-layered Network
Let A be the finite set of all possible connections (arcs) between nodes. I is the finite set of all players. And $L = \{l_1, \cdots, l_M\}$ is the set of layers. $I_m \subset I$ denotes the set of nodes in layer m. Then, a multi-layered network, G, is a subset of $A \times (I \times L) \times (I \times L)$.

Page et al. (2005) give a definition of heterogenous directed networks, which also incorporates the above definition of multi-layered network. Thus, multi-layered network is a special case of general heterogeneous directed networks. Given $x = (i, l_m), i \in V_{l_m}$, $y = (i', l'_m), i' \in V_{l'_m}$, if there is an arc $a \in A$ from x to y, then $G(x, y) := a$. Notice that the domain of function G is not necessarily equal to $(I \times L) \times (I \times L)$. And both approaches are practical in a sense that they allow the domain to be different of the whole set $(I \times L) \times (I \times L)$. This feather turns out to be critical in analyzing multi-layered networks.

For example, in Figure 10, A, B, C, and D are 4 nodes in the set I. The layer "SPVs and Broker-Dealers" does not consists node C. Let layers from top to bottom be l_1 to

[26]See Kivelä et al. (2014).

l_5, and ignore the middle ones which we do not drawn. Thus, $L = \{l_1, \cdots, l_5\}$. Moreover, let dashed arrow be a_1, and solid arrow be a_2. Therefore, $G((D, l_3), (D, l_4)) = a_2$, and $G((D, l_3), (B, l_3)) = a_1$.

In the shadow bank system, broker-dealers vertically integrate their securitization businesses (from origination to funding), lending platforms, and asset management units. Pozsar et al. (2012) list 7 steps of these securitization, lending and investment processes. Each step could be viewed as a layer. Within layer, nodes conducts lending and investment activities with each other as well. An overall view of a shadow banking system is as Figure 10.

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